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Analytical solution for capacitance calculation of a curved patch capacitor that conforms to the curvature of a homogeneous cylindrical dielectric rod

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Abstract

This Letter presents an analytical expression for the capacitance of a curved patch capacitor whose electrodes conform to the curvature of a long, homogeneous, cylindrical dielectric rod. The capacitor is composed of two infinitely long curved electrodes, symmetrically placed about a diameter of the cylinder cross-section. The resulting capacitance per unit length depends on both the dielectric properties of the material under test and the capacitor configuration. A practical capacitance measurement is also presented, with appropriately guarded finite electrodes. Very good agreement between measured and theoretically predicted capacitances were observed, to within 2.4 percent. The analytical result presented in this Letter can be applied for extremely rapid evaluation of rod permittivity from measured capacitance.

Keywords

analytical expressions, analytical results, capacitance calculation, capacitance per unit length, curved electrode, cuved patches, dielectric rods, capacitors, Center for Nondestructive Evaluation

Disciplines

Electrical and Computer Engineering | Materials Science and Engineering

Comments

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Analytical solution for capacitance calculation of a curved patch capacitor that conforms to the curvature of a homogeneous cylindrical dielectric rod

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This Letter presents an analytical expression for the capacitance of a curved patch capacitor whose electrodes conform to the curvature of a long, homogeneous, cylindrical dielectric rod. The capacitor is composed of two infinitely long curved electrodes, symmetrically placed about a diameter of the cylinder cross-section. The resulting capacitance per unit length depends on both the dielectric properties of the material under test and the capacitor configuration. A practical capacitance measurement is also presented, with appropriately guarded finite electrodes. Very good agreement between measured and theoretically predicted capacitances were observed, to within 2.4 percent. The analytical result presented in this Letter can be applied for extremely rapid evaluation of rod permittivity from measured capacitance. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4862434]

Capacitors have been applied to solve many different types of sensing and measurement problems, by measuring the capacitance between two or more conductors.[1] Capacitive touchscreens, based on pressure sensing,[2] are very well known in everyday life. In addition, capacitive techniques find applications in areas, such as, micrometer development,[3] proximity, and position sensing,[4,5] displacement measurement,[6,7] and materials characterization.[8–11]

Previously, we have demonstrated the feasibility of utilizing finite-sized curved patch capacitors for materials characterization of cylindrical structures.[12,13] The sensor modeling was achieved numerically using the method of moments. In this Letter, a two-dimensional analytical solution is provided for the rapid and accurate calculation of the capacitance of curved patch capacitors that conform to the curvature of cylindrical homogeneous dielectric rods. Theoretical derivations in this Letter are extended from discussions on the capacitance between axially slotted open circular cylinders in free space.[14]

Figure 1 shows the configuration of the problem. The capacitor consists of two infinitely long curved patches that are symmetric with respect to the x axis. The radius of the electrodes is equal to unity. The upper electrode is defined by $\phi_0 \le \phi \le \phi_1$, and the lower one is defined by $-\phi_1 \le \phi \le -\phi_0$. These two electrodes are charged to $+1$ V and -1 V, respectively. The material under test is an infinitely long dielectric rod, having the same radius as the curved electrodes. The dielectric constant of the environment is denoted as ϵ_1, and that of the dielectric test-piece is ϵ_2.

Considering the symmetry of the problem, one has only to solve for the potential in the upper half plane in Fig. 1 to obtain the capacitance. The electric potential Ψ resulting from the charged capacitor satisfies the 2D Laplace equation

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)\Psi(\rho,\phi)=0, \qquad (1)$$

and can be expressed as

$$\Psi_i(\rho,\phi)=\sum_{n=1}^{\infty}a_n\rho^{(-1)^i n}\sin n\phi, \qquad i=1,2, \qquad (2)$$

where the sub- and superscripts 1 and 2 correspond to the regions defined by $\rho > 1$ and $\rho \le 1$, respectively. The interface conditions for the potentials at $\rho = 1$ in the upper half plane ($\phi \ge 0$) are

$$\Psi_1(1,\phi)=\Psi_2(1,\phi)=1, \quad \phi\in(\phi_0,\phi_1), \qquad (3)$$

$$\epsilon_1\frac{\partial\Psi_1(\rho,\phi)}{\partial\rho}\bigg|_{\rho=1}=\epsilon_2\frac{\partial\Psi_2(\rho,\phi)}{\partial\rho}\bigg|_{\rho=1}, \quad \phi\in[(0,\phi_0)\cup(\phi_1,\pi)]. \qquad (4)$$

