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Introducing Boolean Semilattices

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Abstract
We present and discuss a variety of Boolean algebras with operators that is closely related to the variety generated by all complex algebras of semilattices. We consider the problem of finding a generating set for the variety, representation questions, and axiomatizability. Several interesting subvarieties are presented. We contrast our results with those obtained for a number of other varieties generated by complex algebras of groupoids.

Keywords
Boolean algebra, BAO, semilattice, Boolean semilattice, Boolean groupoid, canonical extension, equationally definable principal congruence

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The study of Boolean algebras with operators (BAOs) has been a consistent theme in algebraic logic throughout its history. It provides a unifying framework for several branches of logic including relation algebras, cylindric algebras, and modal algebras. From a purely algebraic standpoint, a class of BAOs provides a rich field of study, combining the strength of Boolean algebras with whatever structure is imposed on the operators.

In fact, with all this structure, one might expect that analyzing a variety of BAOs would border on the trivial. The variety of Boolean algebras, after all, is generated by a primal algebra. As such, it is congruence-distributive, congruence-permutable, semisimple, equationally complete, has EDPC,. . . , the list goes on. And yet it turns out that for all but the most degenerate operators, the analysis is anything but simple. The explanation is in the intricate and unexpected interplay between the Boolean operations and the additional operators that arise from standard constructions.

In this paper we consider Boolean algebras with one very simple operator, namely an (almost) semilattice operation, that is, a binary operation that is associative, commutative, and (almost) idempotent. The qualification on idempotence will be explained below. We shall develop some of the arithmetic of these algebras, discuss some representation questions, and pose some problems. While there are no deep results in this work, we hope that it will stimulate further research in this interesting class of algebras.

Peter Jipsen is responsible, for better or worse, for introducing me to representation questions for BAOs. Several of the results presented here are due to him, or jointly to the two of us. Other theorems described here are the result of joint work with Wim Blok in the early 1990s. My interest in algebraic logic in general stems from my long working relationship with Don Pigozzi. Don was my first mentor as a

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professional mathematician. He has played a large role in my subsequent development.

Our universal algebraic terminology and notation follows the book [1]. That reference should be consulted for any notions not defined here. Jipsen's thesis [9], and Jónsson's survey article [12] provide a good introduction to the subject of Boolean algebras with operators. Goldblatt's paper [6] is a detailed study of the complex algebra construction.

1 Complex algebras

We begin with a motivating construction. Let $\mathbf{G} = \langle G, \cdot \rangle$ be an algebra with a single binary operation (a groupoid, in common parlance). We form a new structure, the *complex algebra of* $\mathbf{G}$ by $\mathbf{G}^+ = \langle \mathrm{Sb}(G), \cap, \cup, \sim, \odot, \varnothing, G \rangle$. Here, $\mathrm{Sb}(G)$ is the family of all subsets of G, "$\cap$" and "$\cup$" are the usual operations of intersection and union, ${\sim}X = G - X$ is the complement of the subset X, and $X \odot Y = \{\, x \cdot y : x \in X, y \in Y \,\}$.

The operation "$\odot$" is called a complex operation. In practice, it seems unnecessary to use different notation for an operation and its induced complex operation, so we will generally write $X \cdot Y$ in place of $X \odot Y$. We want to stress that there is nothing special about one binary operation. The complex algebra construction makes sense for any number of operations of any rank. We restrict our attention to groupoids because it already captures the intricacies of the situation. Generalization to arbitrary algebraic structures is straightforward.

The complex algebra is, of course, an expansion of a Boolean algebra. The new operation satisfies several additional identities, namely

$$\begin{aligned} X \cdot \varnothing &= \varnothing, & \varnothing \cdot X &= \varnothing \\ X \cdot (Y \cup Z) &= (X \cdot Y) \cup (X \cdot Z), & (Y \cup Z) \cdot X &= (Y \cdot X) \cup (Z \cdot X)\ . \end{aligned}$$

The first pair of identities assert that the complex operation is *normal,* the latter pair that it is *additive.* We can actually say a bit more, although not in a first-order manner. The Boolean algebra is complete and atomic and the complex operation distributes over arbitrary union, not just finite union.

This is our "ur-example" of a BAO. We formalize it as follows.

Definition 1.1. A *Boolean groupoid* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, ', \cdot, 0, 1 \rangle$ such that $\langle B, \wedge, \vee, ', 0, 1 \rangle$ is a Boolean algebra and "$\cdot$" is an additional binary operation satisfying the identities

$$\begin{gathered} x \cdot 0 \approx 0 \cdot x \approx 0 \\ x \cdot (y \vee z) \approx (x \cdot y) \vee (x \cdot z) \\ (y \vee z) \cdot x \approx (y \cdot x) \vee (z \cdot x). \end{gathered} \tag{1}$$

We shall often use the notation $\mathbf{B}_0$ to denote the Boolean algebra reduct of the Boolean groupoid $\mathbf{B}$ and write $\mathbf{B} = \langle \mathbf{B}_0, \cdot \rangle$.

Since it is defined equationally, the class of Boolean groupoids forms a variety, (that is, a class of algebras closed under subalgebra, homomorphic image, and product) which we denote BG. From our observations above, the complex algebra of every groupoid lies in BG. It is natural to wonder whether the converse could be true: is every Boolean groupoid a complex algebra? A moment's reflection shows that this is impossible on cardinality grounds. There is no complex algebra of cardinality $\aleph_0$, but it is easy to see that there are indeed Boolean groupoids that are countably infinite.

More generally, we can ask whether the complex algebras generate BG as a variety. The answer turns out to be "yes" as we discuss in Sect. 3. In order to demonstrate this, we must develop a technique to extend an arbitrary Boolean groupoid to one that is complete and atomic. We do this in Sect. 2.

Before continuing, we introduce some terminology that we use in the sequel. In the language of Boolean algebras, we write $x-y$ in place of $x \wedge y'$ and

$$x \oplus y = (x-y) \vee (y-x).$$

With this definition we obtain a ring $\langle B, \oplus, \wedge, 0, 1\rangle$ of characteristic 2 from the Boolean algebra $\mathbf{B}_0$.

One important consequence of additivity in a Boolean groupoid is *monotonicity:* if $x_1 \leq x_2$ then $x_1 \cdot y \leq x_2 \cdot y$ and $y \cdot x_1 \leq y \cdot x_2$.

Definition 1.2. Let $p(x_1, x_2, \ldots, x_n)$ and $q(x_1, x_2, \ldots, x_n)$ be terms in the language of groupoids.

1. The identity $p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n)$ is *linear* if each variable occurs exactly once in each of p and q.
2. The identity $p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n)$ is *semilinear* if p has no repeated variables and every variable of q occurs in p. (But q can have repeated variables.)
3. The identity $p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n)$ is *regular* if exactly the same variables appear in p and q. (But each variable may occur any number of times.)

Note that semilinearity is nonsymmetric, that is, $p \approx q$ semilinear does not imply $q \approx p$ semilinear, unless the identity is actually linear. The significance of linear and semilinear identities is delineated in the following proposition whose proof is a simple verification. Regular identities will be addressed in Sect. 2. Figure 1 shows a few familiar identities and their relationship to these properties.

	linear	semilinear	regular
$x(yz) \approx (xy)z$	✓	✓	✓
$xy \approx yx$	✓	✓	✓
$x \approx x^2$		✓	✓
$xy \approx x$		✓	
$(xy)y \approx (yx)y$			✓

Fig. 1 Some linear, semilinear, and regular identities

Proposition 1.3 (Shafaat [18], Grätzer-Whitney [7]). *Let* $\mathbf{G}$ *be a groupoid and* $p \approx q$ *an identity.*

1. *If* $p \approx q$ *is linear then* $\mathbf{G} \models p \approx q \iff \mathbf{G}^{+} \models p \approx q$.
2. *If* $p \approx q$ *is semilinear then* $\mathbf{G} \models p \approx q \iff \mathbf{G}^{+} \models p \leq q$.

By a *partial groupoid* we mean a set with a partially defined binary operation. For example, every subset of a groupoid inherits a partial groupoid structure. We shall say that a partial groupoid, $\mathbf{P}$, satisfies an identity $p(x_1,\dots,x_n) \approx q(x_1,\dots,x_n)$ if, for every $a_1 \dots, a_n \in P$, we have $p(a_1,\dots,a_n)$ is defined in $\mathbf{P}$ if and only if $q(a_1,\dots,a_n)$ is defined in $\mathbf{P}$, and in that case, the two quantities coincide.

2 Duality

In this section we explore the passage from an object to its complex algebra. In particular, we are interested in reversing the process. In order to do this we must, on the groupoid side, expand our attention to ternary relational structures, and on the complex side, extend a Boolean groupoid to one that is complete and atomic.

To begin with, observe that it is quite easy to recover the structure of a groupoid from its complex algebra. For a groupoid $\mathbf{G}$ and $a,b \in G$, we have $a \cdot b = c$ in $\mathbf{G}$ if and only if $\{a\} \odot \{b\} = \{c\}$ in $\mathbf{G}^{+}$. Speaking abstractly, the singletons $\{a\}$, $\{b\}$, and $\{c\}$ are atoms of the Boolean algebra $\mathbf{G}^{+}_{0}$. In the complex algebra, the product of two atoms is always an atom.

Unfortunately, an arbitrary Boolean groupoid may not have any atoms, and even when it does, the product of two atoms need not be an atom. Thus, when we attempt to generalize the passage from complex algebra to groupoid, we obtain, not an algebra, but a ternary relational structure. By a *ternary relational structure* we simply mean a pair $\langle H, \theta \rangle$ in which H is a set and θ is a subset of H^3.

Definition 2.1. Let $\mathbf{B} = \langle \mathbf{B}_0, \cdot \rangle$ be a Boolean groupoid. The *atom structure of* $\mathbf{B}$ is the ternary relational structure $\mathbf{B}_{+} = \langle A, \theta \rangle$ in which A is the set of atoms of $\mathbf{B}_0$ and $\theta = \left\{ (x,y,z) \in A^3 : z \leq x \cdot y \right\}$.

To each groupoid $\mathbf{G} = \langle G, \cdot \rangle$ we can associate the ternary relational structure $\mathbf{G}^{\square} = \langle G, \theta \rangle$ in which $\theta = \{ (x,y,x \cdot y) : x,y \in G \}$. It is easy to verify that $\mathbf{G}^{\square} \cong (\mathbf{G}^{+})_{+}$. In fact, we can extend this association to any partial groupoid $\mathbf{P}$. Notice that in this case, if $x,y \in P$ with $x \cdot y$ undefined, then there will be no triple in $\mathbf{P}^{\square}$ of the form (x,y,z) for any z. Put another way, in the complex algebra $\mathbf{P}^{+}$ we will have $\{x\} \odot \{y\} = \varnothing$.

