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The role of money in two alternative models: When is the Friedman rule optimal, and why?

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1 Introduction

1.1 Overview

Over the last three decades, the workhorse models for macroeconomic theory have been the infinitely-lived representative agent (ILRA) model and the overlapping generations (OG) model. Although both models are widely used for monetary analysis, the differences between them, as models of money, are not well understood. A question that helps illustrate the nature of these differences is the identity of the optimal monetary policy. In most monetary specifications of the ILRA model, one monetary policy, the Friedman rule, is uniquely and unambiguously optimal.[1] Under this policy, the government shrinks the stock of money at a rate that drives the real rate of return on money up to the level of the real return rate on physical assets. In OG models, the question of the optimal monetary policy is more complicated, partly because the term "optimal" has more than one reasonable definition. But by at least one common definition – maximization of steady state utility, sometimes called the "golden rule" – the optimal monetary policy is usually quite different from the Friedman rule.[2]

In this paper, we use the question of the optimal monetary policy to help us identify and study the key difference between ILRA and OG models, as models of money. Our analysis proceeds in two stages. In the first stage, we use a simple OG model to study the welfare implications of different monetary policy rules under two widely used money demand specifications. We find that under the welfare criterion we use, the Friedman rule is not optimal under either specification. Under one specification the optimal policy requires holding the money stock constant, which produces zero inflation. Under the other specification it is optimal for the money stock to increase over time, which produces positive inflation. Under each specification, changing our assumptions about the generational timing of the lump-sum taxes that finance retirement of money produces a qualitative change in the nature of the optimal policy.[3]

In the second stage of our analysis, we identify the key feature of the role of money in OG models that is responsible for the fact that their predictions about the optimality of the Friedman rule are so different from those of ILRA models. To confirm this identification, we devise a government policy regime that eliminates this feature of money's role in the model without changing the conditions under which is it issued, held or traded. We show that if the government adopts this regime, then the Friedman rule is optimal under both of our money demand specifications.

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[1] Friedman (1969); see, for example, Grandmont and Younes (1973) and Townsend (1980) and Kimbrough (1986). Phelps (1973) and others made the case that the Friedman Rule is suboptimal when non-distortionary taxes are excluded from the set of financing alternatives. However, Christiano, Chari, and Kehoe (1986) and Correia and Teles (1996, 1999) derive conditions under which the optimality of the Friedman Rule survives the presence of distortionary taxes.

[2] We confine ourselves to discussing monetary OG models that are structured so that a zero-money-growth policy will produce a steady state in which money is return-dominated by other assets, as is always the case in ILRA models.

[3] When the inflation rate is positive, these taxes become transfers. For expositional simplicity, we will usually refer to them as taxes. The dependence of our results on whether the young or old households pay the lump-sum taxes is noteworthy, in part, because each assumption has been used repeatedly in the literature, usually without much discussion.

1.2 Literature

In OG models, the question about the Friedman rule that has received the most study is whether or not it is *Pareto* optimal. On this question, the literature reaches a consistent conclusion. Wallace (1980), McCallum (1987) and Smith (1991) all find that the Friedman rule is Pareto optimal, even though each author uses a different specification for money demand. In OG models, however, it is possible for a Pareto optimal allocation to deliver lower welfare, for all but a measure-zero subset of the agents (the initial old consumers), than many other feasible allocations. For this reason, an alternative welfare criterion that is widely used for OG models is maximization of the current and future generation's steady state utility. Using this criterion to search for the optimal policy has produced a variety of results, none of which favor the Friedman rule. Weiss (1980) finds that the optimal policy produces positive inflation, but Freeman (1993), using a very similar model, finds that zero inflation is optimal. Smith (2002) and Paal and Smith (2002), using somewhat different models, find that the optimal policy produces higher inflation than the Friedman rule, but they do not attempt characterize the optimal rate as positive, zero, or negative.

The paper in the literature most similar to ours is Freeman (1993). Freeman is also interested in studying the differences between the ILRA model and the OG model regarding the optimality of the Friedman rule. The feature of the two models he focuses on is the presence or absence of preference links between members of different generations. Consequently, he studies an OG specification with a bequest motive that may or may not be active. For the ILRA model, Freeman obtains the standard result: the Friedman rule is optimal. He also finds that the Friedman rule is golden-rule optimal for his OG model, provided the bequest motive is active. When the bequest motive is not active – the case that corresponds to ours, since we make the usual assumption that there is no bequest motive – he finds that the optimal policy is to hold the money stock constant. If the population is constant, as we assume here, then this policy produces zero inflation.

1.3 Our contribution

Our analysis differs from Freeman's in several important ways. First, Freeman uses a money demand specification (money in the utility function) under which consumers hold money voluntarily. Another widely used class of money demand specifications postulates that consumers hold money in order to satisfy a legal restriction. If the nature of the money demand specification matters, then we would like to know how, and why. Consequently, we perform our analysis twice: once using the most widely studied form of legal restrictions money demand (reserve requirements), and once using a new type of voluntary money demand specification (random relocation) that has become very popular in recent years. Second, Freeman assumes that the lump-sum taxes that finance retirement of money are levied on the old consumers. But these taxes could just as easily be levied on the young consumers: again, we want to know whether this would matter, and why. Consequently, for each of our money demand specifications, we study each case.

Not surprisingly, when we study the combination of cases that corresponds to Freeman's – voluntary money demand and taxes on the old consumers – we get results very similar to his. But our legal-restrictions money demand specification produces different results, and we get different results from our voluntary money demand specification if we switch the taxes from the old to the young consumers. Ultimately, we are able to use a single feature of the model to explain all these differences.

The third and most important difference between our paper and Freeman's involves our answer to the central

question addressed by both papers: Why do OG models produce predictions about the optimal monetary policy that are so different from the predictions of ILRA models? Freeman's answer is that the models deliver different predictions because they make different assumptions about preference linkages across agents. In ILRA models, consumers care about the welfare of their descendants – or, at least, the models can be interpreted in this way – while in standard (no bequest) OG models, they do not. Our results, however, indicate that different assumptions about preference linkages are not the principal source of the difference in optimal policy predictions. Instead, the main source of difference is that in OG models, unlike ILRA models, there can be transfers of goods between consumers at different stages of their life cycles (intergenerational transfers). Transactions involving fiat money produce transfers of this type that would not occur if fiat money was valueless or absent. Changes in monetary policy change agents' equilibrium consumption opportunities by changing the scale of these transfers.[4]

We defend this claim in two ways. First, we take a careful look at the role of intergenerational transfers in producing the optimality results we outlined in the first paragraph of this section. We find that the differences between these results can be explained in terms of differences in the roles of intergenerational transfers under the two money demand specifications, and differences in the way that changes in monetary policy change the scale of these transfers. Second, we describe an alternative government fiscal/monetary regime that exactly offsets the intergenerational transfers associated with money holding. Under this regime, net intergenerational transfers are zero, in equilibrium, regardless of the monetary policy. We show that under this regime the Friedman rule is optimal for both money demand specifications, even though intergenerational preference linkages are absent from our model.[5]

2   The Friedman rule in two types of models

2.1   ILRA models

Since the properties of the Friedman rule in ILRA models are well known, we confine ourselves to providing a careful but informal description of the reason the Friedman rule is optimal in these models. For the purposes of this description, we will consider a model in which the source of money demand is that real money balances enter the utility function of the representative agent. However, the description is easy to reformulate for models for other common money demand assumptions.

In the typical ILRA monetary model, the representative agent is endowed with a stock of fiat currency (money) in the first period. In equilibrium, the agent must choose to hold all this money in the first period, and also in all future periods. It does not sell or buy any money, on net, in any period, unless the government is changing the stock of money. In that case, it sells only the money the government wants to buy and retire, or it buys only the new money the government wants to issue and sell. There is no one for the agent to sell money to, or buy money from, except the government.

In the first period, the agent's initial endowment of money finances its money holdings at the end of the period; in later periods, the money the consumer carried over from the preceding period finances its money holdings at

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[4]This aspect of our analysis is related to an analysis conducted by Weil (1991), who uses intergenerational transfers to explain why OG models typically do not produce superneutrality. Although Weil uses a nonstandard version of the OG model, with infinitely lived households, the mechanism driving his results is essentially the same as the mechanism driving ours. Ireland (2003) uses Weil's model to construct a numerical example in which the Friedman rule produces relatively low utility and positive inflation is optimal.

[5]In Freeman (1993), when the bequest motive is active, the offsetting intergenerational transfers take the form of voluntary bequests. As a result, the Friedman rule is optimal without any additional government intervention.

the end of the current period.[6] There is no need for the agent to reduce its consumption, or its holdings of real assets, in order to acquire money. Since there is no inherent tradeoff between purchasing money and purchasing other items that produce current or future utility, such as consumption goods or nonmonetary assets, the optimal steady state is one in which the real value of the agent's money holdings is large enough to allow the agent to reach the point of satiation: the point, that is, at which the marginal utility of its real money balances is zero. But the agent will continue to hold real money balances until their marginal utility is zero only if its opportunity cost of holding real balances is also zero. It follows that in the optimal steady state, the real rate of return on money must be equal to the real rate of return on non-monetary assets, so that the nominal interest rate on the latter is zero.

To equalize the steady state real return rates on money and nonmonetary assets, the government must set the gross growth rate of the nominal money stock at a the value n/X, where n is the gross population growth rate and X is the equilibrium gross real return rate on nonmonetary assets. It accomplishes this by purchasing and retiring money during each period. These purchases are financed by levying a lump-sum tax, each period, on the representative agent. Since the real value of the agent's endowment of money is equal to the present value of this sequence of lump-sum taxes, the money endowment does not add to the agent's net income. And since the agent's opportunity cost of holding money is zero, it faces a consumption-saving decision that is identical to the decision it would face in an otherwise-identical economy without monetary features. Thus, the optimal real allocation in the monetary economy is identical to the real allocation in a non-monetary version of the same economy – that is, a version in which real balances do not appear in the utility function.

If the government does not follow the Friedman rule, choosing a higher money stock growth rate that results in a lower rate of return on money, then the optimal steady state is disturbed in two ways. First, the present value of the sequence of taxes to be levied on the agent is now smaller than the value of the agent's initial real balances at the original (Friedman-rule) price level. So the agent's net wealth has increased relative to its equilibrium net wealth in the Friedman-rule steady state. Second, the agent now incurs a positive opportunity cost from holding money, which has become a relatively low-return asset.   The first effect induces the agent to try to increase its current consumption by spending real balances to buy goods. The second effect induces the agent to spend real balances to buy nonmonetary assets and/or goods. Together, both effects cause the price level to rise. As the price level rises, the real value of the consumer's money endowment falls and the consumer's marginal utility from money starts to rise, if it holds all the money it was endowed with. At some point, this marginal utility rises to the level of the marginal utility from consumption and saving at the original levels of consumption and saving. At that point, equilibrium is restored at the same levels of consumption and saving but lower levels of real money balances and welfare.

2.2 OG models

A good way to begin gaining an understanding of the role of money in a standard OG monetary model is to consider the decision problem of the first agents who make nontrivial decisions – the first generation of two-period-lived consumers, who we may call the initial young. Unlike the ILRA representative agent, these consumers are not endowed with money. Instead, they must purchase it from the agents who *are* endowed with it – the initial old. Since the initial young consumers must trade goods for money, each unit of real money balances they purchase

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[6]In practice, the agent simply continues to hold the money. But we are supposed to imagine the agent spending its money on goods during the period and then using its income to buy replacement money at the end of the period.

reduces their consumption, or their holdings of non-monetary assets, by one unit, relative to an equilibrium in an analogous non-monetary economy. Next period, the initial young consumers sell their money holdings to the young consumers from the next generation, who pay for the money with goods from their incomes or endowments. These goods constitute the "return" on the goods the initial young consumers paid away, last period, to the consumers from the previous generation. They take the place of the return the initial young consumers might have received on the non-monetary assets they passed up when they purchased their money. In subsequent periods, the pattern repeats. Young consumers purchase money, from consumers from the previous generation, using goods they might have used to acquire real assets; they get a return on the money, in the form of a payment from the next generation of consumers, that takes the place of the return they might have received on those assets.

Thus, in a monetary OG model, under a standard monetary regime (see below), consumers that purchase money when they are young and sell it when they are old are participating in a sequence of *intergenerational transfers* And since every unit of goods a consumer devotes to this intergenerational transfer sequence is a unit it cannot use to purchase real assets, there is a genuine tradeoff between purchasing money and purchasing real assets – a tradeoff which is not present in an ILRA model.[7]

The existence of this tradeoff implies that purchasing money has adverse welfare consequences, even under the Friedman rule.[8] The reason for this is that in a steady state, the technological return rate on intergenerational transfers is the rate at which the consumers'aggregate income or endowment grows over time: in our model, this is simply the population growth rate. Consumers may receive a different real return rate on money, but this will be due to government intervention that creates a return distortion. In the OG models we study, the real return rate on physical assets X is fixed and the population growth rate n is lower. Consequently, intergenerational transfers are an inefficient way for consumers to trade current consumption for future consumption. Across steady states, for every unit of goods that consumers devote to intergenerational transfers, instead of acquiring real assets, the economy loses $X - n$ units of consumption.[9]

How does the Friedman rule work in an OG model? As in an ILRA model, the government uses revenue from lump-sum taxes to retire money at a rate that drives the private real return rate on money up to the level of the real rate of return on physical assets, so that the private opportunity cost of holding money is zero. In the OG case, however, holding money means participating in intergenerational transfers, and the social opportunity cost of intergenerational transfers is positive. If consumers face a rate of return on money that is higher than the population growth rate, so that the private opportunity cost of holding money is lower than the social cost of the associated transfers, then they will usually hold an amount of money at which the marginal social cost of the transfers they are participating in exceeds their marginal private benefit from money holding. In equilibrium, they bear the excess social cost in the form of a lower after-tax endowment that reduces their welfare more than

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[7]When the representative agent in an ILRA model makes the decision to hold a unit of real balances during a period, it perceives itself as using part of its current wealth to purchase money, instead of real assets, and thus as facing a tradeoff between the two. But this is an illusion: in equilibrium, the consumer always decides to leave the period with the same amount of money it brought in to the period (or, in the first period, with the same amount of money it was endowed with). The latter money was part of its initial wealth. So if we choose to think of the consumer as purchasing money during the period, then we should also think of this purchase as financed by selling money during the period.

[8]By "has adverse welfare consequences," we mean that the utility a two-period-lived consumer derives from consumption is always lower, in a steady state in a monetary OG model, than it would be in a steady state of an otherwise-identical model without monetary features.

[9]See Freeman (1987), who derives some implications of the inefficiency of intergenerational transfers in an OG model with reserve requirements.

the higher money return rate increases it. Indeed, under both money demand specifications we study, one of the two alternative assumptions about who pays the lump-sum taxes (the young or the old consumers) causes the optimal policy to be to hold the money stock constant, which results in zero inflation. Under this policy, the private return rate on money is equal to the population growth rate and the private opportunity cost of holding money is equal to the social opportunity cost of intergenerational transfers.[10]

Oddly, the tax-timing assumption that makes zero inflation optimal is different across the two money demand specifications. In the reserve-requirements case zero inflation is optimal when the taxes are levied on the old consumers; in the random relocation case it is optimal when they are levied on the young consumers. Again, the explanation for the difference involves intergenerational transfers. In the reserve-requirements case, consumers hold money because they are required to do so, not because it is genuinely useful to them. Consequently, a policy that reduces the size of the intergenerational transfer associated with a unit of real money holdings benefits consumers, *ceteris paribus.* When the taxes are levied on the young consumers a policy of monetary expansion does this. Under this policy, the taxes become transfers. Some of the goods the old consumers would have obtained from the young consumers by selling money are appropriated by the government, through the inflation tax, and returned to the young consumers. These policy-engineered old-to-young transfers partly offset the inefficient young-to-old transfers associated with money holding. However, the inflation that this policy produces exacerbates the return distortion associated with the reserve requirement. So positive inflation is optimal, at the margin, but higher inflation may not always be better than lower inflation.

In the random relocation case, money serves a useful purpose: it allows consumers to obtain some insurance against the bad event of being relocated (see below). Under a standard monetary regime, if the money stock is held constant then the insurance payout consists entirely of intergenerational transfers – more specifically, transfers from young consumers, who purchase money, to old consumers that have been relocated, who turn out to be the only ones who have money to sell. But if money is retired by taxes on the old consumers, then some of the taxes are paid by old consumers who have not been relocated, while all the tax revenue is used to augment the return on the money held by the old consumers who have been relocated. Thus, deflation allows some of the insurance payout to take the form of transfers from old consumers who are *ex post* fortunate to old consumers who are *ex post* unfortunate – a relatively efficient form of insurance that is not possible otherwise. For this reason, negative inflation is always optimal, even though it may act to increase the total amount of inefficient intergenerational transfers. In fact, it is possible for the Friedman rule to be optimal.

2.3 Confirming our hypothesis

The analysis we have just outlined indicates that the differences between the optimality properties of the Friedman rule in OG vs. ILRA models grow out of the fact that intergenerational transfers are possible in OG models but not in ILRA models. Monetary versions of the two models showcase this difference because under standard OG monetary regimes, monetary transactions represent intergenerational transfers, and monetary policies that feature shrinkage or growth in the stock of money can act to augment or partly offset these transfers.