Inserting Eq. (2) into (3) and (4), we find the trigonometric series equations to determine the coefficients a_n

$$\sum_{n=1}^{\infty}na_n\sin n\phi=0, \quad \phi\in[(0,\phi_0)\cup(\phi_1,\pi)], \qquad (5)$$

$$\sum_{n=1}^{\infty}a_n\sin n\phi=1, \quad \phi\in(\phi_0,\phi_1). \qquad (6)$$

These nonsymmetrical triple series equations can be transformed into symmetrical triple series equations and considered in terms of dual series equations[14]

$$\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)b_{2n+1}\sin\left(n+\frac{1}{2}\right)\theta=0, \quad \theta\in(0,\theta_0), \qquad (7)$$

and

$$\sum_{n=0}^{\infty}b_{2n+1}\sin\left(n+\frac{1}{2}\right)\theta=E_0, \quad \theta\in(\theta_0,\pi), \qquad (8)$$

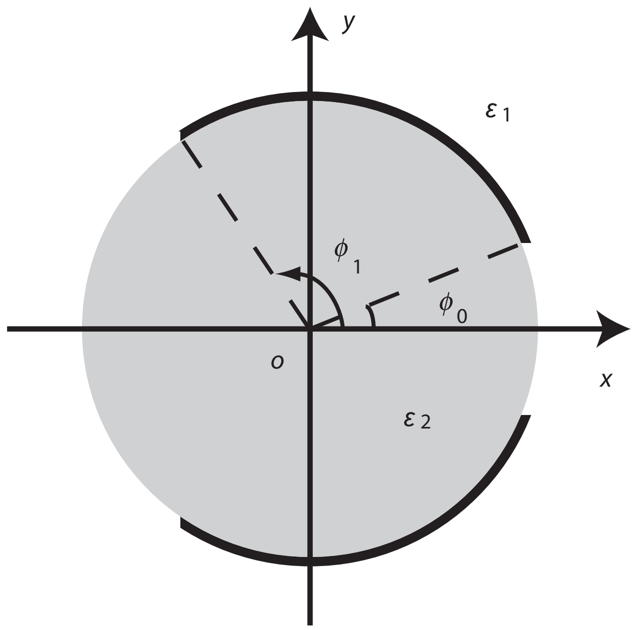

FIG. 1. Configuration of the curved patch capacitor. The symmetric electrodes are defined in the ϕ direction as: $\phi_0 \le \phi \le \phi_1$ (top) and $-\phi_1 \le \phi \le -\phi_0$ (bottom).

where $E_0 = \left(\tan\frac{1}{2}\phi_0 \tan\frac{1}{2}\phi_1\right)^{\frac{1}{2}}$, $\theta = 4\arctan\left[\tan\frac{1}{2}\phi\left(\tan\frac{1}{2}\phi_0 \tan\frac{1}{2}\phi_1\right)^{-\frac{1}{2}}\right]$ and $\theta_0 = 4\arctan\left[\left(\tan\frac{1}{2}\phi_0 \cot\frac{1}{2}\phi_1\right)^{\frac{1}{2}}\right]$. The transformation relationship between a_n and b_n is described in Ref. 14. Coefficients b_{2n+1} in Eqs. (7) and (8) cannot, however, be easily solved through an orthogonality relationship for the sine functions, because of the $(n+1/2)$ term in front of b_{2n+1} in (7). The scheme adopted in this Letter is to transform the dual series Eqs. (7) and (8) into equations containing the product of b_{2n+1} and Legendre functions, with intermediate steps in terms of the product of b_{2n+1} and Jacobi polynomials, and to solve for b_{2n+1} utilizing the orthogonality relationship for Legendre functions. First, Eqs. (7) and (8) are rewritten in terms of Jacobi polynomials $P_n^{(a,b)}$ as

$$\sum_{n=0}^{\infty} \frac{(n+\frac{1}{2})\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} b_{2n+1} P_n^{(\frac{1}{2},-\frac{1}{2})}(z) = 0, \quad z \in (z_0, 1) \tag{9}$$

and

$$\sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} b_{2n+1} \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} (1-z)^{\frac{1}{2}} P_n^{(\frac{1}{2},-\frac{1}{2})}(z) = E_0, \quad z \in (-1, z_0), \tag{10}$$

where $z = \cos\theta$ and $z_0 = \cos\theta_0$, by replacing the sine function with Jacobi polynomials using the identity[14]

$$\sin\left(n+\frac{1}{2}\right)\theta = \frac{\Gamma(\frac{1}{2})\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \sin\frac{\theta}{2} P_n^{(\frac{1}{2},-\frac{1}{2})}(\cos\theta). \tag{11}$$

