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Ore and Chvátal-type Degree Conditions for Bootstrap Percolation from Small Sets

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Abstract

Bootstrap percolation is a deterministic cellular automaton in which vertices of a graph G begin in one of two states, "dormant" or "active". Given a fixed positive integer r, a dormant vertex becomes active if at any stage it has at least r active neighbors, and it remains active for the duration of the process. Given an initial set of active vertices A, we say that G r- percolates (from A) if every vertex in G becomes active after some number of steps. Let m(G, r) denote the minimum size of a set A such that G r-percolates from A.

Bootstrap percolation has been studied in a number of settings, and has applications to both statistical physics and discrete epidemiology. Here, we are concerned with degree-based density conditions that ensure m(G, 2) = 2. In particular, we give an Ore-type degree sum result that states that if a graph G satisfies σ2(G) ≥ n − 2, then either m(G, 2) = 2 or G is in one of a small number of classes of exceptional graphs. (Here, σ2(G) = min{d(x)+ d(y) : xy ∈/ E(G)}.) We also give a Chvátal-type degree condition: If G is a graph with degree sequence d1 ≤ d2 ≤ · · · ≤ dn such that di ≥ i + 1 or dn−i ≥ n − i − 1 for all 1 ≤ i < n , then m(G, 2) = 2 or G falls into one of several specific exceptional classes of graphs. Both of these results are inspired by, and extend, an Ore-type result in [D. Freund, M. Poloczek, and D. Reichman, Contagious sets in dense graphs, to appear in European J. .]

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Bootstrap percolation has been studied in a number of settings, and has applications to both statistical physics and discrete epidemiology. Here, we are concerned with degree-based density conditions that ensure $m(G, 2) = 2$. In particular, we give an Ore-type degree sum result that states that if a graph G satisfies $\sigma_2(G) \geq n-2$, then either $m(G, 2) = 2$ or G is in one of a small number of classes of exceptional graphs. (Here, $\sigma_2(G) = \min\{d(x)+d(y) : xy \notin E(G)\}$.) We also give a Chvátal-type degree condition: If G is a graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ such that $d_i \geq i+1$ or $d_{n-i} \geq n-i-1$ for all $1 \leq i < \frac{n}{2}$, then $m(G, 2) = 2$ or G falls into one of several specific exceptional classes of graphs. Both of these results are inspired by, and extend, an Ore-type result in [D. Freund, M. Poloczek, and D. Reichman, Contagious sets in dense graphs, to appear in *European J. Combin.*]

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1 Introduction

Bootstrap percolation, also known as the *irreversible r-threshold process* [19, 30] or the *target set selection* is a deterministic cellular automaton first introduced by Chalupa, Leath, and Reich [14]. Vertices of a graph are in one of two states, "dormant" or "active." Given an integer r, a dormant vertex becomes active only if it is adjacent to at least r active vertices. Once a vertex is activated, it remains in that state for the remainder of the process.

More formally, consider a graph G and let A denote the initial set of active vertices. For a fixed $r \in \mathbb{N}$, the *r-neighbor bootstrap percolation* process on G occurs recursively by setting $A = A_0$ and for each time step $t \geq 0$,

$$A_t = A_{t-1} \cup \{v \in V(G) : |A_{t-1} \cap N(v)| \geq r\},$$

where $N(v)$ denotes the neighborhood of the vertex v. If all of the vertices of G eventually become active, regardless of order, then we say that A is *r-contagious* or that G *r-percolates from* A. Given G and r, let $m(G, r)$ denote the minimum size of an r-contagious set in G. (Observe that $m(G, r) \geq \min\{r, |V(G)|\}$.)

Originally, bootstrap percolation was studied on lattices by statistical physicists as a model of ferromagnetism [14], and it can also be viewed as a model of discrete epidemiology, wherein a virus or other contagion is being transmitted across a network (cf. [7, 30]). (In the latter context, each vertex is either "infected" or "uninfected".) Further applications include the spread of influence in social networks [16, 24] and market stability in finance [2].

Much attention has been devoted to examining percolation in a probabilistic setting, referred to in [7] as the *random disease problem*. In this setting, the initial activated set A is selected according to some probability distribution. The parameter of interest is then the probability that G r-percolates from A, and in particular determining the threshold probability p for which G almost surely does (or does not) r-percolate when vertices are placed in A independently with probability p. Results have been obtained in this setting for a number of families of graphs, including random regular graphs [8], the Erdős–Rényi random graph $G_{n,p}$ [20, 23], hypercubes [3], trees [6], and grids [1, 4, 5].

In addition, there has recently been interest in extremal problems concerning bootstrap percolation in various families of graphs [10, 28, 29].

The problem has also been studied from the point of view of computational complexity. For $r \geq 3$, determining $m(G, r)$ is NP-complete [19], and determining $m(G, 2)$ is NP-complete even for graphs with maximum degree 4 [13, 25]. Furthermore, it is computationally difficult to approximate $m(G, r)$ [16, 15]. Notice that $m(G, 1)$ is always equal to the number of connected components of G.

