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Abstract

For the normalized Laplacian matrix it is possible for graphs with differing number of edges to have the same spectrum. This leads to the potential for there to be a tree and a non-tree which share the same spectrum and a well-known example of this are star graphs with other complete bipartite graphs. Previous to this work, this was *the* known infinite family with this property. We construct more families of graphs with this property.

1 Introduction

Spectral graph theory studies the relationship between the structure of the graph and the spectrum of matrices associated with the graph. Some of the most actively studied matrices are the adjacency matrix, distance matrix, Laplacian matrix, and the normalized Laplacian matrix. Two graphs G and H are said to be cospectral with respect to a given matrix if the matrices for G and H share the same spectrum (eigenvalues) including multiplicity.

The goal of this paper is to produce graphs which are cospectral for the normalized Laplacian matrix (defined below) but not cospectral for other matrices such as the adjacency or Laplacian. It is known that for the adjacency matrix and the Laplacian matrix that the number of edges can be determined. As a consequence, the only connected graphs which can be cospectral with a tree for the adjacency or Laplacian is another tree. However, for the normalized Laplacian it is possible for a tree to be cospectral with a connected non-tree graph.

As an example, all complete bipartite graphs on n vertices have the same spectrum, namely, $\{0, 1^{(n-2)}, 2\}$. A star $K_{1,n-1}$ is a complete bipartite graph and also a tree. In addition, the star is cospectral with all other complete bipartite graphs, most of which are not trees. An example is shown in Figure 1 of two cospectral graphs, one a tree and one not.

Prior to this work, this is the only infinite family of examples of a tree which is cospectral with a non-tree. In Figure 2, we display the trees that are *not* stars that are cospectral with non-trees for 7, 9, and 10 vertices (and these are all such possible examples on at most 10 vertices). The graphs shown in the figure are highly suggestive in that in many cases we have many vertices that have the same sets of neighbors (these are called twins). We will use facts known about twins and collapsing to smaller graphs to introduce new infinite families and tools for further exploration.

Figure 1: Examples of complete bipartite graphs.