To transform Eq. (10) into an equation containing the product of b_{2n+1} and Legendre functions, multiply both sides of (10) by $(1+z)^{-\frac{1}{2}}(1-z)^{-\frac{1}{2}}$, integrate from -1 to z, and make use of the following identity[14]

$$(1+t)^{\frac{1}{2}} P_n^{(-\frac{1}{2},\frac{1}{2})}(t) = \left(n+\frac{1}{2}\right) \int_{-1}^{t} (1+x)^{-\frac{1}{2}} P_n^{(\frac{1}{2},-\frac{1}{2})}(x)\, dx \tag{12}$$

to obtain

$$\sum_{n=0}^{\infty} b_{2n+1} \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \frac{(1+z)^{\frac{1}{2}}}{n+\frac{1}{2}} P_n^{(-\frac{1}{2},\frac{1}{2})}(z) = E_0 \sqrt{\frac{2}{\pi}} \left(\frac{\pi}{2} + \arcsin z\right), \quad z \in (-1, z_0). \tag{13}$$

Equation (13) can be further written in the form of Abel's integral Eq. (14) as

$$\int_{-1}^{z} \frac{\sum_{n=0}^{\infty} b_{2n+1} P_n(x)}{(z-x)^{\frac{1}{2}}} dx = \sqrt{2} E_0 \left(\frac{\pi}{2} + \arcsin z\right), \quad z \in (-1, z_0), \tag{14}$$

by applying the identity

$$P_n^{(-\frac{1}{2},\frac{1}{2})}(z) = \frac{(1+z)^{-\frac{1}{2}} \Gamma(n+\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} \int_{-1}^{z} \frac{P_n(x)}{(z-x)^{\frac{1}{2}}} dx, \tag{15}$$

where $P_n(z)$ is the Legendre polynomial of order n. The generalized Abel's integral equation is defined as

$$f(x) = \int_a^x \frac{u(\xi)\, d\xi}{(x-\xi)^{\lambda}}, \quad (0 < \lambda < 1), \tag{16}$$

where f is a known function and u is a function to be determined. The inversion formula for Eq. (16) is

$$u(\xi) = -\frac{\sin\lambda\pi}{\pi} \frac{d}{d\xi} \int_{\xi}^{b} \frac{f(x)\, dx}{(x-\xi)^{1-\lambda}}. \tag{17}$$

After making use of the inversion formula for Abel's integral equation for Eq. (14), the following relationship is obtained

$$\sum_{n=0}^{\infty} b_{2n+1} P_n(x) = \frac{2E_0}{\pi} K\left(\sqrt{\frac{1+x}{2}}\right), \quad x \in (-1, z_0), \tag{18}$$

where $K(z)$ is the complete elliptic integral of the first kind. A similar equation containing the product of b_{2n+1} and $P_n(x)$ may be obtained for $x \in (z_0, 1)$ based on Eq. (9). Now, multiply both sides of Eq. (9) by $(1+z)^{-\frac{1}{2}}$, integrate from -1 to z, and make use of the identity[14]

$$(1+t)^{\frac{1}{2}} P_n^{(-\frac{1}{2},\frac{1}{2})}(t) = \left(n+\frac{1}{2}\right) \int_{-1}^{t} (1+x)^{-\frac{1}{2}} P_n^{(\frac{1}{2},-\frac{1}{2})}(x)\, dx \tag{19}$$

to express Eq. (9) as

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} b_{2n+1} P_n^{(-\frac{1}{2},\frac{1}{2})}(z) = F_0 (1+z)^{-\frac{1}{2}}, \quad z \in (z_0, 1), \tag{20}$$

where F_0 is a constant that will be determined later. Next, multiply both sides of Eq. (20) by $(1-z)^{-\frac{1}{2}}$, integrate from z to 1, and make use of the identity[14]

$$(1-t)^{\frac{1}{2}}P_n^{(\frac{1}{2},-\frac{1}{2})}(t) = \left(n+\frac{1}{2}\right)\int_t^1 (1-x)^{-\frac{1}{2}}P_n^{(-\frac{1}{2},\frac{1}{2})}(x)\,dx \quad (21)$$

to obtain

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}(1-z)^{\frac{1}{2}}b_{2n+1}P_n^{(\frac{1}{2},-\frac{1}{2})}(z) = F_0\arccos z, \quad z\in(z_0,1). \quad (22)$$

