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Gergonne's Pile Problem

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Introduction

My Creative Component topic will investigate Magic Tricks, Card Tricks, and Logic Puzzles with mathematical proofs to use to intrigue students so they want to learn mathematical concepts. After nearly five years of teaching in high schools, I have learned that students need to be interested in a topic in order to engage in learning the concepts well. During my teaching experience, I have introduced students to topics through application problems, reviewing fundamental steps, and games. Each method has proven to be helpful in its own way. I believe tricks and puzzles may be an alternate way to engage students in math, without giving them direct instruction from the beginning.

I have taught many students who do not believe that they are good at math, which can cause them to “mentally shut down” the instant that they walk in my classroom door. Introducing topics in a different, interesting, and engaging light can help students to “shrug off” their past misconceptions about where their skills are and to be open to learning instead.

A great example of hope important being open to learning is from working with Science Bound in Des Moines. The instructor asked one student to close their fist and had another student try their hardest to open it. Then after a few moments, the instructor offered another approach asking the student to open their fist. The class then saw that if you don’t want something to happen then you will remain closed off to it, but the instant that you let yourself be open to new possibilities, your path becomes much easier.

During my Summer 2017 courses, Iowa State University Master’s of School Mathematics program provided its students an opportunity to learn and investigate possible Creative Component topics. Until my experiences that day, I wasn’t sure about my topic. One of the topics shared that day showed me how some concepts in mathematics could be learned through student-driven investigations of the topic, rather than explicitly taught through lecture. This inspired me to begin thinking of ideas for engaging students in investigating mathematical concepts instead of the traditional lecture and practice method of teaching mathematics. Although the “I do - We do - You do” (Release of Responsibility) method of instruction is also beneficial, it doesn’t allow students to challenge themselves, work through the learning process, or to enjoy discovering the solutions for themselves. When humans are allowed opportunities to create their own understanding of something, they are more likely to truly understand it and put it into their long term memory. The concepts will also be something that they will understand and remember for more than just one chapter of learning. When students derive their own understanding of a concept they are more likely to make a deep connection to it and thus retain it much longer.

My goal for educating high school students is for them to understand and appreciate math as well as to better prepare them to future learning in higher-level mathematics courses. However at the same time, I understand that not all students will pursue higher level math. So it is important to me, to help my students become more independent in applying their knowledge to appropriate situations. Mathematics needs to be relevant to students’ lives so that they find the purpose in learning it.

During the session, we began with the trick, and then broke it down to figure out how it worked, and extended it to a variety of possibilities. For example: How many decks of cards would you need to be able to create five hidden cards instead of three by following the same pattern? How many in each pile? How would changing how the cards would be flipped affect the trick? I thought that this could be a wonderful way to engage students with various topics. I began reading and investigating magic and card tricks that are based on mathematical principles.

This sparked the idea to investigate card tricks that were based on mathematical principals. I first read The Magic of Math and Mathematics, Magic, and Mystery, the authors provided examples of simple math tricks as well as card tricks that can get students thinking about patterns and relationships between numbers without beginning the conversation with any formulas or vocabulary words.

One of the tricks in particular is Gergonne's Pile Problem. The problem is one that has the potential for many variations, which I will include within the next section. This problem starts with a very simple trick that is easy to learn, perform, and understand - if you keep putting the piles in the middle, eventually the target card will be in the middle. Although the concept on the surface is simple enough, understanding why it must be three rounds and how we can modify the trick takes a little more work. The wonderful thing about using this trick to engage students during the lesson is that they are thinking mathematically without being hindered by a preconceived notion that mathematics is hard or that they are "not good at it." Once I can get students I can break down the barrier of being intimidated by math then they become open to learning. This alone, is a huge step to overcome as a high school math teacher. By the time most students get to high school, their mind is already set to preconceived notions that range from, "Math is hard and I am not good at it" to "Math is easy, I just take numbers and plug them in." Both of these misconceptions are stigmas that as a teacher, I work to overcome.

Gergonne Pile Problem

The Gergonne Pile Problem consists of a pile of 27 of the standard 52 cards. It can be done with other variations, but those will be discussed later. There will be three piles of nine cards each. The dealer will lay out the top card in the first pile, the next card in the second pile, the next in the third pile, the fourth under the first card of the first pile, and the pattern repeats until all 27 cards have been distributed. The spectator will choose one of the 27 cards and indicate to the dealer to which of the three piles that their card is in. The dealer will put the pile with the card that contains the target card in the middle of the three piles. This will force the target card to be narrowed to nine cards. It will be one of the 10th through 18th cards in the entire pack before the second deal.

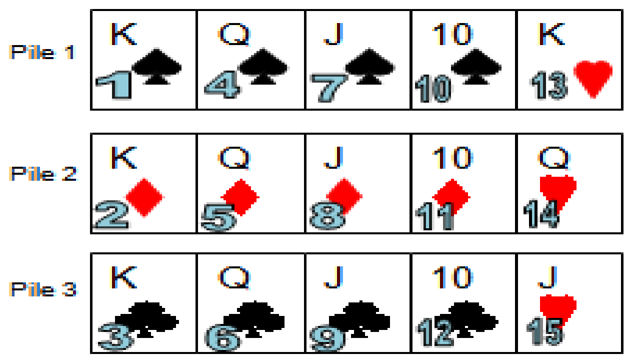
When the pack is dealt the target card will be the fourth, fifth, or sixth cards within a pile. When the spectator indicates the pile the target card is in, the dealer will collect all the piles and then place the pile with the card in the center of the three piles. This will then narrow the target card to one of three cards. It will be one of the 13th through 15th cards in the pack before the third deal.

The dealer will deal out the three piles one more time and the spectator will indicate for a final time as to which pile the card is in. The dealer will collect the cards one final time placing the pile with the target card in it to the middle one last time. This will cause the target card to be in the 14th (middle) position overall.