1.1 Degree-Based Results

In this paper, we are interested in degree-based density conditions that ensure that a graph G will percolate from a small set of initially activated vertices. Freund, Poloczek, and Reichman [21] showed that for each $r \geq 2$, if G has order n and $\delta(G) \geq \frac{r-1}{r}n$, then $m(G, r) = r$. Note that when $r = 2$, this is the same as Dirac's condition for hamiltonicity [18]. Recently, Gunderson [22]

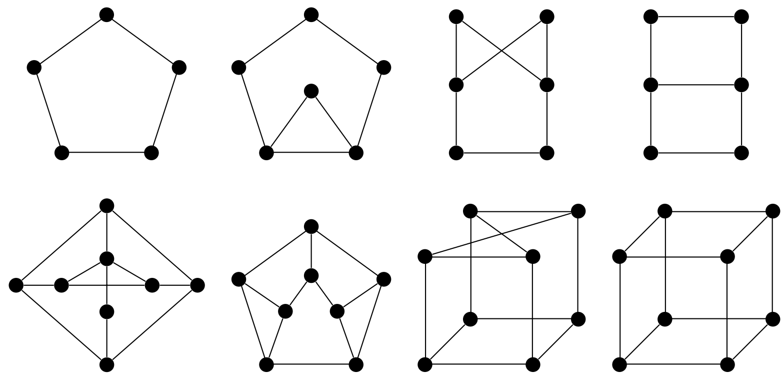

Figure 1: The family $\mathcal{X}$: Small exceptional graphs for Theorem 2.

showed that if $n \geq 30$ and $\delta(G) \geq \lfloor n/2 \rfloor + 1$, then $m(G,3) = 3$, and that for each $r \geq 4$, if n is sufficiently large and $\delta(G) \geq \lfloor n/2 \rfloor + r - 3$, then $m(G,r) = r$. Moreover, both bounds are sharp.

Let $\sigma_2(G)$ denote the minimum degree sum of a pair of nonadjacent vertices in a graph G. Ore [27] proved that every graph G of order $n \geq 3$ that satisfies $\sigma_2(G) \geq n$ is hamiltonian. Freund, Poloczek, and Reichman [21] also showed that Ore's condition is sufficient to ensure that a graph 2-percolates from the smallest possible initially activated set.

Theorem 1 ([21])**.** *Let $n \geq 2$. If G is a graph of order n and $\sigma_2(G) \geq n$, then $m(G,2) = 2$.*

Note that hamiltonicity alone is not sufficient to conclude that a graph G satisfies $m(G,2) = 2$, as $m(C_n, 2) = \lceil n/2 \rceil$, which tends to infinity with n. Rather, Theorem 1 is part of a diverse collection of results that demonstrate that many sufficient density conditions for hamiltonicity imply a much richer structure that allows for stronger conclusions (cf. [11, 12]).

In this paper, we improve Theorem 1 in several ways. First, we characterize graphs of order n with $\sigma_2 \geq n-2$ and $m(G,2) > 2$. These will consist of four infinite families of graphs $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and a finite set of graphs $\mathcal{X}$. The graphs in $\mathcal{X}$ are depicted in Figure 1.

The class $\mathcal{G}_0$ consists of all graphs which are unions of two disjoint non-empty cliques X, Y. Note that X and Y can be of different sizes. Graphs in $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are formed from $\mathcal{G}_0$ by selecting $\{x, x'\} \subseteq X$, $\{y, y'\} \subseteq Y$, adding the edges xy, $x'y'$, and deleting the edges xx' and yy' if they exist (see Figure 2). For simplicity, we have distinguished the cases where $x = x'$ and $y = y'$ ($\mathcal{G}_1$), $x \neq x'$ and $y \neq y'$ ($\mathcal{G}_2$) and $x = x'$ and $y \neq y'$ ($\mathcal{G}_3$). It is easy to see that any graph $G \in \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ containing at least one vertex in each of X and Y that is not adjacent to any vertex in the other set has $\sigma_2(G) = |V(G)| - 2$ and $m(G,2) > 2$.

Theorem 2. *Let G be a graph of order $n \geq 2$ such that G is not in $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ or $\mathcal{X}$. If $\sigma_2(G) \geq n-2$, then $m(G,2) = 2$.*

In particular, Theorem 2 implies that C_5 is the only graph G with $\sigma_2(G) = |V(G)| - 1$ and $m(G,2) > 2$.

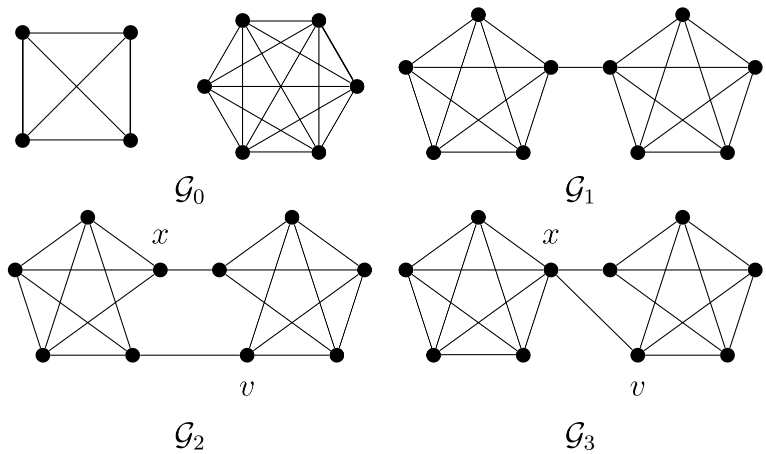

Figure 2: Examples of graphs in $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ for $n = 10$. The labeled vertices in the third and fourth graphs refer to the proof of Theorem 2.

