Boolean Krasner algebras

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Boolean Krasner algebras

Abstract
An algebra $A$ will be called a group action if the basic operations of $A$ are all (unary) permutations of $A$. For the purposes of this discussion, we will disallow nullary operations. But see the remark following the proof of the Theorem.

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This is a pre-print, available at: https://faculty.sites.iastate.edu/cberman/files/inline-files/boolean.pdf.
BOOLEAN KRASNER ALGEBRAS

Clifford Bergman

An algebra \( A \) will be called a *group action* if the basic operations of \( A \) are all (unary) permutations of \( A \). For the purposes of this discussion, we will disallow nullary operations. But see the remark following the proof of the Theorem.

**Theorem.** Let \( B \) be a finite algebra. Then \( B \) is categorically equivalent to a group action if and only if for every \( n \), the lattice \( \text{Sub}(B^n) \) is Boolean.

**Proof.** If \( A \) is a finite group action, then it is easy to verify that the set of subuniverses of \( A^n \) is closed under (set-theoretic) complementation. Consequently, \( \text{Sub}(A^n) \) is a Boolean lattice. If \( B \cong c A \), then \( \text{Sub}(B^n) \cong \text{Sub}(A^n) \), so \( \text{Sub}(B^n) \) is Boolean as well.

Now assume that \( B \) has Boolean subalgebra lattices. Note that, because we are not permitting nullary operations, the empty set is the zero-element of \( \text{Sub}(B^n) \) for every \( n \). Let \( \sigma \) be a unary term operation of \( B \) that minimizes the cardinality of \( \sigma(B) \). It follows that every unary term operation is one-to-one on \( \sigma(B) \). Since \( B \) is finite, some power of \( \sigma \) is idempotent. Thus we can assume, without loss of generality, that \( \sigma \) is idempotent and every unary term operation is injective on \( \sigma(B) \).

Let us show that \( \sigma \) is invertible. Let \( B = \{b_1, b_2, \ldots, b_m\} \). We define \( b = \langle b_1, \ldots, b_m \rangle \in B^m \), and let \( \psi \) denote the subalgebra of \( B^m \) generated by \( b \).

Suppose \( \theta \) is an atom of \( \text{Sub}(B^m) \) and \( \theta \subseteq \psi \). Since \( \theta \) is an atom, it is generated by a single \( m \)-tuple \( x = \langle x_1, \ldots, x_m \rangle \). \( x \in \theta \subseteq \psi \) implies that there is a unary term \( t \) such that \( x = t(B)(b) \), i.e., \( x_i = t(b_i) \) for \( i = 1, 2, \ldots, m \). Applying \( \sigma \) to both sides, \( \sigma(x) = \sigma(t(b)) \). Since \( \theta \) is an atom of \( \text{Sub}(B^m) \), it is generated by \( \sigma(x) \).

Now since the subalgebra lattice is Boolean, every element is a join of the atoms it contains. Therefore, \( \psi = \bigvee_{j=1}^k \theta_j \) where each \( \theta_j \) is an atom. From the previous paragraph, for each \( j \leq k \) there is a unary term \( t_j \) such that \( \theta_j \) is generated by \( \sigma(t_j(b)) \). It follows that

\[
b \in \psi = \bigvee_{j=1}^k \text{Sg}^B(c(t_j(b))) = \text{Sg}^B(c(t_1(b)), c(t_2(b)), \ldots, c(t_k(b))).
\]

Therefore, there is a term \( s \) such that \( b = s(\sigma(t_1(b)), \ldots, \sigma(t_k(b))) \). In other words, \( \sigma \) is invertible.

Let \( A = B(\sigma) \). We claim that \( A \) is a group action. Suppose that \( A \) had a term operation that was not essentially unary. Then there is a \( k \)-ary term \( t_\sigma \) that depends on its first two variables. This means that there are elements \( a_1, a_2, \ldots, a_k, a'_1 \in A \) such that

\[
u = t_\sigma(a_1, a_2, \ldots, a_k) \neq t_\sigma(a'_1, a_2, \ldots, a_k) = u'
\]
and elements $c_1, \ldots, c_k, c'_2$ such that

$$v = t_\sigma(c_1, c_2, \ldots, c_k) \neq t_\sigma(c_1, c'_2, \ldots, c_k) = v'.$$

Now let $\delta_1 = \{ \langle x, x, y, z \rangle : x, y, z \in B \}$ and $\delta_2 = \{ \langle x, y, z, z \rangle : x, y, z \in B \}$. These are subalgebras of $B^4$ (for any algebra $B$). Each quadruple $\langle a_1, a'_1, c_1, c_1 \rangle$, $\langle a_2, a_2, c_2, c'_2 \rangle$, \ldots, $\langle a_k, a_k, c_k, c_k \rangle$ is a member of either $\delta_1$ or $\delta_2$, so they are all members of $\delta_1 \vee \delta_2$. (Join in the lattice $\text{Sub}(B^4)$.) Therefore

$$\langle \sigma t(a_1, a_2, \ldots, a_k), \sigma t(a'_1, \ldots, a_k), \sigma t(c_1, c_2, \ldots, c_k), \sigma t(c_1, c'_2, \ldots, c_k) \rangle \in \delta_1 \vee \delta_2.$$

Since $t_\sigma = \sigma \circ t^*_A$, this means that $\langle u, u', v, v' \rangle \in \delta_1 \vee \delta_2$.

Let $\theta$ be the subalgebra generated by $\langle u, u', v, v' \rangle$. Then $\theta \subseteq \delta_1 \vee \delta_2$. Therefore, by distributivity, $\theta = (\theta \cap \delta_1) \vee (\theta \cap \delta_2)$. But a typical member of $\theta$ is of the form $\langle p(u), p(u'), p(v), p(v') \rangle$ for some unary term $p$ of $B$. Since $p$ is injective on $\text{sigma}(B)$, this element can not be a member of either $\delta_1$ or $\delta_2$. It follows that $\theta = \emptyset$, which is a contradiction.

Therefore $A$ is term equivalent to a unary algebra. The fact that the basic operations are all permutations follows from the fact that every unary operation on $B$ is injective on $\sigma(B)$. □

Remark. If we allow nullary operations, the situation seems to change somewhat. It is crucial to our argument that the zero-element of $\text{Sub}(B^n)$ be the empty set, a condition that fails in the presence of nullary operations. It is usually safe to replace each nullary operation with a constant unary operation. However, that has the side-effect of destroying the “Boolean” condition on the subalgebra lattices.

In the presence of nullary operation, one expects a slightly different characterization. So we conjecture: if $B$ is a finite algebra such that, for every $n$, $\text{Sub}(B^n)$ is Boolean, then $B$ is categorically equivalent to a group-action-with-constants, i.e., a unary algebra in which every operation is either a permutation or constant.

March 1997