To proceed further, we must extend the complex algebra construction to ternary relational structures.

Definition 2.2. Let $\mathbf{H} = \langle H, \psi \rangle$ be a ternary relational structure. The *complex algebra of* $\mathbf{H}$ is the Boolean groupoid $\mathbf{H}^{+} = \langle \mathrm{Sb}(H), \cap, \cup, \sim, \odot, \varnothing, H \rangle$ in which

$$X \odot Y = \{ z \in H : (\exists x \in X)(\exists y \in Y) \ \ (x,y,z) \in \psi \} \ .$$

It is not difficult to verify that for a ternary relational structure $\mathbf{H}$, $(\mathbf{H}^+)_+ \cong \mathbf{H}$, and dually, for a complete and atomic Boolean groupoid $\mathbf{B}$, $(\mathbf{B}_+)^+ \cong \mathbf{B}$.

We still have the problem of a lack of atoms in an arbitrary Boolean groupoid. This was addressed in 1951 by Jónsson and Tarski [13] as an extension of the Stone representation theorem. We do this in two steps. Start with a Boolean groupoid, $\mathbf{B}$. Let $\mathbf{B}^*$ denote the set of ultrafilters (i.e., maximal filters) of $\mathbf{B}_0$. We impose a ternary relational structure on $\mathbf{B}^*$ by defining

$$\theta = \left\{ (U,V,W) \in (B^*)^3 : W \supseteq \{ u \cdot v : u \in U,\ v \in V \} \right\} .$$

Finally we define $\mathbf{B}^\sigma$ to be $\langle B^*, \theta \rangle^+$.

In his exposition [12], Jónsson summarizes the relationship between $\mathbf{B}$ and $\mathbf{B}^\sigma$ as follows (specialized to the case of Boolean groupoids).

Theorem 2.3. *Let* $\mathbf{B}$ *be a Boolean groupoid. There is a unique Boolean groupoid* $\mathbf{B}^\sigma$*, called the* canonical extension of $\mathbf{B}$ *such that*

1. $\mathbf{B}_0^\sigma$ *is a complete and atomic extension of* $\mathbf{B}_0$*;*
2. *For all distinct atoms* p *and* q *of* $(\mathbf{B}^\sigma)$*, there exists* $a \in B$ *such that* $p \leq a$ *and* $q \leq a'$*;*
3. *Every subset of* B *that joins to 1 in* $\mathbf{B}^\sigma$ *has a finite subset that also joins to 1;*
4. *For atoms* p,q *of* $\mathbf{B}^\sigma$*,* $p \cdot q = \bigwedge \{ a \cdot b : a,b \in B,\ a \geq p,\ b \geq q \}$*. The product is extended completely additively to the remainder of* $\mathbf{B}^\sigma$*.*

As an example computation, let $\mathbf{B}$ be a Boolean groupoid, A denote the set of atoms of $\mathbf{B}_0^\sigma$, and let $p \in A$. Then using Theorem 2.3(4) we compute

$$p \cdot 1 = \bigvee_{q \in A} p \cdot q = \bigvee_{q \in A} \bigwedge_{a \geq p} \bigwedge_{b \geq q} a \cdot b = \bigwedge_{a \geq p} \bigvee_{q \in A} \bigwedge_{b \geq q} a \cdot b = \bigwedge_{a \geq p} \bigvee_{q \in A} a \cdot q = \bigwedge_{a \geq p} a \cdot 1 . \quad (2)$$

In practice, it is unnecessary to make reference to B^*. We start from an arbitrary Boolean groupoid $\mathbf{B}$, move first to the canonical extension, $\mathbf{B}^\sigma$, and then to the atom structure $\mathbf{B}^\sigma_+$. This ternary relational structure must serve as an approximation to a groupoid induced by $\mathbf{B}$. The utility of this approximation varies depending upon the particular situation at hand.

A class, or property, preserved by canonical extensions, is called *canonical.* A Boolean groupoid term is called *strictly positive* if it does not involve complementation. One of the deep theorems on the subject is the following.

Theorem 2.4 (Jónsson and Tarski, [13]). *Let* s*,* t*, and* u *be strictly positive terms. Then each of the following is canonical.*

$$s \approx t$$
$$s \approx 0 \rightarrow t \approx u$$
$$s \not\approx 0 \rightarrow t \approx u .$$

Consider now two ternary relational structures $\langle G,\theta\rangle$ and $\langle H,\psi\rangle$. A function $h\colon G\to H$ induces a complete Boolean algebra homomorphism $\overleftarrow{h}\colon \mathbf{H}_0^+\to\mathbf{G}_0^+$ given by $\overleftarrow{h}(X)=\{\,g\in G : h(g)\in X\,\}$. A necessary and sufficient condition for $\overleftarrow{h}$ to be a Boolean groupoid homomorphism is that h be a bounded morphism, as in the following definition.

Definition 2.5. A function $h\colon G\to H$ is a *bounded morphism* between the ternary relational structures $\langle G,\theta\rangle$ and $\langle H,\psi\rangle$ if

$$
\begin{aligned}
(i)\quad & (\forall \mathbf{x}\in G^3)\,\mathbf{x}\in\theta \implies h(\mathbf{x})\in\psi \text{ and}\\
(ii)\quad & (\forall z\in G)(\forall y_1,y_2\in H)\ \ (y_1,y_2,h(z))\in\psi \implies\\
& \qquad (\exists x_1,x_2\in G)\,h(x_1)=y_1,\ h(x_2)=y_2,\ (x_1,x_2,z)\in\theta\ .
\end{aligned}
$$

It is straightforward to verify that h is an injective bounded morphism if and only if $\overleftarrow{h}$ is a surjective Boolean groupoid homomorphism, and h is surjective iff $\overleftarrow{h}$ is injective. Let us study those two special situations a little more closely.

Suppose first that $\langle G,\theta\rangle$ and $\langle H,\psi\rangle$ are ternary relational structures, with $G\subseteq H$. If the inclusion map is a bounded morphism, we call $\langle G,\theta\rangle$ an *inner substructure* of $\langle H,\psi\rangle$. Unwinding Definition 2.5, we have the following characterization.

Lemma 2.6. *$\langle G,\theta\rangle$ is an inner substructure of $\langle H,\psi\rangle$ if $G\subseteq H$ and*

$$
\begin{aligned}
(i)\quad & \theta=\psi\cap G^3 \textit{ and}\\
(ii)\quad & (\forall z\in G)(\forall y_1,y_2\in H)\,(y_1,y_2,z)\in\psi \implies y_1,y_2\in G\ .
\end{aligned}
$$

When these conditions hold, $\langle G,\theta\rangle^+$ is a homomorphic image of $\langle H,\psi\rangle^+$.

Based on the first of the two conditions in the lemma, we often refer to G as an inner substructure of $\langle H,\psi\rangle$ without explicitly mentioning θ.

Now suppose that $\mathbf{G}$ is a groupoid. A subset K is called a *sink* if

$$(x\in K \;\&\; y\in G) \implies (x\cdot y\in K \;\&\; y\cdot x\in K)\ .$$

(It is common to call K an ideal of $\mathbf{G}$, but we wish to avoid conflict with the use of "ideal" in the Boolean algebra context.) Consider $\mathbf{G}$ as a ternary relational structure $\mathbf{G}^\square=\langle G,\theta\rangle$. It follows immediately from Lemma 2.6 that a subset H will be an inner substructure of $\mathbf{G}^\square$ if and only if $G-H$ is a sink.

We now turn to bounded morphic images of a partial groupoid. These correspond to certain quotient structures. Let $\mathbf{P}$ be a partial groupoid, and α an equivalence relation on P. For an element $a\in P$ we write a/α for the equivalence class of a modulo α. We call α a *bounded equivalence* if for all $a,b\in P$ the image of the partial map $p\colon a/\alpha\times b/\alpha\to P$ given by $p(x,y)=x\cdot y$ is a union of α-classes. The bounded equivalence α induces a ternary relational structure $\langle P/\alpha,\psi\rangle$ in which $\psi=\{\,(a/\alpha,\,b/\alpha,\,c/\alpha) : c=a\cdot b\,\}$.

Lemma 2.7. *Let α be a bounded equivalence on the partial groupoid $\mathbf{P}$. Then the natural map $q\colon\mathbf{P}^\square\to\mathbf{P}/\alpha$ is a surjective bounded morphism.*

Proof. We must check the two conditions in Definition 2.5. The first condition is simply the definition of the relation ψ on P/α. For the second, let $z \in P$, y_1/α, $y_2/\alpha \in P/\alpha$ and suppose that $(y_1/\alpha, y_2/\alpha, z/\alpha) \in \psi$. Then there are x_1, x_2, w such that $x_1 \,\alpha\, y_1$, $x_2 \,\alpha\, y_2$, $w \,\alpha\, z$, and $x_1 \cdot x_2 = w$. By assumption, the image of the partial map p on $y_1/\alpha \times y_2/\alpha$ is a union of α-classes. Since w lies in that image and $z \,\alpha\, w$, we must have $z = p(u_1,u_2) = u_1 \cdot u_2$. Hence $q(u_1) = y_1/\alpha$, $q(u_2) = y_2/\alpha$, and $(u_1,u_2,z) \in \theta$. □

The converse of lemma 2.7 is true as well: the kernel of a surjective bounded morphism is a bounded equivalence.

The correspondence $\mathbf{G} \mapsto \mathbf{G}^+$ and $\mathbf{B} \mapsto \mathbf{B}_+$, together with the bounded morphisms and homomorphisms, form the basis of a dual equivalence between the categories of ternary relational structures and of complete and atomic Boolean groupoids. This duality is explored in great detail in [6]. We need only one additional aspect of the duality, which is quite easy to verify.

The coproduct of a family $\langle\langle G_i, \theta_i\rangle : i \in I\rangle$ of ternary relational structures is simply the disjoint union $\langle \dot\bigcup_i G_i, \dot\bigcup_i \theta_i\rangle$. The complex algebra of a disjoint union is isomorphic to the direct product of the complex algebras of the components:

$$\left(\dot{\bigcup_{i\in I}} \mathbf{G}_i\right)^+ \cong \prod_{i\in I} \mathbf{G}_i^+ \ . \tag{3}$$

The isomorphism maps the complex, X, of the disjoint union, to the I-tuple, $\langle X \cap G_i : i \in I\rangle$, in the product. We leave the details to the reader.