Second, we prove a degree sequence condition for $m(G, 2) = 2$. Let G be a graph with degree sequence $d_1 \leq \cdots \leq d_n$. We say that G satisfies *Chvátal's condition* if

$$d_i \geq i + 1 \quad \text{or} \quad d_{n-i} \geq n - i, \quad \forall i, 1 \leq i < \tfrac{n}{2}. \tag{1.1}$$

In [17], Chvátal proved that a graph G of order $n \geq 3$ that satisfies Chvátal's condition is hamiltonian.

Here, we show that, with only a few exceptions, a slightly weaker Chvátal-type condition implies that $m(G, 2) = 2$. We say that a graph G satisfies the *weak Chvátal condition* if

$$d_i \geq i + 1 \quad \text{or} \quad d_{n-i} \geq n - i - 1, \quad \forall i, 1 \leq i < \tfrac{n}{2}, \tag{1.2}$$

and prove the following. We denote the path on k vertices by P_k and the cycle on k vertices by C_k.

Theorem 3. *If G is a graph with degree sequence $d_1 \leq \cdots \leq d_n$ that satisfies the weak Chvátal condition* (1.2)*, then either $m(G, 2) = 2$ or one of the following holds:*

- *G is disconnected,*
- *G contains exactly two vertices of degree one and $G \notin \{P_2, P_3\}$, or*
- *G is C_5.*

Note that the ordinary Chvátal condition (1.1) rules out the last three cases in Theorem 3.

Corollary 4. *If G is a graph with degree sequence $d_1 \leq \cdots \leq d_n$ that satisfies Chvátal's condition* (1.1)*, then $m(G, 2) = 2$.*

Much as Chvátal's Theorem implies Ore's Theorem for hamiltonicity, each of Theorems 2 and 3 and Corollary 4 implies Theorem 1.

1.2 Notation

Let G be a graph, let $U \subseteq V(G)$ and let $v \in V(G)$. We denote by $G[U]$ the subgraph of G induced by U. The notation $\Delta(G)$ means the maximum degree of G and $N(v)$ is theset of neighbors of v. We denote by $N_U(v)$ the set of neighbors in U, that is $N(v) \cap U$. The notation $d(v)$ means the degree of v and $d_U(v)$ is $|N_U(v)|$.

2 Proof of Theorem 2

Proof of Theorem 2. Let G be a graph of order n with $\sigma_2(G) \geq n-2$ that is not in one of the exceptional classes $\mathcal{X}$, $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$, or $\mathcal{G}_3$. Throughout the proof, amongst all subsets of $V(G)$ that can be activated from a starting set of two vertices, let I (for "infected") have maximum size, and let $U = V(G) \setminus I$ denote the set of vertices that remain dormant from this starting set. We repeatedly use the following observation that follows from the maximality of I.

Observation 5. *Each vertex in U has at most one neighbor in I.*

Notice that our assumption on $\sigma_2(G)$ implies that $\Delta(G) \geq (n-2)/2$.

For $n \leq 11$, Nauty [26] was utilized to generate all graphs with $m(G, 2) > 2$, which is precisely the set $\mathcal{X}$. The program we used is available at https://arxiv.org/abs/1610.04499. Thus, we may assume that $n \geq 12$ as we proceed. Further, if G is disconnected, the degree sum condition guarantees that G has exactly two complete components and so $G \in \mathcal{G}_0$, a contradiction. We may therefore assume that G is connected. The following sequence of claims establishes important facts about the size and structure of I and U.

Claim 6. $|I| > \frac{n}{2}$ *and $G[U]$ is complete.*

Proof. First we show that $|I| \geq 4$. Suppose for a contradiction that $|I| < 4$. If G contains a triangle T, then since G is connected some vertex $y \in V(T)$ has a neighbor y' in $G - T$. So, if we initially activate y' and some $x \in V(T) \setminus \{y\}$, then at least 4 vertices are activated.

If G contains a C_4, we can activate the entire cycle starting with either pair of nonadjacent vertices. Hence we may suppose that G contains neither a triangle nor C_4.

Let w be a vertex with $d(w) = \Delta(G) \geq (n-2)/2 \geq 4$. As G is triangle-free, $N(w)$ is independent. Let x and y be distinct neighbors of w. As G contains no C_4, we have $N(x) \cap N(y) = \{w\}$, which means that $d(x) + d(y) \leq n-3$, a contradiction. So, we may assume that $|I| \geq 4$.

Next we establish that $|I| \geq |U|$. It suffices to show that $m(G[U], 2) = 2$, which would imply that $|U| \leq |I|$ since $|I|$ is maximum among all sets activated by two vertices. To that end, let u and v be nonadjacent vertices in U and recall that every vertex in U has at most one neighbor in I. Consequently, as $|I| \geq 4$,

$$d_U(u) + d_U(v) \geq d(u) - 1 + d(v) - 1 \geq n - 4 \geq |U|.$$

Thus, $m(G[U], 2) = 2$ by Theorem 1, so $|U| \leq |I|$. Therefore, the first condition $|I| > \frac{n}{2}$ holds unless $|U| = |I| = \frac{n}{2}$. We will deal with this case after establishing that $G[U]$ is complete.