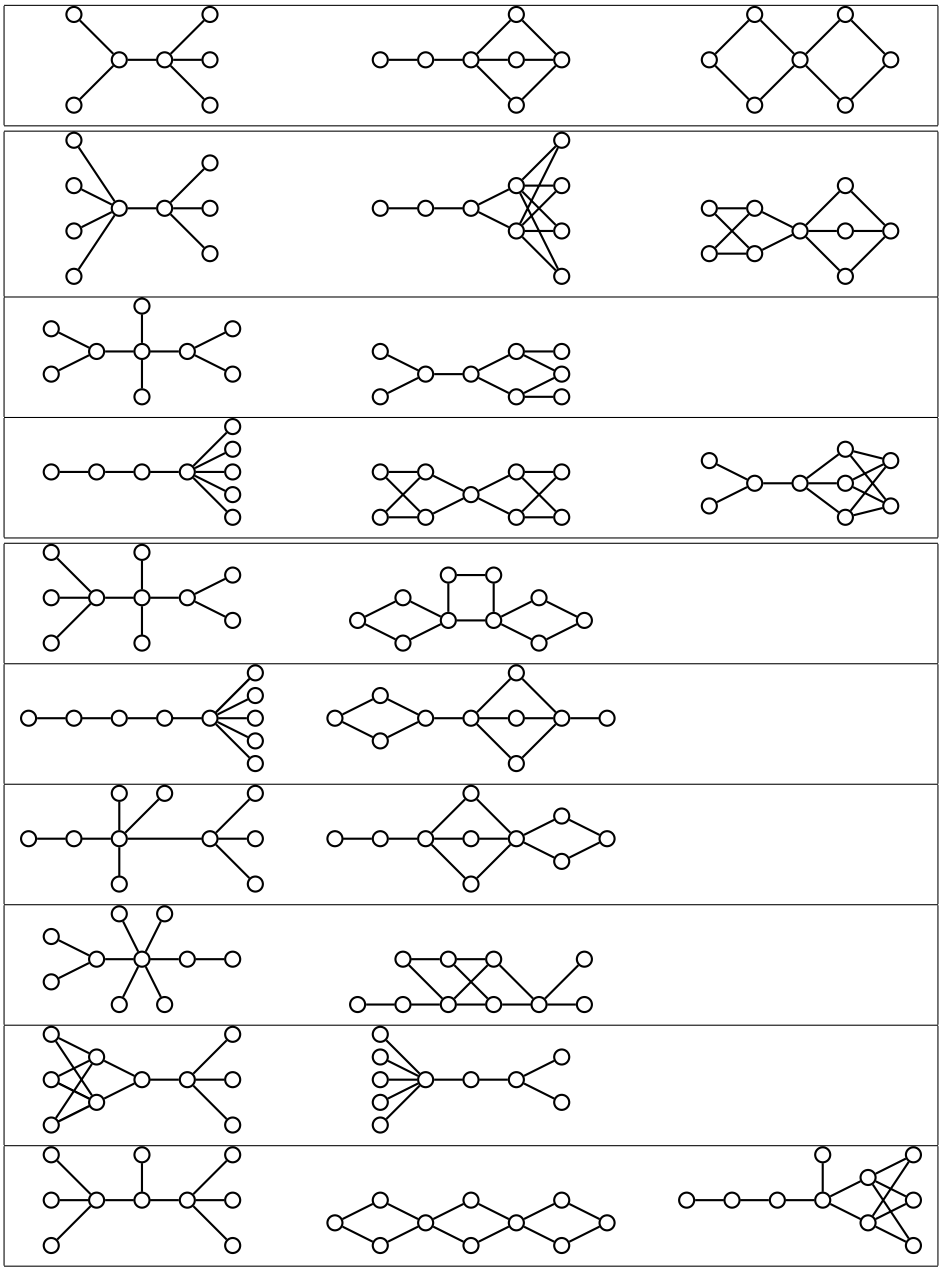

Figure 2: Some trees and corresponding cospectral non-tree pairs for $n = 7, 9, 10$.

2 Preliminaries

As mentioned in the previous section, there is a number of different matrices to represent graphs, notably, the adjacency matrix, the Laplacian matrix, the signless Laplacian matrix and the normalized Laplacian matrix. Each of these types of matrices comes with a set of strengths and weaknesses. For instance, the spectrum of an adjacency matrix tells whether or not the graph is bipartite. A tree cannot be cospectral with a connected non-tree with respect to the adjacency, Laplacian and signless Laplacian matrices.

We will look for simple graphs with the desired property, i.e., graphs without loops or multiple edges. In the process of examining the spectrum of these graphs we will make use of graphs which have weighted edges (which can be interpreted as graphs with multiple edges).

Given a weight function on edges w, such that $w(u, v) > 0$ if and only if $u \sim v$, otherwise $w(u, v) = 0$, the adjacency matrix A is defined by $A_{uv} = w(u, v)$. We let the weight function on the vertices satisfy $w(u) \geq 0$ for all u. The degree function is defined by $d(u) = w(u) + \sum_v w(u, v)$. For simple graphs $w(u) = 0$ for all vertices u.

2.1 Normalized Laplacian matrix

With the above notions, we can now define the normalized Laplacian matrix.

Definition 1. Normalized Laplacian matrix

$$\mathcal{L}_{uv} = \begin{cases} \dfrac{-w(u,v)}{\sqrt{d(u)d(v)}} & \text{if } u \neq v \text{ and } u \sim v \\ \dfrac{d(u) - w(u,u)}{d(u)} & \text{if } u = v \text{ and } d(u) \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

In the special case where none of the vertices has degree 0, the above definition is equivalent to $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$, where A is the adjacency matrix of the graph.

2.2 Twin subgraphs

We will use the following idea to simplify the computation necessary in establishing cospectral tree and non-tree pairs.

Definition 2. Let G be a simple graph and let $V_1, V_2, \dots, V_k$ be disjoint subsets of the vertices such that $G[V_1], \dots, G[V_k]$ (i.e., the induced subgraphs on the V_i) are isomorphic with the isomorphism $\pi_i : V_1 \to V_i$. Further, these subgraphs connect in a consistent manner, i.e., for $u \notin V_1 \cup \cdots \cup V_k$ then $u \sim v$ for $v \in V_1$ if and only if $u \sim \pi_i(v)$ for $2 \leq i \leq k$. Then $G[V_1], \dots, G[V_k]$ are twin subgraphs.

We know the following facts about eigenvalues of G and its twin subgraphs.

Theorem 1 (Butler [1])**.** *Let G be a simple graph with twin subgraphs coming from $V_1, \dots, V_k$. Then the eigenvalues of G can be found using the following weighted graphs:*

Multiplicity $k-1$ for each of the eigenvalues arising from the graph $G[V_1]$ with vertex weights added so that the degrees agree with the degrees in G.

Figure 3: The vertices in the dashed rectangle are twins

Multiplicity 1 for each of the eigenvalues arising from the graph obtained by deleting $V_2 \cup \cdots \cup V_k$ *and changing the edge weight of each edge incident to a vertex in* V_1 *to* k.

Using the above theorem, we can "collapse" graphs into smaller parts while preserving all the information about the spectrum. We will primarily focus on graphs which collapse after removing twin subgraphs to weighted path of length at most five.