The dealer can reveal the target card in the middle of the deck however they would like to provide a flare.

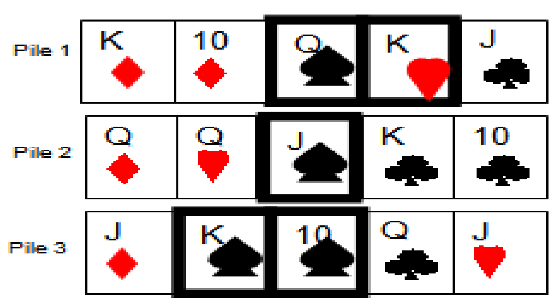
Here is an example with 15 cards:

This is the initial deal. The spectator can pick one of the 15 cards from any of the three piles that each has five cards. Let's say the spectator chooses the King of Hearts. We want to put it as the middle of the 15 cards, which would be the 8th card. Using the same method as described above, we will always put the pile with the target card as the center of the three piles after each deal.



We would put the Pile 1 with the target card (the King of Hearts) in the middle of the three piles after the initial deal. Let's put Pile 2 on top, Pile 1 in the middle and then Pile 3 on the bottom. As Pile 2 and Pile 3 do not have the target card, it doesn't matter which pile goes on top or bottom. Then redistribute the 15 cards. When we deal the cards again, we know our card is one of the 6th through 10th cards that we distribute. This is because there was a pile of five cards above group and five cards below the group.

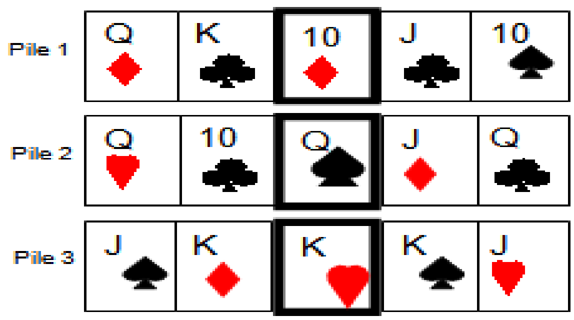
After our first iteration we now have:



We would put Pile 1 with the target card in the middle of the three piles after the second deal. Let us put Pile 2 on top, Pile 1 in the middle and then Pile 3 on the bottom. As mentioned earlier, we could also put Pile 3 on top and Pile 2 on bottom and this will not affect the placement of the target card.

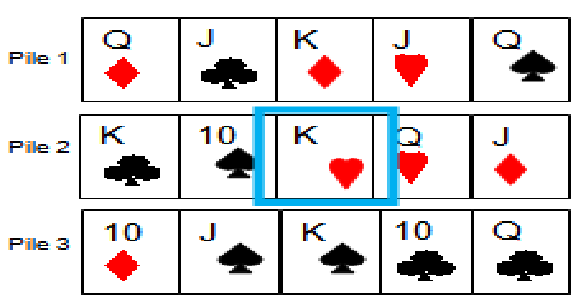
When we redistribute the cards, we know our card is one of the 7th through 9th cards that we distribute. This is because the 6th through 10th cards were either 2nd, 3rd or 4th in their pile, therefore our target card will be the (5+2)th, (5+3)th, or (5+4)th in the entire pack.

After the second iteration we have the order below:



We would put the Pile 3 with the target card as the middle of the three piles after the second deal. Let's put Pile 1 on top, Pile 3 in the middle and then Pile 2 on the bottom. When we put the cards back out we know our card is specifically the 8th card. This is because since the 7th through 9th cards are all the third card out of the five in the pile, we will get our target card to be the (5+3)th card in the pack.

In our hand we have the cards in the final order:



Proof of Gergonne Pile Problem

Let us consider the usual case with 27 cards, which are dealt into three piles of nine cards each. We will prove that the position of the card in the final trick can be determined by the order in which the piles are taken up and reorganized after each of the three deals.

Let a represent the order in which the pile with the target card is taken up after the first deal and reorganized into the new pack. Let b represent the order the pile is taken up after the second deal and let c be the order the pile is taken up after the third deal. The value of a, b, and c can be 1, 2, or 3. This represents the 1st, 2nd or 3rd pile within the pack.

Ball, Rouse W.W. & Coxeter, H.S.M. (1974) wrote a nice proof that creates an equation we can use easily for our original and first variation. "Suppose that, after the first deal, the pile containing the selected card is taken up ath: then (i) at the top of the pack there are a - 1 piles each containing nine cards; (ii) next there are 9 cards, of which one is the selected card; and (iii) lastly there are the remaining cards of the pack. The cards are dealt out now for the second time: in each pile the bottom 3(a - 1) cards will be taken from (i), the next three cards will be from (ii), and the remaining 9-3a cards from (iii).

Suppose that the pile now indicated as containing the selected card is taken up bth: then (i) at the top of the pack are 9(b-1) cards; (ii) next are 9-3a cards; (iii) next are three cards, of which one is the selected card; and (iv) lastly are the remaining cards of the pack. The cards are dealt out now for the third time: in each pile the bottom 3(b-1) cards will be taken from (i), the next 3 - a cards will be taken from (ii), the next card will be one of the three cards in (iii), and the remaining 9 - 3b+ a cards are from (iv).

Hence, after this deal, as soon as the pile is indicated, it is know that the card is the (9 - 3b + a)th card from the top of that pile. If the process is continued by taking up this pile as the *c*th, then the selected card will come out in the place 9(c - 1) + (9 - 3b + a) + 1 from the top, that is, will come out as the (9c - 3b + a)th card.

