Categorical equivalence and central relations

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Categorical equivalence and central relations

Abstract
A finite algebra is called preprimal if its clone of term operations is a coatom in the lattice of clones. These algebras are of interest both as a generalization of primal algebras, and as a toehold in the difficult analysis of the lattice of clones on a finite set.

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A finite algebra is called *preprimal* if its clone of term operations is a coatom in the lattice of clones. These algebras are of interest both as a generalization of primal algebras, and as a toehold in the difficult analysis of the lattice of clones on a finite set.

Two algebras $A$ and $B$ are called *categorically equivalent* if the varieties $V(A)$ and $V(B)$ are equivalent as categories via a functor mapping $A$ to $B$. We write $A \equiv_c B$ to indicate this relationship. It was shown in [5] that if $A$ is primal, then $A \equiv_c B$ if and only if $B$ is primal. This was extended to preprimal algebras in [3] and [4]. Unfortunately, the treatment of one case—that of central relations—is incorrect in the second paper and is difficult to follow in the first. The purpose of this note is to provide a clear and straightforward argument for this one case.

Let $e$ be an equivalence relation on \{1, 2, ..., $h$\}, for some positive integer $h$, and let $A$ be a set. We let

$$\delta_e = \{(x_1, x_2, \ldots, x_h) \in A^h : (i, j) \in e \Rightarrow x_i = x_j \}.$$  

Relations of the form $\delta_e$ are called *generalized diagonals*, and are invariant under every operation on $A$. We will call $e$ nontrivial if $\delta_e \neq A^h$.

**Definition.** Let $A$ be a finite set, $h$ a positive integer, and $\rho \subseteq A^h$. Then $\rho$ is an *$h$-ary central relation* on $A$ if

- for every nontrivial $e$, $\delta_e \subseteq \rho$;
- for every permutation $\sigma$ of $\{1, 2, \ldots, h\}$, $(x_1, x_2, \ldots, x_h) \in \rho \implies (x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma h}) \in \rho$;
- there is a nonempty subset $Z(\rho)$ of $A$ such that $Z(\rho) \times A^{h-1} \subseteq \rho$.

The set $Z(\rho)$ is called the *center* of $\rho$.

Let $\Theta$ be a set of relations on a set $A$. By $\mathcal{P}(\Theta)$ we mean the clone of all operations preserving every member of $\Theta$. Let us call a finite algebra $A$ of *$h$-ary central type* if the clone of term operations on $A$ is equal to $\mathcal{P}(\rho)$ for some $h$-ary central relation $\rho$ on $A$. Unary central type is a special case of subalgebra-primality, which is considered in [2]. For the remainder of this paper we restrict to the case that $h > 1$.

Let $h$ be an integer greater than 1, and $B_h = \{0, 1, 2, \ldots, h\}$. We define

$$\nu_h = \{(x_1, x_2, \ldots, x_h) \in B_h^h : \{x_1, \ldots, x_h\} \neq \{1, 2, \ldots, h\}\}.$$
(Equivalently, \( \nu_h \) is the set of those \( h \)-tuples containing at least one component equal to 0, or at least one pair of equal components.) Note that \( \nu_h \) is the unique central relation on \( B_h \) such that \( Z(\nu_h) = \{0\} \). Finally, let \( B_h = \langle B_h, \mathcal{P}(\nu_h) \rangle \).

**Theorem.** Let \( h \) be an integer greater than 1 and let \( A \) be of \( h \)-ary central type. Then \( A \cong B_h \).

**Proof.** Let \( \rho \) be the \( h \)-ary central relation on \( A \) guaranteed by the hypothesis. By McKenzie’s theorem [6, Corollary 6.1], it suffices to find an invertible idempotent term \( s \) on \( A \) such that \( A(s) \) is weakly isomorphic to \( B_h^n \). By Theorem 2.1 (and the remarks following the proof) of [2] and Lemma 2.4 of [4], this is equivalent to finding an operation \( s \in \mathcal{P}(\rho) \) such that

1. For all \( x \in A \), \( s(s(x)) = s(x) \);
2. \( \mathcal{P}(\rho) \) contains an \((h+1)\)-ary near unanimity term;
3. For every pair \( \theta, \psi \) of distinct subalgebras of \( A^h \), \( s(\theta) \neq s(\psi) \);
4. The relational structures \( \langle s(A), s(\rho) \rangle \) and \( \langle B_h, \nu_h \rangle \) are isomorphic.

In 3 and 4, by \( s(\theta) \) we mean \( \{ (s(x_1), \ldots, s(x_h)) : (x_1, x_2, \ldots, x_h) \in \theta \} \).

Since \( \rho \neq A^h \), there is \( a = (a_1, a_2, \ldots, a_h) \in A^h - \rho \). From the definition of central relation, for every \( 1 \leq i < j \leq n \) we have \( a_i \neq a_j \) and \( a_i \notin Z(\rho) \).

Since the center is nonempty, we fix an element \( a_0 \) of \( Z(\rho) \).