Suppose $G[U]$ is not a complete graph, and let u and v be nonadjacent vertices in U. Then, as $|U| \leq \frac{n}{2}$,

$$d(u) + d(v) \leq 2(|U| - 1) \leq n - 2,$$

which is a contradiction unless equality holds. If equality holds, then U induces a complete graph on exactly $\frac{n}{2}$ vertices minus a matching M, where every vertex of M has a neighbor in I. Notice that in this case, $G[U]$ percolates from any choice of two vertices in U (since $\frac{n}{2} \geq 5$). Activating two neighbors of a vertex of M – one in I and one in U – results in at least $|U| + 1$ activated vertices, a contradiction to $|I| = \frac{n}{2}$ being maximum. Therefore, $G[U]$ must be a complete graph.

We finally return to the case where $|U| = |I| = \frac{n}{2}$. Let $v \in I$ have a neighbor u in U. For any z in $U \setminus \{u\}$, initially activating $\{v, z\}$ leads to (at least) the activation of $U \cup \{v\}$, contradicting the maximality of $|I|$ and establishing Claim 6. □

Partition I into sets I_0 and I_1, where I_1 is the set of vertices of I with at least one neighbor in U, so that vertices in I_0 have no neighbors in U. Since $|I| > |U|$, and no vertex in U has more than one neighbor in I, there exists a vertex $w \in I_0$. Let $u \in U$ and observe that

$$n - 2 \leq d(w) + d(u) \leq |I| - 1 + |U| = n - 1. \tag{2.1}$$

The bound on $d(w)$ in (2.1) has the following useful consequences.

Observation 7. *Each vertex in I_0 has at most one non-neighbor in I. Furthermore, if any three vertices in I are activated, then all of I_0 will be activated in the following step of the percolation.*

Claim 8. *Every vertex in U has exactly one neighbor in I.*

Proof. By Observation 5, it suffices to show that each vertex in U has at least one neighbor in I. Suppose otherwise, so that there exists $z \in U$ with no neighbors in I, and therefore there are at least two vertices w_1 and w_2 in I_0. Then

$$n - 2 \leq d(w_i) + d(z) \leq |I| - 1 + |U| - 1 = n - 2$$

for $i \in \{1, 2\}$. Hence $d(w_1) = d(w_2) = |I| - 1$ and w_1, w_2 are adjacent to all vertices of I. Let $v \in I$ and $u \in U$ be adjacent vertices. If we initially activate $\{w_1, u\}$, this in turn would activate at least $I \cup \{u\}$, contradicting the maximality of $|I|$. Consequently, every vertex in U has a neighbor in I, establishing Claim 8. □

Claim 9. $|I_0| \geq 2$.

Proof. As observed above, I_0 is non-empty, so suppose for a contradiction that $|I_0| = 1$. It follows from Claims 6 and 8 that

$$|I| > |U| \geq |I_1| = |I| - 1,$$

which is a contradiction unless $|U| = |I| - 1$.

Because $|U| = |I_1|$, Claim 8 implies that there is a perfect matching between U and I_1. Also, I_1 cannot be an independent set, or else for all $a, b \in I_1$, $d(a) + d(b) \leq 4 < n - 2$, a contradiction. So, let x_1 and x_2 be adjacent vertices of I_1 and let u_1 and u_2 be their respective neighbors in U. If we initially activate $\{u_1, x_2\}$, then x_1 will also become active. Furthermore, by Claim 6, U is a clique, so all of U will become active, for a total of at least $|U| + 2 = |I| + 1$ active vertices. This contradiction completes the proof of Claim 9. □

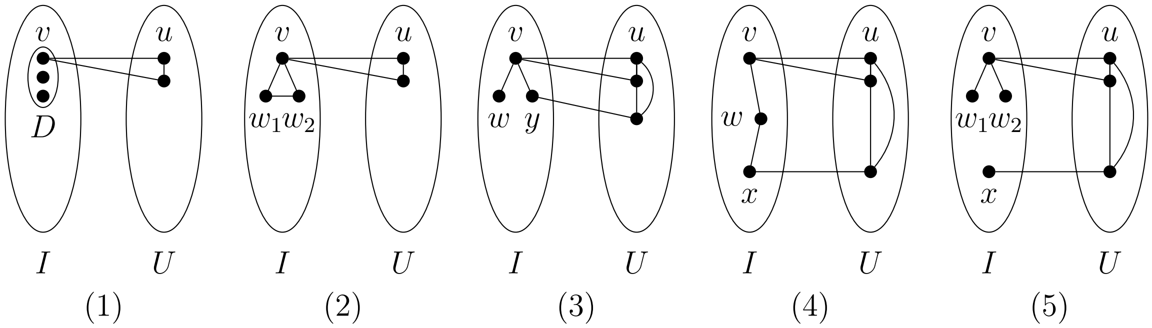

Figure 3: The situation in Claim 10.