Let $a = x + 1,\ b = 3 - y,\ c = z + 1$; then x, y, and z may only have the values of 0 (top pile), 1 (middle pile), or 2 (bottom pile). In this case Gergonne's equation takes the form of $9z + 3y + x = n - 1$. Hence, if $n - 1$ is expressed in the ternary scale of notation, *x, y*, and *z* will be determined." (pp.242 - 43)

Specifically, for this original case with 27 cards, $x = y = z = 1$ since we will place the card as the middle of the 3 piles each time. $9 * 1 + 3 * 1 + 1 * 1 = 9 + 3 + 1 = 13 = 14 - 1$.Therefore our card will be in the 14th position each time when you place the pile with the target card in the middle pile after each deal. Additionally, $14 - 1 = 13$ represented in ternary scale as111_3, this verifies the Pile Problem explanation of placing the pile containing the target card in the middle of the stack each time.

Variations

1. How can we move the target card to a specified position within the pack of 27 cards?

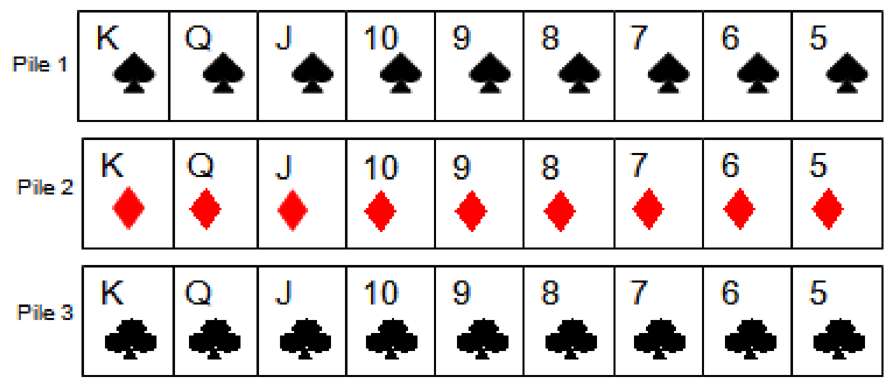
If we want to put a target card into the *n*th position, we will solve for *x, y,* and *z*, where $x, y, z \in \{0,1,2\}$ such that$9z + 3y + x = n - 1$. In the example given below we place our card in the 16th position: we have $n - 1 = 15 = 9 + 6 + 0 = 1 * 3^2 + 2 * 3^1 + 0 * 3^0$ where *x* = 0, *y* = 2, and *z* = 1. As mentioned above, we could use $n - 1 = 15 = 120_3$ to determine the values of *x*, *y,* and *z* as well. Using these values, we would take the pile with the target card and place it on top the first round, bottom the second round and in the middle the third round. This will cause 15 cards to be on top of it and make it "magic" when you flip over the 16th card to be their target card. This table below illustrates all values of *x*, *y*, and *z* as to put the target card into any of the 27 positions within the pack.

n-1	*n-1 in ternary*	$n - 1 = 9z + 3y + x$		
		0 = Top, 1 = Middle, 2 = Bottom		
		z	***y***	***x***
0	000_3	0	0	0
1	001_3	0	0	1
2	002_3	0	0	2
3	010_3	0	1	0
4	011_3	0	1	1
5	012_3	0	1	2
6	020_3	0	2	0
7	021_3	0	2	1
8	022_3	0	2	2
9	100_3	1	0	0
10	101_3	1	0	1

11	102_3	1	0	2
12	110_3	1	1	0
13	111_3	1	1	1
14	112_3	1	1	2
15	120_3	1	2	0
16	121_3	1	2	1
17	122_3	1	2	2
18	200_3	2	0	0
19	201_3	2	0	1
20	202_3	2	0	2
21	210_3	2	1	0
22	211_3	2	1	1
23	212_3	2	1	2
24	220_3	2	2	0
25	221_3	2	2	1
26	222_3	2	2	2

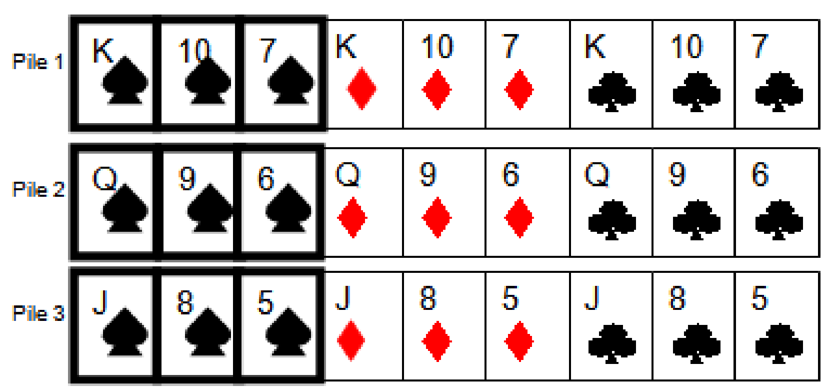
Here is an example. Each of the three piles has nine cards. The spectator will pick one of the 27 cards and a position for it to end up in. Let's say the spectator chooses the Queen of Spades and wants the target card to be in the 16th (n = 16) position overall.

As per above, the dealer will put the pile on the top, then the bottom, then the middle:

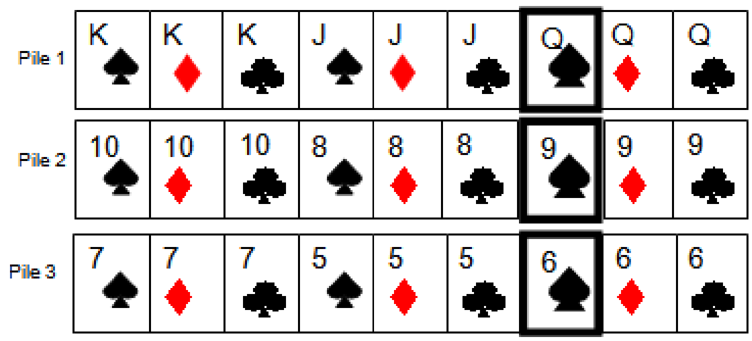


Putting the Pile 1 with the target card as the top pile will make the target card in one of the 1st through 9th positions overall and one of the first three cards within each pile. We will put Pile 2 pile next and Pile 3 on the bottom.