Let $\mathbf{P}$ be a partial groupoid. We can extend $\mathbf{P}$ to a total groupoid $\overline{\mathbf{P}}$ by adjoining a new element, ∞, to P, and defining $x \cdot y = \infty$ whenever $x, y \in \overline{P}$ and their product is undefined in $\mathbf{P}$. This construction is surprisingly robust. It preserves associativity, commutativity, idempotence; in fact, any *regular identity*, as in Definition 1.2. See [17] for the importance of these identities.

It is immediate from Lemma 2.6 that $\mathbf{P}^\square$ is an inner substructure of $\overline{\mathbf{P}}^\square$. Thus, every partial groupoid is an inner substructure of a groupoid. And conversely, every inner substructure of a groupoid is itself a partial groupoid.

Now suppose that $\langle \mathbf{G}_i : i \in I\rangle$ is a family of groupoids (or even partial groupoids). Then the disjoint union is a partial groupoid, $\mathbf{G}$, which can be extended to a total groupoid, $\overline{\mathbf{G}}$. Taking complex algebras, and applying the duality principles that we have developed, we have a surjective Boolean groupoid homomorphism, h:

$$\overline{\mathbf{G}}^+ \xrightarrow{h} \left(\dot{\bigcup_{i\in I}} \mathbf{G}_i\right)^+ \cong \prod_{i\in I} \mathbf{G}_i^+ \ .$$

We summarize these observations as a theorem.

Theorem 2.8. *Let Σ be a set of regular identities and $\mathscr{K}$ be the variety of groupoids defined by Σ.*

1. *Every partial groupoid that satisfies Σ can be embedded as an inner substructure of a groupoid in $\mathscr{K}$.*
2. $\mathsf{P}(\mathscr{K}^+) \subseteq \mathsf{H}(\mathscr{K}^+)$.

3 Representation of Boolean Groupoids

We now return to our examination of the relationship between the finitely based variety, BG, of Boolean groupoids, and the class of complex algebras of groupoids. Our discussion is lifted almost verbatim from [9, Theorem 3.20].

Lemma 3.1. *Every Boolean groupoid,* $\mathbf{B}$*, can be embedded into* $\mathbf{P}^+$ *for some partial groupoid,* $\mathbf{P}$.

Proof. In light of Theorem 2.3, we can assume that $\mathbf{B}$ is complete and atomic. Let $\langle A, \psi\rangle = \mathbf{B}_+$. Thus A is the set of atoms of the Boolean algebra $\mathbf{B}_0$ and

$$\psi = \left\{ (u,v,w) \in A^3 : w \leq u \cdot v \right\} .$$

Let $P = A \times A$. Fix a surjection $g \colon P \to P$, and let $p_1 \colon P \to A$ denote the first projection, i.e., $p_1(x,y) = x$. We define a partial binary operation on P by

$$(a,b)\cdot(c,d) = g(b,d) \quad \text{if } p_1 g(b,d) \leq a \cdot c .$$

In this definition, both the computations of $a \cdot c$ and $p_1 g(b,d) \leq a \cdot c$ take place in $\mathbf{B}$.

We claim that $p_1 \colon \mathbf{P} \to \mathbf{A}$ is a surjective, bounded morphism. If this is so, then by our observations following Definition 2.5, $\overleftarrow{p}_1$ embeds $\mathbf{B} = \mathbf{A}^+$ into $\mathbf{P}^+$, proving the lemma.

Clearly p_1 is surjective. To verify the two conditions in Definition 2.5, observe that if $(a,b)\cdot(c,d) = (u,v)$ then $u = p_1 g(b,d) \leq a \cdot c$. Consequently, $(a,c,u) \in \psi$. Thus the first requirement holds.

For the second, let $(u,v) \in P$, $a, c \in A$, and $(a,c,u) \in \psi$. By the definition of ψ, $u \leq a \cdot c$. By the surjectiveness of g, there is a pair $(b,d) \in P$ such that $g(b,d) = (u,v)$. Then $p_1(a,b) = a$, $p_1(c,d) = c$, and $(a,b)\cdot(c,d) = (u,v)$ as desired. □

Theorem 3.2. *Every Boolean groupoid,* $\mathbf{B}$*, lies in* $\mathsf{SH}(\mathbf{G}^+)$ *for some groupoid* $\mathbf{G}$*. If* B *is finite, then* G *can be taken to be finite as well.*

Proof. By Lemma 3.1, $\mathbf{B}$ can be embedded into $\mathbf{P}^+$ for a partial groupoid, $\mathbf{P}$. By Theorem 2.8, $\mathbf{P}$ is an inner substructure of a total groupoid $\mathbf{G}$. Therefore $\mathbf{P}^+$ is a homomorphic image of $\mathbf{G}^+$. Thus $\mathbf{B} \in \mathsf{SH}(\mathbf{G}^+)$. If B is finite, then, in the proof of Lemma 3.1, A is finite, so P, hence G, is finite as well. □

As a result, we see that the variety generated by all complex algebras of groupoids is axiomatized by the identities of Boolean algebras, together with those of (1.1). In particular, it is a finitely based variety. If we write G for the variety of groupoids, and G^+ for the class of complex algebras of groupoids, we can state this relationship succinctly as follows.

Corollary 3.3. $\mathsf{BG} = \mathsf{SH}(\mathsf{G}^+) = \mathsf{V}(\mathsf{G}^+)$.

A Boolean groupoid is *commutative* if the binary operator is commutative. We have analogous statements to the results above for the commutative case.

Theorem 3.4. *Every commutative Boolean groupoid lies in* $\mathsf{SH}(\mathbf{G}^+)$ *for some commutative groupoid,* $\mathbf{G}$*. Consequently, the variety of commutative Boolean groupoids is generated by the complex algebras of all commutative groupoids.*

Proof. The construction of the partial groupoid $\mathbf{P}$ in Lemma 3.1 must be modified to make it commutative. Let A be the set of atoms as before. Choose a set W of cardinality $2|A|$. (If A is infinite, we can simply take $W = A$.) Let $P = A \times W$. Fix a surjective function $g\colon W \times W \to P$ such that $g(x,y) = g(y,x)$. (This is possible because $|P| \leq \frac{1}{2}|W \times W|$.) Now we define the partial binary operation on P just as before

$$(a,b)\cdot(c,d) = g(b,d) \quad \text{if } p_1 g(b,d) \leq a\cdot c$$

but note that now, a and c lie in A, while b and d lie in W. Thus $p_1 g(b,d) \in A$. The remainder of the argument now proceeds as before. □

These last two results can be looked at in a couple of different ways. On the one hand, two fairly natural varieties of BAOs (Boolean groupoids and commutative Boolean groupoids) are shown to be generated by an easy-to-characterize class of complex algebras. Following Jónsson, [12], we might call the complex algebras of groupoids the *primary models* of the system defined in Definition 1.1. Viewed this way, Theorems 3.2 and 3.4 are *generation theorems:* the variety of (commutative) Boolean groupoids can be generated by the complex algebras of all (commutative) groupoids.

On the other hand, we can consider the two theorems of this section to be providing a finite axiomatization for two naturally occurring varieties of algebras, namely the varieties generated by complex algebras of groupoids and of commutative groupoids. And not just any axiomatization. The axiom sets consist of only the identities that "must" be included: the axioms for Boolean algebras, additivitiy, normality, and (in the commutative case), the commutative law. Note that the commutative law is linear. According to Proposition 1.3 it is preserved by the passage to complex algebras so it must be present in the axiomatization.

The simplicity of this axiomatization tells us that in the passage to the complex algebra of a groupoid, there are no "unexpected" interactions between the complex operation and the Boolean operations. Peter Jipsen first presented this topic to the author in a seminar in 1991, in the context of semigroups, rather than groupoids. Note that the associative law is also linear. Thus, "of course," this author thought, "there will be no unexpected interactions between the semigroup operation and the Boolean ones, besides associativity."[1] How wrong that was!

Theorem 3.5 (Jipsen, [11]). *The variety generated by all complex algebras of semigroups is not finitely based.*

Jipsen's theorem is in striking contrast to Corollary 3.3. Not only do "unexpected" interactions exist, but there are infinitely many. In fact, at the time this paper

[1] This author also recalls Don Pigozzi rolling his eyes and proclaiming "You have no idea what you are getting yourself into."

is being written it is unknown whether the variety generated by all complex algebras of semigroups even has a decidable equational base.

Somewhat stronger than a generation theorem is a *representation* theorem. Let $\mathscr{V}$ be a variety of Boolean groupoids, and $\mathscr{K}$ a finitely axiomatizable class of ternary relational structures. We say $\mathscr{V}$ is *representable by* $\mathscr{K}$ if $\mathscr{V} = \mathsf{SP}(\mathscr{K}^+)$. At this time it is not known whether Corollary 3.3 or Theorem 3.4 can be strengthened to representations.

Thus we are presented with a wealth of possible problems that we can pose in the following general framework. Let $\mathscr{V}$ denote a finitely based variety of Boolean groupoids, and $\mathscr{K}$ a finitely axiomatizable class of ternary relational structures (preferably groupoids).

A generation problem. Given $\mathscr{V}$, find a class $\mathscr{K}$ so that $\mathscr{V} = \mathsf{V}(\mathscr{K}^+)$.
A representation problem. Given $\mathscr{V}$, does there exist a class $\mathscr{K}$ so that $\mathscr{V} = \mathsf{SP}(\mathscr{K}^+)$?
A finite basis/decidability problem. Given $\mathscr{K}$, is $\mathsf{V}(\mathscr{K}^+)$ finitely based/decidable?

Problem 3.6. Is BG represented by the class of all groupoids?

We have positive answers to these questions in a couple of other interesting cases.

Theorem 3.7. 1. (Bergman) *Let* Lz *denote the variety of left-zero semigroups. Then* $\mathsf{V}(\mathsf{Lz}^+)$ *is finitely based. This variety is representable by left-zero semigroups.*

2. (Jipsen) *Let* Rb *denote the variety of rectangular bands. Then* $\mathsf{V}(\mathsf{Rb}^+)$ *is finitely based. This variety is representable by rectangular bands.*

We were surprised to discover that the two varieties of complex algebras in the above theorem are term-equivalent to the varieties of diagonal-free cylindric algebras of dimensions 1 and 2, respectively. Note also that both Lz and Rb satisfy the associative law. So it is not the associative law *per se* that is responsible for destroying the finite axiomatizability of the complex algebras in Theorem 3.5. The situation is apparently more subtle. Recently, Peter Jipsen announced the following theorem.

Theorem 3.8. *Let* IG *(respectively* CIG*) denote the variety of idempotent (respectively commutative and idempotent) groupoids. Then* $\mathsf{V}(\mathsf{IG}^+)$ *coincides with the variety of Boolean groupoids satisfying the additional identity* $x \leq x^2$*. Similarly,* $\mathsf{V}(\mathsf{CIG}^+)$ *is equal to the variety of commutative Boolean groupoids satisfying* $x \leq x^2$.