On the other hand, the fact that the OG model permits intergenerational transfers does not necessarily mean

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[10]Stated a bit differently, in the OG standard-regime case, the social opportunity cost of providing money is the difference between the physical-asset return rate and the population growth rate, so it is optimal for the private opportunity cost of holding money to be the same. In the ILRA model, by contrast, the social opportunity cost of providing money is zero – the seminal observation of Friedman (1969) – so it is optimal for the private opportunity cost of holding money to be zero.

that these transfers always occur. In fact, intergenerational transfers typically require some form of government intervention. As we have seen, one frequently studied form of government intervention – issuing fiat money to the initial old consumers – can support monetary equilibria that feature intergenerational transfers. It may be possible, however, to structure a monetary regime that produces equilibria that do not have this feature. For our purposes, such a regime would be quite interesting, because it would allow us to conduct a simple test of our hypothesis about the source of the non-optimality of the Friedman rule in monetary OG models. If we are right, then under a monetary regime that produces equilibria with no intergenerational transfers, the Friedman rule should be optimal.

Since we are particularly interested in comparing OG models of money to their ILRA counterparts, the type of no-intergenerational-transfers monetary regime that would be the most interesting to us would be a regime that made the conditions under which money is acquired and held as similar as possible to those of monetary ILRA models. We construct a regime of this sort, which we call the "ILRA simulator" (or simply "simulator") regime   The simulator regime is designed to mimic a key feature of typical ILRA monetary regimes, which is that the representative agent is endowed with money initially, rather than purchasing it from other agents, and then holds it throughout its life without ever selling it, at least on net. Under our simulator regime, each two-period-lived consumer receives a lump-sum transfer when it is young. In equilibrium, this transfer is just large enough to finance the consumer's purchases of money: it takes the place of the money endowment the infinitely-lived representative agent starts out with. The transfers are financed by lump-sum taxes on the old consumers, including the initial old. Although young consumers continue to purchase their money from old consumers, the fact that these purchases are financed entirely, in equilibrium, by transfers from the old consumers means that the purchases do not represent intergenerational transfers. Of course, since an OG consumer lives for only two periods, it will never be rational for it to be holding money at the end of its second period, as it would in a monetary ILRA model. Instead, it must sell its money before the end of the period. But since the proceeds of this sale are exactly absorbed, in equilibrium, by the lump-sum tax the old consumer pays to the government, the consumer's money holdings never finance any future consumption – just as in an equilibrium in a monetary ILRA model. In effect, each OG consumer is endowed with money when it is young, uses it, for whatever purpose makes it useful, during its two periods of life, and surrenders it, at the end of its life, without obtaining any goods in exchange.

2.4   Outline

In the remainder of the paper, we proceed as follows. In section 3, we lay out a simple monetary overlapping generations model, describing two alternative monetary regimes (standard and simulator) but retaining generality about the source of money demand. In section 4, we lay out a reserve requirements money demand specification; in section 5 we replace it with a random relocation specification. In each of these sections, we begin our analysis by studying the welfare properties of monetary policy under the standard monetary regime. We do this first for the case of the taxes on or transfers to the young consumers, and then for the other case. At the end of each section, we study the welfare properties of monetary policy under the simulator regime. In section 6, we present a brief summary and some concluding remarks. The proofs of our major results appear in an appendix.

3 The model: general features

We begin our analysis by describing features of our model that are common to both of our money demand specifications. Time is discrete: the periods are dated $t = 1, 2, \ldots$. At the beginning of each period, a unit-mass continuum of two-period-lived consumers is born: the consumers born at the beginning of period t are the members of "generation t". These consumers are identical to each other within a generation, and they are identical across generations except for their birthdates. In the first period, the population also includes a unit-mass continuum of identical consumers that live for only one period: these are the "initial old" consumers.

There is a single good that may be consumed or stored. Each two-period-lived consumer is endowed with ω units of the good when it is young and nothing when it is old.[11] The initial old consumers have no endowments of goods.

Consumers derive satisfaction from consuming units of the good during their two periods of life. The preferences of the two-period lived consumers are representable by an intertemporal expected utility function U with the form

$$U(c_{1t}, c_{2,t+1}) = u(c_{1t}) + \beta \, E_t \, u(c_{2,t+1}) \tag{1}$$

for $\beta > 0$. Here c_{1t} represents the consumption of a representative consumer generation t when it is young, and $c_{2,t+1}$ represents the consumption of the same consumer when it is old. Note that $c_{2,t+1}$ may be state-contingent. We assume that the atemporal utility function $u(c)$ satisfies $u'(c) > 0$, $u''(c) < 0$ and $u'(c) \to \infty$ as $c \to 0$, for $c > 0$. We also make the more specific assumption that $u(c) = \left[c^{1-\gamma} - 1 \right] / (1 - \gamma)$, with $\gamma > 0$.[12] In one case, we assume $\gamma = 1$, which is $u(c) = \ln c$.

Each period, an indeterminate number of perfectly competitive, zero-cost financial intermediaries (banks) offer their liabilities (deposits) to young consumers. A bank that is organized at date t acquires its portfolio at that date and then liquidates and closes at date $t + 1$. Under our legal-restrictions money demand specifications, we assume that all consumer saving must be intermediated; under the other specification, this situation will arise endogenously. The assets available to the banks are goods, which they may store, and fiat currency (money). If $\kappa > 0$ units of the good are placed in storage at any date $t \geq 1$, then $X \kappa$ units are recovered from storage at date $t + 1$, where $X > 1$. Let $x \equiv X - 1$.

The quantity of money in circulation at the end of period $t \geq 1$, per young consumer, is denoted h_t. Each initial old consumer is endowed with $h_0 > 0$ units of money. The market for money is perfectly competitive. The price level at date t – the price of a unit of the consumption good in units of money – is denoted p_t. We confine ourselves to the study of equilibria in which p_t is finite for all $t \geq 1$. In such an equilibrium, the gross real rate of return on money acquired at date t is $R_t^m \equiv p_t/p_{t+1}$; let $r_t^m \equiv R_t^m - 1$. The gross inflation rate is $\Pi_t \equiv 1/R_t^m = p_{t+1}/p_t$ and the net inflation rate is $\pi_t \equiv \Pi_t - 1$.

The government conducts monetary policy by changing the nominal stock of money at gross rate $Z > 0$ per period, starting at date 1 or date 2. Let $z \equiv Z - 1$. If $z > 0$ then the government issues additional money, each period, and uses it to purchase goods; it distributes these goods through equal lump-sum transfers to the current young consumers, or the current old consumers, but not both. If $z < 0$ then the government purchases money

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[11] It is equivalent to think of the endowment as goods or as productive time. In the latter interpretation, the y goods is interpreted as the factor payment received for inelastically supplying this productive time to the labor-only production technology.

[12] Propositions 3 and 6 below hold under much more general preference assumptions, and we prove Proposition 2 without using our specific assumption about the form of u.

from consumers, each period, in order to withdraw it from circulation. It obtains the goods it uses to make these purchases by levying equal lump-sum taxes on the current young consumers, or the current old consumers, but not both. We refer to these taxes or transfers as the government's "policy taxes" or "policy transfers" in order to distinguish them from the "regime taxes and transfers" that the government collects and distributes under one of the two monetary regimes we study.

We will study two types of monetary policy arrangements. They are distinguished by which age group receives the policy transfers, or pays the policy taxes, and by whether the changes in the money supply start at date 1 or date 2. We will let i be an index variable with a value of 1 or 2. If $i = 1$ then the money supply starts changing at date 1 (that is, $h_1 \neq h_0$, unless $z = 1$) and the young consumers receive the policy transfers or pay the policy taxes, starting at date 1. If $i = 2$ then the money supply starts changing at date 2 (so that $h_1 = h_0$ but $h_2 \neq h_1$, unless $z = 1$) and the old consumers receive the transfers or pay the taxes, starting at date 2. Thus, if $i = 1$ then

$$h_t = Z\, h_{t-1} \tag{2}$$

for all $t \geq 1$, while if $i = 2$ then $h_1 = h_0$ and equation (2) holds for all $t \geq 2$.

The value of the policy tax or transfer, per consumer that receives it, is denoted τ_t. A positive value represents a tax and a negative value represents a transfer. Under our assumptions, the government's budget constraint for its policy taxes or transfers is

$$\tau_t = \frac{h_t - h_{t-1}}{p_t} = \left(1 - \frac{1}{Z}\right)\frac{h_t}{p_t}\,. \tag{3}$$

Note that this constraint holds for all $t \geq 1$.

As we have just indicated, the government may also levy taxes or transfers for reasons that are connected to the monetary regime, but do not involve financing money retirement or distributing seigniorage revenue. If this happens, then the young consumers will receive a transfer and the old consumers will pay a tax. Let $\mathrm{T}_{1t} \leq 0$ denote the "regime transfer" to each current young consumer at date t, and let $\mathrm{T}_{2t} \geq 0$ denote the "regime tax" on each current old consumer. The government's budget constraint for regime taxes or transfers is simply

$$\mathrm{T}_{1t} + \mathrm{T}_{2t} = 0\,. \tag{4}$$

Young consumers' deposits in banks are denoted d_t. The banks divide their deposits between stored goods $k_t \geq 0$ and real balances of fiat money $m_t \geq 0$, so that

$$d_t = m_t + k_t\,. \tag{5}$$

Since banks are perfectly competitive and face zero non-interest costs, the average real return rate they pay on their deposits must equal the average real return rate they receive on their assets. The average rate of return on bank deposits, is denoted $\overline{R}_t^d$. Banks' zero-profits condition is

$$\overline{R}_t^d = \frac{R_t^m\, m_t + X\, k_t}{d_t}\,. \tag{6}$$

Banks will supply deposits perfectly elastically at this average return rate, so we are free to view d_t as a consumer decision.

Since money may contribute to consumers' welfare in ways that go beyond the fact that it is an asset, there may be equilibria in which $R_t^m < X$ and $m_t > 0$. If $R_t^m > X$, however, then $k_t = 0$. In addition, it should

be emphasized that $\overline{R}_t^d$ is an *average* gross rate of return: it is not necessarily the gross return rate received by each member of generation t (see section 5). All young consumers from a given generation face identical circumstances at the time they make their consumption and saving (deposit) decisions, and at the time banks make their money demand and storage decisions. So we may think of choice variables like d_t, m_t and k_t both as average values and as values for each young consumer. However, under one of our money demand specifications, consumers may not be identical *ex post*, and the banks may not pay them identical returns.

In the market for money, banks' demand money on behalf of their depositors, and the government supplies it. The money market clearing-clearing condition is

$$m_t = \frac{h_t}{p_t} \quad \forall t \geq 1. \tag{7}$$

Each two-period-lived consumer faces the following budget constraints:

$$c_{1t} + d_t \leq \omega - (2 - i) \tau_t - \text{T}_{1t} \tag{8}$$

$$c_{2,t+1} \leq R_t^d d_t - (i - 1) \tau_{t+1} - \text{T}_{2,t+1}. \tag{9}$$

The deposit return rate R_t^d is the rate received by a particular depositor, which may be state contingent; thus, $c_{2,t+1}$ may also be state contingent. Given our assumptions about consumers' preferences, these constraints will be met with equality in equilibrium.

At date 1, the initial old consumers sell their money endowments to the banks and/or the government. They may also receive a transfer from or pay a tax to the government. The budget constraint of an initial old consumer is simply

$$c_{21} \leq \frac{h_0}{p_1} - \text{T}_{21}, \tag{10}$$

where c_{21} represents its consumption. These consumers are assumed to prefer more consumption to less, so this constraint will also be met with equality in equilibrium.

We will study two different monetary regimes in this economy. Under the *standard regime*, there are no regime taxes or transfers: $\text{T}_{1t} = \text{T}_{2t} = 0$. This regime comes in two varieties: one in which $i = 1$, and one in which $i = 2$; we will study both varieties. Under the *ILRA simulator regime*, we have

$$\text{T}_{1t} = -m_t^* \tag{11}$$

$$\text{T}_{2t} = m_t^* \tag{12}$$

where m_t^* represents a young consumer's equilibrium level of real money balances under the regime. Under this regime, the policy transfer is always paid to the old consumers ($i = 2$).

4 Legal restrictions (reserve requirements) money demand

Under this money demand specification, the government requires banks to hold at least θ units of real money balances for each unit of goods they place in storage, where $\theta \in (0, 1]$. It follows that

$$m_t \geq \theta d_t. \tag{13}$$

Since the banks maximize their profits, in equilibrium we must have

$$m_t = \theta d_t \text{ when } R_t^m < X. \tag{14}$$

Thus,

$$\overline{R}_t^d = (1 - \theta) X + \theta R_t^m \text{ when } R_t^m \leq X, \tag{15}$$

and

$$\overline{R}_t^d = R_t^m \text{ when } R_t^m \geq X. \tag{16}$$

Since this money demand specification is entirely nonstochastic, we have

$$R_t^d = \overline{R}_t^d \tag{17}$$

for all members of all generations $t \geq 1$.

4.1 The standard monetary regime

Throughout the remainder of this paper, we will study the properties of *competitive equilibria*. For this money demand specification, and for values of γ, X, h_0, and θ that satisfy our assumptions, a competitive equilibrium under the standard monetary regime consists of [1] government policy choices $Z^* > 0$ and $i^* = 1$ or 2, a sequence of policy taxes or transfers $\{\tau_t^*\}_{i=1}^\infty$ and regime transfers and taxes $\{T_{1t}^*, T_{2t}^*\}_{t=1}^\infty$, and, respectively, a money supply sequence $\{h_t^*\}_{t=1}^\infty$, [2] a finite, positive price level sequence $\{p_t^*\}_{t=1}^\infty$, an average real deposit rate sequence $\{\overline{R}_t^{d*}\}_{t=1}^\infty$ and a consumer real deposit rate $\{R_t^{d*}\}_{t=1}^\infty$, [3] a non-negative consumption choice c_{21}^* for the initial old consumers, a non-negative consumption sequence $\{(c_{1t}^*, c_{2,t+1}^*)\}_{t=1}^\infty$ and a non-negative real deposit sequence $\{d_t^*\}_{t=1}^\infty$ for the two-period-lived consumers, and [4] non-negative bank storage quantities and real money holdings sequences $\{k_t^*\}_{t=1}^\infty$ and $\{m_t^*\}_{t=1}^\infty$. At date 1, c_{21}^* must satisfy budget constraint (10) with equality. For each date $t \geq 1$, the consumption and deposit quantities $(c_{1t}^*, c_{2,t+1}^*)$ and d_t^* must maximize the utility of the two-period-lived consumers (1) subject to budget constraints (8) and (9), the storage and real money holdings quantities k_t^* and m_t^* must satisfy the bank budget constraint (5) as well as the reserve requirement conditions (13) and (14), and the money supply and price level values h_t^*, h_{t-1}^* (or h_0, if $t = 1$) and p_t^*, p_{t+1}^*, the deposit rates $\overline{R}_t^{*d}$ and R_t^{d*} and the policy tax or transfer τ_t^* must jointly satisfy the government policy rule (2), the money market clearing condition (7), the deposit rate conditions (15) and (16), and the government budget constraint (3). Finally the regime transfers and taxes T_{1t}^*, and T_{2t}^* must satisfy $T_{1t}^* = T_{2t}^* = 0$.

We will also confine our analysis to the study of competitive equilibria that are *stationary*: we will refer to these as "steady state equilibria" or "steady states." For this specification and this monetary regime, a stationary equilibrium is an equilibrium for which there are values c_1^*, c_2^*, d^*, τ^*, m^*, k^*, R^{m*} and R^{d*} such that $c_{1t}^* = c_1^*$, $c_{2,t+1}^* = c_2^*$, $d_t = d^*$, $\tau_t = \tau^*$, $m_t = m^*$, $k_t = k^*$, $p_t^*/p_{t+1}^* = R^{m*}$ and $R_t^d = R^{d*}$, for all $t \geq 1$.

It is readily seen that $R^{m*} = 1/Z^*$ in any steady state, and that the choice of Z^* influences other variables in the model only by determining the value of R^{m*}. Thus, the government's choice of a money growth rate can be viewed as a choice of a rate of real return on money, or equivalently, of an inflation rate.

Our first result characterizes the optimal monetary policy when $i = 1$, so that the policy taxes or transfers are paid by or to the young consumers. (Here and henceforth, unless we indicate otherwise, when we say "optimal" we mean "steady-state utility maximizing.")

Proposition 1 *Suppose $i = 1$. There are positive-inflation ($Z^* > 1$) policies that support steady state equilibria which produce higher levels of welfare, for the two-period lived consumers, than any policy that produces zero or negative inflation ($Z^* \leq 1$).*

Note that Proposition 1 does not say there is necessarily an optimal inflation rate. The reason for this is that in some specifications, increasing the inflation rate always increases the steady-state utility of the two-period-lived consumers. In the relatively simply case where $\gamma = 1$, we can describe these specifications precisely:

Corollary 1 *Suppose $i = 1$ and $\gamma = 1$, so that $u(c) = \ln c$. If*

$$x < \frac{1}{1+\beta}\frac{\theta}{1-\theta}\,,\tag{18}$$

then a steady state equilibrium in which the (net) inflation rate is

$$\widetilde{\pi} \equiv \frac{(1-\theta)\,x}{\frac{\theta}{1+\beta} - (1-\theta)\,x}\tag{19}$$

produces higher welfare, for the two-period-lived consumers, than any consumer holds, than a steady state equilibrium with a lower inflation rate.

In this case, if the net storage return rate x is sufficiently small then there is a finite optimal inflation rate. But if x is relatively large, then the (unattainable) optimal policy would be one in which the money supply growth rate was infinite, the inflation rate was infinite, and the gross real return rate on money was zero.