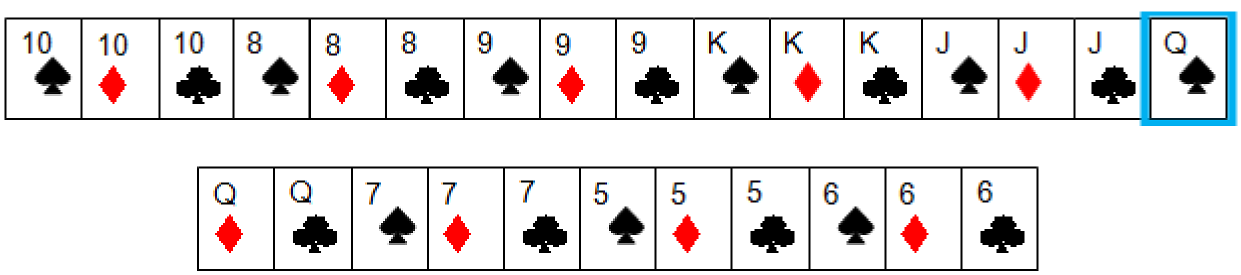
After we deal out for the second time, we get the following distribution of the cards:



Next the dealer will put Pile 2 with the target card on the bottom, Pile 2 on top and the Pile 3 in the middle. This will shift the target card into the first three of the bottom nine cards, in other words the 7th card in one of the three piles or the 19th through 21st in the pack. Once the spectator indicates the pile their card is in, the dealer would know which of those three cards is the target card. Additionally, since all 3 cards are in the 7th position, no matter which is the spectator's card, when we put whichever pile as the middle in the pack, the target card will be shifted to the (9 + 7)th = 16th position overall.



Finally, the dealer will put Pile 1 with the target card into the middle. The dealer will place the Pile 2 on top and the Pile 3 on bottom. The target card will now be in the final (9+7)th = 16th position in the entire pack as shown below.



2. General case - What if we had m^x cards, where $x \geq 2$ and place it in the nth position, where $n \in \{1, 2, \ldots, m_x\}$?

With m^x cards, we would have m piles of m^{x-1} cards each. We would complete x rounds in order to move the card to the specified position. Similar to the original and the first variation, the nth position could be found using base m counting for $n - 1$.

Using an example with $10^3 = 1000$ cards, it will become clearer how our placement of cards work and why base *m* counting is used. We know that we have 10 piles of 100 cards each and it will take us $x = 3$ iterations to move our card to the indicated position.

Let's say we want to place our target card into the 752nd position. This would mean we want 751 cards on top of it, which is 751_{10}. This would indicate we pick up the pile with the target card 2nd, since $a_n \in \{0, 1, 2, \dots, 9\}$. This will shift our target card to be one of the 11th through 20th cards within any of the newly sorted piles of 100 cards each. Next, the indicated pile with the target card will be taken up 6th. Because that means 5 groups of 100 cards each plus the 10 cards that are now distributed throughout the 10 piles will come before the target card. The target card will shift to the 52nd position within any of the piles. Then once the pile with the target card is indicated, the pile will be taken up 8th, so that 7 groups of 100 cards go first, and then the pile in which our target card is 52nd, thus making it in the 752nd position in the entire pack.

After the first deal, we took the piles up in such a way to indicate our $(n-1)$*'s* ones' value. Then after the second deal, we took up the piles in such a way to shift our card into the tens' value position within any of the 10 piles. This made it simple for the final step to place the card in the specified hundreds' value position. Therefore we get the following equation using base 10, $n - 1 = z_2 * 10^2 + z_1 * 10^1 + z_0 * 10^0$. This can also be written as $n - 1 = [z_2 z_1 z_0]_{10}$.

If we have m^x cards we can place the target card in the *n*th overall position after *x* iterations by using $n - 1 = [z_x z_{x-1} \dots z_1 z_0]_m$ to indicate how to place the piles after each of the *x* iterations. The z_0 value narrows our card to one of m^{x-1} cards (i.e. if we have 1000 cards and $z_0 = 2$, it will give us the cards that will have the same ones' values: 002, 012, …, 982, 992).

Next, since the m^{x-1} cards are distributed over the *m* piles, once we pick up using z_1 the cards will be narrowed to one of m^{x-2} cards (i.e. if we continue with 1000 cards, where $z_0 = 2$, and now $z_1 = 7$, it will narrow to all the cards that have 72 in the tens and ones' place: 072, 172, … , 972). Each iteration, the position of the card will be narrowed by a factor of *m* and will eventually, after *x* iterations, be narrowed to one specific position overall.

3. What about if we have 3*p* cards where *p* is odd?

As mentioned before, if we have m^x cards we can use base *m* to determine our position values, however, when we have not exactly m^x cards, we can still place our card in the exact middle using the same method. Why does this work?

Take the first example, it showed if we have 15 cards and you place the pile in the middle, then the middle, then the middle, you will move the card to the middle position. Since we have $3^2 < 15 < 3^3$ we will need 3 iterations to move our card to the specified position.

If we use the original idea of base 3 to determine our positioning, $8 - 1 = 7 = 021_3$, this disagrees with placing the card in the middle each time. The reason our previous method does not work here is because 15 is not a power of 3. The aforementioned method is only applicable if the number of cards is m^x, where $m \geq 2$ and $x \geq 2$.