Similarly, Eq. (22) can be expressed in the form of Abel's integral equation as

$$\int_z^1 \frac{\sum_{n=0}^{\infty} b_{2n+1}P_n(x)}{\sqrt{x-z}}dx = \sqrt{\pi}F_0\arccos z, \quad z\in(z_0,1) \quad (23)$$

by making use of the identity[14]

$$P_n^{(\frac{1}{2},-\frac{1}{2})}(z) = \frac{(1-z)^{-\frac{1}{2}}\Gamma(n+\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)}\int_z^1 \frac{P_n(x)}{(x-z)^{\frac{1}{2}}}dx. \quad (24)$$

The following relationship:

$$\sum_{n=0}^{\infty} b_{2n+1}P_n(x) = \sqrt{\frac{2}{\pi}}F_0K\left(\sqrt{\frac{1-x}{2}}\right), \quad x\in(z_0,1) \quad (25)$$

is obtained by applying the inversion formula of Abel's integral equation to Eq. (23). Equations (18) and (25) have to be continuous at $x=z_0$ because of the continuity condition for the electric potential. Therefore, it is found that

$$F_0 = \sqrt{\frac{2}{\pi}}\left(\tan\frac{\phi_0}{2}\tan\frac{\phi_1}{2}\right)^{\frac{1}{2}}K\left(\sqrt{\frac{1+z_0}{2}}\right)\Big/K\left(\sqrt{\frac{1-z_0}{2}}\right). \quad (26)$$

The coefficients b_{2n+1} are determined by applying the orthogonality relationship for Legendre polynomials to Eqs. (18) and (25). The result is

$$b_{2n+1} = \left(\tan\frac{\phi_0}{2}\tan\frac{\phi_1}{2}\right)^{\frac{1}{2}}\left\{\left(n+\frac{1}{2}\right)K\left(\sqrt{\frac{1-z_0}{2}}\right)\right\}^{-1}P_n(z_0). \quad (27)$$

Knowing the coefficients b_{2n+1}, the electric potential in space is obtained via (2) and (5)–(8).

In order to solve for the capacitance, the surface charge density $\sigma_s(\phi)$ on the electrodes is derived. Applying the interface condition for the tangential components of electric flux density $\mathbf{D}$ at $\rho=1$, which can be derived from Eq. (2), gives rise to

$$\sigma_s(\phi) = (\epsilon_1+\epsilon_2)\sum_{n=1}^{\infty} na_n\sin n\phi. \quad (28)$$

Since an explicit expression is given for b_n instead of a_n, Eq. (28) is expressed in terms of b_n as

$$\sigma_s(\theta) = (\epsilon_1+\epsilon_2)\left(\cos^2\frac{\theta}{4}+\tan\frac{\phi_0}{2}\tan\frac{\phi_1}{2}\sin^2\frac{\theta}{4}\right)\times\left(\tan\frac{\phi_0}{2}\tan\frac{\phi_1}{2}\right)^{-1}\sum_{n=1}^{\infty} nb_n\sin\frac{n\theta}{2} \quad (29)$$

based on the transformation relationship[14] between a_n and b_n. Next, insert Eq. (27) into (29) and express θ in terms of ϕ to express the electrode surface charge density in terms of the original capacitor configuration. After some straightforward manipulation, it is found that

$$\sigma_s(\phi) = \frac{\epsilon_1+\epsilon_2}{K(\sqrt{1-t^2})}\frac{\sin\frac{1}{2}(\phi_0+\phi_1)}{\sqrt{(\cos\phi_0-\cos\phi)(\cos\phi-\cos\phi_1)}}, \quad (30)$$

where $t=\sin\frac{\phi_1-\phi_0}{2}/\sin\frac{\phi_1+\phi_0}{2}$. The surface charge distribution on the lower electrode ($\phi\le 0$) is equal and opposite to that expressed in Eq. (30).