We begin our discussion of Proposition 1 by explaining why the Friedman rule is not steady-state optimal, as it would be in the analogous ILRA model. The Friedman rule has the advantage of causing the private return rate on deposits to be equal to the return rate on storage. Thus, given the goods the consumer loses by devoting goods to intergenerational transfers, instead of storing them, it makes the right decision regarding consumption vs. saving of the remainder of its goods. Since the goods devoted to the first intergenerational transfers in the sequence benefit the initial old consumers, Friedman-rule steady states are *Pareto* optimal [Wallace (1980), Smith (1991)]. But they are very bad for the two-period-lived consumers, who pay dearly, period after period, for the lost returns on the goods that were transferred to the initial old consumers instead of being stored, and that are transferred again to the old consumers, next period, instead of being stored, and so on. The two-period-lived consumers face a heavily-subsidized deposit return rate – a rate that ignores the low return rate on the portion of their banks' assets that must be devoted to intergenerational transfers, because it takes the form of reserves of money. The consumers' saving decision is privately optimal, given the magnitude of the lump-sum tax that funds the deposit-return subsidy. But they do not internalize the effect of their saving decision on the size of the tax, so they make a decision that results in a large tax and a gives them a low level of welfare. The tax is large because it is very costly for the government to pay a net return rate of x on money whose true net return rate is zero, (the intergenerational-transfer rate), and because the artificially high deposit return rate encourages the consumers to save more, which forces their banks to hold more money.

When $i = 1$, this problem with the Friedman rule is exacerbated by the fact that the government finances money retirement by a tax on the young consumers. Since this tax is used to augment the return on the money held by the old consumers, it is yet another young-to-old intergenerational transfer. Thus, under the Friedman rule, holding one unit of real money forces consumers to participate in an intergenerational transfer that is larger than one unit of goods, which means that it imposes an opportunity cost that is larger than $X - n$ units of goods.

If we imagine the government choosing rates of money supply contraction lower than the Friedman-rule rate, so that the net private return rate on money remains positive but gets smaller and smaller than x, then all these sources of consumer-decision distortion attenuate. The subsidy on the money return rate decreases, and the size

of intergenerational transfer per unit of money holding also decreases: both these effects contribute to reducing the tax. In addition, the gap between the private and social return rates on deposits falls as the return rate on money gets closer to the intergenerational-transfer rate.

When the money supply is constant and the gross real return rate on money is unity (the population growth rate), the private return rate on deposits is the same as the social return rate, given the reserve requirement and the fact that holding reserves of money means participating in intergenerational transfers. In addition, one real unit of money represents an intergenerational transfer of exactly one unit. So why isn't this zero-inflation policy optimal? The reason is that if the government allows the money supply to increase, then the taxes on the young consumers become transfers that are funded by reducing the return on the money held by the old consumers (revenue from seigniorage). These old-to-young intergenerational transfers partly offset the transfers in the opposite direction that are associated with reserve holding.[13] Holding one real unit of money now requires participating in a net intergenerational transfer of less than one unit. In this respect, the policy has the same beneficial effect as a reduction in the reserve ratio (and a deflationary policy has the opposite effect: see below).

However, monetary expansion has another effect that is not beneficial. The "inflation tax" that funds the transfer to the young consumers reduces the real return rate on money, causing the consumer to face a deposit return rate that is "too low": lower than the rate that would be associated with the reduced effective reserve ratio if the real money return rate remained equal to the underlying return rate on intergenerational transfers.[14] This return distortion causes consumers to save less than it is socially optimal for them to save, at that ratio. In some specifications, the benefit from the lower reserve ratio always exceeds the harm caused by the return distortion, so that higher inflation always improves welfare. In others, the latter effect eventually outweighs the former, so there is a positive optimal inflation rate.

The corollary to Proposition 1 establishes that in the special case of logarithmic utility ($\gamma = 1$), if the rate of return on storage is sufficiently low, relative to the required reserve ratio and the rate of time preference, then there is a unique positive, finite inflation rate that is optimal for the two-period-lived consumers. Otherwise, the optimal inflation rate is infinite, so that the optimal gross real rate of return on money is zero. The intuition here is as follows: if the rate of return on storage is high, then the opportunity cost of intergenerational transfer is also very high. Consequently, a policy that reduces the size of the intergenerational associated with the reserve requirement tends to be very beneficial to the two-period-lived consumers. However, when the reserve ratio is high, an increase in the inflation rate produces a large increase in the return distortion. And when the discount factor is high, so that these consumers care deeply about their future consumption, this distortion is very costly, because the saving decision being distorted is very important for their welfare.

Our results are consistent with the results of closely related research on optimal seigniorage in OG models. Freeman (1987) studies an economy identical to ours, except that the reserve ratio is not fixed and the government

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[13]If the inflation rate was infinite, so that the real rate of return on currency was zero, then the two transfers would exactly offset each other, so that the net intergenerational transfer would be zero.

[14]Here, we use the term effective as follows: let $\widehat{\theta} = \theta \frac{1}{\frac{1-\theta}{R^m} + \theta}$ where $\widehat{\theta}$ is the "effective reserve ratio." It captures the income effect of the government's monetary policy, which, in a graphical exposition would capture the tendency for money supply growth to rotate the equilibrium budget line around the fixed endowment bundle. It is readily seen that for any reserve ratio θ and any real money return rate $R^m < X$, the competitive equilibrium allocation must lie along a line in (c_1, c_2) space, emanating from the endowment bundle $(\omega, 0)$, with a slope of $-\widehat{R}^d$, where $\widehat{R}^d = (1 - \widehat{\theta})X + \widehat{\theta}$. The consumer's actual consumption choice bundle lies at the point along the equilibrium budget line at which the consumer's marginal rate of substitution is equal to R^d [see equation (15)]. With $R^m > 1$, the equilibrium budget line lies inside the zero-inflation budget line everywhere except at the endpoint $(\omega, 0)$, indicating the adverse income effect of the government's monetary policy.

must finance a real purchase via seigniorage. He shows that the steady-state optimal choices for the reserve ratio and money growth rate are the lowest ratio feasible and infinity (a hyperinflation), respectively. We can use our income-and-substitution effect decomposition to explain Freeman's result. For a given amount of saving, the return distortion (adverse substitution effect) caused by financing the purchase through seigniorage is independent of the reserve ratio. A high ratio permits a low inflation rate, and vice-versa, but the deposit rate is the same, in both cases. However, a low reserve ratio produces a smaller adverse income effect, because the enforced intergenerational transfer is smaller. Thus, the optimal reserve ratio is the smallest ratio feasible. At this ratio, the government simply confiscates the reserves from the banks and uses them to finance its purchase, giving the banks (and, ultimately, the consumers) no return.[15]

Our discussion of Proposition 1 indicates that if $i = 2$, so that the government's monetary policy choice has no impact on the size of the effective reserve ratio, then the optimal monetary policy should be zero money growth, which produces zero inflation. Proposition 2 confirms this prediction:

Proposition 2 *If $i = 2$, then a policy of zero inflation ($Z^* = 1$) supports a steady state equilibrium that produces a higher level of welfare, for the two-period lived consumers, than any other policy.*

When $i = 2$, the taxes that finance the retirement of money are imposed on the same consumers that receive the increased real return rate on money: the old consumers. Consequently, monetary contraction does not increase size of the of net transfer from the young to the old consumers, per unit of real money balances, and monetary expansion does not increase it. Monetary policy no longer has an income effect: the effective reserve ratio is policy-invariant, and every steady-state equilibrium consumption bundle lies along the same equilibrium budget line. When $R^m > 1$, the budget line facing the consumers is steeper than the equilibrium budget line and the consumers oversave, relative to the reserve-ratio-constrained optimum: when $R^m < 1$ the reverse is true, and the consumers undersave.

It should be emphasized, however, that the fact changes in monetary policy do not change the size of the intergenerational transfer per unit of money holding does not mean that monetary policy has no effect on the level of intergenerational transfers, or that the link between money and intergenerational transfers has no effect on the identity of the optimal monetary policy. Under a contractionary policy the real money return rate rises above the intergenerational-transfer rate. As a result, the deposit return rate facing consumers does not accurately reflect the opportunity cost of the intergenerational transfers necessitated by the reserve requirement, given the monetary regime. Typically, consumers respond by saving more, and thus by holding more reserves (through their banks) than they should: more, that is, than they would save and hold under a constant-money-supply policy where the deposit rate reflected the actual opportunity cost of the transfers. The consumer will choose the intertemporal consumption bundle that is steady-state optimal only if the unit gross return rate on intergenerational transfers is accurately priced into the deposit rate: that is, only in a zero-inflation steady state where $R^m = 1$ and the deposit rate is $(1 - \theta) X + \theta (1)$.[16]

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[15] Our Proposition 1 is even more closely related to an optimal-seigniorage result obtained by Bhattacharya and Haslag (2001): their paper is the direct ancestor of this paper. They use the same model, with a fixed reserve ratio, to study a situation in which the government must finance a real purchase via some combination of first-period lump-sum taxes and inflation tax. They find that the optimal combination always includes some use of the inflation tax.

[16] In Weiss (1980), the tax or transfer is also imposed on or paid to the old consumers. However, he finds that positive inflation is optimal. The reason for this is that he uses a model with neoclassical production and capital. In that model, when an increase in current inflation rate causes the current generation of consumers to acquire more capital (as in our model, thinking of stored

As we shall now see, if the monetary regime is modified in a way that breaks the link between money holding and intergenerational transfers then the identity of the optimal policy changes dramatically.

4.2   The simulator regime

The ILRA simulator regime is an alternative monetary (or monetary/fiscal) regime for an OG model. The difference between the standard regime and the simulator regime is that under the latter, the government gives a lump-sum transfer to the young consumers that is financed by a lump-sum tax on the old consumers. There continues to be a separate tax or transfer that funds currency retirement or distributes revenue from money creation: this tax or transfer, which we refer to as the "policy tax or transfer," is always paid or received by the old consumers. Although the regime transfers and taxes are lump-sum, from the viewpoint of the consumers, the government chooses their values so that, given the structure of the economy and its choice for the money growth rate, the regime transfer has the same value as the equilibrium real money holdings of the young consumers. It follows that the sum of the regime tax and the policy tax or transfer is equal to the value of the money held by the old consumers. Thus, the under the simulator regime, the government creates an old-to-young transfer that is designed to exactly offset the young-to-old transfer associated with monetary transactions. In equilibrium, the net intergenerational transfer is zero, regardless of the monetary policy.[17]

A key feature of the simulator regime is that although it changes the nature of monetary transactions in a fundamental way, it does not change the form of these transactions in any way at all. Young consumers still purchase most of their money from old consumers, and old consumers sell most of their money to young consumers. Under both regimes, the government may conduct monetary policy by contracting or expanding the money supply, with money retirement financed by lump-sum taxes and the proceeds of money issues funding lump-sum transfers. In the former case, old consumers sell some of their money to the government, and in the latter case, young consumers purchase some of their money from the government, just as in the standard regime. In addition, the reserve requirement functions in exactly the same way, under both regimes: banks must use a fixed fraction of their deposits to purchase money (reserves) from the old consumers and (possibly) the government

The definition of a competitive equilibrium under this regime is identical to the definition presented above, except that we require $i^* = 2$, and the regime transfers and taxes T_{1t}^* and T_{2t}^* must satisfy conditions (11) and (12) for all $t \geq 1$; for a stationary equilibrium, we must have $\text{T}_{1t}^* = \text{T}_1^* = -m^*$ and $\text{T}_{2t}^* = \text{T}_2^* = m^*$ for all $t \geq 1$.

As we have indicated, one of our purposes in constructing the simulator regime is to mimic, in an OG model, the conditions under which money is acquired and held in a standard monetary ILRA model. In our OG model, under the simulator regime, each young consumer receives a transfer of goods from the government that is just large enough to finance its desired purchases of money. In a standard monetary ILRA model, the representative agent is endowed with the money it decides to hold in its first period; in equilibrium: this is equivalent to the agent receiving an initial transfer of goods from the government and then using these goods to buy money from the government (so that the government's net expenditure is zero). Thus, the two models share the feature

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goods as capital), the income of the next generation of consumers is increased, because their wage rate rises. Thus, current capital accumulation provides external benefits for members of future generations. Starting from a steady state with zero inflation, a small increase in the inflation rate makes the current generation of consumers worse off, as in our Proposition 2. But it initiates a path of capital accumulation that converges to a new steady state with higher utility. In our endowment-storage model, by contrast, an increase in current storage provides no benefits for members of future generations. Freeman (1993) gets the same result we do, because his model has nonlinear storage rather than capital.

[17] We do not argue that it would be possible for a real-world government to impose such a regime. We study it in order to make a point about the nature of monetary transactions in OG models.

that no consumer actually gives up goods in order to acquire money, although every consumer perceives itself as having the option to do so. In the OG model, under the simulator regime, each old consumer pays a tax that is just large enough, in equilibrium, to absorb the goods it obtains when it sells its money. In effect, it holds the money during its second (and last) period of life without getting any goods in exchange for it. In an equilibrium in a monetary ILRA model, the agent simply holds its money from period to period; equivalently, it sells its beginning-of-period money holdings during the period and uses the goods it obtains from the sale to buy an equal quantity of money to carry in to the next period. It never sells money to finance consumption or investment.

As we have also indicated, once we restructure the monetary regime in a way that produces equilibria without net intergenerational transfers, the optimal monetary policy becomes the Friedman rule:

Proposition 3 *Under the ILRA simulator regime, any Friedman rule ($Z^* = 1/X$) steady state produces higher utility, for the two-period-lived consumers, than the steady state under any other monetary policy. Moreover, the consumption allocation in any Friedman rule steady state is identical to the allocation in an unrestricted steady state (a steady state when the reserve ratio is zero).*

Under the simulator regime, the social opportunity cost of holding money is zero. Consumers can hold money without participating in intergenerational transfers, on net, so purchases of money do not reduce the quantity of goods available for consumption or investment (storage). Under the Friedman rule, the transfer the consumers receive when they are young is equal, in present value, to the total tax (policy tax plus regime tax) they pay when they are old. So they increase their saving, relative to the non-monetary steady state, by the amount of the tax, leaving their intertemporal consumption plan unaltered. And since the private opportunity cost of holding money is zero, they are content to devote all this extra saving to holding money, leaving them undisturbed by the reserve requirement.

If the government chooses a monetary policy that produces a real money return rate lower than the Friedman-rule rate, then the consumer faces a deposit return rate that is lower than the rate of return on storage. Since net intergenerational transfers are zero, the equilibrium consumption choice must continue to lie along the consumer's non-monetary budget constraint, which has an absolute slope equal to the storage return rate. But the consumer chooses the bundle along that constraint at which its marginal rate of substitution is equal to the deposit rate, causing it to consume too much during its first period of life and too little during the second, relative to the social optimum. This tendency to choose excess current consumption is reinforced by the fact that the consumer receives a transfer, when it is young, that is larger, in present value, than the tax it pays when it is old. So it increases its saving, relative to the non-monetary case, by less than the amount of the transfer, causing its current consumption to rise.

Proposition 3 is the same result we would get in an ILRA model with a reserve requirement and storage or capital. The decisions the representative agent makes in that model are very close analogues of the decisions the OG consumers make under the simulator regime, if they face the same monetary policies. Thus, from the viewpoint of monetary analysis, studying the OG model under the simulator regime is not very different from studying an ILRA model. We have seen, however, that under the standard regime, both our welfare results and the consumers' decisions are very different from their simulator-regime counterparts. And since the only difference between the two regimes is the presence vs. absence of intergenerational transfers, it is clear that the presence of these transfers in most monetary OG models is the underlying source of the difference between their predictions and the predictions of monetary ILRA models.

5   Voluntary (random relocation) money demand

In this section, we study a version of our model in which banks hold money because it helps them allay problems caused by a spatial separation friction. This money demand specification was introduced by Champ, Smith and Williamson (1996) and Schreft and Smith (1997).[18]   As we have indicated, we study this "random relocation" specification for two reasons.   First, it allows us to show that our most important results – the suboptimality of the Friedman rule under the standard monetary regime, and the optimality of the Friedman rule under the simulator regime – do not depend on the assumption that money derives its value from a legal restriction. The second reason grows out of the fact that under the standard regime, the optimal policies for this money demand specification are different from the optimal policies for the reserve requirements specification. These differences turn out to be due to differences in the role of intergenerational transfers in the two specifications. Exploring them helps us deepen our understanding of the role of these transfers in monetary OG models.

The economy has two distinct regions. Half the consumers born at any date begin their lives in one region, and half in the other. The initial old consumers are also equally divided across the two regions. At the end of the first period of their lives, a fraction ϕ of the young consumers in each region are relocated to the other region, where they will spend the second period of their lives. This event occurs after the consumers have made their consumption and saving decisions. Each young consumer knows the value of ϕ when it makes these decisions, but it does not know whether it will be relocated.

Involuntary relocation is the only way consumers can travel from one region to the other. Goods cannot be transported across regional boundaries, and neither can information. Thus, if a young consumer stored goods before it was relocated, or if it purchased a claim to the returns on goods stored by a bank in its region, then there is no way it can obtain the goods that are recovered from storage. In addition, banks in one region will not honor claims issued by banks in the other region: there is no way for the banks to verify the validity of such claims, and there is no way for them to obtain compensation from the banks that issued the claims. However, if a consumer is holding money when it is relocated, then it can take the money with it to the other region, where it is indistinguishable from the money already circulating in that region. Moreover, after a consumer finds out it will be relocated, it can visit its bank, if it has one, before the relocation occurs. If the consumer owns a claim to money held by the bank, then the bank can pay off the claim.