Claim 10. *Let $v \in I_1$ have at least two neighbors in U and let u be one such neighbor. Also, let D be a subset of I containing at least three vertices, including v, and let $x \in I_1 \setminus \{v\}$. The following hold:*

(1) There is no set of size 2 that activates $U \cup D$;

(2) $N_I(v)$ is an independent set;

(3) If there is a vertex y in $N_I(v) \cap I_1$, then y is the only neighbor of v in I;

(4) v and x have no common neighbor;

(5) $|N_I(v)| = 1$, $I_1 = \{v, x\}$, x has exactly one neighbor in U and v is adjacent to every other vertex of U.

Proof. Before we begin, it is useful to note that if u and v are activated, then so too will be all of U, as $G[U]$ is complete. Also, the proofs of *(1)–(5)* are illustrated in Figure 3.

(1): Suppose that $U \cup D$ can be activated starting from two vertices a and b. By Observation 7, all of I_0 is activated. Thus, the set of vertices activated starting with $\{a, b\}$ contains $U \cup I_0 \cup \{v\}$. Since $|U| \geq |I_1|$ by Claim 8, we obtain a contradiction to the maximality of $|I| = |I_0 \cup I_1|$.

(2): Suppose otherwise, and let w_1 and w_2 be adjacent vertices in $N_I(v)$. Initially activate $\{w_1, u\}$. In the first three steps, all of $\{u, v, w_1, w_2\} \cup U$ is activated, contradicting *(1)* with $D = \{v, w_1, w_2\}$.

(3): Assume otherwise, that v has two neighbors y and w in I, where y has a neighbor in U. Initially activating $\{w, u\}$ activates v in the first step, and U in the step that follows. Consequently, y is activated, contradicting *(1)* with $D = \{v, w, y\}$.

(4): Let x be in $I_1 \setminus \{v\}$, as given, and assume that w is a common neighbor of v and x. Initially activating $\{w, u\}$ then activates v, x and the entirety of U in four iterations, again contradicting *(1)*.

(5): Suppose first that w_1 and w_2 are distinct neighbors of v in I. By *(2)*, *(3)* and *(4)*, they are nonadjacent, they have no neighbors in U, and neither is adjacent to x. Furthermore, because $w_1, w_2 \notin I_1$, both vertices are distinct from x. Then $d(w_1) + d(u) \leq |I| - 3 + |U| = n - 3$, a contradiction. Hence $|N_I(v)| \leq 1$.

By Claim 9, $|I_0| \geq 2$, so there exists $w \in I_0 \setminus N(v)$. By applying the assumption $\sigma_2(G) \geq n-2$ to the nonadjacent vertices v and w, we obtain

$$|I| - 2 + d(v) \geq d(w) + d(v) \geq n - 2 = |U| + |I| - 2.$$

Hence $|U| \leq d(v)$. On the other hand $d(v) \leq |U \setminus N(x)| + 1$, so $d(v) = |U|$. It follows that v is adjacent to all but one vertex of U, so that $I_1 = \{v, x\}$ and x has exactly one neighbor in U. This completes the proof of Claim 10. □

Let v and x be as in the statement of Claim 10. By Claims 6 and 10*(5)*, U is a clique, x has exactly one neighbor in U, and v is adjacent to every other vertex of U.

If v and x are not adjacent, let z be the (only) neighbor of v in I. By Claim 10*(5)*, $z \in I_0$, and by Claim 10*(4)*, z is not adjacent to x. It follows from Observation 7 that all other pairs of vertices in $I_0 \cup \{x\}$ are adjacent (cf. Figure 2). Thus, $G \in \mathcal{G}_2$.

If v and x are adjacent, then Claim 10*(5)* and Observation 7 imply that $I_0 \cup \{x\}$ is a clique. It follows that $G \in \mathcal{G}_3$.

In either case, we have a contradiction, the final one needed to complete the proof of Theorem 2. □

3 Proof of Theorem 3

Proof of Theorem 3. For $n \leq 12$, Theorem 3 was verified using Nauty [26], so throughout the proof, we may assume that $n \geq 13$. The program we used is available at https://arxiv.org/abs/1610.04499. Suppose then that G is a graph of order n that satisfies the weak Chvátal condition (1.2). Further, by way of contradiction, suppose that $m(G, 2) > 2$ and that G is connected and has at most one vertex of degree 1. Also, let $V(G) = \{v_1, v_2, \ldots, v_n\}$, where $d(v_i) = d_i$ and $d_i \leq d_j$ whenever $i \leq j$, and let L be the set of vertices of degree at least $\frac{n-1}{2}$.

If G satisfies the full Chvátal condition (1.1), one of the following must hold:

- $d(v) \geq \frac{n}{2}$ for all $v \in L$ and $|L| > \frac{n}{2}$ or
- $d(v) > \frac{n}{2}$ for all $v \in L$ and $|L| \geq \frac{n}{2}$.

As one would expect, the weak Chvátal condition (1.2) results in slightly weaker conclusions. However, (1.2) still implies that $|L| \geq n/2$. Also, notice that if n is even, then $d(v) \geq \frac{n}{2}$ for all $v \in L$, and if n is odd, then $|L| \geq \frac{n+1}{2}$.