Since we are talking in multiples of 3, we can perform this trick with every odd multiple of three. That means from 9 up to 15 cards we are adding 6 cards, and therefore adding 3 to each side to maintain the middle. Therefore instead of adding 1 + 3 to find the middle, we will add 1 + 3 + 3 to get the 7. Since $7 = n - 1$, this agrees with *n* = 8 as the middle position of the 15 cards.

# of cards	**Difference from 3^2**	$\frac{Difference}{2}$	**Middle (n)**	$n-1$	**Sum**
9	0	0	5	4	1 + 3 + 0
15	+6	+3	8	7	1 + 3 + 3
21	+12	+6	11	10	1 + 3 + 6
# of cards	**Difference from3^3**	$\frac{Difference}{2}$	**Middle (n)**	$n-1$	**Sum**
27	0	0	14	13	1 + 3 + 9 + 0
33	+6	+3	17	16	1 + 3 + 9 + 3
39	+12	+6	20	19	1 + 3 + 9 + 6
45	+18	+9	23	22	1 + 3 + 9 + 9
51	+24	+12	26	25	1 + 3 + 9 + 12
57	+30	+15	29	28	1 + 3 + 9 + 15
63	+36	+18	32	31	1 + 3 + 9 + 18
69	+42	+21	35	34	1 + 3 + 9 + 21
75	+48	+24	38	37	1 + 3 + 9 + 24

For 3p cards, where p > 0, we can place it in the middle using a variation of the original equation. Since $3^x < 3p < 3^{x+1}$ where x > 1, we will use the equation below up to $z_{x-1} * 3^{x-1}$ then we need to add the difference from 3p to3^xdivide by 2.

$$n - 1 = z_0 * 3^0 + z_1 * 3^1 + \cdots + z_{x-1} * 3^{x-1} + z_x * \frac{3p-3^x}{2} \text{ where } z_0 = z_1 = \ldots = z_x = 1$$

Let's say we have 93 cards and want to put the target card in the center. $3^4 < 93 < 3^5$ and $3p = 93$ which means we will need five iterations to place the cards and the middle is at $\frac{93+1}{2} = 47$:

$$46 = 1 + 3 + 9 + 27 + \frac{93 - 3^4}{2}$$

$$= 1 + 3 + 9 + 27 + \frac{12}{2}$$

$$= 1 + 3 + 9 + 27 + 6$$

4. What if we have some multiple of 3 cards and want to put our target card in the *n*th position?

Harrison et al (1997) investigated the dynamic relationship between the position the card was in before an iteration of the trick and where it will be after *m* iterations through. First however, we need to specify how we can represent the position of the card with 3p cards. Using some of their ideas, assume we have a deck containing 3p cards dealt into 3 piles of p cards each, where $p \geq 2$. The case where $p = 1$ is trivial. Imagine the pack of 3p cards are arranged in a 3 by p matrix. Let k represent the position of the target card within the pile$(1 \leq k \leq p)$. Also, let z represent the number of piles picked up before the pile with the target card when the cards are re-collected. Since we pick up z piles before the target card's pile whenever we re-collect, it follows that there are $zp + k - 1$cards before the target card. When we redistribute the cards, those

$zp + k - 1$ cards are distributed into the 3 piles before the target card is dealt again; therefore $\frac{zp+k-1}{3}$ positions within each pile are filled. Our card will be the $\left(\frac{zp+k-1}{3} + 1\right)$th position within the pile. (pp. 105-06)

Now that we have the representation of the card within a pile, using the dynamic relationship that Harrison et al (1997) discussed, let *p* be a fixed positive integer and let $r = zp + k - 1$, and write the function $f(k,r) = \left\lfloor \frac{k+r+2}{3} \right\rfloor$.

For $j \geq 1$, we have

$$f^{j}(k,r) = \left\lfloor \frac{k}{3^{j}} + \frac{r+2}{2}\left(1 - \frac{1}{3^{j}}\right) \right\rfloor,$$

where $f^{j}(k,r)$denotes the *j*th iteration of the function $f(k,r)$.

Proof: When j=1, we have $\frac{k}{3^1} + \frac{r+2}{2}\left(1 - \frac{1}{3^1}\right) = \frac{k+r+2}{3}$, and so the initial case holds. We now proceed by induction. Assuming that the theorem holds from some positive integer *j*, we see that

$f^{j+1}(k,r) = f\left(\left\lfloor \frac{k}{3^{j}} + \frac{r+2}{2}\left(1 - \frac{1}{3^{j}}\right) \right\rfloor\right)$

$$= \left\lfloor \frac{\left\lfloor \frac{k}{3^{j}} + \frac{r+2}{2}\left(1 - \frac{1}{3^{j}}\right) \right\rfloor + r + 2}{3} \right\rfloor$$

$$= \left\lfloor \frac{\left\lfloor \frac{k}{3^{j}} + \frac{r+2}{2}\left(1 - \frac{1}{3^{j}}\right) + \frac{r+2}{2}(2) \right\rfloor}{3} \right\rfloor$$

$$= \left\lfloor \frac{k}{3^{j+1}} + \frac{r+2}{2}\left(1 - \frac{1}{3^{j+1}}\right) \right\rfloor$$

so this holds for *j+1*, and the proof follows by induction. (p. 108)

Using the function from above, we can show that after x iterations, the target card will only be in the middle position within a given pile.

$$f^x(k,r) = \left\lfloor \frac{\mathrm{k}}{3^{\mathrm{x}}} + \frac{\mathrm{r}+2}{2}\left(1 - \frac{1}{3^{\mathrm{x}}}\right) \right\rfloor$$