Under the circumstances, there are two strategies a consumer can use to enable it to consume if it is relocated. It can acquire and hold money on its own, or it can purchase a claim – possibly, relocation-contingent – to money held by a bank. Similarly, a consumer can enable itself to consume in its second period, if it is not relocated, by storing goods on its own, or by purchasing a claim to the returns on goods stored by a bank. Such a claim is inherently contingent, as it is uncollectable if the consumer is relocated. Of course, if a consumer stores good on its own then it must abandon those goods if it is relocated.

It is readily seen that under these assumptions, it is optimal for consumers to save exclusively by purchasing deposits issued by banks. A consumer may redeem its deposit at the end of its first period, after it finds out whether it will be relocated but before relocation occurs, or during its second period, provided it is not relocated. We assume that the banks can observe whether or not consumers who attempt to redeem their deposits early will be relocated, making it possible for them to offer deposit contracts that permits early redemption only by

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[18] These authors acknowledge their debt to Townsend (1987) and related research on limited communication and spacial separation as a source of demand for fiat money.

consumers that will be relocated. The value and form of the deposit return depends on whether the deposit is redeemed early or late. We let R_{nt} represent the gross real deposit return rate received by a consumer from generation t that redeems early. This return is paid in goods during the consumer's second period of life. Its bank may obtain these goods by recovering goods from storage or by selling money holdings. We let R_{rt} represent the deposit rate for a consumer from generation t that redeems early. This return is paid in the form of money, which the consumer takes to its new region and uses, next period, to purchase goods.

The budget constraints of the two-period lived consumers are now (8) plus a contingent versions of (9):

$$c_{2n,t+1} = R_{nt}^d d_t - (i-1)\tau_{t+1} - \mathrm{T}_{2,t+1}. \tag{20}$$

$$c_{2r,t+1} = R_{rt}^d d_t - (i-1)\tau_{t+1} - \mathrm{T}_{2,t+1}. \tag{21}$$

The consumer chooses d_t in order to maximize its intertemporal expected utility

$$E_t\{U(c_{1t}, c_{2,t+1})\} = u(c_{1t}) + \beta\left[(1-\phi)u(c_{2n,t+1}) + \phi u(c_{2r,t+1})\right]. \tag{22}$$

Given the deposits d_t entrusted to them by the young consumers, banks choose non-negative values m_t and k_t subject to (5). Since the banks are competitive and have zero operating costs, the deposit returns they offer must satisfy the zero-profits conditions

$$R_{nt}^d = \frac{X k_t + (1-\mu_t) R_t^m m_t}{(1-\phi) d_t} \tag{23}$$

$$R_{rt}^d = \frac{\mu_t R_t^m m_t}{\phi d_t} \tag{24}$$

Here $\mu_t \in [0,1]$ represents the fraction of banks' money holdings m_t that they pay to early withdrawers. To compete successfully for deposits, the banks must choose m_t, k_t, and μ_t, subject to (5), (25)-(26) and (22)-(23) in order to maximize the consumers' second-period expected utility

$$V(c_{2,t+1}) \equiv E_t\{u(c_{2,t+1})\} = (1-\phi)u(c_{2n,t+1}) + \phi u(c_{2r,t+1}). \tag{25}$$

Under this money demand specification, the fraction of banks' deposits that they use to purchase money may depend on the size of consumers' deposits d_t, while the return rates R_{nt}^d and R_{rt}^d on which consumers base their deposit decision depend on this fraction. In equilibrium, the value of d_t that the banks take as given is the optimal value for the consumers, given the values of R_{nt}^d and R_{rt}^d produced when the banks choose m_t and k_t optimally based on that value.

If $R_t^m < X$ then it is optimal for the banks to choose $\mu_t = 1$, so that they pay out all their money holdings to early redeemers (who will be relocated). In this case, the zero-profits return conditions reduce to

$$R_{nt}^d = \frac{X k_t}{(1-\phi) d_t} \quad \Rightarrow \quad R_{nt}^d d_t = \frac{X}{1-\phi} k_t \tag{26}$$

$$R_{rt}^d = \frac{R_t^m m_t}{\phi d_t} \quad \Rightarrow \quad R_{rt}^d d_t = \frac{R_t^m}{\phi} m_t \tag{27}$$

If $R_t^m = X$ then it is optimal for the banks to choose m_t, k_t, and μ_t so that $R_{nt} = R_{rt} = X$. In this case, m_t and k_t are not uniquely determined, although we must have $m_t \geq \phi d_t$. If $R_t^m > X$ then it is optimal for the banks to choose $m_t = d_t$, so that $R_{nt} = R_{rt} = R_t^m$.

19

5.1 The standard monetary regime

Again, we study competitive equilibria. For this money demand specification, a competitive equilibrium can be defined as in section 4.1, with the following revisions: The relocation probability ϕ replaces the reserve ratio θ. The deposit return rate R_t^{d*} is state-contingent: $R_t^{d*} = (R_{nt}^{d*}, R_{rt}^{d*})$. For the two-period-lived consumers, $c_{2,t+1}^*$ is also state-contingent: $c_{2,t+1}^* = (c_{2n,t+1}^*, c_{2n,t+1}^*)$. These consumers' second-period budget constraints are now conditions (22)-(23), which replace condition (9), and they maximize their expected utility given by function (24), the explicit expected-utility version of (1). The list of bank choice variables now includes $\{\mu_t^*\}_{t=1}^{\infty}$, where we require $\mu_t^* \in [0,1]$ for all $t \geq 1$. For each $t \geq 1$, the values μ_t^*, m_t^* and k_t^* maximize objective (27), given d_t^* and conditions (25)-(26) and (22)-(23); conditions (13) and (14) are discarded. The contingent deposit return rates.$(R_{nt}^{d*}, R_{rt}^{d*})$ must satisfy conditions (25)-(26), which replace conditions (15) and (16). For a stationary equilibrium, $\mu_t^* = \mu^*$, $(c_{2n,t+1}^*, c_{2r,t+1}^*) = (c_{2n}^*, c_{2r}^*)$ and $(R_{nt}^{d*}, R_{rt}^{d*}) = (R_n^{d*}, R_r^{d*})$ for all $t \geq 1$.

The biggest difference between this money demand specification and the specification we studied in section 4 is that under this specification, money is useful to consumers, even though there are no legal restrictions that require or encourage its use. In this physical environment, money is the only asset that is useful to consumers that have been relocated. For this reason, banks will hold money, voluntarily, even when it has a lower return rate than storage. A related difference involves the timing of the usefulness of money to consumers. Under our legal restrictions specification, it is the *ex ante* (first period) real value of consumers' money holdings that matters for the purpose of satisfying the reserve requirement. The *ex post* real value of these holdings determines their usefulness as a store of value, but not as required reserves. Under our voluntary money demand specification, money derives its usefulness from the fact that consumer who have been relocated can use it to purchase goods when they are old. So the extent of its usefulness depends entirely on its *ex post* value.[19]

As we have seen, under the standard monetary regime zero inflation has presumption as the optimal policy. If transactions important for consumers' welfare are difficult or impossible to conduct without holding money, which is true under virtually any money demand specification, and if holding money requires participating in intergenerational transfers, which is true under the standard regime, then it seems natural for the optimal return rate on money to be the technological return rate on intergenerational transfers. However, under our legal restrictions specification, where it is the *ex ante* real value of the money that matters for satisfying the reserve requirement, a monetary policy featuring inflation-financed transfers to the young consumers ($i = 1$) reduces the size of the net intergenerational transfer per unit of real money holdings, and thus the social opportunity cost of holding that unit, without reducing the unit's effectiveness in satisfying the requirement. Stated differently, the distinctive usefulness of money depends on the magnitude of m, but the size of the intergenerational transfer depends on the magnitude of $m + \tau$. Consequently, a policy that reduces $m + \tau$ without reducing m (or without reducing m as much) may be welfare-improving, at the margin.

Under the voluntary money demand specification we study in this section, the *ex post* real value of the money is the only thing that matters to its holders. As a result, reducing the size of the net intergenerational transfer associated with holding money reduces the money's usefulness to exactly the same extent: both depend on the

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[19]This difference is not an inherent feature of legal restrictions vs. voluntary money demand specifications. It is possible to construct a legal restrictions specification in which the *ex post* real value of the money matters for the purpose of meeting the restriction, and it is possible to construct a voluntary specification in which the usefulness of money as anything other than a store of value depends on its *ex ante* real value.

magnitude of $m + \tau$.[20] We can infer that under this specification, zero inflation should be optimal when $i = 1$. And this turns out to be the case:

Proposition 4 *Suppose $i = 1$. A policy of zero inflation ($Z^* = 1$) supports a steady state that produces a higher level of welfare, for the two-period lived consumers, than the steady state supported by any other policy.*

When $i = 1$, the old consumers who are relocated obtain all the goods they consume from the young consumers, which is to say, through intergenerational transfers. Old consumers who have been relocated can obtain goods only by selling money. Young consumers are the only ones who purchase money, and they also pay the taxes that augment the return on money (if $R^m > 1$). The case $i = 2$ (taxes on the old) is different, in this regard, and it gives us a new type of welfare result. The distinctive characteristic of this case is that under a negative-inflation policy ($R^m > 1$) the taxes that are levied on the old consumers in order to augment the return on money allow money to play a limited role as a device for *intra*generational transfers: transfers from old consumers who have not been relocated to old consumers who have been relocated. Old consumers who have not been relocated pay some of these taxes, while old consumers who have not been relocated hold all the money whose return is increased. Transfers between the old consumers are a relatively efficient way to insure young consumers against the possibility of being relocated. They allow consumers who have been fortunate to compensate consumers who have been unfortunate, and they do not require substituting intergenerational transfers for storage.

If money derives its usefulness as insurance only from its role as a device for arranging intergenerational transfers, as is the case when $i = 1$, then the optimal rate of return on money is the technological return rate on intergenerational transfers, and zero inflation is optimal. If $i = 2$, however, then when $R^m > 1$ money has additional insurance usefulness as a device for arranging intragenerational transfers, and this aspect of its usefulness becomes more important as the real return rate on money increases. We can infer that the optimal inflation rate should be negative, rather than zero. Again, this prediction turns out to be correct. In fact, for some parameter combinations, the optimal monetary policy is the Friedman rule.

Proposition 5 *Suppose $i = 2$. [A] There are negative-inflation policies ($Z^* < 1$) that support steady state equilibria that produce higher levels of expected utility, for the two-period lived consumers, than the policy that produces zero inflation ($Z^* = 1$). [B] Let $m(R^m)$ represent the equilibrium money demand function. The condition $m'(X) > 0$ is sufficient for the Friedman rule to produce lower steady state utility than some policies that produce lower inflation, and failure of this condition is necessary for the Friedman rule to maximize steady state utility. [C] For given values of the other parameters of the model, it is always possible to choose a value γ high enough so that $m'(X) < 0$. If $\gamma \geq 1$, then it is always possible to choose X or β high enough so that $m'(X) < 0$. But if $\gamma < 1$, then there may be no values of X or β such that $m'(X) < 0$. [D] It is possible to construct examples involving $\gamma \gtreqless 1$ in which a policy that produces a negative inflation rate higher than the Friedman-rule rate ($1/X < Z^* < 1$) maximizes steady state utility, and it is possible to construct examples involving $\gamma \gtreqless 1$ in which the Friedman rule ($Z^* = 1/X$) maximizes steady state utility.*

Note that the equilibrium money demand function is obtained by starting with the consumer money demand function, which includes τ and R^m among its arguments, solving the equilibrium condition $\tau = (R^m - 1) m(R^m, \tau)$ for τ, and then substituting the solution into $m(R^m, \tau)$.

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[20] Note that in a steady state, government budget constraint (3) can be written $(R^m - 1) m = \tau$, which implies $R^m m = m + \tau$.

Proposition 5 is more complicated, and much more highly qualified, than the other formal results in this paper. The reason for this is that we have been unable to obtain global results about the characteristics of the indirect expected utility function $U(R^m)$ [see the appendix]. We can show that $U'(R^m) > 0$ at $R^m = 1$, which implies that zero inflation is not steady state optimal. But we cannot show that $U'(R^m) > 0$ for $R^m \in (0, 1)$, which means that we cannot rule out the possibility that there are specifications in which positive inflation is optimal. Similarly, we can show that for some specifications, $U'(R^m) > 0$ at $R^m = X$, which suggests that the Friedman rule may be optimal. But we cannot show that if $U(R^m)$ has this property then it is monotone increasing over $R^m \in (1, X)$. So we cannot rule out the possibility that there are specifications in which $U'(X) > 0$ but the Friedman rule is not optimal.[21] On the other hand, we have been unable construct numerical examples in which $U(R^m)$ is anything other than monotone increasing on $R^m \in (0, 1)$, or in which the Friedman rule fails to be optimal when $U'(X) > 0$. So we strongly suspect that negative inflation is always optimal, and that $U'(X) > 0$ [and thus, $m'(X) < 0$] is necessary and sufficient for the Friedman rule to be optimal.

Proposition 5 provides a counterexample to the claim, which is sometimes made, that the Friedman rule is never steady-state optimal in standard-regime monetary OG models. It is the only case in this paper where the Friedman rule can be optimal under the standard regime, and it is the only case of this type we know of in the literature. Paradoxically, however, the condition $m'(X) < 0$, which is necessary [and sufficient?] for the optimality of the Friedman rule, helps illustrate the logic behind its usual failure to be optimal. When this condition holds, a marginal increase in the inflation rate, starting from the Friedman-rule rate, actually increases the level of equilibrium real money balances, which means that it increases the size of one component of the transfer from young to old consumers that occurs under the standard regime.[22] As we have seen, the tendency of increases in the inflation rate to reduce the size of this transfer explains why the optimal policy typically features an inflation rate higher than the Friedman-rule rate. So it seems natural that cases in which the Friedman rule is optimal involve situations in which consumer money demand behavior works against this tendency.[23]

Inspection of condition (65) from the proof of Proposition 5, which is a necessary and sufficient condition for $m'(X) < 0$, may seem to indicate that constructing economies in which the Friedman rule is optimal requires choosing extreme combinations of parameters. In the log-preferences case ($\gamma = 1$) the condition reduces to $X - 1 > [\beta(1 - \phi)]^{-1}$, which seems to require rather large values of X for plausible values of β and ϕ. Suppose, for example, that we choose $X = 1.06$, $\beta = 0.96$, and $\phi = 0.1$, which seem superficially plausible. For these values, the condition fails badly: the threshold value of X, denoted $\overline{X}$ in the proof, is almost 1.16.

There are, however, at least three reasons why the case for the Friedman rule is much stronger than this example suggests. First, in OG models the plausible values of X are much larger than they might seem, and the plausible values of β are much smaller. The values we used in the preceding paragraph are annual values, and a period in an OG model represents something on the order of 30 years! If we choose $X = (1.06)^{30} \doteq 5.74$ and $\beta = (0.96)^{30} \doteq 0.294$, the baseline values for the three examples in the proof of Proposition 5, then the log-case version of condition (65) holds, and the Friedman rule is optimal when $\gamma = 1$.

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[21] In the log-preferences case [$u(c) = \ln c$] we can show that $U(R^m)$ is monotone increasing on $(0, R^m)$, which means that some negative inflation rate is optimal. But we cannot characterize $U(R^m)$ on $R^m \in (1, X)$, which means that we cannot state a definitive condition under which the Friedman rule is optimal.

[22] The other component of this transfer, the tax that the young households pay to augment the old household's return on money, may fall or rise in size, depending on whether $\tau'(X) = m(X) + (X - 1)m'(X)$ is negative or positive.

[23] Our results also suggest that the Friedman rule may be optimal, under the standard regime, under other money demand specifications in which it is the *ex post* value of real money balances that matters to households. One example might be a specification in which a portion of households' second-period consumption must be financed by money acquired in the first period.

A second point is that condition (65) becomes easier to meet very quickly as γ rises above unity. In one of our examples, we report that if γ is increased to 1.02, holding β at $(0.96)^{30}$ and ϕ at 0.1, then the threshold value of X falls from 5.74 to 2.54 (1.032, annualized). Indeed, if $\gamma \geq 1.1$ then condition (65) holds for all $X > 1$. Since the empirical literature that tries to estimate γ frequently turns up values between 2 and 4, these calculations suggest that Friedman-rule optimality is quite plausible.

The third and final point, which is also evident from our examples, is that even when condition (65) fails by a wide margin, the optimal inflation rate is often very close to the Friedman-rule rate, and the indirect expected utility function, which is usually rising rapidly at $R^m = 1$, is typically quite flat between the two rates. Thus, a policymaker who chose the Friedman rule instead of the optimal policy would usually come quite close to the optimal level of expected utility – much closer, in most cases, than if he chose an inflation rate close to zero.

5.2 The simulator regime

Under this money demand specification, the simulator regime works in the same way it did under the reserve requirements specification, with one exception. Each young consumer continues to receive a transfer that is equal, in equilibrium, to the value of the money it purchases, and each old consumer alive at the same date continues to pay a tax with the same value as the transfer. However, the tax is no longer equal to the *ex ante* (before appreciation or depreciation) value of the money held by the banks on behalf of each old consumer. Under this new specification, the old consumers hold money directly – unlike the reserve requirements specification, where the banks hold money on behalf of the old consumers, sell the money to the young consumers, and pay the goods to the old consumers as part of the return on their deposits. However, the only old consumers who hold money are the ones have been relocated. The tax is equal to the average *ex ante* value of the old consumers' money holdings. Thus, it is larger than the *ex ante* value of the money holdings of the consumers who have not been relocated (which is zero), and it is smaller than the *ex ante* value of the money holdings of the consumers who have been relocated.