Let I have maximum size among sets that can be actived by starting from two active vertices and that satisfy $I \cap L \neq \emptyset$. Furthermore, let $U = V \setminus I$, $L_I = L \cap I$ and $L_U = L \cap U$. Notice that Observation 5, which says that each vertex in U has at most one neighbor in I, continues to hold.

Claim 11. $|I| \leq \frac{n+1}{2}$.

Proof. Suppose, towards a contradiction, that $|I| > \frac{n+1}{2}$, so that $|U| < \frac{n-1}{2}$. Let $u \in U$ and note that since u has at most one neighbor in I, $d(u) \leq |U|$, which implies that $d(v_{|U|}) \leq |U|$. The weak Chvátal condition (1.2) then implies that $d(v_{n-|U|}) \geq n - |U| - 1$. Let $X = \{v_{n-|U|}, \ldots, v_n\}$, so that $d(v) \geq n - |U| - 1 = |I| - 1 > |U|$ for all $v \in X$. Note that $|X| = |U| + 1$.

As $|U| \leq |I| - 2$, we have $n - |U| - 1 > |U|$, which means that $X \subseteq I$. Suppose that there are vertices v and w in X with no neighbors in U, implying that they have degree equal to $|I| - 1$ in I. Because G is connected, there exists $u \in U$ with a neighbor in I. Initially activating $\{v, u\}$ then activates w in the second round and consequently activates $I \cup \{u\}$, which contradicts the maximality of $|I|$. Thus there is at most one vertex $v \in X$ with no neighbors in U. Indeed, since $|X| = |U| + 1$, v is the unique member of X that has no neighbors in U and, by Observation 5, every $y \in X \setminus \{v\}$ has exactly one neighbor in U. Hence, there is a perfect matching between U and $X \setminus \{v\}$.

Let $w \in X \setminus \{v\}$ and let $u \in U$ be adjacent to w. Initially activating $\{v, u\}$ activates w in the first round, followed by all vertices in $I \cap N(w)$, since they are also adjacent to v. By the maximality of $|I|$ and the fact that $|N(w) \cap I| \geq |I| - 2$, there is exactly one vertex $z \in I \setminus \{w\}$ that is not adjacent to w. Moreover, z must be adjacent only to v in I, otherwise $I \cup \{u\}$ would be activated.

If z had a neighbor in U, then $z \in X$. Hence, $2 \geq d(z) \geq |I| - 1$, which implies that $|I| \leq 3$ and thus $n \leq 5$, a contradiction. Hence, z is a vertex of degree one. Also, because $z \notin X$, all of X becomes active. If u has a neighbor u' in U, then u' has a neighbor in X, and hence also becomes active since all of X is activated. This would contradict the maximality of $|I|$. Hence u is also a vertex of degree one, a contradiction to the assumption that G has at most one vertex of degree one. This concludes the proof of Claim 11. □

Claim 12. $|I| < \frac{n+1}{2}$.

Proof. Assume otherwise. By Claim 11, n is odd, $|I| = \frac{n+1}{2}$ and $|U| = \frac{n-1}{2}$.

First we show that $G[I]$ does not contain two universal vertices. Suppose for a contradiction that x and y are two vertices of I, each with $|I| - 1$ neighbors in I. By the connectivity of G, there exists an edge zu, where $z \in I$ and $u \in U$. By symmetry, assume $x \neq z$. Initially activating $\{x, u\}$ activates z in the first step. After the second step, y is activated. This activates $I \cup \{u\}$, contradicting the maximality of $|I|$.

For $i = (n-3)/2$, the weak Chvátal condition (1.2) states that either $d_i \geq i+1 = (n-3)/2+1 = (n-1)/2$ or $d_{n-i} \geq n - i - 1 = n - (n-3)/2 - 1 = (n+1)/2$. This implies there exists L' such that

(A) $d(v) \geq \frac{n-1}{2}$ for all $v \in L'$ and $|L'| \geq \frac{n-1}{2} + 3$ or

(B) $d(v) \geq \frac{n+1}{2}$ for all $v \in L'$ and $|L'| \geq \frac{n-1}{2}$.

If (B) holds, Observation 5 implies that $L' \subset I$. Moreover, every vertex in L' must have at least one neighbor in U. Since $|U| = |L'|$, every vertex in $|L'|$ has exactly one neighbor in U and thus every vertex in L' is universal in $G[I]$. Since $|L'| \geq 2$, this contradicts that $G[I]$ has at most one universal vertex.

Now we assume that (A) holds. Since $|I| = \frac{n+1}{2}$ and $|L'| \geq \frac{n+1}{2} + 2$, there are at least two vertices, call them u and v, in $L' \cap U$. Denote the neighbors of u and v in I by u_I and v_I, respectively; we first show that every vertex in I has at most one neighbor in U. Suppose otherwise, so that there exists $x \in I$ with at least two distinct neighbors a and b in U. If $u_I \neq x$, then initially activating $\{u_I, v\}$ eventually activates $U \cup \{u_I, x\}$, contradicting the maximality of $|I|$. Hence we may assume that $u_I = v_I = x$. If x has a neighbor $y \in I$, then initially activating $\{y, v\}$ eventually activates $U \cup \{y, x\}$, again contradicting the maximality of $|I|$. Consequently, we may then assume that x has no neighbors in I; this implies, however, that x was in the initially activated set and therefore $|I| = 2$ and thus $n \leq 3$, a contradiction.