Motivated by all of this, a natural next class to investigate is that of complex algebras of semilattices. This turns out to be a rich field of study in and of itself, and constitutes the remainder of this paper. The doctoral dissertation [16] contains a similar analysis of the variety generated by complex algebras of semigroups. We close this discussion with several open problems.

4 Boolean semilattices

We now turn to our primary object of study, namely complex algebras of semilattices. Let Sl denote the variety of semilattices, that is, groupoids satisfying

$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$$
$$x \cdot y \approx y \cdot x$$
$$x \cdot x \approx x\ .$$

These are the identities of *associativity, commutativity, and idempotence* respectively. As before, we can form the complex algebra of any semilattice, and consider the variety generated by all such complex algebras: $\mathbf{HSP}(\mathsf{Sl}^+)$. Once again, we are faced with fascinating questions about this variety: Can we find an axiomatization? Is it finitely axiomatizable? Is the equational theory even decidable?

Unfortunately, we don't know the answers to any of these questions. The evidence suggests that they are all negative. As an approximation to the theory, we assemble a short list of identities, all of which are easily seen to hold in Sl^+, and derive some interesting algebraic properties.

To begin with, we have the axioms for Boolean groupoids listed in (1). Guided by Proposition 1.3 we add both the associative and commutative laws. They are linear, so are inherited by the complex algebras. Idempotence is semilinear. Thus we add the identity $x \leq x \cdot x$, which is called the *square-increasing law.*

Definition 4.1. A *Boolean semilattice* is a Boolean groupoid (Definition 1.1) satisfying the additional axioms

$$\begin{array}{ll} \mathrm{bsl}_1 & x \cdot (y \cdot z) \approx (x \cdot y) \cdot z \\ \mathrm{bsl}_2 & x \cdot y \approx y \cdot x \\ \mathrm{bsl}_3 & x \leq x \cdot x \end{array}$$

The variety of Boolean semilattices will be denoted BSl.

We introduce the term "Boolean semilattice" with no small amount of trepidation. Is this the right definition for such a natural piece of terminology? Only time will tell. Our axiomatization has the merit of being short, natural (in light of Proposition 1.3), and equational. As we shall demonstrate in the next few pages, a number of interesting consequences of these axioms can be derived that demonstrate the strength and interest of this system. However, it is certainly possible that further research will suggest additional identities that should be added to the above set.

Since every semilattice is idempotent, it is reasonable to expect that the term "Boolean semilattice" should imply idempotence as well, that is, that bsl_3 should be replaced by the stronger identity $x \approx x \cdot x$. However it is not hard to see that the complex algebra of a semilattice, $\mathbf{S}$, satisfies this stronger identity if and only if $\mathbf{S}$ is linearly ordered. In fact, as we show in Sect. 5, the variety defined by that stronger identity is generated by the complex algebras of all linear semilattices. For this reason, we chose to define Boolean semilattice using the square-increasing law.

As we already noted, the complex algebra of every semilattice is a Boolean semilattice. Thus we have $\mathbf{V}(\mathsf{Sl}^+) \subseteq \mathsf{BSl}$. Conversely, if $\mathbf{G}$ is a Boolean groupoid and $\mathbf{G}^+ \in \mathsf{BSl}$, then $\mathbf{G}$ must be a semilattice. To see this, note that in $\mathbf{G}^+$, the product of two atoms is an atom. Thus, by bsl_1 and bsl_2, $\mathbf{G}$ is associative and commutative. Further, if $a, b \in G$ and $a \cdot a = b$, then, in $\mathbf{G}^+$ we have $\{a\} \subseteq \{b\}$ by bsl_3, so $a = b$.

It is easy to see that each of the 3 identities are independent from the others by considering the complex algebra of a groupoid that is either associative or not, commutative or not, etc.

We list next several additional identities and other formulae that are consequences of the definition of Boolean semilattice. These are useful in practice.

Proposition 4.2. *Every Boolean semilattice satisfies the following formulae.*

$$1 \cdot 1 \approx 1 \tag{4}$$
$$x \wedge y \leq x \cdot y \tag{5}$$
$$x \cdot y \cdot 1 \approx (x \cdot 1) \wedge (y \cdot 1) \tag{6}$$
$$x \cdot ((x \cdot 1) - x) \leq x^2 \vee ((x \cdot 1 - x)^2 \; . \tag{7}$$

In fact, bsl_3 *can be replaced by* (5)*.*

Proof. By the square-increasing law, $1 \leq 1 \cdot 1 \leq 1$, proving (4). In any Boolean semilattice, $x \wedge y \leq (x \wedge y) \cdot (x \wedge y) \leq x \cdot y$ by monotonicity. Thus (5) holds. Conversely, bsl_3 can be derived from (5) by taking $x = y$.

For (6), first observe that $x \cdot y \cdot 1 \leq x \cdot 1 \cdot 1 = x \cdot 1$ and similarly $x \cdot y \cdot 1 \leq y \cdot 1$. Thus $x \cdot y \cdot 1 \leq (x \cdot 1) \wedge (y \cdot 1)$. On the other hand by (5), $(x \cdot 1) \wedge (y \cdot 1) \leq x \cdot 1 \cdot y \cdot 1 = x \cdot y \cdot 1$ by bsl_1–bsl_3.

Let us derive (7). First, by monotonicity, $x \cdot ((x \cdot 1) - x) \leq x \cdot 1$. Second, $x \cdot 1 - x = (x \cdot 1) \wedge x'$ by definition. Note that

$$x \vee ((x \cdot 1) \wedge x') = (x \vee (x \cdot 1)) \wedge (x \vee x') = x \cdot 1 \; .$$

Hence $x \cdot ((x \cdot 1) - x) \leq x \vee ((x \cdot 1) - x) \leq x^2 \vee ((x \cdot 1) - x)^2$. □

As we have already noted, $\mathbf{V}(\mathsf{Sl}^+) \subseteq \mathsf{BSl}$. It was, of course, our hope that these two varieties would coincide. Alas, that is not the case. We present two examples. Consider first the identity

$$x \wedge (y \cdot 1) \leq x \cdot y \; . \tag{8}$$

This identity is easily seen to hold in $\mathbf{S}^+$, for any semilattice, $\mathbf{S}$. However, let $\mathbf{H}$ denote the ternary relational structure $\langle \{a, b\}, \theta \rangle$ in which

$$\theta = \{(a,a,a), (a,b,b), (b,a,b), (b,b,a), (b,b,b)\} \; .$$

One can conveniently represent this relation with the multiplication table

$\cdot$	a	b
a	a	b
b	b	1

This table can be thought of as a subset of the multiplication table for $\mathbf{H}^+$. Since this particular complex algebra has two atoms, $a \vee b = 1$. The remainder of the table can be deduced from normality and additivity. $\mathbf{H}^+$ is easily checked to be associative, commutative, and, square-increasing. We see that $\mathbf{H}^+$ fails to satisfy (8) with $x = a$ and $y = b$.

As a second example, let $\mathbf{A}$ be the 8-element Boolean groupoid, with atoms $\{a, b, c\}$ that multiply as follows:

	a	b	c
a	a	a	a
b	a	$a \vee b$	$b \vee c$
c	a	$b \vee c$	1

The algebra $\mathbf{A}$ satisfies bsl_1–bsl_3, so $\mathbf{A} \in \mathsf{BSl}$. In fact it also satisfies (8). However, $\mathbf{A}$ fails to satisfy the identity

$$x \cdot \tau \leq (x \cdot z \wedge v) \cdot y \vee (x \cdot z - v) \cdot \tau \tag{9}$$

with τ shorthand for $u \wedge (y \cdot z)$. It is a simple computation to verify that the complex algebra of any semilattice satisfies equation (9). Thus $\mathbf{A} \notin \mathbf{V}(\mathsf{Sl}^+)$.

These examples were relatively easy to find, involving algebras with 2 or 3 atoms. It certainly suggests to us that it will be possible to find longer and longer identities that fail in larger and larger finite algebras. Based on this, we conjecture that the answer to the following finite basis problem is ‘no’.

Problem 4.3. Is $\mathbf{V}(\mathsf{Sl}^+)$ finitely based? Is the equational theory decidable?

A useful source of tools for attacking Problem 4.3 might be [8]. Perhaps there is more hope for a positive answer to one of the following problems. (But see Theorem 7.2.)

Problem 4.4. Is either BSl or $\mathbf{V}(\mathsf{Sl}^+)$ generated by its finite members?

Problem 4.5. Is there a finitely axiomatizable class, $\mathscr{K}$, of ternary relational structures, such that $\mathsf{BSl} = \mathbf{V}(\mathscr{K}^+)$?

Algebraic theory of Boolean semilattices

Let $\downarrow x$ denote the term $x \cdot 1$. Notice that for a semilattice $\mathbf{S}$ and $X \subseteq S$, the complex $\downarrow X$ is the downset (i.e., the ideal) generated by X. (We view the semilattice operation to be the greatest lower bound.) This operator plays a key role in the structure theory of Boolean semilattices.

Proposition 4.6. *In any Boolean semilattice, '$\downarrow$' yields a closure operator, that is, for* $\mathbf{B} \in \mathsf{BSl}$ *and* $x, y \in B$, $x \leq {\downarrow}x = {\downarrow}{\downarrow}x$, *and* $x \leq y \implies {\downarrow}x \leq {\downarrow}y$.

Proof. $x \leq x \cdot x \leq x \cdot 1 = {\downarrow}x$ by bsl_3 and monotonicity. Also

$$ {\downarrow}{\downarrow}x = (x \cdot 1) \cdot 1 = x \cdot (1 \cdot 1) = x \cdot 1 = {\downarrow}x $$

by associativity and (4). Finally, if $x \leq y$ then ${\downarrow}x = x \cdot 1 \leq y \cdot 1 = {\downarrow}y$, again, by monotonicity. ◻

An element x of a Boolean semilattice is called *closed* if $x = {\downarrow}x$. By normality, we always have ${\downarrow}0 = 0$ and by identity (4), ${\downarrow}1 = 1$. Thus 0 and 1 are always closed elements.

It is well-known that if θ is a congruence relation on a Boolean algebra $\mathbf{B}_0$, then $I = 0/\theta$ is an ideal of $\mathbf{B}_0$. Conversely, every ideal, I, gives rise to a congruence by defining $\theta_I = \left\{ (x, y) \in B_0^2 : x \oplus y \in I \right\}$. This correspondence provides a lattice isomorphism between the congruences and ideals of $\mathbf{B}_0$. It can be extended to Boolean groupoids, indeed, to BAOs in general, as follows.