The definition of a competitive equilibrium is modified in exactly the same way as under the reserve requirements specification.

Again, the optimal monetary policy is also the Friedman rule, and again, the consumption allocation under the Friedman rule is identical to the equilibrium consumption allocation in an analogous economy without monetary features (an economy in which no consumers are relocated):

Proposition 6 *Under the ILRA simulator regime, any Friedman rule ($Z^* = 1/X$) steady state produces higher utility, for the two-period-lived consumers, than the steady state under any other monetary policy. Moreover, the consumption allocation in any Friedman rule steady state is identical to the allocation in a zero-transactions-costs steady state (a steady state when the relocation probability is zero).*

As we noted above, the optimal solution to the problem created by the relocation friction is an insurance arrangement under which old consumers that have not been relocated make payments to old consumers that have been relocated.[24] Implementing this insurance scheme directly is not feasible: old consumers who are relocated cannot receive payments from insurance firms in their original locations, and they cannot credibly

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[24] Our results do not extend to models in which intergenerational transfers play an indispensable insurance role, such as the models studied by Sargent (1987, Ch. 8). In these models, imposing the simulator regime deprives money of value and reduces steady state welfare, relative to the standard regime.

reveal to firms in the other location, or even to the government, that they have been relocated. But the simulator regime allows these payments to occur indirectly. As we have also noted, only old consumers who have been relocated carry money into their second life-periods. Abstracting from new money that might be issued by the government, the young consumers purchase all their money from these consumers. These purchases represent a large transfer of goods from the young consumers to the old consumers who have been relocated. The young consumers are compensated for the goods they pay to the old relocated consumers by the transfer they receive from the government. However, this transfer is financed by equal taxes on both groups of old consumers. On net, the *inter*generational transfer from the young consumers to the old consumers is zero, but there is a net *intra*generational transfer from the old unrelocated consumers to the old relocated consumers. Under the Friedman rule, the insurance is complete: both groups of old consumers face a return rate of X. In addition, the present value of the future taxes the consumers pay when they are old is equal to the value of the transfer they receive when they are young, so they use the transfer to increase their saving, by holding money. As a result, the consumption allocation is the same as the allocation in an otherwise-identical economy with no relocation friction

When the inflation rate is higher than the Friedman-rule rate, the insurance becomes incomplete: old consumers that are relocated get a lower return rate than the other old consumers, because money has a lower return rate than storage. Unless consumers are quite risk averse, the attractiveness of money holding declines, and so, to a lesser extent, does the attractiveness of saving. On the other hand the present value of the future taxes to be levied on the consumer (regime tax plus policy tax) is now smaller than the value of the regime transfer. The consumer perceives its net wealth as increased, relative to the Friedman-rule steady state; consequently, it wants to increase its consumption in both periods. In the typical case, the net effect of these changes on desired real money holdings is negative: at the old value of the regime transfer, desired money holdings would fall short of the value of the transfer. So the value of regime transfer must fall: as it does, desired real money holdings fall more slowly, until the two are equalized at a lower level. The price level rises until the real money supply is equal to the real demand for money. The new steady state is suboptimal both because second-period consumption is now risky and because the decrease in the expected return rate on saving (caused by the decreased real money return rate) leads the consumer to save less, and consume more in the first-period, than it would save and consume in the absence of the return distortion.

6 Conclusion

6.1 Summary

In this paper, we identify the basic difference between the infinitely-lived representative agent (ILRA) model and the overlapping generations (OG) model, as models of money. The point of departure for our investigation is the difference between the predictions of these two models regarding the nature of the optimal monetary policy. In most monetary ILRA models, the optimal monetary policy is the Friedman rule. Under this policy, the government contracts the stock of money at a rate that drives the real rate of return on fiat money up to the level of the real return rate on nonmonetary assets, so that the nominal interest rate on those assets is zero. Although there has been much less research on the optimal monetary policy in overlapping generations models, the literature that does exist suggests that the policy that maximizes steady state welfare (the most widely used welfare criterion) almost always produces a real return rate on money that is lower than the Friedman rule rate. Very often, the optimal policy produces zero inflation, or even positive inflation, instead of Friedman-rule

deflation.

In our formal analysis, we study a simple, linear-storage overlapping generations model under two different money demand specifications: a legal restrictions specification (reserve requirements) and a transactions costs specification (random relocation). Throughout our analysis, the government conducts monetary policy by increasing or decreasing the money supply at a constant rate and using lump-sum transfers or taxes to distribute seigniorage revenue or finance money retirement. These transfers/taxes may be paid to/levied on the young consumers or the old consumers. Initially, the government introduces money into the economy in the same way that has been studied in most of the literature: we call this the "standard monetary regime." Under our first money demand specification, we find, as we expected, that the monetary policy that maximizes steady state welfare does indeed produce a real money return rate that is lower than the Friedman rule rate. However, the exact nature of the optimal policy depends on which age group receives or pays the transfers or taxes. If it is the young consumers, then the optimal inflation rate is positive; if it is the old consumers, then the optimal inflation rate is zero. Under our second money demand specification, we find that if the transfers/taxes go to/from the young consumers, then zero inflation is optimal. However, if they go to/from the young consumers, then negative inflation is optimal; in some cases, moreover, the optimal policy is the Friedman rule.

A careful examination of these results reveals that there is a single underlying reason why they are different from each other, and also different from the analogous results in ILRA models. The difference is the fact that under the standard monetary regime, when consumers that acquire, hold and sell fiat money they are participating in intergenerational transfers. These transfers, which are inherently absent from ILRA models, have important effects on real allocations because they force consumers to divert resources away from real investment (here, storage), a more productive method for converting current income into future consumption. In a standard-regime monetary OG model, policy-induced changes in the real return rate on money affect consumers' welfare in two ways that have no analogues in ILRA models. First, changing the opportunity cost of holding money tends to change the real value of consumers' money holdings, which changes the scale of the intergenerational transfers. Second, if the transfers or taxes that accompany changes in the money supply are paid to or by the young consumers, then changes in the rate of return of money change the magnitude of the intergenerational transfer per unit of real money holdings, causing the change in the scale of the intergenerational transfers to be larger or smaller than it would be otherwise.

In order to confirm the accuracy of our diagnosis, we construct an alternative monetary/fiscal regime that allows money to remain useful and valued, but eliminates its role as a vehicle for intergenerational transfers. We call this regime the "ILRA simulator regime" because it introduces money into the OG model in essentially the same way it is usually introduced into ILRA models. We find that under the simulator regime, the Friedman rule is the optimal monetary policy under either of our money demand specifications.

6.2   Implications, empirical relevance, and future research

We hope our results serve to highlight a fact that has been revealed by other researchers but has never, in our view, received sufficient attention: the predictions of monetary OG models are systematically and importantly different from the predictions of monetary ILRA models, even when the OG models permit money to be dominated in rate of return. We have also identified the source of this difference: in monetary OG models, under the monetary regimes usually studied, monetary transactions represent intergenerational transfers.

Our results point to a number of specific issues about which OG models offer predictions and prescriptions quite different from those of ILRA models. The most obvious of these is that monetary ILRA models usually predict that the Friedman rule is optimal, while monetary OG models usually do not. Thus, monetary OG models may help explain why we do not see any countries conducting their monetary policies so as to create significant deflation, and we see few countries paying interest on bank reserves. Along the same lines, monetary ILRA models usually predict that inflation is costly, while monetary OG models often predict that positive inflation is optimal. Thus monetary OG models may help explain why most countries – even countries that are viewed as quite conservative, monetarily – conduct their monetary policies in ways that produce average inflation rates that are well above zero.

Another difference in predictions, which is closely related, concerns the magnitude of the costs of deviating from the optimal monetary policy. Monetary ILRA models usually predict that the "cost of inflation" is positive but extremely small. The costs are driven entirely by distortions in the return rates on money and other assets that are produced or exacerbated by inflation. But the prediction that the level of the inflation rate does not matter very much is hard to reconcile with the controversies inflation has often produced in our society. Monetary OG models provide an additional source of welfare costs and benefits from policy-induced changes in the inflation rate that is not present in ILRA models: these changes may be associated with changes in the volume of inefficient intergenerational transfers. This new source of costs and benefits has the potential to produce much larger estimates of the magnitudes of the costs or the benefits. We have done some preliminary work along these lines: our results indicate that in OG models, the costs or benefits of a given change in the inflation rate under a standard monetary regime are much larger than in otherwise-identical models under a simulator regime (which, as we have seen, produces result analogous to those from ILRA models).[25]

Predictions about the welfare costs or benefits of changes in monetary policy are usually closely related to predictions about the effects of monetary policy on variables like the real interest rate, the capital stock, and the level of output. Models that produce large welfare effects typically predict that the effects of policy changes on these variables are fairly large, and vice versa. There is a small but growing body of evidence which indicates that persistent changes in monetary policy may indeed have the power to produce persistent and significant changes in the values of these variables.[26] Monetary OG models are capable of delivering predictions consistent with this evidence.[27] We hope the results reported in this paper will help stimulate additional research, using calibrated OG models, on the nature and magnitude of the effects of persistent changes in monetary policy on welfare and on the values of key macroeconomic variables.

Of course, monetary policy effects that are driven by changes in the volume of intergenerational transfers are likely to be empirically important only if intergenerational transfers are empirically important. Many readers are likely to suspect that the importance of intergenerational transfers in OG models depends on the assumption that

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[25] Bullard and Russell (2004) study an OG model in households live for 55 periods. They calibrate the model to produce steady states consistent with postwar U.S. averages. Unlike our model, their model is calibrated to produce steady states that are dynamically inefficient: as a result, increases in the inflation rate always reduce welfare. Their estimates of the costs of increasing the inflation rate are an order of magnitude larger than most estimates produced by ILRA models.

[26] See, for example, Bullard and Keating (1995), King and Watson (1997), Ahmed and Rogers (2000), Rapach (2003) and Rapach and Wohar (2003).

[27] Espinosa and Russell (1998, 2002) use an "unpleasant arithmetic" analysis to explain why it is possible for changes in monetary policy to have relatively large effects in OG models with money, bonds, and a nontrivial government budget constraint. Bullard and Russell (2004) construct a calibrated multi-period version of this model; they report large welfare effects (see above) and also fairly large real effects on real interest rates, investment, and output.

consumers live for only two periods. It is important for us to point out that this is not the case. Bullard (1992) and Bullard and Russell (1999) have shown that intergenerational transfers remain important on OG models even when consumers live for many periods.[28] In addition, it is sometimes argued that government liabilities are not necessary to facilitate intergenerational transfers, because these transfers are conducted via altruistic gifts and bequests. However, there is a wealth of empirical evidence which indicates that the types of gifts and bequests that would eliminate the need for government-facilitated intergenerational transfers are not important empirically.[29]

Our results may also suggest that there is a need for further research on properties of different monetary regimes and the ways in which they influence the effects of monetary policy rules or actions. We have shown that in OG models, the choice of a monetary regime may have effects on the predictions of a monetary policy analysis that are at least as important as the effects of the choice of a money demand specification.

To our knowledge, the question of the best way to model real-world monetary regimes has not been addressed in any very systematic fashion. Under the standard regime, fiat money is issued by a government that does not manage it in any way except to control its supply. In the modern United States, however, fiat money is issued by a central bank (the Fed) that acquires a dollar's worth of assets – principally, Treasury securities – for every dollar of its money liabilities. Under these circumstances, the nature of the monetary regime depends largely on the backing of these Treasury securities. If they are unbacked – if the Treasury intends to roll them over forever, using earnings turned over to it by the Fed to cover the interest expense – then the Treasury might as well have issued the money directly, as it did during and after the Civil War, and the U.S. monetary regime is appropriately modeled as a standard regime. But if the securities are fully backed by future tax revenue, so that the Fed is a genuine financial intermediary, then the U.S. monetary regime is appropriately modeled as simulator regime: it can be shown that the simulator regime is equivalent to a regime featuring intermediated money.[30] We do not know how to use observations to distinguish one possibility from the other, but our results indicate that it may be an important question for future research.

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[28] More specifically, they show that it is possible for plausibly calibrated multi-period OG models to equilibria in which outside government liabilities (fiat currency or unbacked debt) have value in the absence of transactions frictions. This happens because these liabilities become the vehicles for intergenerational transfers that could not occur otherwise.

[29] For a brief survey of this evidence, see Bullard and Russell (1999).

[30] Proofs of this assertion, for both our money demand specifications, are available from the authors. Under a standard intermediated-money regime, consumers purchase money from a central bank in exchange for goods, and the bank uses these goods to acquire physical assets. The bank redeems the currency, next period, out of the returns on these assets. Consequently, consumers can purchase and liquidate money without participating in intergenerational transfers. See Freeman and Haslag (1996) for an example of an economy with intermediated money in which an open market operation undoes the transfer to the initial old that accompanies a policy implementing paying interest on required reserves..

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Proof of Proposition 1:

In any steady state equilibrium, we can view a consumer as choosing c_1, c_2 and d to maximize

$$U(c_1, c_2) = u(c_1) + \beta\, u(c_2)\,, \tag{28}$$

the nonstochastic steady-state version of (1), subject to

$$c_1 = \omega - \tau - d \tag{29}$$
$$c_2 = R^d\, d\,, \tag{30}$$

the steady-state versions of (8) and (9) under the standard regime. The government chooses τ so that

$$\tau = (R^m - 1)\, m\,, \tag{31}$$

which is the steady state version of (3).

In a steady state with a binding reserve requirement ($R^m < X$), the banks choose m and R^d by

$$m = \theta\, d \text{ and} \tag{32}$$
$$R^d = (1 - \theta)X + \theta\, R^m\,, \tag{33}$$

which are the steady-state versions of (14) and (15). We can view the first-order condition

$$u'(\omega - \tau - d) = \beta\, R^d\, u'(R^d\, d)\,, \tag{34}$$

along with the equation for the banks' choice of R^d, as implicitly defining the consumer deposit demand function $d(R^m, \tau)$. [Note that we can use (35) to write $R^d(R^m)$, with $(R^d)' = \theta$.] The equation

$$\tau = (R^m - 1)\, \theta\, d(R^m, \tau)\,, \tag{35}$$

obtained by combining (33) and (34), then implicitly defines the equilibrium functions $\tau(R^m)$ and $d(R^m)$. The associated indirect utility function is $U(R^m) \equiv u(\omega - \tau(R^m) - d(R^m)) + \beta\, u(R^d(R^m)\, d(R^m))\,.$

We have

$$
\begin{aligned}
U'(R^m) &= -u'(\omega - \tau - d)\,[\tau'(R^m) + d'(R^m)] + \beta\, u'(R^d\, d)\,[\theta\, d + R^d\, d'(R^m)] \\
&= d'(R^m)\,[-u'(\omega - \tau - d) + \beta\, R^d\, u'(R^d d)] + \theta\, d\, \beta\, u'(R^d\, d) - u'(\omega - \tau - d)\, \tau'(R^m) \\
&= \theta\, d\, \beta\, u'(R^d\, d) - u'(\omega - \tau - d)\, \tau'(R^m)
\end{aligned}
$$

using first order condition (36). We also have, using (37),

$$\tau'(R^m) = \theta\, d(R^m) + (R^m - 1)\, \theta\, d'(R^m)\,. \tag{36}$$

Notice that when $R^m = 1$ we have $\tau' = \theta\, d > 0$ and $U' = \theta\, d\,[\beta\, u'(R^d\, d) - u'(\omega - \tau - d)]$. But first-order condition (36) allows us to rewrite this equation as $U' = \beta\, \theta\, d\, u'(R^d\, d)\,[1 - R^d]$. Thus, $U' < 0$, since $R^d > 1$.

More generally, we have

$$
\begin{aligned}
U'(R^m) &= \theta\, d\, \beta\, u'(R^d\, d) - u'(\omega - \tau - d)\, \tau'(R^m) \\
&= \theta\, d\, \beta\, u'(R^d\, d) - u'(\omega - \tau - d)\, [\theta\, d + (R^m - 1)\, \theta\, d'(R^m)] \\
&= \theta\, \{d\,[\beta\, u'(R^d\, d) - u'(\omega - \tau - d)] - u'(\omega - \tau - d)\, (R^m - 1)\, d'(R^m)\} \\
&= \theta\, \{d\,[\beta\, u'(R^d\, d) - \beta\, R^d\, u'(R^d\, d)] - \beta\, R^d\, u'(R^d\, d)\, (R^m - 1)\, d'(R^m)\} \\
&= \theta\, \beta\, u'(R^d\, d)\, \{d\,[1 - R^d] - R^d\, (R^m - 1)\, d'(R^m)\}\,, \tag{37}
\end{aligned}
$$

using (36) and (38). For relatively low inflation ($R^m > 1$), the first term in the final expression in set brackets is negative. The second term will also be negative if $d'(R^m) > 0$, which will be true for small values of γ (see below). But if $\gamma \geq 1$ then $d'(R^m) < 0$ (see below). In that case, we have offsetting effects.