The capacitance per unit length C is calculated using the formula $C=Q/V$, where Q is the total charge per unit length on each electrode and V is the potential difference between the two electrodes. Q can be obtained by integrating Eq. (28) with respect to ϕ from 0 to π and following the same transformation method by which $\sigma_s(\phi)$ was obtained. Note that the identity

$$\sum_0^{\infty}\frac{P_n(z_0)}{n+\frac{1}{2}} = K\left(\sqrt{\frac{1+z_0}{2}}\right) \quad (31)$$

is used in deriving Q. Finally, the capacitance per unit length C for a curved patched capacitor that conforms to the curvature of a cylindrical dielectric rod is obtained as

$$C = (\epsilon_1+\epsilon_2)K(t)/K\left(\sqrt{1-t^2}\right). \quad (32)$$

A practical capacitance measurement setup based on the described theory is shown in Fig. 2. Electrodes 1 and 2 are driving and pick-up electrodes, whereas 3 and 4 are the guard electrodes. Guard electrodes are introduced to

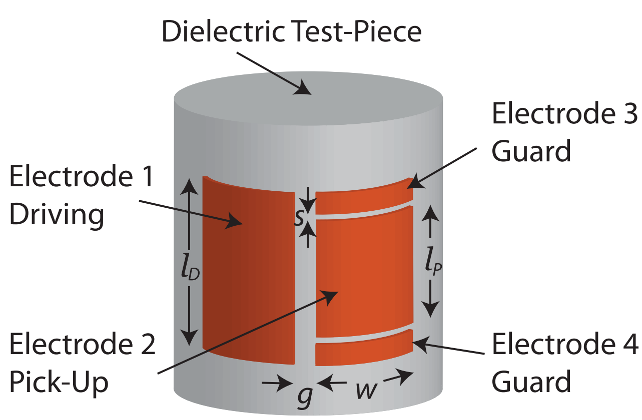

FIG. 2. A curved patch capacitor in surface contact with a dielectric test-piece. All the capacitor electrodes have a width w. The gap between the driving and pick-up electrodes and the separation between the guard and pick-up electrodes are denoted g and s. The lengths of the driving and pick-up electrodes are denoted l_D and l_P, respectively. The length of the guards is $(l_D - l_P - 2s)/2$.

TABLE I. Parameters of the dielectric test-pieces and curved patch capacitors used in benchmark experiments. Capacitor dimensions are $l_D = 80.24 \pm 0.01$ mm, $l_P = 40.00 \pm 0.01$ mm, $g = 1.00 \pm 0.01$ mm, and $s = 0.12 \pm 0.01$ mm except where indicated (see Fig. 2).

				Calculated	Measured	
Test-piece material	Test-piece diameter (mm)	Measured test-piece permittivity	Electrode width (mm)	C (pF)	C (pF)	Relative difference (%)
Tecaform™	19.08 ±0.01	3.77 ±0.05	10.00 ±0.01	2.44	2.48 ±0.01	1.6
Acrylic	19.03 ±0.01	2.88 ±0.05	20.00 ±0.01	2.47	2.53 ±0.01	2.4
Teflon®	19.10 ±0.01	2.23 ±0.05	15.00 ±0.01	1.85	1.89 ±0.01	2.2

eliminate fringing field effects not considered in the 2D model. In the measurements, the guard electrodes are kept at the same potential as the pick-up electrode, so that the electric fields go straight from the driving electrode to pick-up electrode without bending out of the planes of constant z. Benchmark experiments were carried out to verify the theory. Three groups of capacitors of different dimensions were fabricated using photolithography, and attached to cylindrical dielectric rods of different materials and diameters (Table I). Capacitance measurements were performed in free space at 1 MHz and room temperature using an Agilent LCR meter E4980A and an Agilent probe 16095A. The high potential probe pin was placed in contact with the driving electrode and the low potential pin (virtual ground) on the pick-up electrode. The guard electrodes were connected electrically to the guard port of the LCR meter (virtual ground). More details on sensor fabrication, test-piece information, and measurement procedures can be found in Ref. 12. Table I lists the calculated and measured C, for the three measurement configurations, showing very good agreement between measured and theoretically predicted capacitances (to within 2.4%). Note that the calculated capacitances were obtained easily by multiplying the electrode length l_P with Eq. (32).

To summarize, an analytical solution for the capacitance of a curved patched capacitor in surface contact with a homogeneous cylindrical dielectric rod has been derived. Very good agreement between theoretically predicted and measured capacitances was observed. Results described in this Letter may be applied for rapid evaluation of rod permittivity from measured capacitance.

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