Definition 4.7. Let $\mathbf{B}$ be a Boolean groupoid, and I an ideal of $\mathbf{B}_0$. Then I is a *congruence ideal of* $\mathbf{B}$ if, for some $\theta \in \mathrm{Con}(\mathbf{B})$, we have $I = 0/\theta$.

Proposition 4.8 (Jipsen, [9]). *Let* $\mathbf{B}$ *be a Boolean groupoid, and I and ideal of* $\mathbf{B}_0$. *Then I is a congruence ideal of* $\mathbf{B}$ *if and only if* $x \in I$ *implies* $x \cdot 1 \in I$ *and* $1 \cdot x \in I$. *There is a lattice isomorphism between the congruences and the congruence ideals of* $\mathbf{B}$.

Corollary 4.9. *Let* $\mathbf{B}$ *be a Boolean semilattice.*

1. *Let I be an ideal of* $\mathbf{B}_0$. *Then I is a congruence ideal of* $\mathbf{B}$ *if and only if* $x \in I \implies {\downarrow}x \in I$.
2. *Let* $a \in B$. *Then the smallest congruence ideal of* $\mathbf{B}$ *containing a is*

$$ ({\downarrow}a] = \{ x \in B : x \leq {\downarrow}a \} . $$

An element a such that $(a]$ is a congruence ideal is called a *congruence element.* It follows from the above corollary, that on a Boolean semilattice, the congruence elements are precisely the closed elements. If $\mathbf{S}$ is a semilattice, then the congruence elements of $\mathbf{S}^+$ are the downsets of $\mathbf{S}$.

It is easy to see that if x and y are congruence elements in any Boolean semilattice, then so are $x \vee y$ and $x \cdot y$. In fact, in the lattice of congruence ideals, $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \cdot y]$, for congruence elements x and y. Thus the principal congruence ideals form a sublattice of the lattice of all congruence ideals.

Recall from the discussion following Lemma 2.6 the definition of a sink in a groupoid. We noted there that the inner substructures of a groupoid coincide with the complements of the sinks. In the case of a semilattice, the sinks are precisely the downsets, and the complements of the downsets are the upsets. We state this formally.

Lemma 4.10. *Let* $\mathbf{S}$ *be a semilattice. The inner substructures of* $\mathbf{S}^{\Box}$ *are the upsets of* $\mathbf{S}$.

We reiterate that every partial semilattice, $\mathbf{P}$, is an upset, hence an inner substructure, of a semilattice, simply by adjoining a smallest element to P. Lemma 4.10 can be generalized somewhat.

Proposition 4.11. *Let* $\mathbf{B}$ *be a complete and atomic Boolean semilattice, and let* c *be a closed element of* $\mathbf{B}$. *Then* $U = \{ z \in B_+ : z \leq c' \}$ *is an inner substructure of* $\mathbf{B}_+$.

Proof. Write $\mathbf{B}_+ = \langle A, \psi \rangle$. We need to check the condition in Lemma 2.6. Let $z \in U$ and $y_1, y_2 \in A$. The condition $(y_1, y_2, z) \in \psi$ is equivalent to $z \leq y_1 \cdot y_2$. Suppose that $y_1 \notin U$. Then, since y_1 is an atom, $y_1 \leq c$, hence $z \leq y_1 \cdot y_2 \leq c \cdot 1 = c$, since c is closed. But this implies $z \leq c \wedge c' = 0$, which is false. Similarly, $y_2 \in U$. $\Box$

In a landmark series of papers, [14, 2, 3, 4, 5], Don Pigozzi, together with Wilem Blok and Peter Köhler, developed the notion of equationally definable principal congruences (EDPC). Varieties with EDPC exhibit remarkable properties. The variety of Boolean semilattices has EDPC, and provides a very interesting case study in its application.

Definition 4.12. A variety, $\mathscr{V}$, has EDPC if there are 4-variable terms $p_i(x,y,z,w)$, and $q_i(x,y,z,w)$, for $i = 1, \ldots, n$, such that for every $\mathbf{A} \in \mathscr{V}$ and every $a,b,c,d \in A$

$$(c,d) \in \mathrm{Cg}^{\mathbf{A}}(a,b) \iff \mathbf{A} \vDash p_i(a,b,c,d) = q_i(a,b,c,d), \text{ for } i = 1, \ldots, n\,.$$

Theorem 4.13. *The variety* BSl *has EDPC.*

Proof. Let $\mathbf{B}$ be a Boolean semilattice, $a,b,c,d \in B$. Then from the theory of Boolean algebras we know that $(c,d) \in \mathrm{Cg}^{\mathbf{B}}(a,b)$ iff $(c \oplus d, 0) \in \mathrm{Cg}^{\mathbf{B}}(a \oplus b, 0)$. From our observations above, in a Boolean semilattice, this latter condition is equivalent to $c \oplus d \leq \downarrow(a \oplus b)$. Thus, in the definition of EDPC, we can take $n = 1$, $p_1(x,y,z,w) = (z \oplus w) \wedge ((x \oplus y) \cdot 1)$ and $q_1(x,y,z,w) = z \oplus w$. $\Box$

Every variety with EDPC is congruence distributive and has the congruence extension property. Of course the first of these holds in any variety of BAOs. But the second is significant.

Corollary 4.14. *The variety* BSl *has the* congruence extension property (CEP). *That is, for every* $\mathbf{C} \leq \mathbf{B} \in \mathsf{BSl}$ *and* $\theta \in \mathrm{Con}(\mathbf{C})$, *there is* $\bar{\theta} \in \mathrm{Con}\,\mathbf{B}$ *such that* $\bar{\theta} \cap C^2 = \theta$.

It is actually quite easy to see from Corollary 4.9 that BSl has the congruence extension property. Suppose that $\mathbf{C} \leq \mathbf{B}$. For a congruence ideal, I, on $\mathbf{C}$, let $J = \{ x \in B : (\exists y \in I)\, x \leq y \}$. It is easy to see that J is an ideal of $\mathbf{B}_0$ and that $J \cap C = I$. To apply Corollary 4.9, let $x \in J$. By definition, there is $y \in I$ with $x \leq y$. Then $\downarrow x \leq \downarrow y \in I$ since I is assumed to be a congruence ideal.

An important application of the congruence extension property is the following relationship which is useful in understanding the generation of varieties. The proof is a straightforward verification.

Corollary 4.15. *Let $\mathscr{K}$ be a class of algebras with the congruence extension property. Then $\mathsf{HS}(\mathscr{K}) = \mathsf{SH}(\mathscr{K})$.*

Let us turn now to a consideration of subdirect irreducibility. Recall that an algebra is *subdirectly irreducible* if it is nontrivial and has a smallest nontrivial congruence, called the *monolith.* Subdirectly irreducible algebras form the basic building blocks for analyzing varieties. The notion tends to disappear from view in the study of Boolean algebras because the only subdirect irreducible is the 2-element algebra. However the situation for Boolean semilattices is radically different.

Lemma 4.16. *A Boolean semilattice is subdirectly irreducible if and only if it has a smallest nonzero closed element.*

Proof. Let $\mathbf{B}$ be a subdirectly irreducible Boolean semilattice and let I be the congruence ideal associated with the monolith. Choose any $a \in I$, $a \neq 0$ and let $c = \downarrow a$. Note that c is a nonzero closed element. Since I is a congruence ideal, $c \in I$, so $(c] \subseteq I$. But by the minimality of I, $(c] = I$. Now, if b is any nonzero closed element, then $(b]$ is a congruence ideal, so $(c] \subseteq (b]$, which is to say, $c \leq b$. □

Proposition 4.17. *Let $\mathbf{S}$ be a semilattice. Then $\mathbf{S}^+$ is subdirectly irreducible if and only if $\mathbf{S}$ has a lower bound. In particular, every finite semilattice has a subdirectly irreducible complex algebra.*

Proof. Recall that the closed elements of $\mathbf{S}^+$ are the downsets of $\mathbf{S}$. The smallest nonempty downset of a semilattice (if it exists) will always be of the form $\{a\}$, where a is the lower bound. □

It is usually easier to work with congruence ideals rather than congruences. We will frequently consider the monolith to the the smallest nonzero congruence ideal on a subdirectly irreducible Boolean semilattice.

Theorem 4.18. *Let $\mathbf{B}$ be a subdirectly irreducible Boolean semilattice. Then $\mathbf{B}^\sigma$ is subdirectly irreducible.*

Proof. By Lemma 4.16, $\mathbf{B}$ has a smallest nonzero closed element, a. Thus, for every $x \in B - \{0\}$, $x \cdot 1 \geq a$. Let y be an atom of $\mathbf{B}^\sigma$. Since $\mathbf{B}$ is a subalgebra of $\mathbf{B}^\sigma$, the condition $a = a \cdot 1$ continues to hold in $\mathbf{B}^\sigma$. By Equation (2)

$$y \cdot 1 = \bigwedge \{ x \cdot 1 : y \leq x \in B \} \geq a \ .$$

Therefore a generates the monolith of $\mathbf{B}^\sigma$. □

Two concepts related to subdirect irreducibility are simplicity and finite subdirect irreducibility. A nontrivial algebra $\mathbf{A}$ is *simple* if $\mathrm{Con}(\mathbf{A})$ has exactly 2 elements. $\mathbf{A}$ is *finitely subdirectly irreducible* if, for any two congruences θ and ψ on $\mathbf{A}$, $\theta > 0 \ \& \ \psi > 0 \implies \theta \wedge \psi > 0$. Finally, we call a Boolean groupoid *integral* if $x > 0 \ \& \ y > 0 \implies x \cdot y > 0$.

Proposition 4.19. *Let $\mathbf{B}$ be a Boolean semilattice.*

1. $\mathbf{B}$ *is finitely subdirectly irreducible if and only if it is integral.*
2. $\mathbf{B}$ *is simple if and only if* $x \neq 0 \implies \downarrow x = 1$.
3. $\mathbf{B}$ *simple implies* $\mathbf{B}^\sigma$ *simple.*

Proof. Suppose that $\mathbf{B}$ is finitely subdirectly irreducible and that $x \cdot y = 0$. Then $0 = x \cdot y \cdot 1 = (x \cdot 1) \cdot (y \cdot 1) = \downarrow x \cdot \downarrow y$. Consequently $(\downarrow x] \wedge (\downarrow y] = (0]$. Then by our assumption, either $\downarrow x = 0$, which implies $x = 0$, or $\downarrow y = 0$, so $y = 0$. Thus $\mathbf{B}$ is integral.