For $U'(R^m) < 0$ we need $d(R^m)\left[1 - R^d\right] < R^d\left(R^m - 1\right)d'(R^m)$. Note that since $R^d \geq R^m$, if $R^m > 1$ then $R^d > 1$, in which case this condition is

$$d'(R^m) \;>\; \frac{d(R^m)\left[1 - R^d\right]}{R^d\left(R^m - 1\right)}, \quad \text{or}$$

$$-R^d \frac{d'(R^m)}{d(R^m)} \;<\; \frac{R^d - 1}{R^m - 1}. \tag{38}$$

It is readily seen that

$$d(R^m) = \frac{\omega\left(\beta R^d\right)^{\frac{1}{\gamma}}}{R^d + \left[\theta R^m + (1 - \theta)\right]\left(\beta R^d\right)^{\frac{1}{\gamma}}} = \frac{\omega}{\beta^{-\frac{1}{\gamma}}\left(R^d\right)^{1 - \frac{1}{\gamma}} + \left[\theta R^m + (1 - \theta)\right]},$$

and

$$d'(R^m) = \frac{-\theta\,\omega\left[\left(\beta^{-\frac{1}{\gamma}}\right)\left(1 - \frac{1}{\gamma}\right)\left(R^d\right)^{-\frac{1}{\gamma}} + 1\right]}{\left[\beta^{-\frac{1}{\gamma}}\left(R^d\right)^{1 - \frac{1}{\gamma}} + \left\{\theta R^m + (1 - \theta)\right\}\right]^2}.$$

It follows that

$$-\frac{R^d d'(R^m)}{d(R^m)} = \frac{\theta R^d\left[\left(1 - \frac{1}{\gamma}\right) + \left(\beta R^d\right)^{\frac{1}{\gamma}}\right]}{R^d + \left[\theta R^m + (1 - \theta)\right]\left(\beta R^d\right)^{\frac{1}{\gamma}}}.$$

So we need to show that

$$\frac{\theta R^d\left[\left(1 - \frac{1}{\gamma}\right) + \left(\beta R^d\right)^{\frac{1}{\gamma}}\right]}{R^d + \left[\theta R^m + (1 - \theta)\right]\left(\beta R^d\right)^{\frac{1}{\gamma}}} < \frac{R^d - 1}{R^m - 1}$$

for $R^m > 1$.

The preceding expression can be shown to be equivalent to

$$\left(\beta R^d\right)^{\frac{1}{\gamma}} + R^d\left(1 - \frac{1}{\gamma}R^d\right) < \left[(1 - \theta)X + \theta\right]\left[\left(\beta R^d\right)^{\frac{1}{\gamma}} + R^d\left(1 - \frac{1}{\gamma}\right)\right].$$

If $\gamma \geq 1$, then this expression is equivalent to

$$\frac{\left(\beta R^d\right)^{\frac{1}{\gamma}} + R^d\left(1 - \frac{1}{\gamma}R^d\right)}{\left(\beta R^d\right)^{\frac{1}{\gamma}} + R^d\left(1 - \frac{1}{\gamma}\right)} < (1 - \theta)X + \theta,$$

which follows from $R^d > 1$. The same is true if $\gamma < 1$ but $\left(\beta R^d\right)^{\frac{1}{\gamma}} + R^d\left(1 - \frac{1}{\gamma}\right) > 0$. If $\gamma < 1$ and $\left(\beta R^d\right)^{\frac{1}{\gamma}} + R^d\left(1 - \frac{1}{\gamma}\right) < 0$, then the inequality in the expression is reversed. In this case, both the numerator and the denominator are negative. The LHS expression can be written

$$\frac{\left[\left(\beta R^d\right)^{\frac{1}{\gamma}} + R^d\right] - \frac{1}{\gamma}\left(R^d\right)^2}{\left[\left(\beta R^d\right)^{\frac{1}{\gamma}} + R^d\right] - \frac{1}{\gamma}R^d}.$$

Note that $R^m > 1$ implies $R^d > (1 - \theta)X + \theta$ So it would be sufficient to show that

$$\frac{\left[\left(\beta R^d\right)^{\frac{1}{\gamma}} + R^d\right] - \frac{1}{\gamma}\left(R^d\right)^2}{\left[\left(\beta R^d\right)^{\frac{1}{\gamma}} + R^d\right] - \frac{1}{\gamma}R^d} \;>\; R^d, \quad \text{or}$$

$$\left[\left(\beta R^d\right)^{\frac{1}{\gamma}} + R^d\right] - \frac{1}{\gamma}\left(R^d\right)^2 \;<\; \left[\left(\beta R^d\right)^{\frac{1}{\gamma}} + R^d\right]R^d - \frac{1}{\gamma}\left(R^d\right)^2,$$

which is equivalent to $R^d > 1$. So we have our result.

Note that our result that $U'(R^m) < 0$ when $R^m \geq 1$ applies to initial steady states that satisfy $R^m < X$, and to marginal decreases in R^m starting from an initial Friedman-rule steady state in which $R^m = X$ and $m = \theta d$. However, it does not apply to marginal increases in R^m starting from the latter steady state, and it does not apply to initial Friedman-rule steady states in which $m > \theta d$ – steady states in which the reserve requirement is not binding. In addition, it does not apply to initial steady states in which $R^m > X$ and $m = d$ – steady states in money dominates storage in return rate. Thus, in order to show that positive inflation is always steady-state optimal, we need to show that [1] any steady state with $R^m = X$ delivers higher steady state utility than any steady state with $R^m > X$, and [2] the steady state with $R^m = X$ and $m = \theta d$ delivers higher steady state utility than any other steady state with $R^m = X$.

[1] The Friedman rule vs. higher money return rates

Under money return rates higher than the Friedman-rule rate ($R^m > X$), it is optimal for the banks to choose $m = d$, and it follows that $R^d = R^m$. The consumer budget constraints then become $c_1 = \omega - \tau - m$ and $c_2 = R^m \, m$. The first-order condition is

$$u'(\omega - \tau - m) = \beta \, R^m \, u'(R^m \, m) \, ; \tag{39}$$

it implicitly defines the consumer money demand function $m(R^m, \tau)$. The government chooses τ by (33), as before. The equation $\tau = (R^m - 1) \, m(R^m, \tau)$ then implicitly defines the equilibrium functions $\tau(R^m)$ and $d(R^m)$. The associated indirect utility function is $U(R^m) \equiv u(\omega - \tau(R^m) - m(R^m)) + \beta \, u(R^m \, m(R^m))$.

Now

$$U'(R^m) = -u'(c_1) \, [m'(R^m) + \tau'(R^m)] + \beta \, u'(c_2) \, [m(R^m) + R^m \, m'(R^m)]$$

using first-order condition.(41). In addition,

$$\tau'(R^m) = (R^m - 1) \, m'(R^m) + m(R^m) \tag{40}$$

so

$$
\begin{aligned}
U'(R^m) &= -u'(c_1) \, [m'(R^m) + \{(R^m - 1) \, m'(R^m) + m(R^m)\}] + \beta \, u'(c_2) \, [m(R^m) + R^m \, m'(R^m)] \\
&= [-u'(c_1) + \beta \, u'(c_2)] \, [m(R^m) + R^m m'(R^m)] \, .
\end{aligned}
$$

When $R^m > 1$, first-order condition (41) implies $-u'(c_1) + \beta \, u'(c_2) < 0$.It follows that $U'(R^m) < 0$ iff $m(R^m) + R^m \, m'(R^m) > 0$.

Differentiating first-order condition (41) with respect to R^m gives us

$$
\begin{aligned}
-u''(c_1) \, [m'(R^m) + \tau'(R^m)] &= \beta \, [u'(c_2) + R^m \, u''(c_2) \, \{m + R^m m'(R^m)\}] \\
-u''(c_1) \, [m'(R^m) + \{(R^m - 1) \, m'(R^m) + m\}] &= \beta \, [u'(c_2) + R^m \, u''(c_2) \, \{m + R^m m'(R^m)\}] \\
-u''(c_1) \, [R^m m'(R^m) + m] &= \beta \, [u'(c_2) + R^m \, u''(c_2) \, \{m + R^m m'(R^m)\}]
\end{aligned}
$$

[using equation (42) for $\tau'(R^m)$] or

$$
\begin{aligned}
-m'(R^m) \, [u''(c_1) + \beta \, R^m \, u''(c_2)] &= \beta \, u'(c_2) + m \, [u''(c_1) + \beta \, R^m \, u''(c_2)] \\
m(R^m) + R^m \, m'(R^m) &= \frac{-\beta \, u'(c_2)}{u''(c_1) + \beta \, R^m \, u''(c_2)} > 0 \, .
\end{aligned}
$$

So we have our result.

Note that our result implies that the Friedman-rule ($R^m = X$) steady state with $m = d$ produces higher steady state utility than any steady state with $R^m > X$ (and $m = d$). It remains to be shown that the Friedman-rule steady state with $m = \theta d$ produces higher steady state utility than any Friedman-rule steady state with $m > \theta d$.

[2] Comparing multiple FR equilibria

Under the Friedman rule, $R^m = X = R^d$. The consumer budget constraints are $c_1 = \omega - \tau - d$ and $c_2 = X d$. The first-order condition is

$$u'(\omega - \tau - d) = \beta \, X \, u'(X \, d) \, . \tag{41}$$

This condition implicitly defines the function $d(\tau)$. The banks choose m and k so that

$$m + k = d, \tag{42}$$

which is the steady-state version of (5); they are indifferent between different combinations of m and k that satisfy this constraint. The government chooses τ by

$$\tau = (X - 1)\,m. \tag{43}$$

We can think of (46) as defining the equilibrium function $\tau(m)$, and we can then define the equilibrium functions $d(m) \equiv d(\tau(m))$ and $k(m) = d(m) - m$ [using (44)]. It follows that

$$
\begin{aligned}
c_1(m) &= \omega - \tau(m) - d(m) = \omega - (X - 1)\,m - \{m + k(m)\} = \omega - X\,m - k(m) \\
c_2(m) &= X\,\{m + k(m)\}.
\end{aligned}
$$

The indirect utility function is $U(m) \equiv u(\omega - X\,m - k(m)) + \beta\,u(X\,\{m + k(m)\})$. We have

$$
\begin{aligned}
U'(m) &= -u'(c_1)\,[X + k'(m)] + \beta X\,u'(c_2)\,[1 + k'(m)] \\
&= [-u'(c_1) + \beta X\,u'(c_2)]\,k'(m) + X\,[-u'(c_1) + \beta\,u'(c_2)] \\
&= X\,[-u'(c_1) + \beta\,u'(c_2)],
\end{aligned}
$$

using first-order condition (43). But condition (43) also implies $-u'(c_1) + \beta\,u'(c_2) < 0$, and it follows that $U'(m) < 0$. This means that the Friedman-rule steady state that delivers the highest steady-state utility is the one with the lowest value of m.

It remains to be shown that the steady state with the lowest value of m is the one in which $m = \theta\,d$. It suffices, for this purpose, to show that $d'(m) < 0$, which implies that the ratio m/d increases with m. Differentiating first-order condition (43) with respect to m produces $-u''(c_1)\,[\tau'(m) + d'(m)] = \beta\,X^2\,u''(c_2)\,d'(m)$. Since (45) gives us $\tau'(m) = X - 1$, this equation implies

$$d'(m) = \frac{-u''(c_1)\,(X - 1)}{u''(c_1) + \beta\,X^2\,u''(c_2)} < 0.$$

$\square$

Proof of Corollary 1:

When $\gamma = 1$, so that $u(c) = \ln c$, the consumption and deposit demand functions are

$$
\begin{aligned}
c_1 &= \frac{1}{1 + \beta}\,(\omega - \tau) \\
c_2 &= R^d\,\frac{\beta}{1 + \beta}\,(\omega - \tau) \\
d &= \frac{\beta}{1 + \beta}\,(\omega - \tau).
\end{aligned}
$$

We are interested only in situations where $R^m < X$, so that the reserve requirement is binding. In these cases R^d and m satisfy equations (34) and (35). We can rewrite equation (33), which describes the government's choice of τ, as

$$\tau = (R^m - 1)\,\theta\,d = (R^m - 1)\,\theta\,\frac{\beta}{1 + \beta}\,(\omega - \tau).$$

Define $r^m \equiv R^m - 1$. Then

$$
\begin{aligned}
\tau &= r^m\,\theta\,\frac{\beta}{1 + \beta}\,(\omega - \tau) \Leftrightarrow \frac{\omega}{\tau} - 1 = \frac{1}{r^m\,\theta\,\frac{\beta}{1+\beta}} \\
&\Leftrightarrow \frac{\omega}{\tau} = \frac{1}{r^m\,\theta\,\frac{\beta}{1+\beta}} + 1 = \frac{1 + r^m\,\theta\,\frac{\beta}{1+\beta}}{r^m\,\theta\,\frac{\beta}{1+\beta}} = \frac{(1 + \beta) + r^m\,\theta\,\beta}{r^m\,\theta\,\beta}.
\end{aligned}
$$

33

Thus

$$\tau = \omega \, \frac{r^m \, \theta \, \beta}{(1+\beta) + r^m \, \theta \, \beta} \quad \text{and}$$

$$\omega - \tau = \frac{\omega \, (1+\beta)}{(1+\beta) + r^m \, \theta \, \beta} \, .$$

It follows that

$$c_1 = \frac{\omega}{(1+\beta) + r^m \, \theta \, \beta} \, .$$

$$c_2 = \frac{\beta \, \omega}{(1+\beta) + r^m \, \theta \, \beta} \, [(1-\theta) \, X + \theta \, R^m]$$

$$= \frac{\beta \, \omega}{(1+\beta) + r^m \, \theta \, \beta} \, [(1-\theta) \, X + \theta \, (1+r^m)] \, .$$

The indirect utility function can be written

$$\begin{aligned} U(r^m) &= \log c_1 + \beta \log c_2 \\ &= \log \frac{\omega}{(1+\beta) + r^m \, \theta \, \beta} + \beta \log \frac{\beta \, \omega}{(1+\beta) + r^m \, \theta \, \beta} \, [(1-\theta) \, X + \theta \, (1+r^m)] \\ &= (\log \omega + \beta \log \beta \, \omega) - (1+\beta) \log [(1+\beta) + r^m \, \theta \, \beta] \\ &\quad + \beta \log [(1-\theta) \, X + \theta \, (1+r^m)] \, m \, . \end{aligned}$$

We have

$$U'(r^m) = -\frac{\theta \, \beta \, (1+\beta)}{(1+\beta) + r^m \, \theta \, \beta} + \frac{\theta \, \beta}{(1-\theta) \, X + \theta \, (1+r^m)} \, .$$

Thus, $U'(r^m)$ is proportional to

$$\frac{1}{(1-\theta) \, X + \theta \, (1+r^m)} - \frac{1}{1 + r^m \, \theta \, \frac{\beta}{1+\beta}} \, .$$

It follows that $U'(r^m) < 0$ iff

$$1 + r^m \, \theta \, \frac{\beta}{1+\beta} < (1-\theta) \, X + \theta \, (1+r^m) \quad \Leftrightarrow \quad -r^m \, \frac{\theta}{1+\beta} < (1-\theta) \, x \, .$$

This is always true for $r^m \geq 0$, but it may not be true for $r^m < 0$.

For $U' = 0$ we need $r^m = r^m_{\max} \equiv -x \, (1+\beta) \, \frac{1-\theta}{\theta}$. Note that $U' < 0$ for $r^m > r^m_{\max}$ and $U' > 0$ for $r^m < r^m_{\max}$. However, equilibrium requires $r^m > -1$. For $r^m_{\max} > -1$ we need $x < \frac{1}{1+\beta} \frac{\theta}{1-\theta}$, which is condition (18) from the statement of the proposition. If x is too large to satisfy this inequality, then the optimal policy is a hyperinflation: that is, increasing the inflation rate will always increase steady state utility. Finally since $\pi = \frac{1}{1+r^m} - 1$, if condition (18) holds then the optimal net inflation rate is given by

$$\pi_{\max} = \frac{1}{1 - x \, (1+\beta) \, \frac{1-\theta}{\theta}} - 1 = \frac{(1-\theta) \, x}{\frac{\theta}{1+\beta} - (1-\theta) \, x} \, .$$

$\square$

Proof of Proposition 2:

In this case, the consumers maximize objective (30) subject to the constraints $c_1 = \omega - d$ and $c_2 = R^d \, d - \tau$. The remainder of the setup is the same as in the proof of Proposition 1. The first-order condition is

$$u'(\omega - d) = \beta \, R^d \, u'(R^d \, d - \tau) \, , \tag{44}$$

and we proceed, as in the proof of Proposition 1, to define the equilibrium functions $\tau(R^m)$ and $d(R^m)$. The associated indirect utility function is $U(R^m) \equiv u(\omega - d(R^m)) + \beta\, u(R^d(R^m)\, d(R^m) - \tau(R^m))$. Thus

$$
\begin{aligned}
U'(R^m) &= -u'(\omega - d)\, d'(R^m) + \beta\, u'(R^d\, d - \tau) \left[\theta\, d(R^m) + R^d d'(R^m) - \tau'(R^m))\right] \\
&= d'(R^m)\left[-u'(\omega - d) + \beta\, R^d\, u'(R^d\, d - \tau)\right] + \theta\, d\, \beta\, u'(R^d\, s - \tau) - \beta\, u'(R^d\, d - \tau)\, \tau'(R^m) \\
&= \beta\, u'(R^d\, d - \tau)\left[\theta\, d - \tau'(R^m)\right]
\end{aligned}
$$

using first order condition (46). Using equation (38) for $\tau'(R^m)$ then gives us

$$
U'(R^m) = \beta\, u'(R^d\, d - \tau)\left[(1 - R^m)\, \theta\, d'(R^m)\right].
$$

Clearly, $U'(R^m) = 0$ when $R^m = 1$. We need to show that $d'(R^m) > 0$; if this is the case, then $U'(R^m) < 0$ when $R^m > 1$ and $U'(R^m) > 0$ when $R^m < 1$, so that steady-state utility is maximized at $R^m = 1$.