Consider then initially activating $\{u_I, v\}$, which activates $U \cup \{u_I\}$. As $|U \cup \{u_I\}| = |I|$, $U \cup \{u_I\}$ has the same properties as I. In particular, this implies that u_I has at most one neighbor in I. Therefore, $d(u_I) = 2$, and by symmetry, $d(v_I) = 2$.

Since every vertex in I has at most one neighbor in U, $\Delta(G) \leq \frac{n+1}{2}$. The weak Chvátal condition (1.2) therefore implies that the initial part of the degree sequence of G term wise dominates $2, 3, 4, \ldots$. In particular, there is at most one vertex of degree at most 2 in G, contradicting the existence of u_I and v_I. $\square$

Claim 13. $|I| < \frac{n}{2}$.

Proof. Assume otherwise. By Claim 12, $|I| = |U| = \frac{n}{2}$, which implies that n is even. If $u \in L_U$, then $d(u) \geq \frac{n}{2}$. So, by Observation 5, u is adjacent to all other vertices in U and has exactly one neighbor in I. If $|L_U| \geq 2$, let $u, x \in L_U$, let $v \in N_I(u)$ and initially activate $\{v, x\}$. It follows that all of U is activated by the end of the second round. Because this activates $U \cup \{v\}$, it contradicts the maximality of I.

Suppose then, that $|L_U| \leq 1$, so that $|L_I| \geq \frac{n}{2} - 1$. As n is even, every vertex in L has degree at least $\frac{n}{2}$. Hence every vertex in L_I has at least one neighbor in U. Since the number of edges between I and U is at most $\frac{n}{2}$, there is at most one vertex in L_I with more than one neighbor in U and all the other vertices of L_I are complete to I.

Because $|L_I| - 1 \geq \frac{n}{2} - 2 \geq 2$, there are vertices v and w in I such that each vertex is adjacent to all of I except for itself and such that v has a neighbor u in U. Activating u and w results in the activation of $I \cup \{u\}$, contradicting the maximality of I. This concludes the proof of Claim 13. $\square$

Let

$$p = \frac{n}{2} - |I| \geq \frac{1}{2}. \tag{3.1}$$

Notice that p is an integer if n is even.

Claim 14. $|L_U| \geq 3$.

Proof. If $p \geq 3$, the claim follows from $|L_I| \leq |I| \leq \frac{n}{2} - 3$ and $|L_I| + |L_U| \geq \frac{n}{2}$. So, we let $p \leq 2$ and assume for a contradiction that $|L_U| \leq 2$. We distinguish the following three cases based on the parity of n and the value of p.

Case 1: n is even.

Since n is even, (3.1) implies that $p \in \{1,2\}$. Also, if $v \in L_I$, then $d(v) \geq n/2$. It follows from (3.1) that $d_U(v) \geq \frac{n}{2} - (|I|-1) = p+1$. Furthermore, every vertex in U has at most one neighbor in I, so $|U| \geq |L_I|(p+1)$. Since $|L_U| \leq 2$, we get $|L_I| \geq |L| - 2 \geq \frac{n}{2} - 2$. Therefore,

$$\frac{n}{2} - 2 \leq |L_I| \leq \frac{|U|}{p+1} = \frac{n/2+p}{p+1}.$$

However, as $p \in \{1,2\}$ and thus $n \leq 10$, this is a contradiction.

Case 2: n is odd and $p > \frac{1}{2}$.

Since n is odd and $p > \frac{1}{2}$, for every $v \in L_I$, we have $d_U(v) \geq p + \frac{1}{2}$. Every vertex in U has at most one neighbor in I, so $|U| \geq |L_I|(p+1/2)$. Since $|L| \geq \frac{n+1}{2}$, we obtain $|L_I| \geq \frac{n-3}{2}$. Therefore,

$$\frac{n-3}{2} \leq |L_I| \leq \frac{|U|}{p+1/2} = \frac{n/2+p}{p+1/2}.$$

However, as $p \in \{\frac{3}{2}, \frac{5}{2}\}$ and $n > 9$, this inequality fails, a contradiction.

Case 3: n is odd and $p = \frac{1}{2}$.

In this case, $|I| = \frac{n-1}{2}$ and, by assumption, $|L_I| \geq |L| - 2 \geq \frac{n-1}{2} - 1$.

Every vertex in L_I has degree at least $\frac{n-1}{2}$, and therefore must have a neighbor in U. Since every vertex in U has at most one neighbor in I, there are at most $\frac{n+1}{2}$ edges between L_I and U. Hence there are at most two vertices in L_I with more than one neighbor in U. If there are two such vertices in L_I, then both have exactly two neighbors in U. If there is exactly one such vertex in L_I, then it has at most three neighbors in U.

Consequently, I is a complete graph of order at least 5 except for either a single edge or two incident edges. As each vertex in L_I has at least one neighbor in U, it is straightforward to select two vertices that, when initially activated, activate all of I and at least one vertex in U, a contradiction. This concludes the proof of Claim 14. $\square$

Claim 15. *L_U is a clique.*

Proof. Assume otherwise, and let u and v be nonadjacent vertices in L_U. We claim that initially activating u and v generates a contradiction to the maximality of I.