Since $1 \leq k \leq p$

$$\left\lfloor \frac{1}{3^{\mathrm{x}}} + \frac{\mathrm{r}+2}{2}\left(1 - \frac{1}{3^{\mathrm{x}}}\right) \right\rfloor \leq \left\lfloor \frac{\mathrm{k}}{3^{\mathrm{x}}} + \frac{\mathrm{r}+2}{2}\left(1 - \frac{1}{3^{\mathrm{x}}}\right) \right\rfloor \leq \left\lfloor \frac{\mathrm{p}}{3^{\mathrm{x}}} + \frac{\mathrm{r}+2}{2}\left(1 - \frac{1}{3^{\mathrm{x}}}\right) \right\rfloor$$

The difference in each of these expressions is:

$$\left\lfloor \frac{1}{3^{\mathrm{x}}} \right\rfloor \leq \left\lfloor \frac{\mathrm{k}}{3^{\mathrm{x}}} \right\rfloor \leq \left\lfloor \frac{\mathrm{p}}{3^{\mathrm{x}}} \right\rfloor$$

Since $p \leq 3^{x-1}$

$$\left\lfloor \frac{1}{3^{\mathrm{x}}} \right\rfloor \leq \left\lfloor \frac{\mathrm{k}}{3^{\mathrm{x}}} \right\rfloor \leq \left\lfloor \frac{3^{\mathrm{x}-1}}{3^{\mathrm{x}}} \right\rfloor$$

$$0 \leq \left\lfloor \frac{\mathrm{k}}{3^{\mathrm{x}}} \right\rfloor \leq 0$$

Therefore, no matter the k, p, or z values, the card can only be in one position within the pile after x iterations.

If we know $3^{x-1} < p \leq 3^x$, $x + 1$ iterations are necessary to place the target card to a specified position. After the first iteration, one pile will be indicated as having the target card, therefore narrowing the card to *p* possible cards. After the second iteration, those *p* cards will be distributed through the three piles, therefore there will be at most 3^{x-1} of those cards in each pile. After the third iteration, our target card will be at most 3^{x-2} cards that could be the target card. After the *x*th iteration, there will be at most 3 cards that could be the target card. Finally, after *x+1* iterations, there will be only 1 card left that could be the target card of the 3*p* cards.

This means that no matter the *n* value we can place our target card into the *n*th position in $x + 1$ iterations. The values multiplied with the z values are based around the value of p and 3^x.

According to Chang (2015), “Although the 27 Card Trick is not an overly difficult puzzle to crack in itself, it provides a very interesting method is which information can be sorted. The further generalizations that have been proven show the many ways in which this method can be used to organize the information by powers, we can reposition any piece of information at incredibly fast speeds.” (p. 5)

Algebra 2 - (10,11,12) - Modular Arithmetic

Objectives:	Students will be able to understand and apply bases, exponents, and modular arithmetic.
Course Name	Algebra 2 (10th through 12th)
Estimated Time	3 - 45 minute classes
Prerequisite Knowledge	- Remainders - Division - Base - Exponent - Power
Vocabulary	- Modulus - Remainder
Materials Needed	- Standard Deck of Playing Cards for each group - Day 1, Day 2, and Day 3 Reflection Sheets - Modulo and Base Worksheets
Common Core Content Standards	HSF.LE.B.5 Interpret the parameters in a linear or exponential function in terms of a context. HSA.APR.B.2 Understand the relationship between zeros and factors of polynomials. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.
Iowa Standards for Mathematical Practices	3. Construct viable arguments and critique the reasoning of others. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.

Launch: *How will you engage students in the content for the day?*

Day 1: Teacher will demonstrate Gergonne's Pile Problem to class, revealing the target card by spelling out "T-H-I-S I-S Y-O-U-R C-A-R-D" and flipping the card on the letter "D" to reveal the target card.

Day 2: Teacher will demonstrate the variation of Gergonne's Pile Problem where you place the card into any position, not just the middle.

Day 3: Teacher will ask for students to explore the relationship between the power and the number of piles and iterations to discover how they could extend the trick to different numbers of cards.

Explore: *How will students explore the content for the day?*

1. Day 1:
    - *5 minutes:* Teacher will demonstrate the Pile Problem.
    - *10-15 minutes:* In groups of 2-4, students will be given 27 cards and try to understand Gergonne's Pile Problem. They need to know how many cards there are and what position their card ended up in. They will figure out the pattern of middle, middle, middle.
    - *5 - 10 minutes:* **Class discussion:** various groups will explain their hypotheses. Class will decide which method works and venture as to why.
    - *5 minutes:* Teacher will then guide to explanation of trick.
    - *10 minutes:* **Group discussion:** Would this work for 15 cards? 24 cards?
    - *5 minutes:* **Reflection**: What amounts of cards will this work?

2. Day 2:
    - *5 minutes:* Teacher will then show variation of trick where the chosen card will end in any position.
    - *10 minutes:* **Group discussion:** Students will try to figure it out.
        - How did the teacher know which position to place the piles each time?
    - *5-10 minutes:* Class discussion.
    - *5 minutes:* Teacher explains about base 3 and modulo.
    - *15 - 20 minutes:* Students practice finding various mods and numbers in base 3.
    - *5 minutes:* **Reflection:** How do modulo and base 3 help to understand the original and variation of the trick? Did you change from what you thought yesterday?

3. Day 3:
    - *5-10 minutes:* **Group discussion:** How do we know how many times to go through to get our card into the intended position (ex. Why three times through with 27 cards)?
    - *10 - 20 minutes:* Review and practice base 3 and modulo, discuss other bases. What relationship do modulo and bases have?
    - *10 minutes:* **Group discussion:** If we have a different number of cards (ex. 9, 16, or 25) how would our trick change?
    - *5 minutes:* **Reflection:** What relationship is there between the power that represents the number of cards and how the trick is performed?

Summary/Close of the Lesson:

How will you close your lesson and bring student understanding to a close for the day?