2.3 The characteristic polynomial

The characteristic polynomial of the normalized Laplacian, which we denote by $\phi(x) = \det(xI - \mathcal{L})$ gives the eigenvalues of a graph. Namely, the roots of the characteristic polynomial consist the spectrum of the graph. Two graphs are cospectral if and only if the characteristic polynomials are the same. Note that for simple graphs we have $\phi(x)$ is a polynomial with rational coefficients, which is a bit surprising given that our entries have square roots.

3 Inifinite Families

Based on what we have seen in small examples, we are going to look at graphs where we can collapse down the twins and the resulting (weighted) graph is a path. These types of graphs have the following general form.

Definition 3. Given a sequence $\mathbf{s} = (s_1, s_2, \ldots, s_k)$, let $G_\mathbf{s}$ be the graph on $s_1 + s_2 + \ldots + s_k$ vertices where vertices $s_1 + \ldots + s_i + 1$ through $s_1 + \ldots + s_i + s_{i+1}$ are adjacent to vertices $s_i + \ldots s_{i+1} + 1$ though $s_1 + \ldots + s_{i+1} + s_{i+2}$.

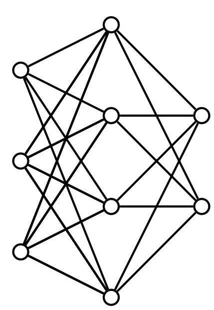

Figure 4: The graph $G_\mathbf{s}$ for $\mathbf{s} = [3, 4, 2]$.

This can be thought of as a sequence of compete bipartite graphs glued together sequentially. See Figure 4 for example. The key observation is that all of the s_i vertices are twins, so upon removing twins, each set collapses down to a single vertex and the resulting graph becomes a path. In particular, the spectrum is 1 many times (for the twins) together with the spectrum of the (short weighted) path to which it collapses.

We now look at one special case.

Proposition 1. *Let* $\mathbf{s} = (2k, 1, 1, 2k+1)$ *and* $\mathbf{s}_\ell = (k, k+1-\ell, 1, k+1+\ell, k)$ *then* $G_{\mathbf{s}}$ *and* $G_{\mathbf{s}_\ell}$ *are cospectral for the normalized Laplacian.*

Proof. For convenience we can work with the normalized adjacency matrix (essentially removing the 1s on the diagonal). Using collapsing and twins, the characteristic polynomial of the original graph is the product of the number of twin vertices removed together with the characteristic polynomial of the (fixed length) weighted path.

Collapse $4k-1$ twins from the first graph and $4k-2$ twins from the second (each twin contributing a contribution of 0 to the spectrum of the normalized adjacency matrix). This results in two weighted paths (on four and five vertices, respectively). We now can use the characteristic polynomial to compare the two spectrums.

For $G_{\mathbf{s}}$ this gives

$$x^{4k-1}\left(x^4 - \frac{2k+1}{k+1}x^2 + \frac{k}{k+1}\right).$$

For $G_{\mathbf{s}_\ell}$ this gives

$$x^{4k-2}\left(x^5 - \frac{2k+1}{k+1}x^3 + \frac{k}{k+1}x\right).$$

These are the same characteristic polynomials, so we can conclude the original graphs are cospectral. □

Note that $(2k, 1, 1, 2k+1)$ is always a tree. In general we have that $(a, 1, \ldots, 1, b)$ is always a tree for any choice of a and b. Moreover, if any entry *not* at the end is greater than 1, then the result is a non-tree. So the preceding proposition establishes infinitely many examples of trees being cospectral with non-trees.

This pair is not unique, and Table 1 shows some other (far from all) sequences which produce cospectral graphs. The first item in the table is the one given in the proposition. Proofs for other sequences are identical.

4 Concluding remarks

We have constructed infinite families of trees and cospectral non-trees. As seen in Table 1, there are many such examples. A natural question that arises from it is how is the ratio of trees with cospectral non-tree mate to all trees changes as the number of vertices increases. A related problem was answered by Osborne [2] who showed that almost all trees have a cospectral mate with respect to the normalized Laplacian (so that the ratio is 1). Whether the analogous statement, almost all trees have a cospectral non-tree mate, holds requires further investigation. However, we have the data for cases where the number of vertices is small in Table 2, and it suggests that the above statement is false, as the number of trees seems to increase at the higher rate than the number of trees with non-tree cospectral mates as the number of vertices increases.

Table 1: Lists of sequences which produce cospectral graphs. (Note any values of k, ℓ are allowed as long as the entries in the sequence are positive.)

