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David A. Hennessy
Iowa State University

Harvey E. Lapan
Iowa State University, hlapan@iastate.edu

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Market Cycles for a Non-Storable Product under Adjustment Costs[1]

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by

David A. Hennessy
Professor
Department of Economics
578C Heady Hall
Iowa State University
Ames, IA 50011-1070

Harvey E. Lapan
University Professor
Department of Economics
283 Heady Hall
Iowa State University
Ames, IA 50011-1070

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When adjustment costs are present, cyclical preference and technology heterogeneities in a product's markets induce cycles in production. We exploit cyclic and dihedral group invariances in an industry's cost technology to describe these patterns. We show when equilibrium cyclical pricing and production patterns are ordered according to demand patterns. Our approach allows us to identify periods when prices may fall below unit costs, net of adjustment costs. Social welfare preferences over cyclical demand and supply heterogeneities are identified. We study the particulars of cycle dynamics when demand is linear and adjustment costs are quadratic. The analysis is developed for when external trade is impossible and when it is possible.

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Electronic information for correspondence with contact author

David A. Hennessy
Ph: US 515-294-6171, Fax: US 515-294-6336, e-mail: hennessy@iastate.edu

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1. INTRODUCTION

CHRISTMAS COMES BUT ONCE A YEAR, and with it a boom in the sale of airline tickets, holiday bric-a-brac, and fine foods throughout much of the world. For many items, late Fall is by far the year's most important sales period but airline travel has other important seasons, depending on national holidays, local customs, and climate. Demand for personal tax services peaks just prior to a common national filing date. In regions with extreme continental climates, sales in the construction, hospitality, wedding, hobby, and recreation sectors follow the weather. Food consumption baskets reflect the season. Demand for daycare services change with the local school calendar, while public spending can depend on election cycles. Electricity demand has a daily cycle, overlaying a weekly cycle, overlaying an annual cycle. Commerce beats to cyclic rhythms that originate in nature, institutional timetables, culture, and religion. Yet the adjustments necessary to adapt over a cycle come at a cost.

Economics has long emphasized the word 'cycle' in describing the evolution of equilibrium, as in Grandmont (1985). The focus in this literature has been on cycles in behavior given a homogeneous market environment. It has not been largely on the microeconomic modeling of cycles due to heterogeneities in preference and technology primitives.[2] This is a pity given the importance of seasonal cycles in product and factor markets. Beaulieu and Miron (1990), for example, have found that seasonality accounted for between 50% and 70% of variation in U.S. manufacturing production and shipments over the years 1967-87, as well as about 50% of national variation in employment and hours worked. Prices and wages tended to be much less seasonal, most likely due in part to temporal arbitrage for storable goods. Beaulieu and Miron also found that there tended to be more than one local production peak for the United States within the yearly economic cycle. Braun and Evans (1998) have identified seasonal cycles in total factor productivity growth, where productivity grows fastest in the fourth quarter.

The motivation for our paper is the belief that, while no economy is immune from random shocks, much can be learned by identifying regularities as a market or economy moves through time. We focus on non-storable goods and services such as hospitality and professional services, electricity, and perishable produce, where temporal arbitrage is of limited relevance. For these markets, the paper's intent is to develop a market equilibrium model that accommodates cycles in exogenous demand and supply heterogeneities.

It is hoped that the methods used will facilitate researchers seeking to understand life-cycle and non-market behavior, as well as optimal and observed macroeconomic policy where cyclic behavior can arise. Inventory cycles are investigated in Laroque (1989), where the presence of inventories under tardy price adjustments can induce a business cycle. Alesina, Roubini, and Cohen (1997), to name one of many works, consider the interactions between the parameters of political mechanisms and economic policy. Gale (1996) and Kiminori (1999) show how innovation and market power can interact to generate cycle-like behavior in a growth model. Cycle-like behavior in the economy can also occur as the result of an exogenous shock, such as Abel's (2003) baby-boom increase in a random birth process. The work most similar in spirit to ours is the literature following a paper by Grandmont (1985), e.g., Bhattacharya and Russell (2003), where deterministic cycles are modeled with a dynamical system representation of an economy. In that literature the cycles arise endogenously from asymmetries due to agent age, and the analyses rely on methods from chaos theory.