Differentiating the first-order condition (46) with respect to R^m produces

$$
\begin{aligned}
-u''(c_1)\, d'(R^m) &= \beta\, \theta\, u'(c_2) + \beta\, R^d\, u''(c_2)\left[\theta\, d + R^d\, d'(R^m) - \tau'(R^m)\right] \\
&= \beta\, \theta\, u'(c_2) + \beta\, R^d\, u''(c_2)\left[R^d d'(R^m) + (1 - R^m)\, \theta\, d'(R^m)\right] \\
&= \beta\, \theta\, u'(c_2) + \beta\, R^d\, u''(c_2)\left[X\,(1 - \theta) + \theta\right] d'(R^m).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
-\beta\, \theta\, u'(c_2) &= d'(R^m)\left\{\beta\, R^d\, u''(c_2)\left[X\,(1 - \theta) + \theta\right] + u''(c_1)\right\} \\
d'(R^m) &= \frac{-\beta\, \theta\, u'(c_2)}{\beta\, R^d\, u''(c_2)\left[X\,(1 - \theta) + \theta\right] + u''(c_1)} > 0.
\end{aligned}
$$

Again, however, our result does not show that the zero inflation steady state produces higher steady state utility than Friedman-rule steady states with $m > \theta\, d$ or than the money-only steady states with $R^m > X$. To accomplish this, we need to establish the analogues of items [1] and [2] from the proof of Proposition 1.

[1] The Friedman rule vs. higher money return rates

Again, when $R^m > X$ the banks choose $R^d = R^m$ and $m = d$. The consumer budget constraints are $c_1 = \omega - m$ and $c_2 = R^m\, m - \tau$. The first-order condition is

$$
u'(\omega - m) = \beta\, R^m\, u'(R^m\, m - \tau). \tag{45}
$$

Obtaining $\tau(R^m)$ and $d(R^m)$ as before, we have $U(R^m) \equiv u(\omega - m(R^m)) + \beta\, u(R^m\, m(R^m) - \tau(R^m))$, so

$$
\begin{aligned}
U'(R^m) &= -u'(c_1)\, m'(R^m) + \beta\, u'(c_2)\left[m + R^m\, m'(R^m) - \tau'(R^m)\right] \\
&= -u'(c_1)\, m'(R^m) + \beta\, u'(c_2)\left[m + R^m\, m'(R^m) - \{(R^m - 1)\, m'(R^m) + m\}\right] \\
&= \left[-u'(c_1) + \beta\, u'(c_2)\right] m'(R^m) \\
&= u'(c_1)\left[\frac{1}{R^m} - 1\right] m'(R^m),
\end{aligned}
$$

using equation (42) for $\tau'(R^m)$, and also first-order condition (47). Thus when $R^m > X$ we have $U'(R^m) < 0$ iff $m'(R^m) > 0$.

Differentiating first-order condition (46) with respect to R^m produces

$$
\begin{aligned}
-u''(c_1)\, m'(R^m) &= \beta\left[u'(c_2) + R^m\, u''(c_2)\left\{m + R^m\, m'(R^m) - \tau'(R^m)\right\}\right] \\
&= \beta\left[u'(c_2) + R^m\, u''(c_2)\left\{m + R^m\, m'(R^m) - \left[(R^m - 1)\, m'(R^m) + m\right]\right\}\right] \\
&= \beta\left[u'(c_2) + R^m\, u''(c_2)\, m'(R^m)\right],
\end{aligned}
$$

using equation (42) for $\tau'(R^m)$. Thus,

$$
m'(R^m) = \frac{-\beta\, u'(c_2)}{u''(c_1) + \beta\, R^m\, u''(c_2)} > 0.
$$

35

[2] Comparing multiple FR equilibria

Again, under the Friedman rule $R^m = X = R^d$. The consumer budget constraints are $c_1 = \omega - d$ and $c_2 = X\,d - \tau$. The first-order condition is

$$u'(\omega - d) = \beta\,X\,u'(X\,d - \tau). \tag{46}$$

This condition implicitly defines the function $d(\tau)$. The rest of the setup is as in the corresponding part of the proof of Proposition 1. Again, we obtain $d(m)$ and $k(m)$ from $d(\tau)$ and the government tax function $\tau(m)$. In this case we have

$$
\begin{aligned}
c_1(m) &= \omega - d(m) = \omega - \{m + k(m)\}\\
c_2(m) &= X\,\{m + k(m)\} - \tau(m) = X\,\{m + k(m)\} - (X-1)\,m = X\,k(m) + m\,.
\end{aligned}
$$

So $U(m) \equiv u(\omega - \{m + k(m)\}) + \beta\,u\,(X\,k(m) + m)$ and

$$
\begin{aligned}
U'(m) &= -u'(c_1)\,[1 + k'(m)] + \beta\,u'(c_2)\,[X\,k'(m) + 1]\\
&= [-u'(c_1) + \beta\,X\,u'(c_2)]\,k'(m) - u'(c_1) + \beta\,u'(c_2)\\
&= -u'(c_1) + \beta\,u'(c_2) < 0\,,
\end{aligned}
$$

using first-order condition (49). Again, the Friedman-rule steady state that delivers the highest steady-state utility is the one with the lowest value of m, and, again, it remains to be shown that the steady state with the lowest value of m is the one in which $m = \theta\,d$.

In this case, however, it turns out that d increases with m, so we need to show directly that m/d increases with m. It will be sufficient to show that $d'(m) < 1$. Differentiating first-order condition (#) with respect to m produces

$$-u''(c_1)\,d'(m) = \beta\,X\,u''(c_2)\,[X\,d'(m) - \tau'(m)]\,.$$

Again, (45) gives us $\tau'(m) = X - 1$, and thus

$$d'(m) = \frac{\beta\,X\,u''(c_2)\,(X-1)}{u''(c_1) + \beta\,X^2\,u''(c_2)} = \frac{X-1}{\frac{1}{\beta\,X}\frac{u''(c_1)}{u''(c_2)} + X}\,.$$

If $X > 1$, as we assume, then $0 < d'(m) < \frac{X-1}{X} = 1 - \frac{1}{X} < 1$. $\square$

Proof of Proposition 3:

Under the simulator regime, a stationary competitive equilibrium in which the reserve requirement is exactly satisfied can be characterized as values R^m, τ, T_1 and T_2 chosen by the government, values c_1, c_2, and d, chosen by the consumers, and values R^d, m and k chosen by the banks, such that c_1, c_2, and d maximize a two-period-lived consumer's utility (30) subject to the budget constraints

$$
\begin{aligned}
c_1 + d &= \omega - T_1 \tag{47}\\
c_2 &= R^d\,d - \tau - T_2\,, \tag{48}
\end{aligned}
$$

the steady-state versions of (8) and (9) under the simulator regime. The government's choices of τ, T_1 and T_2 satisfy (33) plus

$$
\begin{aligned}
T_1 &= -m \tag{49}\\
T_2 &= m\,, \tag{50}
\end{aligned}
$$

the steady-state versions of (11) and (12). The banks' choices of R^d, m and k satisfy (34), (35) and (44).

In order to show that demonstrate that a Friedman-rule stationary equilibrium under the simulator policy is steady-state optimal, we will show that it coincides with an allocation that might be chosen by a social planner. Recall that our definition of a stationary competitive equilibrium requires it to be implementable starting from date 1, when the initial old consumers have no endowments.. Consequently, in any such equilibrium, the net

transfer from the young consumers to the old consumers must be non-negative. In constructing an optimal stationary allocation as a "target" for stationary equilibrium allocations, we impose the same constraint on the social planner. The planner can take goods from young consumers' endowments and allocate them for consumption by either group of consumers, or for storage. However, she must allocate any storage returns to old consumers: otherwise, the allocation would not be implementable starting from date 1, when there can be no storage returns. Since $X > 1$, if the planner is not concerned about the welfare of the initial old consumers, then it does not make sense for her to allocate any goods to them, since the consumption of future old consumers can be financed, less expensively, via storage. Thus, a necessary condition for any steady-state allocation implementable from date 1 to be optimal is that all goods consumed by old consumers are derived from storage returns.

Given these constraints, the social planner's problem for choosing the steady-state optimal allocation is to maximize the utility of a two-period-lived consumer (30) subject to

$$c_1 + k = \omega \tag{51}$$
$$c_2 = X k. \tag{52}$$

The planner's consolidated budget constraint, obtained by solving the second constraint for k and substituting the solution into the first constraint, is

$$c_1 + \frac{c_2}{X} = \omega. \tag{53}$$

Let c_1^*, c_2^*, k^* denote the solutions to this problem.

Returning to stationary competitive equilibria under the simulator regime, the intertemporal budget constraint of a consumer, obtained by solving the constraint (50) for d and substituting the solution into constraint (49), is

$$c_1 + \frac{c_2}{R^d} = (\omega - \mathrm{T}_1) - \frac{\tau + \mathrm{T}_2}{R^d}. \tag{54}$$

Suppose the government chooses $\widehat{R}^m = X$; it follows from (35) that $\widehat{R}^d = X$ Let $\widetilde{m}$ represent the solution to the equation $m = \theta (\omega + m - c_1^*)$, so that

$$\widetilde{m} = \frac{\theta}{1 - \theta} (\omega - c_1^*). \tag{55}$$

Suppose the government chooses $\widehat{\tau}$, $\widehat{\mathrm{T}}_1$ and $\widehat{\mathrm{T}}_2$ so that $\widehat{\tau} = (X - 1) \widetilde{m}$, $\widehat{\mathrm{T}}_1 = -\widetilde{m}$, and $\widehat{\mathrm{T}}_2 = \widetilde{m}$. The consumers' intertemporal budget constraint becomes

$$c_1 + \frac{c_2}{X} = (\omega + \widetilde{m}) - \frac{(X - 1) \widetilde{m} + \widetilde{m}}{X} = \omega,$$

which is constraint (55). It follows that the consumers will choose $\widehat{c}_1 = c_1^*$, $\widehat{c}_2 = c_2^*$, and $\widehat{d} = \omega + \widetilde{m} - c_1^*$. Suppose the banks choose $\widehat{m} = \theta \widehat{d}$ and $\widehat{k} = \widehat{d} - \widehat{m}$, satisfying (34) and (44). Then it is readily seen that $\widehat{m} = \widetilde{m}$ and $\widehat{k} = k^*$. In addition, we have $\widehat{\tau} = (X - 1) \widehat{m}$, $\widehat{\mathrm{T}}_1 = -\widehat{m}$, and $\widehat{\mathrm{T}}_2 = \widehat{m}$, so that (33), (51) and (52) are satisfied. So we have a steady state simulator equilibrium that generates a consumption allocation $(\widehat{c}_1, \widehat{c}_2)$ that is identical to the social planner's steady-state optimal allocation (c_1^*, c_2^*). It follows that this equilibrium is the optimal steady state simulator equilibrium. $\square$

Proof of Proposition 4:

When $R^m < X$, we can view a consumer as choosing c_1, c_{21}, c_{22}, m and k to maximize

$$E_1\{U(c_1, c_2)\} = u(c_1) + \beta [(1 - \phi) u(c_{21}) + \phi u(c_{22})], \tag{56}$$

the steady-state stochastic version of (1), subject to

$$c_1 = \omega - \tau - m - k \tag{57}$$
$$c_{21} = \frac{X}{1 - \phi} k \tag{58}$$
$$c_{22} = \frac{R^m}{\phi} m. \tag{59}$$

37

The first-order conditions are

$$u'(c_1) = \beta\, X\, u'(c_{21}) \tag{60}$$

$$u'(c_1) = \beta\, R^m\, u'(c_{22}), \tag{61}$$

where c_1, c_{21} and c_{22} are given by (59)-(61). These conditions implicitly define the consumer money and storage demand functions $m(R^m, \tau)$ and $k(R^m, \tau)$.

As always the government chooses τ according to (33). This equation and the function $m(R^m, \tau)$ implicitly define the equilibrium function $\tau(R^m)$, which can then be used, in conjunction with $m(R^m, \tau)$ and $k(R^m, \tau)$, to define the equilibrium functions $m(R^m)$ and $k(R^m)$. The associated indirect expected utility function is

$$U(R^m) = u(\omega - \tau(R^m) - m(R^m) - k(R^m)) + \beta\left[(1-\phi)\,u(\frac{X}{1-\phi}\,k(R^m)) + \phi\,u(\frac{R^m}{\phi}\,m(R^m))\right].$$

We have

$$
\begin{aligned}
U'(R^m) &= -u'(c_1)\left[\tau'(R^m) + m'(R^m) + k'(R^m)\right] + \beta\left\{X\,u'(c_{21})\,k'(R^m) + u'(c_{22})\left[m + R^m m'(R^m)\right]\right\} \\
&= k'(R^m)\left[-u'(c_1) + \beta\,X\,u'(c_{21})\right] + m'(R^m)\left[-u'(c_1) + \beta\,R^m\,u'(c_{22})\right] \\
&\quad -u'(c_1)\,\tau'(R^m) + \beta\,m\,u'(c_{22}) \\
&= -u'(c_1)\,\tau'(R^m) + \beta\,m\,u'(c_{22}),
\end{aligned}
$$

using the first-order conditions.(62) and (63).

In addition, $\tau'(R^m)$ is given by (42), so

$$
\begin{aligned}
-u'(c_1)\,\tau'(R^m) + \beta\,m\,u'(c_{22}) &= -u'(c_1)\left[(R^m - 1)\,m'(R^m) + m\right] + \beta\,m\,u'(c_{21}) \\
&= m\left[-u'(c_1) + \beta\,u'(c_{22})\right] - u'(c_1)(R^m - 1)\,m'(R^m).
\end{aligned}
$$

Thus,

$$U'(R^m) = m\left[-u'(c_1) + \beta\,u'(c_{22})\right] - u'(c_1)(R^m - 1)\,m'(R^m).$$

When $R^m = 1$, both RHS terms are zero, so we have $U'(R^m) = 0$. [Note that when $R^m = 1$, the term in square brackets is the second first-order condition .] In addition, since f.o.c. (63) gives us $u'(c_{22}) = u'(c_1)/(\beta\,R^m)$, we have

$$
\begin{aligned}
U'(R^m) &= m\left[-u'(c_1) + \frac{u'(c_1)}{R^m}\right] - u'(c_1)(R^m - 1)\,m'(R^m) \\
&= u'(c_1)\left[m\left(\frac{1}{R^m} - 1\right) - (R^m - 1)\,m'(R^m)\right] \\
&= u'(c_1)\left[\frac{m}{R^m}\,(1 - R^m) + (1 - R^m)\,m'(R^m)\right] \\
&= u'(c_1)\,(1 - R^m)\left[\frac{m}{R^m} + m'(R^m)\right].
\end{aligned}
$$

Thus, to establish that zero inflation is optimal we to need to show that $\frac{m}{R^m} + m'(R^m) > 0$, which implies $U'(R^m) < 0$ when $R^m > 1$ and $U'(R^m) > 0$ when $R^m < 1$.

Now

$$\frac{m}{R^m} + m'(R^m) > 0 \Leftrightarrow m'(R^m) > -\frac{m}{R^m} \Leftrightarrow -\frac{R^m}{m}m'(R^m) < 1.$$

We have

$$m(R^m) = \frac{\omega\,\phi\,\beta^{\frac{1}{\gamma}}\,(R^m)^{\frac{1}{\gamma}-1}}{1 + (1-\phi)\,\beta^{\frac{1}{\gamma}}\,(X)^{\frac{1}{\gamma}-1} + \phi\,\beta^{\frac{1}{\gamma}}\,(R^m)^{\frac{1}{\gamma}}}$$

and

$$m'(R^m) = \frac{\omega\,\phi\,\beta^{\frac{1}{\gamma}}\left(\frac{1}{\gamma}-1\right)(R^m)^{\frac{1}{\gamma}-2}\,\frac{\omega\,\phi\,\beta^{\frac{1}{\gamma}}(R^m)^{\frac{1}{\gamma}-1}}{m(R^m)} - \omega\,\phi\,\beta^{\frac{1}{\gamma}}\,(R^m)^{\frac{1}{\gamma}-1}\left(\frac{1}{\gamma}\right)\phi\,\beta^{\frac{1}{\gamma}}\,(R^m)^{\frac{1}{\gamma}-1}}{\left[1 + (1-\phi)\,\beta^{\frac{1}{\gamma}}\,(X)^{\frac{1}{\gamma}-1} + \phi\,\beta^{\frac{1}{\gamma}}\,(R^m)^{\frac{1}{\gamma}}\right]^2}.$$

It can then be shown that

$$-\frac{R^m}{m(R^m)} m'(R^m) = \left[\left(\frac{1}{\gamma}\right) \left\{ \frac{1}{\omega} R^m\, m(R^m) - 1 \right\} + 1 \right].$$

So we need to show that

$$\left[\left(\frac{1}{\gamma}\right) \left\{ \frac{1}{\omega} R^m\, m(R^m) - 1 \right\} + 1 \right] < 1 \quad \Leftrightarrow \quad R^m\, m(R^m) < \omega\,.$$

The latter inequality is

$$R^m\, \frac{\omega\,\phi\,\beta^{\frac{1}{\gamma}}\,(R^m)^{\frac{1}{\gamma}-1}}{1 + (1-\phi)\,\beta^{\frac{1}{\gamma}}\,(X)^{\frac{1}{\gamma}-1} + \phi\,\beta^{\frac{1}{\gamma}}\,(R^m)^{\frac{1}{\gamma}}} < \omega \quad \Leftrightarrow \quad \frac{\phi\,\beta^{\frac{1}{\gamma}}\,(R^m)^{\frac{1}{\gamma}}}{1 + (1-\phi)\,\beta^{\frac{1}{\gamma}}\,(X)^{\frac{1}{\gamma}-1} + \phi\,\beta^{\frac{1}{\gamma}}\,(R^m)^{\frac{1}{\gamma}}} < 1\,,$$

which is equivalent to $1 + (1-\phi)\,\beta^{\frac{1}{\gamma}}\,(X)^{\frac{1}{\gamma}-1} > 0$. So we have our result.