Day 1: Students will discuss in groups how this trick could be done with different numbers of cards. They will then formalize in a short reflection what pattern they believe is needed for this trick to work.

Day 2: Students will discuss in groups how modulo and base 3 work together to make this trick work. The students will end class by writing a short reflection on the relationship and make any modifications to what they wrote at the end of the previous class.

Day 3: Students will discuss in groups how modulo and various bases can help if a different number of cards is used. The students will end class by writing a short reflection on the relationship between the base and exponent that represent the number of cards and how many piles and iterations are necessary? Could there be multiple ways? (Ex. $16 = 2^4 = 4^2$)

Extension(s):

1. What if we want to place the card anywhere within the pack and the number of cards is a multiple of, but not a power of *n*?
2. Modular Arithmetic
3. Modular Encryptions

Check for Understanding: *How will you assess students throughout and at the end of the lesson?*

Each day the students will work in groups and at the end write a reflection over the lesson. The students will have small tasks to complete throughout the lesson as well.

Strategies to Support English Learners:

- Guided Notes with vocabulary and prompts for discussion.
- Visual representations of vocabulary and examples of modulo and base conversion.
- Small group discussion.
- When possible, group with a student with same language background for support.

Key Ideas:

Key Ideas	**Teacher Strategies**
Modulos as a remainder	Modulo takes out (or adds) multiples of that number until we have a value $0 \leq r < m$.

	Modulo gives us the remainder from division.
Base 3	Refer students back to base 10. First, is the ones which represent 10^0, then we have the tens which represent 10^1, and the exponent continues to increase. For base 3, it is similar, but with 3's instead. Including a couple of illustrations of various numbers worked out would be beneficial. It is a representation of a number.
b^x as a power, where b represents the number of piles and x is the number of iterations necessary for the Pile Problem.	While talking about $27 = 3^3$ explain what each 3 represents. Then talk about if we have $9 = 3^2$cards, we would still have 3 piles, but would need only to complete 2 iterations instead of 3.

Guiding Questions: *Focus on the mathematics and using open-ended questioning*

Good questions to ask	**Possible student responses**
When the original trick was performed, what did we need to know in order to figure out the pattern?	How many cards there were and what place was the card at the end of the trick.
Why wouldn't the original trick work with 16 cards?	Since we put the card in the middle for the original trick and 16 doesn't have a real middle, we can't do it the same.
Why do we go through 3 iterations instead of 2 or 4?	We would have to have 3 iterations because there are 27 cards and since $27 = 3^3$, we have 3 piles that get sorted 3 times.
What is the relationship between modulo 3 and base 3?	Modulo is just the remainder when I divide by 3, base 3 gives me more information about the number and how small or big it is.

Misconceptions:

Possible Misconceptions	**Teacher Action**
15 mod 3 = 5	Modulo is our remainder, not the quotient.

Only 2 iterations are needed for 15 cards since $15 < 3^3$	Since $3^2 < 15 < 3^3$, we need more than 2 iterations to get our target card into the middle position.
$27\ mod\ 3 \equiv\ 0$ means 27 = 0 base 3	Remember: modulo is our remainder. Since we have 0, that means 3 goes into 27 an exact amount of times. To write in base 3, we use our powers of 3 to determine how many times that 3 goes into 27.

Day 1: Exploring Gergonne's Pile Problem Reflection Sheet

1. What questions do you have about Gergonne's Pile Problem to understand it?

2. Write down any observations about what you think happens.

3. Write down ideas from 2 other students for how the trick works. Do you think they could be right? Explain your reasoning.
   1.
   2.

4. Now that the teacher explained Gergonne's Pile Problem, write in your own words, how the trick works.

5. Do you think the method you just described could work for 15 cards? 24 cards? Some other number of cards? Explain your reasoning.

6. Reflection: How can you know if you can do this trick with *n* cards? What do you need to know about *n*?

Day 2: Exploring Base 3 and Modulus Reflection Sheet

1. Compare Gergonne's Pile Problem from last class to the version you just saw. What is similar? What is different?

2. Now that you have talked with your group, how do you think this version works?

3. Compare ideas from the class to your idea.

4. Now that the teacher explained this version, write in your own words how it works.

5. Reflection:

a. What does 13 mod 3mean? What is a modulus?
b. How do you write 24 in base 3? Explain how you figured it out.
c. How do modulo and base 3 help to understand the original and variation of the trick?
d. Did you change from what you thought yesterday? If so, How?

Day 3: What happens when the number of cards change?

1. Review from the last 2 classes: How do we know how many piles to make and how many times we need to repeat for the trick? (This back to the trick with 27 cards)

2. After some more practice with modulo and writing in base 2, 3, 4, and 5, look back at your reflection parts 5a and 5b from last class. Did your understanding of modulo and writing in base 3 change? Explain.

3. Use what you wrote today from #1 and #2 to explain if and how you can complete the two versions of the trick using 9 cards, 16 cards, 25 cards and 243 cards.

4. Reflection: What relationship is there between the power that represents the number of cards and how the trick is performed?

5. Summary: Explain what modulo is. Give 2 examples. Explain how to write numbers in base 3. Give one example less than 27 and one greater than 81.

Day 2: Modulo and Base 3 Practice Sheet

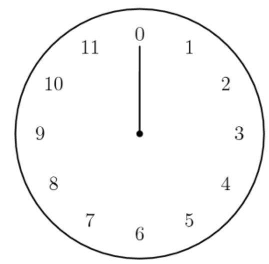
Review:

$$\frac{A}{B} = Q + R$$

A represents the ________________
B represents the ________________
Q represents the ________________
R represents the ________________

<u>Definition:</u> **Modulo** is a function where given two positive numbers A and B, A modulo B (shortened to A mod B) is the remainder of A divided by B. B would also be referred to as the **modulus**. A mod B = R.