Conversely, suppose that $\mathbf{B}$ is integral and that I and J are nonzero congruence ideals of $\mathbf{B}$. Then there are nonzero closed elements $x \in I$ and $y \in J$. We have $x \cdot y \leq x \cdot 1 = x \in I$ and similarly, $x \cdot y \leq y \cdot 1 \in J$. By integrality, $0 \neq x \cdot y \in I \cap J$. This is enough to show that $\mathbf{B}$ is finitely subdirectly irreducible.

Part (2) follows easily from Corollary 4.9. Part (3) follows from Theorem 2.4 since the terms in part (2) are all strictly positive. □

Corollary 4.20. *Let* $\mathbf{S}$ *be a semilattice. Then* $\mathbf{S}^+$ *is finitely subdirectly irreducible.* $\mathbf{S}^+$ *is simple iff* $|S| = 1$.

Proof. The complex algebra of a groupoid is always integral. If $\mathbf{S}^+$ is simple, then the only downset of $\mathbf{S}$ is S itself, so $\mathbf{S}$ must be trivial. □

In fact, this corollary, as well as Proposition 4.17 holds whenever $\mathbf{S}$ is a partial semilattice.

Discriminator algebras

The *discriminator* on a set A is the ternary operation

$$d_A(x,y,z) = \begin{cases} z & \text{if } x = y \\ x & \text{if } x \neq y. \end{cases}$$

A nontrivial algebra, $\mathbf{A}$, is a *discriminator algebra* if d_A is a term operation of $\mathbf{A}$. Discriminator algebras have powerful structure. They are simple, every nontrivial subalgebra is again a discriminator algebra, and they generate an arithmetical variety. As an example, every finite field is a discriminator algebra.

A variety is called a *discriminator variety* if there is a single term that induces the discriminator on every subdirectly irreducible member. The varieties of Boolean algebras, relation algebras, and cylindric algebras (of a fixed dimension) are examples of discriminator varieties.

The discriminator is a kind of “if-then-else” operation on a set. Because of its connection to propositional logic, it is perhaps not surprising that on a Boolean algebra with operators, there is a convenient shortcut to building a discriminator term. We define the *unary discriminator* on a Boolean algebra $\mathbf{B}_0$ to be the function

$$c(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0. \end{cases}$$

The ternary and unary discriminators are interdefinable by

$$c(x) = d(0,x,1)' \text{ and } d(x,y,z) = \big(x \wedge c(x \oplus y)\big) \vee \big(z \wedge c(x \oplus y)'\big) \ .$$

Thus, a Boolean algebra with operators has a term defining the (ternary) discriminator if and only if it has a term defining the unary discriminator.

Proposition 4.19 tells us that every simple Boolean semilattice is a discriminator algebra, with unary discriminator $c(x) = \downarrow x$. The variety BSl, of all Boolean semilattices, is not a discriminator variety since there are many subdirectly irreducible algebras that are not simple. However, BSl has a largest discriminator subvariety, which is easily described (see also [10, 15]).

Theorem 4.21. *Let* $\mathsf{BSl_D}$ *be the subvariety of* BSl *defined by the identity*

$$(x \cdot 1)' \cdot 1 \approx (x \cdot 1)' \ . \tag{10}$$

$\mathsf{BSl_D}$ *is a discriminator variety, in fact, it is the largest discriminator subvariety of* BSl. $\mathsf{BSl_D}$ *is generated by the class of all simple Boolean semilattices.*

Proof. Let $\mathbf{B}$ be a subdirectly irreducible member of $\mathsf{BSl_D}$, with minimal nonzero congruence ideal, M. Let a be a nonzero element of M. From equation (10), the element $b = (a \cdot 1)'$ is closed, consequently $I = (b]$ is a congruence ideal. If $b \neq 0$, then by the minimality of M, we must have $M \subseteq I$, so $a \leq b$. But then $a \cdot 1 \leq b \cdot 1 = b = (a \cdot 1)'$ which is impossible as $a > 0$. Consequently, we must have $b = 0$, which is to say, $\downarrow a = 1$. Thus by Proposition 4.21, $\mathbf{B}$ is simple. As we have already argued that every simple algebra is a discriminator, we conclude that $\mathsf{BSl_D}$ is a discriminator variety.

On the other hand, let D be any discriminator subvariety of BSl. Then each of its subdirectly irreducible algebras is simple. It is easy to see that every simple Boolean semilattice satisfies equation (10). Consequently, $\mathsf{D} \subseteq \mathsf{BSl_D}$. ☐

Equation (10) says that the complement of a closed element is closed. From this we obtain another property that is characteristic of discriminator varieties—numerous direct decompositions. If $\mathbf{B} \in \mathsf{BSl_D}$ and a is any closed element of $\mathbf{B}$, then we have the decomposition $\mathbf{B} \cong \mathbf{B}/(a] \times \mathbf{B}/(a']$.

Equation (10) also implies that if $\mathbf{S}$ is a nontrivial semilattice, then $\mathbf{S}^+ \notin \mathsf{BSl_D}$. For, the closed elements of $\mathbf{S}^+$ are the downsets of $\mathbf{S}$. And the complement of a downset is never a downset.

Thus we have to look harder for primary models for $\mathsf{BSl_D}$. Here is one interesting class of such structures. Let Tot denote the class of all ternary relational structures $\langle H, H^3 \rangle$ (i.e., total relations) for any set H.

Theorem 4.22. *Every member of* Tot^+ *is a simple Boolean semilattice.* $\mathbf{V}(\mathsf{Tot}^+)$ *is the subvariety of* $\mathsf{BSl_D}$ *defined by the identity* $x \cdot y \cdot 1 \approx x \cdot y$. *This subvariety is represented by* Tot.

Proof. It is straightforward to verify that any member of Tot^+ satisfies bsl_1–bsl_3. Let $\mathbf{H} \in \mathsf{Tot}$. Then for any $a,b,c \in H$, we have $(a,b,c) \in H^3$, hence $c \leq a \cdot b$ in $\mathbf{H}^+$. Since c represents an arbitrary atom of the complete and atomic $\mathbf{H}^+$, we conclude that $a \cdot b = 1$. Since a and b are themselves arbitrary atoms we deduce that for any $x > 0$ and $y > 0$ in $\mathbf{H}^+$, $x \cdot y = 1$. In particular, $\downarrow x = 1$, so by Proposition 4.19, $\mathbf{H}^+$ is simple. Furthermore, we conclude that $\mathbf{H}^+ \vDash x \cdot y \cdot 1 \approx x \cdot y$ since if either x or y is 0 then both sides of the identity are 0.

Let $\mathscr{W}$ be the subvariety of $\mathsf{BSl_D}$ defined by the identity $x \cdot y \cdot 1 \approx x \cdot y$. By the previous paragraph and Theorem 4.21, $\mathbf{V}(\mathsf{Tot}^+) \subseteq \mathscr{W}$. We shall show that, conversely, $\mathscr{W} \subseteq \mathbf{SP}(\mathsf{Tot}^+)$. suppose that $\mathbf{A}$ is a subdirectly irreducible member of $\mathscr{W}$. It is enough to show that $\mathbf{A} \in \mathbf{S}(\mathsf{Tot}^+)$. Since $\mathsf{BSl_D}$ is a discriminator variety containing $\mathbf{A}$, we must have $\mathbf{A}$ simple. Therefore, by Theorem 4.18, $\mathbf{A}^\sigma$ is simple, hence $\mathbf{A}^\sigma \in \mathsf{BSl_D}$. Since $x \cdot y \cdot 1 \approx x \cdot y$ is a strictly positive identity satisfied by $\mathbf{A}$, by Theorem 2.4 we get $\mathbf{A}^\sigma \vDash x \cdot y \cdot 1 \approx x \cdot y$. Hence $\mathbf{A}^\sigma \in \mathscr{W}$.

Now, for any three atoms a,b,c of $\mathbf{A}^\sigma$, we have $a \cdot b > 0$ (since simple algebras are integral), so $a \cdot b = \downarrow(a \cdot b) = 1$. Thus $c \leq a \cdot b$. This means that the ternary relational structure $(\mathbf{A}^\sigma)_+$ is a total relation. Therefore $\mathbf{A}^\sigma \in \mathsf{Tot}^+$. Since $\mathbf{A}$ is a subalgebra of $\mathbf{A}^\sigma$, we get $\mathbf{A} \in \mathbf{S}(\mathsf{Tot}^+)$. □

The class $\mathbf{V}(\mathsf{Tot}^+)$ is a proper subvariety of $\mathsf{BSl_D}$. The algebras $\mathbf{B}_2$ and $\mathbf{B}_3$ in Figure 2 of Sect. 7 are both simple (so they lie in $\mathsf{BSl_D}$) but fail to satisfy the identity $x \cdot y \cdot 1 \approx x \cdot y$ with $x = y = a$. Thus the question of a nice class of generators for $\mathsf{BSl_D}$ remains open.

Problem 4.23. Is $\mathsf{BSl_D} = \mathbf{V}(\mathscr{K}^+)$ for some finitely axiomatizable class, $\mathscr{K}$, of ternary relational structures?

Finally, notice that equation (10) is not strictly positive. Thus we can not apply Theorem 2.4 to conclude that $\mathsf{BSl_D}$ is closed under canonical extension. However, let $\mathbf{B}$ be a subdirectly irreducible member of $\mathsf{BSl_D}$. Then $\mathbf{B}$ is simple, hence $\mathbf{B}^\sigma$ is simple, so $\mathbf{B}^\sigma \in \mathsf{BSl_D}$. This suggests the following question.

Problem 4.24. Is $\mathsf{BSl_D}$ canonical?

5 Linear Semilattices

It would seem, based on a rational naming convention, that a "Boolean semilattice" should always satisfy the identity $x \cdot x \approx x$. However, as we explain in this section, this identity is too strong to be of much use.

In fact, for a semilattice $\mathbf{S}$, $\mathbf{S}^+ \vDash x^2 \approx x$ precisely when $\mathbf{S}$ is linearly ordered. To see this, observe that for $X \subseteq S$, the condition $X \cdot X = X$ is equivalent to X being a subsemilattice of $\mathbf{S}$. Thus $\mathbf{S}^+ \vDash x^2 \approx x$ says that every subset is a subsemilattice, and this in turn holds exactly when $\mathbf{S}$ is linearly ordered.

We shall call a Boolean semilattice *idempotent* if it satisfies the identity $x^2 \approx x$. Let LS denote the class of linearly ordered semilattices. We have just argued that every member of LS^+ is idempotent. Thus every member of $\mathbf{V}(\mathsf{LS}^+)$ is idempotent. In this section, we shall establish the converse. Let us write IBSl for the variety of idempotent Boolean semilattices.

Lemma 5.1 (Bergman-Jipsen). *The following identities hold in* IBSl.