Finally, when $R^m \geq X$, this money demand specification is equivalent to the analogous reserve requirements specification, with ϕ, the money-deposit ratio in a Friedman-rule steady state with $\mu = 1$, playing essentially the same role as θ. And we have seen that, under the reserve requirements specification, the steady state equilibrium with $R^m = X$ and $m = \theta\,d$ produces higher steady state utility than any other steady state equilibrium with $R^m = X$, and the steady state equilibrium with $R^m = X$ and $m = d$, produces higher steady state utility than any steady state equilibrium with $R^m > X$. $\square$

Proof of Proposition 5:

When $R^m < X$, we can view a consumer as choosing c_1, c_{21}, c_{22}, m and k to maximize objective (58) subject to

$$c_1 = \omega - m - k \tag{62}$$

$$c_{21} = \frac{X}{1-\phi} k - \tau \tag{63}$$

$$c_{22} = \frac{R^m}{\phi} m - \tau\,. \tag{64}$$

The first-order conditions have the same form as in the proof of Proposition 4, except that c_1, c_{21}, c_{22} are given by (64)-(66). These conditions implicitly define the consumer money and storage demand functions $m(R^m, \tau)$ and $k(R^m, \tau)$.

Again, the government chooses τ by (33). This equation and the function $m(R^m, \tau)$ implicitly define the equilibrium function $\tau(R^m)$, which can then be used, in conjunction with $m(R^m, \tau)$ and $k(R^m, \tau)$, to define the equilibrium functions $m(R^m)$ and $k(R^m)$. The associated indirect expected utility function is

$$U(R^m) = u(\omega - m(R^m) - k(R^m)) + \beta \left[(1-\phi)\, u(\frac{X}{1-\phi} k(R^m) - \tau(R^m)) + \phi\, u(\frac{R^m}{\phi} m(R^m) - \tau(R^m)) \right].$$

Now

$$\begin{aligned}
U'(R^m) &= -u'(c_1)\,[m'(R^m) + k'(R^m)] \\
&\quad + \beta\,\{u'(c_{21})\,[X\,k'(R^m) - (1-\phi)\,\tau'(R^m)] + u'(c_{22})\,[m + R^m\,m'(R^m) - \phi\,\tau'(R^m)]\} \\
&= \frac{\partial k}{\partial R^m}\,[-u'(c_1) + \beta\,X\,u'(c_{21})] + m'(R^m)\,[-u'(c_1) + \beta\,R^m\,u'(c_{22})] \\
&\quad - \beta\,[(1-\phi)\,u'(c_{21}) + \phi\,u'(c_{22})]\,\tau'(R^m) + \beta\,m\,u'(c_{22}) \\
&= \beta\,\{m\,u'(c_{22}) - [(1-\phi)\,u'(c_{21}) + \phi\,u'(c_{22})]\,\tau'(R^m)\}
\end{aligned}$$

using the revised version of first-order conditions (62) and (63), and $\tau'(R^m)$ is given by (42). So

$$\begin{aligned}
&m\,u'(c_{22}) - [(1-\phi)\,u'(c_{21}) + \phi\,u'(c_{22})]\,\tau'(R^m) \\
&= m\,u'(c_{22}) - [(1-\phi)\,u'(c_{21}) + \phi\,u'(c_{22})]\,[(R^m - 1)\,m'(R^m) + m] \\
&= -[(1-\phi)\,u'(c_{21}) + \phi\,u'(c_{22})]\,[(R^m - 1)\,m'(R^m)] - (1-\phi)\,m\,[u'(c_{21}) - u'(c_{22})]\,.
\end{aligned}$$

39

Thus,

$$U'(R^m) = \beta \left\{ (1-\phi)\, m\, [u'(c_{22}) - u'(c_{21})] - [(1-\phi)\, u'(c_{21}) + \phi\, u'(c_{22})]\, [(R^m - 1)\, m'(R^m)] \right\}.$$

The first-order conditions imply

$$u'(c_{22}) = \frac{u'(c_1)}{\beta\, R^m} = \frac{\beta\, X\, u'(c_{21})}{\beta\, R^m} = \frac{X}{R^m}\, u'(c_{21}),$$

so

$$
\begin{aligned}
U'(R^m) &= \frac{\beta}{R^m}\, u'(c_{21}) \left\{ (1-\phi)\, (X - R^m)\, m - [(1-\phi)\, R^m + \phi\, X]\, [(R^m - 1)\, m'(R^m)] \right\} \\
&= \frac{u'(c_1)}{R^m X} \left\{ (1-\phi)\, (X - R^m)\, m - [(1-\phi)\, R^m + \phi\, X]\, [(R^m - 1)\, m'(R^m)] \right\}.
\end{aligned}
$$

In the case $R^m = 1$, we have

$$U'(1) = \frac{u'(c_1)}{X}\, (1-\phi)\, (X - 1)\, m(1) > 0,$$

so a marginal increase in R^m (a marginal decrease in the inflation rate) causes utility to rise. **(This completes the proof of Part [A].)** In the case, $R^m = X$, we have

$$U'(X) = -\frac{u'(c_1)}{X}\, [(X - 1)\, m'(X)].$$

So $U'(X)$ has the opposite sign from $m'(X)$. If $m'(X) > 0$ then $U'(X) < 0$ and the Friedman rule is not optimal. But if $m'(X) < 0$ then $U'(X) > 0$ and the Friedman rule may be optimal. **(This completes the proof of Part [B].)**

Now

$$
\begin{aligned}
m(R_m) &= \phi \omega \frac{X\, (\beta R_m)^{\frac{1}{\gamma}}}{\phi X \left[1 + (\beta R_m)^{\frac{1}{\gamma}} \right] + R_m (1-\phi) \left[X + (\beta X)^{\frac{1}{\gamma}} \right] + \beta^{\frac{1}{\gamma}} (R_m - 1)\, \phi\, (1-\phi) \left[(R^m)^{\frac{1}{\gamma}} - X^{\frac{1}{\gamma}} \right]} \\
&= \frac{X \omega}{X \left[(\beta R^m)^{-\frac{1}{\gamma}} + 1 \right] + \beta^{-\frac{1}{\gamma}} R_m^{1 - \frac{1}{\gamma}} \frac{(1-\phi)}{\phi} \left[X + (\beta X)^{\frac{1}{\gamma}} \right] + (R_m - 1)\, (1-\phi) \left[1 - \left(\frac{X}{R^m} \right)^{\frac{1}{\gamma}} \right]}
\end{aligned}
$$

Define $\Psi(R^m)$ implicitly by $m(R_m) = X\, \omega / \Psi(R^m)$. It follows that $m'(R_m) < 0$ iff $\Psi'(R^m) > 0$. We have

$$
\begin{aligned}
\Psi'(R^m) &= -\frac{1}{\gamma} \beta X\, (\beta R^m)^{-\frac{1}{\gamma} - 1} \\
&\quad + (1-\phi) \left\{ \frac{1}{\phi} \left(1 - \frac{1}{\gamma} \right) (\beta R^m)^{-\frac{1}{\gamma}} \left[X + (\beta X)^{\frac{1}{\gamma}} \right] \left[1 - \left(\frac{X}{R^m} \right)^{\frac{1}{\gamma}} \right] + \frac{1}{\gamma} \left(\frac{R^m - 1}{R^m} \right) \left(\frac{X}{R^m} \right)^{\frac{1}{\gamma}} \right\}
\end{aligned}
$$

and thus

$$\Psi'(X) = -\frac{1}{\gamma}\, (\beta X)^{-\frac{1}{\gamma}} + (1-\phi) \left\{ \frac{1}{\phi} \left(1 - \frac{1}{\gamma} \right) \left[X\, (\beta X)^{-\frac{1}{\gamma}} + 1 \right] + \frac{1}{\gamma} \left(\frac{X - 1}{X} \right) \right\}.$$

So $\Psi'(X) > 0$ iff

$$\Omega(\gamma, X, \beta) \equiv \frac{1}{(\beta X)^{\frac{1}{\gamma}}} \left\{ (\gamma - 1)\, \frac{1 - \phi}{\phi}\, X - 1 \right\} + (1-\phi) \left\{ \frac{\gamma - 1}{\phi} + \left(1 - \frac{1}{X} \right) \right\} > 0. \tag{65}$$

Clearly, $\lim_{\gamma \to \infty} \Omega(\gamma, X, \beta) = \infty$. When $\gamma \geq 1$, $\lim_{X \to \infty} \Omega(\gamma, X, \beta) = \infty$ and $\lim_{\beta \to \infty} \Omega(\gamma, X, \beta) = (1-\phi) \left\{ \frac{\gamma - 1}{\phi} + \left(1 - \frac{1}{X} \right) \right\} > 0$. However, if $\gamma < 1$ then $\lim_{X \to \infty} \Omega(\gamma, X, \beta) = (1-\phi) \left[\frac{\gamma - 1}{\phi} + 1 \right]$, which is negative for $\gamma < 1 - \phi$, and

$\lim_{\beta \to \infty} \Omega(\gamma, X, \beta) = (1 - \phi)\left\{\frac{\gamma - 1}{\phi} + \left(1 - \frac{1}{X}\right)\right\}$, which is negative for $\gamma < (1 - \phi) + \frac{\phi}{X}$. **(This completes the proof of Part [C].)**

Again, the proof that the Friedman-rule steady state with $m/d = \phi$ delivers higher steady state utility than [2] any other Friedman-rule steady state, and [1] and steady state with $R^m > X$, proceeds as in the proof of Proposition 2.

Our examples are calibrated in a very casual way, based on the assumption that a period represents 30 years. We choose $X = (1.06)^{30} \doteq 5.7435$ unless otherwise indicated, $\beta = (0.96)^{30} \doteq 0.29386$ and $\phi = 0.1$. We let $\overline{X}$ represent the value of X at which $\Omega(\gamma, X, \beta) = 0$.

Example 1 *This example involves two specifications in which $\gamma = 1$. In these specifications, we have $\overline{X} = 1 + [\beta(1 - \phi)]^{-1}$. In our baseline specification the function $\Omega(\gamma, X, \beta)$ is monotone increasing in X, with $\overline{X} \doteq 4.7811 < X \doteq 5.7435$. (The annualized value of $\overline{X}$ is 1.054.) We have $m'(X) \doteq -0.00002$ and $U'(X) \doteq 0.00002$. The function $U(R^m)$ is monotone increasing on $(0, X)$ and the Friedman rule is optimal. The function $m(R^m)$ peaks at $\widetilde{R}^m \doteq 5.2403 < X$ (annualized value 1.057); it is monotone increasing on $(0, \widetilde{R}^m)$ and monotone decreasing on $(\widetilde{R}^m, X)$.*

If we choose $X = (1.05)^{30} \doteq 4.3219 < \overline{X}$ then $m'(X) \doteq 0.00001$ and $U'(X) \doteq -0.00001$. In this case, the function $U(R^m)$ peaks at $\widehat{R}^m \doteq 4.3131 < X$ (annualized value 1.050); it is monotone increasing on $(0, \widehat{R}^m)$ and monotone decreasing on $(\widehat{R}^m, X)$. So the Friedman rule is not optimal. The function $m(R^m)$ is monotone increasing on $(0, X)$.

Example 2 *This example involves two specifications in which $\gamma = 0.99$. In our baseline specification, the function $\Omega(\gamma, X, \beta)$ is monotone increasing in X, with $\overline{X} \doteq 8.4339 > X \doteq 5.7435$. (The annualized value of $\overline{X}$ is 1.074.) We have $m'(X) \doteq 0.00003$ and $U'(X) \doteq -0.00003$. The function $U(R^m)$ peaks at $\widehat{R}^m \doteq 5.7113 < X$ (annualized value 1.060); it is monotone increasing on $(0, \widehat{R}^m)$ and monotone decreasing on $(\widehat{R}^m, X)$. So the Friedman rule is not optimal. The function $m(R^m)$ is monotone increasing on $(0, X)$.*

If we choose $X = (1.08)^{30} \doteq 10.063 > \overline{X}$ then $m'(X) \doteq -0.00001$ and $U'(X) \doteq 0.00001$. In this case, The function $U(R^m)$ is monotone increasing on $(0, X)$ and the Friedman rule is optimal. The function $m(R^m)$ peaks at $\widetilde{R}^m \doteq 9.4437 < X$ (annualized value 1.078); it is monotone increasing on $(0, \widetilde{R}^m)$ and monotone decreasing on $(\widetilde{R}^m, X)$.

Example 3 *This example involves two specifications in which $\gamma = 1.02$. In our baseline specification the function $\Omega(\gamma, X, \beta)$ is monotone increasing in X, with $\overline{X} \doteq 2.5361 < X \doteq 5.7435$. (The annualized value of $\overline{X}$ is 1.032.) We have $m'(X) \doteq -0.00009$ and $U'(X) \doteq 0.00010$. The function $U(R^m)$ is monotone increasing on $(0, X)$ and the Friedman rule is optimal. The function $m(R^m)$ peaks at $\widetilde{R}^m \doteq 3.3009 < X$ (annualized value 1.041); it is monotone increasing on $(0, \widetilde{R}^m)$ and monotone decreasing on $(\widetilde{R}^m, X)$.*

If we choose $X = (1.02)^{30} \doteq 1.8114 < \overline{X}$ then $m'(X) \doteq 0.00020$ and $U'(X) \doteq -0.00012$. In this case, the function $U(R^m)$ peaks at $\widehat{R}^m \doteq 1.7971 < X$ (annualized value 1.020); it is monotone increasing on $(0, \widehat{R}^m)$ and monotone decreasing on $(\widehat{R}^m, X)$. So the Friedman rule is not optimal. The function $m(R^m)$ is monotone increasing on $(0, X)$.

(This completes the proof of Part [D].) $\square$

Proof of Proposition 6:

Under the simulator regime, a stationary competitive equilibrium can be characterized as values R^m, τ, T_1 and T_2 chosen by the government, values c_1, c_2, and d, chosen by the consumers, and values R_n^d, R_r^d m and k chosen by the banks, such that c_1, c_2, and d maximize a two-period-lived consumer's expected intertemporal utility (58) subject to the budget constraints

$$c_1 + d = \omega - \text{T}_1 \tag{66}$$

$$c_{2n} = R_n^d \, d - \tau - \text{T}_2 \tag{67}$$

$$c_{2r} = R_r^d \, d - \tau - \text{T}_2 \,, \tag{68}$$

the banks choose R_n^d, R_r^d, m, k and μ to maximize a two-period-lived consumers' expected second-period utility

$$V(c_2) \equiv (1 - \phi)\, u(c_{2n}) + \phi\, u(c_{2r}) \tag{69}$$

subject to (69) and (70) as well as (44) and

$$R_n^d = \frac{X\,k + (1 - \mu)\,R^m\, m}{(1 - \phi)\, d} \tag{70}$$

$$R_r^d = \frac{\mu\, R^m\, m}{\phi\, d}. \tag{71}$$

In addition, the government's choices of τ, T_1 and T_2 satisfy (33), (51) and (52).

As we saw in the proof of Proposition 3, in the absence of any relocation friction, a social planner's problem for choosing the steady-state optimal allocation is to maximize the utility of a two-period-lived consumer subject to (54) and (54); the associated combined budget constraint is (55). Let c_1^*, c_2^*, k^* denote the solutions to this problem.

Returning to stationary competitive equilibria under the simulator regime, suppose the government chooses $\widehat{R}^m = X$. It is then readily seen that the banks maximize their objective function (71) by choosing μ, m and k in any way so that so that $R_n^d = .R_r^d = X$. The second-period budget constraints of a consumer collapse to the single constraint

$$c_2 = X\, d - \tau - T_2 \tag{72}$$

Now let $\widetilde{m}$ be chosen by a revised version of (57), with ϕ replacing θ. Suppose the government chooses $\widehat{\tau}$, $\widehat{T}_1$ and $\widehat{T}_2$ such that $\widehat{\tau} = (X - 1)\, \widetilde{m}$, $\widehat{T}_1 = -\widetilde{m}$, and $\widehat{T}_2 = \widetilde{m}$. Given these choices, the combined budget constraint of a consumer, whose general form was (56), becomes

$$c_1 + \frac{c_2}{X} = (\omega + \widetilde{m}) - \frac{(X - 1)\, \widetilde{m} + \widetilde{m}}{X} = \omega,$$

which is constraint (55). The rest of the proof proceeds exactly as in the proof of Proposition 3. $\square$