Think of time, where after 12 hours, we reset and start counting again at 1. Once we get to our modulus, the values reset to 0, because the remainder is again 0.



<u>Example:</u> 15 mod 12
15/12 = 1 remainder 3,
therefore 15 mod 12 = 3

Practice Problems finding remainders using various modulus

1. 16 mod 3
2. 10 mod 3
3. 57 mod 3
4. 22 mod 3
5. 3 mod 3
6. 17 mod 4
7. 15 mod 2
8. 100 mod 6
9. 29 mod 5
10. 30 mod 4
11. 18 mod 1
12. 99 mod 4

Changing to Ternary, Tertiary, Base 3

Some of you have heard of binary (base 2) and all of you make use of it every time you use your phone, a calculator or a computer. Binary is used to complete many calculations in a very short amount of time. Base 2 is made up of 0's and 1's. The decimal system (base 10) uses 0's, 1's, 2's, 3's, 4's, 5's, 6's, 7's, 8's and 9's. Each of these systems uses 0 through one less than the base value. That is because, like in modulo, the value resets to each time it goes in an exact amount of times.

You have been learning and thinking in base 10 for most of your life. 234 can be broken up in base 10 into the ones, the tens, and the hundreds part of the number. It would be written as $234 = 2 * 100 + 3 * 10 + 4 * 1$ or as it relates to base 10 is $234 = 2 * 10^2 + 3 * 10^1 + 4 * 10^0$ (recall: $b^0 = 1$). In short, it will be written as 234_{10}.

We were going to focus on base 3. We know the values could be 0, 1, or 2.

Recall: $3^0 = 1, 3^1 = 3, 3^2 = 9, 3^3 = 27$, ...

<u>Example:</u> 22 in base 3

We start with the largest power less than 22, which is 9.

9 goes into 22 twice with a remainder of 4.

Next power is 3, which goes into 4 once with 1 left over.

The final power is 1, which goes into 1 once.

$22 = 2 * 9 + 1 * 3 + 1 * 1 = 211_3$

<u>Example:</u> 38 in base 3

27 goes into 38 once with a remainder of 11

9 goes into 11 once with a remainder of 2

3 does not go into 2

1 goes into 2 twice.

$38 = 1 * 27 + 1 * 9 + 0 * 3 + 2 * 1 = 1102_3$

Practice Problems: Convert each to base 3 Recall: $3^0 = 1, 3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81$, ...	Answer
<u>EXAMPLE: 50:</u> 81 is too big. 27 goes into 50 once, with a remainder of 23 9 goes into 23 twice, with a remainder of 5 3 goes into 5 once, with a remainder of 2 1 goes into 2 twice, with a remainder of 0. $50 = 1 * 3^3 + 2 * 3^2 + 1 * 3^1 + 2 * 3^0$	1212_3
11	
15	
23	
37	
42	
68	
86	
100	

Day 3: Modulo and Base 2, 3, 4, and 5 Practice Sheet

More practice with modulo

Practice Problems:

1. 5 mod 2
2. 10 mod 3
3. 17 mod 7
4. 23 mod 4
5. 50 mod 6
6. 18 mod 5
7. 22 mod 2
8. 8 mod 3

Practice Problems: Convert each of the following to base 2 Recall: $2^0 = 1, 2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16, \ldots$	Answer
9	
18	
37	
Practice Problems: Convert each of the following to base 3 Recall: $3^0 = 1, 3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81, \ldots$	Answer
8	
22	

52	
Practice Problems: Convert each of the following to base 4 Recall: $4^0 = 1, 4^1 = 4, 4^2 = 16, 4^3 = 64, 4^4 = 256$, ...	Answer
23	
43	
67	
Practice Problems: Convert each of the following to base 5 Recall: $5^0 = 1, 5^1 = 5, 5^2 = 25, 5^3 = 125, 5^4 = 625$, ...	Answer
19	
29	
62	

Conclusion

In my initial exposure to Gergonne's Pile Problem, I learned of the basis on the ternary system. The variety of uses of the ternary system made me interested to understand more about this original problem and begin looking into variations including the generalization to any number of cards. My hope while investigating I would learn more about the problem that would help to cross over into a high school lesson that was based around the original and one or more variations of the problem.

Throughout my exploration of Gergonne's Pile Problem and the variations, I have discovered relationships that go beyond the ternary system, including the dynamic function Harrison et al discussed in their article. For part of my exploration, I was stuck on the idea of figuring out how each one worked instead of looking at the whole to determine how I will know that it can work. Once I refocused my efforts on exploring different ways to prove that it is possible to perform the trick with different amounts of cards, I was able to see the relationships using the various systems, exploring a variety of equations, and the dynamic function.

When thinking about the lesson I would create for the students, it was clear to me that the explanation using the ternary system would be best. I determined incorporating a discussion of the modulo operation would also tie in nicely to exponential and polynomial function standards. I determined that starting with base 10 then extending to base 3 would be best, since not all students had experience with different base systems before. It is important for students to understand how base 10 works for them to be able to relate it to base 3 and others. When learning about base 3, I thought it would be an ideal time for students to be introduced to the modulo operation. I would like to spend more time exploring the pile problem in particular with the students, but I was not sure if this would be possible at the high school level other than the exploration of the ternary system.

I am grateful that I found such an interesting topic that has so many extensions worth exploring. I enjoy card tricks and think that games, magic and card tricks are wonderful ways for students to be introduced to mathematics. It gives the students a chance to explore something unfamiliar in an approach that is potentially less daunting for some. One of my favorite parts of starting a lesson with a game or a trick is that the students can't help but to naturally be curious and ask questions that can lead to a deeper understanding that they would have never thought of if presented in lecture.

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