As every vertex in U has at most one neighbor in I, both u and v have at least $\frac{n-1}{2} - 1$ neighbors among the other $\frac{n}{2} + p - 2$ vertices of U. Let r be the number of vertices in U that are common neighbors of u and v. By counting edges from u and v to the remainder of U, we obtain

$$2\left(\frac{n-1}{2} - 1\right) \leq d_U(u) + d_U(v) \leq 2r + \big(|U \setminus \{u,v\}| - r\big) \leq 2r + \left(\frac{n}{2} + p - 2 - r\right),$$

which implies that

$$\frac{n}{2} - p - 1 \leq r.$$

Together with $\{u,v\}$, a total of $r + 2 \geq \frac{n}{2} - p + 1 = |I| + 1$ vertices become activated by the second round, contradicting the maximality of $|I|$ and proving Claim 15. $\square$

Claim 16. *Vertices in L_U have no neighbors in I.*

Proof. Suppose for a contradiction that $v \in I$ and $u \in L_U$ are adjacent. Let w and z be two vertices in L_U aside from u, which exist by Claim 14. Initially activate the set $\{v, w\}$. In the first round, u is activated, and in the second round, the remainder of L_U, including z, is activated.

By counting edges from u, w, and z to the remainder of U, and letting r denote the number of vertices adjacent to at least two of u, w or z, we get

$$3\left(\frac{n-1}{2}-3\right) \leq d_U(u)-2+d_U(z)-2+d_U(w)-2 \leq 3r+\big(|U\backslash\{u,v,w\}|-r\big) \leq 3r+\Big(\frac{n}{2}+p-3-r\Big),$$

which implies that

$$\frac{n}{2}-\frac{p}{2}-\frac{15}{4} \leq r.$$

When we include u, v, w, and z, we see that there are at least $\frac{n}{2}-\frac{p}{2}+\frac{1}{4} \geq \frac{n}{2}$ activated vertices. This contradicts the maximality of $|I| = \frac{n}{2} - p$ and concludes the proof of Claim 16. $\square$

To finish the proof, let u, w, $z \in L_U$ and initially activate the set $\{z, w\}$, so that u is activated in the first round. Now we count edges from $\{u, w, z\}$ to $U \setminus \{u, w, z\}$, again letting r denote the number of vertices in $U \setminus \{u, w, z\}$ that are adjacent to at least two vertices in $\{u, w, z\}$. By Claim 16,

$$3\left(\frac{n-1}{2}-2\right) \leq d_U(u)-1+d_U(z)-1+d_U(w)-1 \leq 3r+\big(|U\backslash\{u,v,w\}|-r\big) \leq 3r+\Big(\frac{n}{2}+p-3-r\Big).$$

Hence,

$$\frac{n}{2}-\frac{p}{2}-\frac{11}{4} \leq r.$$

Together with u, w, and z, we get at least $\frac{n}{2}-\frac{p}{2}+\frac{1}{4} \geq \frac{n}{2} > |I|$ activated vertices, the final contradiction to the maximality of $|I|$ necessary to complete the proof of Theorem 3. $\square$

Conclusion

Theorem 3 gives a sharp degree condition that ensures that a graph G satisfies $m(G, 2) = 2$, in that it provides a class of graphs that demonstrates the sharpness of the weak Chvátal condition for this property. However, Chvátal-type conditions are often shown to be best possible in a different manner, which gives rise to a perhaps challenging open problem related to the work in this paper.

Let $S = (d_1, \ldots, d_n)$ and $S' = (d'_1, \ldots, d'_n)$ be sequences of real numbers. We say that S *majorizes* S', and write $S \succeq S'$, if $d_i \geq d'_i$ for every i. A sufficient degree condition $\mathcal{C}$ for a graph property $\mathcal{P}$ is *monotone best possible* if whenever $\mathcal{C}$ does not imply that every realization of a degree sequence π has property $\mathcal{P}$, there is some graphic sequence $\pi' \succeq \pi$ such π' has at least one realization without property $\mathcal{P}$. Note that it is possible that $\pi' = \pi$.

The Chvátal condition is a monotone best possible degree condition for hamiltonicity [17], and [9] is a thorough survey of monotone best possible degree criteria for a number of graph properties. However, it is easy to show that it is not the case that either the Chvátal condition (1.1) or the weak Chvátal condition (1.2) is monotone best possible for the property $m(G,2)=2$.

To see that the Chvátal condition is not monotone best possible, consider the graphic sequence

$$\pi = (i^i, (n-i-1)^{n-2i}, (n-1)^i),$$

where $2 \leq i < \frac{n}{2}$ and the exponents represent the multiplicities of the terms of π. The unique realization of π is $K_i \vee (\overline{K_i} \cup K_{n-2i})$. However, any sequence π' such that $\pi' \succeq \pi$ must have at least two vertices of degree $n-1$, implying every realization G of π' has $m(G,2)=2$. If n is even, the sequence

$$\pi = (i^i, (n-i-2)^{n-2i}, (n-1)^i),$$

suffices to show that the weak Chvátal condition is also not monotone best possible for the property $m(G,2)=2$.

This gives rise to the following problem:

Problem 1. Determine a monotone best possible degree condition for the property "$m(G,2)=2$".

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