1. $x \wedge y \leq x \cdot y \leq x \vee y$;
2. $x \wedge (y \cdot 1) \leq x \cdot y$;
3. $x \cdot y \approx (x \wedge (y \cdot 1)) \vee (y \wedge (x \cdot 1))$.

Proof. $x \wedge y \leq x \cdot y$ holds in any Boolean semilattice, by Proposition 4.2(5). By idempotence and additivity,

$$x \vee y = (x \vee y)^2 = x^2 \vee (x \cdot y) \vee y^2 \geq x \cdot y$$

proving (1).

For (2),

$$\begin{aligned} x \wedge (y \cdot 1) = x \wedge y \cdot (x \vee x') = x \wedge ((x \cdot y) \vee (x' \cdot y)) \leq x \wedge ((x \cdot y) \vee (x' \vee y)) \\ = (x \wedge (x \cdot y)) \vee (x \wedge x') \vee (x \wedge y) \leq (x \cdot y) \vee (x \wedge y) = x \cdot y \end{aligned}$$

where (1) is used in the first inequality and the last equality.

Finally, $(x \wedge (y \cdot 1)) \vee (y \wedge (x \cdot 1)) \leq x \cdot y$ follows from (2). Conversely, by (1), monotonicity, and distributivity

$$\begin{aligned} &x \cdot y \leq (x \cdot 1) \wedge (y \cdot 1) \wedge (x \vee y) = \\ &\quad \big((x \cdot 1) \wedge (y \cdot 1) \wedge x\big) \vee \big((x \cdot 1) \wedge (y \cdot 1) \wedge y\big) = (x \wedge (y \cdot 1)) \vee (y \wedge (x \cdot 1)) \,. \end{aligned}$$

□

The third identity in the above lemma can be written

$$x \cdot y \approx (x \wedge {\downarrow}y) \vee (y \wedge {\downarrow}x). \tag{11}$$

Thus an idempotent Boolean semilattice is term-equivalent to its closure-reduct.

Lemma 5.2. *let* **B** *be an idempotent Boolean semilattice. Then for atoms a, b,*

$$a \cdot b = \begin{cases} a & \text{if } {\downarrow}a < {\downarrow}b \\ a \vee b & \text{if } {\downarrow}a = {\downarrow}b \\ b & \text{if } {\downarrow}a > {\downarrow}b \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose that ${\downarrow}a < {\downarrow}b$. Then $a < {\downarrow}b$ and $b \not\leq {\downarrow}a$ so $b \wedge {\downarrow}a = 0$, since b is an atom. Consequently $a \cdot b = a$ by (11). The second and third alternatives are argued similarly. Finally, if ${\downarrow}a$ and ${\downarrow}b$ are incomparable then $a \not\leq {\downarrow}b$ and $b \not\leq {\downarrow}a$. Then from (11), $a \cdot b = 0$. □

Lemma 5.3. *Let* **B** *be an atomic, subdirectly irreducible, idempotent Boolean semilattice. Then* $\mathrm{Con}(\mathbf{B})$ *is linearly ordered.*

Proof. We shall first show that the closed elements of **B** are linearly ordered. Suppose that b and c are incomparable, closed elements. By atomicity, there are atoms b_0 and c_0 such that $b_0 \le b$, $b_0 \not\le c$, $c_0 \le c$, and $c_0 \not\le b$. If $\downarrow b_0 \le \downarrow c_0$ then (since c is closed)

$$b_0 \le \downarrow b_0 \le \downarrow c_0 \le c$$

which is a contradiction. Similarly $\downarrow c_0 \not\le \downarrow b_0$, i.e., $\downarrow b_0$ and $\downarrow c_0$ are incomparable. Therefore by Lemma 5.2, $b_0 \cdot c_0 = 0$. But **B** is subdirectly irreducible, hence integral (by Proposition 4.19), which is a contradiction. Thus our original elements b and c must be comparable.

Now we address the statement in the lemma. Because of the correspondence between congruences and congruence ideals, it is enough to show that for any two congruence ideals I and J, either $I \subseteq J$ or $J \subseteq I$. So assume instead that there are elements $b \in I - J$ and $c \in J - I$. By Lemma 4.9, $\downarrow b \in I$ and, since $b \le \downarrow b$, we have $\downarrow b \notin J$. Similarly $\downarrow c \in J - I$. By our deductions above, either $\downarrow b \le \downarrow c$ or $\downarrow c \le \downarrow b$. But then either $\downarrow b \in J$ or $\downarrow c \in I$, which is a contradiction. ☐

Theorem 5.4. *Let* **B** *be a complete, atomic, idempotent Boolean semilattice, and suppose that* $\mathrm{Con}(\mathbf{B})$ *is linearly ordered. Then* $\mathbf{B} \in \mathsf{S}(\mathsf{LS}^+)$.

Proof. Let A be the set of atoms of $\mathbf{B}_0$. Fix a linear ordering, $\trianglelefteq$, on A. Let $S = \{\, (\downarrow a, n, a) : a \in A, n \in \mathbb{N} \,\}$ ordered lexicographically. That is

$$(\downarrow a, n, a) < (\downarrow b, m, b) \text{ if } \quad \begin{array}{l} \downarrow a < \downarrow b \text{ computed in } \mathbf{B}, \text{ or} \\ \downarrow a = \downarrow b \;\;\&\;\; n < m \text{ or} \\ \downarrow a = \downarrow b \;\;\&\;\; n = m \;\;\&\;\; a \triangleleft b\,. \end{array}$$

Because of our assumption on the congruence lattice of **B**, the closed elements are linearly ordered. So with this definition, **S** becomes a linearly ordered meet-semilattice.

Write $\mathbf{S}^{\square} = \langle S, \theta \rangle$ and $\mathbf{B}_+ = \langle A, \psi \rangle$. Recall that

$$\theta = \{\, (u, v, u \cdot v) : u, v \in S \,\}$$
$$\psi = \{\, (a, b, c) : c \le a \cdot b \,\}\ .$$

Define $h \colon \mathbf{S}^{\square} \to \mathbf{B}_+$ by $h(\downarrow a, n, a) = a$. Clearly h is surjective. We shall show that h is a bounded morphism. From our comments in Sect. 2 it will follow that $\overleftarrow{h}$ embeds $\mathbf{B} = (\mathbf{B}_+)^+$ into $\mathbf{S}^+$, thereby proving the theorem.

We apply Definition 2.5. To verify the first condition, let $(u, v, u \cdot v) \in \theta$, say, $u = (\downarrow a, n, a)$ and $v = (\downarrow b, m, b)$. Since **S** is linear, we can assume that $u \le v$, so $u \cdot v = u$. Then $(h(u), h(v), h(u \cdot v)) = (a, b, a)$. The condition $u \le v$ implies $\downarrow a \le \downarrow b$. By Lemma 5.2 we must have $a \le a \cdot b$, so $(a, b, a) \in \psi$.

For the second condition in the definition of bounded morphism, let $a, b \in A$, $u = (\downarrow c, n, c) \in S$, and assume that $(a, b, h(u)) \in \psi$. This implies that $c \le a \cdot b$. By

Lemma 5.2 we must have $c = a$ or $c = b$. If $c = a$ then $\downarrow a \le \downarrow b$ and $u = (\downarrow a, n, a)$. Take $v = u$ and $w = (\downarrow b, n+1, b)$. Then $v \le w$ in **S**, so $(v, w, u) \in \theta$, satisfies the condition. On the other hand, if $c = b$, take $v = (\downarrow a, n+1, a)$, and $w = u = (\downarrow b, n, b)$. Then $w \le v$, so $(v, w, u) \in \theta$ again satisfies the condition. ☐

Corollary 5.5 (Bergman-Blok). *The variety of idempotent Boolean semilattices is equal to* $\mathsf{SP}(\mathsf{LS}^+)$.

Proof. At the beginning of the section we verified that LS^+ is contained in IBSl, from which one inclusion of the theorem follows. We must verify that every idempotent Boolean semilattice lies in $\mathsf{V}(\mathsf{LS}^+)$. For this, it suffices to show that every subdirectly irreducible member of IBSl lies in $\mathsf{S}(\mathsf{LS}^+)$.

So let **A** be a subdirectly irreducible, idempotent Boolean semilattice, and let $\mathbf{B} = \mathbf{A}^\sigma$. Since the identities defining IBSl are strictly positive, **B** is itself an idempotent Boolean semilattice. By Theorem 4.18, **B** is subdirectly irreducible as well. And of course **B** is complete and atomic.

Then by Lemma 5.3, $\mathrm{Con}(\mathbf{B})$ is linearly ordered, and therefore by Theorem 5.4, $\mathbf{B} \in \mathsf{S}(\mathsf{LS}^+)$. Since **A** is a subalgebra of **B**, the result follows. ☐

Thus we have a satisfactory resolution to the representation problem: the finitely based variety IBSl is represented by the (finitely axiomatizable) class of linearly ordered semilattices. In fact, the variety IBSl is term-equivalent to the variety S4.3 of modal algebras via the interpretations $\Diamond x = x \cdot 1$ and $x \cdot y = (\Diamond x \wedge y) \vee (x \wedge \Diamond y)$. From this equivalence it follows from known results that IBSl has only countably many subvarieties, each of which is finitely axiomatizable and generated by its finite members.

6 Semilattice Representability

Let us return to the relationship between the members of BSl and the complex algebras of semilattices. An integral Boolean semilattice is called *semilattice representable* if it can be embedded into $\mathbf{S}^+$ for some semilattice **S**. In this section we shall simply say "representable" instead of "semilattice representable." It may also be of interest to determine whether a finite Boolean semilattice can be embedded into the complex algebra of a finite semilattice. When this occurs we say that the Boolean semilattice is *finitely representable.*

Lemma 6.1. *Let* **B** *be a Boolean semilattice, and* $r \in B$*. Suppose that* $\downarrow r = 1$*. Then for any homomorphism* $h\colon \mathbf{B} \to \mathbf{S}^+$ *for a semilattice,* **S**, *the complex* $h(r)$ *must contain all maximal elements of* **S**.