Define the unary operation \( s \) on \( A \) by

\[
s(x) = \begin{cases} 
  x & \text{if } x \in \{a_0, a_1, \ldots, a_h\} \\
  a_0 & \text{otherwise.}
\end{cases}
\]

Our first task is to prove that \( s \) preserves \( \rho \). So let \( x = (x_1, \ldots, x_h) \in \rho \) and \( y = (y_1, \ldots, y_h) = s(x) \). If some pair of components of \( x \) are equal, then the corresponding pair of components of \( y \) are equal, thus \( y \in \rho \). So suppose that the components of \( x \) are pairwise distinct. Since no permutation of \( a \) is in \( \rho \), there must be an \( i \leq h \) such that \( y_i = s(x_i) = a_0 \), so \( y \in \rho \). We conclude that \( s \in \mathcal{P}(\rho) \).

It is obvious from its definition that \( s \) is idempotent (i.e., condition 1 holds) and \( s(A) = \{a_0, a_1, \ldots, a_h\} \). Furthermore, the mapping \( a_i \mapsto i \), for \( i = 0, 1, 2, \ldots, h \) satisfies condition 4. It is well-known that every central relation admits a near-unanimity term. In fact, we can obtain such a term by defining \( m(x_0, x_1, \ldots, x_h) \) to be \( a_0 \) whenever the “near-unanimity” conditions do not apply.

We now consider condition 3. For \( x \in A^h \), let \( e(x) = \{(i, j) : x_i = x_j \} \). We require the following Lemma.

**Lemma.** Let \( x \in A^h \), and let \( \theta \) be the subalgebra of \( A^h \) generated by \( x \). If \( e(x) \) is nontrivial, then \( \theta = \delta_{e(x)} \). If \( e(x) \) is trivial, then \( \theta = \rho \) if \( x \in \rho \), else, \( \theta = A^h \).

**Proof of Lemma.** Let \( y \) be any element of the subalgebra that, according to the statement of the Lemma, is supposed to be equal to \( \theta \). Define a unary
operation \( f \) by \( f(x_i) = y_i \), for \( i = 1, 2, \ldots, h \), and \( f(w) = a_0 \) otherwise. Since \( e(y) \supseteq e(x) \), \( f \) is well-defined. It suffices to prove that \( f \in \mathcal{P}(\rho) \).

So let \( z \in \rho \). If \( z \) has a pair equal components, then so does \( f(z) \), so \( f(z) \in \rho \). Thus we can assume that the components of \( z \) are pairwise distinct. If, for some \( i \leq h, z_i \not\in \{x_1, \ldots, x_h\} \), then \( f(z_i) = a_0 \in Z(\rho) \), so again, \( f(z) \in \rho \). The only remaining possibility is that \( z \) is a permutation of \( x \). In that case, \( x \in \rho \), so \( y \in \rho \). Since \( f(z) \) is a permutation of \( y \), we conclude that \( f(z) \in \rho \).

Now we verify condition 3. Let \( \theta, \psi \) be subalgebras, and assume that \( \theta \not\subseteq \psi \). Therefore, there is a join-irreducible subalgebra \( \mu \) such that \( \mu \subseteq \theta \) and \( \mu \not\subseteq \psi \). Since every join-irreducible subalgebra is 1-generated, it follows from the Lemma that either \( \mu \) is a generalized diagonal, or \( \mu = \rho \). For each of the possibilities, we show, via the Lemma, that there is a generator \( x \) of \( \mu \) with \( x \in s(A)^h \). Then \( x \in s(\theta) - s(\psi) \) as desired.

If \( \mu = A^h \), we take \( x = (a_1, a_2, \ldots, a_h) \). If \( \mu = \rho \), take \( x = (a_0, a_2, \ldots, a_h) \). If \( \mu = \delta_e \) with \( e \) nontrivial. Choose any \( x \in \{a_1, \ldots, a_h\}^h \) with \( e(x) = e \).

\[ \square \]

Remarks. It follows from the Lemma that if \( A \) is of \( h \)-ary central type, then the only join-irreducible members of \( \text{Sub}(A^h) \) are generalized diagonal relations and, possibly, the central relation \( \rho \). In fact, if \( h = 2 \), then \( \rho \) is indeed join-irreducible. However, it is easy to show that if \( h > 2 \) then there are distinct nontrivial equivalence relations \( e \) and \( e' \), such that \( \delta_e \vee \delta_{e'} = \rho \).

The notion of a \( c \)-minimal algebra was introduced in [1]. A finite \( c \)-minimal algebra, if it exists, is unique in its categorical equivalence class. It follows from the proof of the Theorem that every \( B_h \) is \( c \)-minimal. As a corollary we obtain: If \( h \neq k \) then \( B_h \not\subseteq c B_k \). If \( A \) is \( h \)-ary central and \( A' \) is \( k \)-ary central, then \( A \not\subseteq c A' \). No algebra can be both \( h \)-central and \( k \)-central.

References


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