Tree	**Non-tree**
$(2k, 1, 1, 2k+1)$	$(k, k+1-\ell, 1, k+1+\ell, k)$
$(2k-1, 1, 1, 2k)$	$(k-\ell, k+\ell, 1, k+1-\ell, k-1+\ell)$
$(k+1, 1, 1, k+2)$	$(1, k+1, 1, 2, k)$
$(2k, 1, 1, 2k+1)$	$(k, 2, 1, 2k, k)$
$(2k, 1, 1, 2k+1)$	$(k, 1, 1, 2k+1, k)$
$(2k, 1, 1, 2k+1)$	$(k, 2k-1, 1, 3, k)$
$(4k-2, 1, 1, 4k-1)$	$(2k-1, k+\ell, 1, 3k-\ell, 2k-1)$
$(2, 1, 1, 3k)$	$(2k-1, 2, k, k+1, 2k-1)$
$(3k-1, 1, 1, 3k)$	$(k, 2k, 1, k+1, 2k-1)$
$(3k-1, 1, 1, 3k)$	$(1, 3k-1, 1, 2, 3k-2)$
$(3k-1, 1, 1, 4k-1)$	$(2k-1, k+1, 1, 3k-1, 2k-1)$
$(3k-1, 1, 1, 4k-1)$	$(k, 3k-1, 1, k+1, 3k-2)$
$(3k-1, 1, 1, 4k-1)$	$(2k-1, k, 1, 3k, 2k-1)$
$(k+2, 1, 1, k+3)$	$(1, k, 1, 2, k+3)$
$(2k+1, 1, 1, 2k+2)$	$(k, k+2, 1, k+1, k+1)$
$(2k+1, 1, 1, 2k+2)$	$(k, 1, 1, k+1, 2k+2)$
$(2k+1, 1, 1, 2k+2)$	$(1, 2k-1, 1, 2, 2k+2)$
$(3k, 1, 1, 3k+1)$	$(k, 2k+1, 1, k+1, 2k)$
$(3k, 1, 1, 3k+1)$	$(1, 3k, 1, 2, 3k-1)$
$(3k, 1, 1, 3k+1)$	$(k, k, 1, k+1, 3k+1)$
$(3k, 1, 1, 3k+1)$	$(1, 3k-2, 1, 2, 3k+1)$
$(4k-2, 1, 1, 4k-1)$	$(2k-1, k, 1, 3k, 2k-1)$
$(6k-4, 1, 1, 6k-3)$	$(3k-2, 2k, 1, 4k-2, 3k-2)$
$(6k-4, 1, 1, 6k-3)$	$(3k-2, k+1, 1, 5k-3, 3k-2)$
$(6k-4, 1, 1, 6k-3)$	$(k, 5k-3, 1, k+1, 5k-4)$
$(6k-4, 1, 1, 6k-3)$	$(3k-2, 4k-3, 1, 2k+1, 3k-2)$
$(6k-4, 1, 1, 6k-3)$	$(3k-2, 2k-1, 1, 4k-1, 3k-2)$
$(6k-4, 1, 1, 6k-3)$	$(3k-2, k, 1, 5k-2, 3k-2)$
$(6k-4, 1, 1, 6k-3)$	$(3k-2, 2k, 1, 4k-2, 3k-2)$
$(8k-6, 1, 1, 8k-5)$	$(4k-3, 3k-1, 1, 5k-3, 4k-3)$
$(8k-6, 1, 1, 8k-5)$	$(4k-3, k+1, 1, 7k-5, 4k-3)$
$(8k-6, 1, 1, 8k-5)$	$(3k-2, 5k-3, 1, 3k-1, 5k-4)$
$(2, 1, 1, 12k-9)$	$(4k-3, k+1, 2k-1, k+1, 4k-3)$
$(2, 1, 1, 12k-9)$	$(4k-3, k, 2k-1, k+2, 4k-3)$
$(2, 1, 1, 6k-3)$	$(2k-1, k, k, 3, 2k-1)$
$(2, 1, 1, 6k-3)$	$(2k-1, 2-\ell, k, k+1+\ell, 2k-1)$

Conjecture 1. *For the normalized Laplacian, most trees do not have a cospectral mate that is a connected non-tree.*

Table 2: Number of trees with a non-tree cospectral mate.

# vertices	7	8	9	10	11	12	13	14	15
# trees	11	23	47	106	235	551	1301	3159	7741
# w/ a non-tree pair	2	1	4	7	11	5	59	10	105

This conjecture is trivially true for the adjacency and Laplacian matrix. Answering it for the normalized Laplacian matrix will improve our understanding of the matrix. (Note it is even possible for the conjecture to be false!)

References

[1] S. Butler, Algebraic aspects of the normalized Laplacian, in *Recent Trends in Combinatorics*, A. Beveridge, J. Griggs, L. Hogben, G. Musiker, and P. Tetali, eds., Springer, 2016, 295–315.

[2] S. Osborne, Cospectral bipartite graphs for the normalized Laplacian, Ph.D. Dissertation, Iowa State Univeristy, Ames, 2013.