Proof. Let $R = h(r) \subseteq S$. Then $\downarrow r = 1$ implies that the downset generated by R is all of S. Thus if u is a maximal element of **S**, then for some $x \in R$, $u \le x$. By maximality, $u = x \in R$. ☐

Corollary 6.2. *Let* $\mathbf{B}$ *be a Boolean semilattice,* $r \in B$*. Suppose that* $\downarrow r = \downarrow(r') = 1$*. Then there is no homomorphism from* $\mathbf{B}$ *to* $\mathbf{S}^+$ *for any semilattice with a maximal element. In particular* $\mathbf{B}$ *is not finitely representable.*

Proof. Let $h\colon \mathbf{B} \to \mathbf{S}^+$ be a homomorphism. By Lemma 6.1, both $h(r)$ and $h(r') = h(r)'$ must contain all maximal elements. Since these sets are disjoint, $\mathbf{S}$ has no maximal elements. □

Corollary 6.3. *No simple Boolean semilattice is finitely representable.*

Proof. Follows from Proposition 4.19 and Corollary 6.2. □

Recall that every partial semilattice is an inner substructure (i.e., an upset) of a semilattice. It is easy to see that the proofs of Lemma 6.1 and Corollary 6.2 remain valid when $\mathbf{S}$ is only a partial semilattice. Thus no simple Boolean semilattice can be embedded into the complex algebra of an upset of a semilattice.

Finally, we make one observation that may be useful in addressing Problems 4.3 and 4.4. Since the identities defining semilattices are regular, we can apply Theorem 2.8 to obtain $\mathsf{P}(\mathsf{Sl}^+) \subseteq \mathsf{H}(\mathsf{Sl}^+)$ and then Corollary 4.15 yields

$$\mathsf{V}(\mathsf{Sl}^+) = \mathsf{HSP}(\mathsf{Sl}^+) = \mathsf{HS}(\mathsf{Sl}^+) = \mathsf{SH}(\mathsf{Sl}^+).$$

7 Varieties of Boolean semilattices

The lattice of subvarieties of BSl is itself a rich and complex structure. At this time, we content ourselves with a few simple observations.

Because of normality and square-increasingness, $\{0,1\}$ forms a subalgebra of any nontrivial Boolean semilattice, in which $0\cdot 0 = 0\cdot 1 = 1\cdot 0 = 0$ and $1\cdot 1 = 1$. This algebra can be represented as $\mathbf{1}^+$, in which $\mathbf{1}$ represents a 1-element semilattice. Consequently, this algebra generates the smallest nontrivial subvariety of BSl. This subvariety is defined, relative to BSl, by the identity $x\cdot y \approx x \wedge y$. Thus, this subvariety is term-equivalent to the variety of Boolean algebras.

There are seven 4-element Boolean semilattices. Two of them are $\mathbf{1}^+ \times \mathbf{1}^+$ and $\mathbf{2}^+$, where $\mathbf{2}$ represents the 2-element semilattice. Figure 2 describes the product of the two atoms, a and b on each of the 7 algebras.

The algebra $\mathbf{A}$ in the figure is identical to the complex algebra $\mathbf{H}^+$ discussed in conjunction with equation (8). As we demonstrated at that time, $\mathbf{A}$ is not semilattice representable. $\mathbf{B}_1$ can be embedded into $\mathbf{S}^+$, where $\mathbf{S}$ is the 3-element, nonlinear semilattice. The remaining three algebras are not finitely representable, by Corollary 6.2. However it is not hard to show that each can be represented on an infinite semilattice.

$\mathbf{1}^+ \times \mathbf{1}^+$ of course lies in the variety generated by $\mathbf{1}^+$. Both $\mathbf{2}^+$ and $\mathbf{B}_2$ are idempotent, so they lie in IBSl. $\mathbf{A}$, $\mathbf{B}_2$, $\mathbf{B}_3$, and $\mathbf{B}_4$ are simple, so they lie in $\mathsf{BSl}_{\mathsf{D}}$. (In fact, $\mathbf{B}_4 \in \mathsf{Tot}^+$, see Theorem 4.22.) Finally, since all six (except for $\mathbf{1}^+ \times \mathbf{1}^+$) are subdirectly irreducible and have the same finite size, by Jónsson's lemma (see for

	$a\cdot a$	$b\cdot b$	$a\cdot b$
$\mathbf{1}^+\times\mathbf{1}^+$	a	b	0
$\mathbf{2}^+$	a	b	a
$\mathbf{A}$	a	1	b
$\mathbf{B}_1$	a	1	a
$\mathbf{B}_2$	a	b	1
$\mathbf{B}_3$	a	1	1
$\mathbf{B}_4$	1	1	1

Fig. 2 The 4-element Boolean semilattices, with atoms a and b

example [1, Cor. 5.13]) they must generate pairwise incomparable varieties. All five generate varieties that cover $\mathsf{V}(\mathbf{1}^+)$.

Problem 7.1. Determine all covers of $\mathsf{V}(\mathbf{1}^+)$ in the lattice of subvarieties of BSl. Does every subdirectly irreducible, 8-element Boolean semilattice contain a 4-element subalgebra?

We have already observed that the variety of Boolean semilattices has EDPC. In [2], Blok and Pigozzi discuss the significance of quotients via compact congruences. For a class $\mathscr{K}$ of algebras we write

$$\mathsf{H}_\omega(\mathscr{K}) = \{\,\mathbf{B}/\theta : \mathbf{B}\in\mathscr{K},\ \theta \text{ a compact congruence of } \mathbf{B}\,\}\ .$$

In a Boolean semilattice, compact congruences correspond precisely to closed elements. In a semilattice, $\mathbf{S}$, a closed element of $\mathbf{S}^+$ is precisely a downset, D, of $\mathbf{S}$. The complex $S-D$ is an upset, which is to say, an inner substructure of $\mathbf{S}$. The resulting quotient, $\mathbf{S}^+/[D)$ is isomorphic to the complex algebra $(\mathbf{S}-D)^+$.

Let $\mathbf{A}$ be a member of a fixed variety, $\mathscr{V}$. The algebra $\mathbf{A}$ is called a *splitting algebra* (relative to $\mathscr{V}$), if $\mathscr{V}$ has a largest subvariety excluding $\mathbf{A}$. This variety, if it exists, is denoted $\mathscr{V}/\mathbf{A}$, and is called the *conjugate variety to* $\mathbf{A}$. The conjugate variety is defined by a single equation (relative to $\mathscr{V}$) called the conjugate equation. Blok and Pigozzi prove that if $\mathscr{V}$ has EDPC, then every finitely presented, subdirectly irreducible algebra in $\mathscr{V}$ is a splitting algebra, with conjugate variety

$$\mathscr{V}/\mathbf{A} = \{\,\mathbf{B}\in\mathscr{V} : \mathbf{A}\notin\mathsf{SH}_\omega(\mathbf{B})\,\}\ . \tag{12}$$

In particular, if $\mathscr{V}$ has finite similarity type, which is the case for Boolean semilattices, then every finite subdirectly irreducible algebra is splitting.

As an application of this idea, we offer the following. Let $\mathsf{Sl}_{\mathrm{fin}}$ denote the class of finite semilattices.

Theorem 7.2. $\mathsf{V}({\mathsf{Sl}_{\mathrm{fin}}}^+)\neq\mathsf{V}(\mathsf{Sl}^+)$.

Proof. Let $\mathbf{B}_2$ be the 4-element algebra in Figure 2. We have already observed that $\mathbf{B}_2$ is finite and simple, hence splitting. Suppose $\mathbf{S}$ is a semilattice and $\mathbf{B}_2\in\mathsf{SH}_\omega(\mathbf{S}^+)$. Then $\mathbf{B}_2$ is a subalgebra of $\mathbf{C}^+$ in which $\mathbf{C}$ is an inner substructure, i.e., an upset, of $\mathbf{S}$. By the remark following Corollary 6.3, $\mathbf{C}$, hence $\mathbf{S}$, must be infinite.

Therefore, by Equation (12), $\mathsf{Sl}^+_{\text{fin}} \subseteq \mathbf{V}(\mathsf{Sl}^+)/\mathbf{B}_2$. Since the latter class is a variety, $\mathbf{V}({\mathsf{Sl}_{\text{fin}}}^+) \subseteq \mathbf{V}(\mathsf{Sl}^+)/\mathbf{B}_2$. Since $\mathbf{V}(\mathsf{Sl}^+)/\mathbf{B}_2$ obviously omits $\mathbf{B}_2$ itself, it must be a proper subvariety of $\mathbf{V}(\mathsf{Sl}^+)$. □

We close with a construction of $2^{\aleph_0}$ distinct subvarieties of $\mathbf{V}(\mathsf{Sl}^+)$. Several other constructions are known. For example, it is known that there are uncountably many varieties of closure algebras, and this can be transformed into a construction for Boolean semilattices.

For any positive integer n, let A_n denote an antichain of size n, and let $\mathbf{Y}_n$ be the semilattice obtained from A_n by adjoining a new least element, z. It is easy to see that the only upsets of $\mathbf{Y}_n$ are Y_n itself and sets of the form A_k for some $k \leq n$.

Clearly, a bounded morphic image of $\mathbf{A}_k$ is of the form $\mathbf{A}_l$ for $l \leq k$. Also, no proper bounded morphic image of $\mathbf{Y}_n$ is a semilattice. To see this, we use Lemma 2.7. Suppose that α is a proper, nontrivial, bounded equivalence on $\mathbf{Y}_n$. There must be distinct elements a,b,c with $(a,b) \in \alpha$, $(a,c) \notin \alpha$ and $a \in A_n$. If $b = z$ then the set $a/\alpha \cdot c/\alpha$ is not a union of α-classes, since it contains b but not a. This contradicts Lemma 2.7. Hence $b \neq z$, so the ternary relation on $\mathbf{Y}_n/\alpha$ contains $(a/\alpha, a/\alpha, z/\alpha)$ which is impossible in a semilattice.

Since $\mathbf{Y}_n$ is a lower-bounded semilattice, $\mathbf{Y}_n^+$ is subdirectly irreducible (Proposition 4.17). Applying duality to the previous two paragraphs, we deduce that

$$n \neq m \implies \mathbf{Y}_m^+ \notin \mathbf{SH}(\mathbf{Y}_n^+). \tag{13}$$

From this and the Blok-Pigozzi relationship (12), we obtain the following.

Proposition 7.3. *Let S be a set of natural numbers and define $\mathscr{V}_S = \mathbf{V}\{\,\mathbf{Y}_n^+ : n \in S\,\}$. Then $\mathbf{Y}_m^+ \in \mathscr{V}_S$ if and only if $m \in S$. Consequently, $\{\,\mathscr{V}_S : S \subseteq \mathbb{N}\,\}$ forms an uncountable family of subvarieties of $\mathbf{V}(\mathsf{Sl}^+)$.*

Proof. If $m \notin S$ then by (12) and (13), $\mathscr{V}_S \subseteq \mathbf{V}(\mathsf{Sl}^+)/\mathbf{Y}_m^+$. Since $\mathbf{Y}_m^+$ is finite and subdirectly irreducible, it is a splitting algebra, so this latter class is a variety. □

The proof of 7.3 actually shows something stronger. The variety $\mathbf{V}(\mathsf{Sl}^+_{\text{fin}})$ has uncountably many subvarieties.

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