Our approach is to emphasize the symmetries inherent in a market under adjustment costs when there are periodic demand-side and supply-side heterogeneities. In contrast to the dynamical system approach, where the idea is in the background but not used, we will apply group theory to provide an understanding of these symmetries. By showing how adjustment costs impose bounds on the evolution of equilibrium market prices and quantities over a cycle, our model allows us to indicate when the intensity of demand in a period matches the period's equilibrium consumption level. It shows too that adjustment costs have a quite targeted effect on

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[2] Some macroeconomics of seasonal cycles have been investigated in detail by Miron (1996).

prices and quantities over a cycle. Price and quantity dispersion can be affected, but their means need not change.

The model also provides a clarification on how the gains from smoothing production over time trade off with the gains from greater dispersion in production conditions over the cycle. Adjustment costs suggest that instability in the production environment should be detrimental to the amount of surplus a market generates. Yet standard convexity arguments suggest that more dispersion should be beneficial because one can adjust production appropriately. We use group theory to parse the difference and find that a precisely qualified increase in the dispersion of market conditions increases the surplus generated over the cycle. Of course, the ability to engage in spatial arbitrage matters in all of this. We will analyze an environment where spatial arbitrage, or external trade, is not possible and also where it is.

The main body of the paper commences by specifying the market model absent external trade opportunities. After providing an overview of the tools we use to model symmetries, we analyze equilibrium. We identify and characterize production and pricing cycles, as well as welfare effects over the primitive sources of seasonal heterogeneities. We develop a precise solution under linear demand and quadratic adjustment costs. The opportunity to trade outside the market is then introduced.

2. MODEL

The market is competitive and there are N production periods within the cycle. For cultural, climatic, or other reasons, demand varies across these periods. These demand heterogeneities are captured by parameter vector $\alpha \in \mathbb{R}^N$, the reals in N dimensions, with entries $\alpha_n, n \in \{0,1, \dots, N-1\} \equiv \Omega_N$. Period inverse demand, aggregated over consumers, is given as $p_n(q_n) = \alpha_n - A(q_n), n \in \Omega_N$, where $q_n \in \overline{\mathbb{R}}_+$ (i.e., non-negative) is the amount consumed in the period and $A(q_n)$ is strictly increasing. No other technical assumptions on demand are required.

Period markets clear in the absence of storage. The industry's cost function is comprised of two parts. There is a period-specific cost function with unit period shifters, $\beta_n q_n + Y(q_n), n \in \Omega_N$, where β_n is labeled the period unit cost and $Y(q_n): \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ is a weakly increasing and weakly convex function. Coordinate β_n is the n+1st entry in vector $\beta \in \mathbb{R}^N$. There are also adjustment costs. As is standard in market models involving adjustment, aggregate adjustment costs are of form $B(q_n - q_{n-1})$ where the function is convex and continuously differentiable with $B(0) = 0$.[3] Note that closure of the cycle requires adjustment costs for the first period to be $B(q_0 - q_{N-1})$; it should not matter when the cycle is initiated.

Summing, industry costs over the cycle are

$$(2.1) \quad \sum_{n=0}^{N-1} Y(q_n) + \sum_{n=0}^{N-1} \beta_n q_n + \sum_{n=0}^{N-1} B(q_n - q_{n-1}),$$

where it is understood that $q_{-1} = q_{N-1}$. We will at times require some additional structure on $B(q_n - q_{n-1})$. Function $B(\cdot)$ is centrally symmetric, or $B(\cdot) \in CS$, whenever $B(\delta) \equiv B(-\delta)\ \forall \delta \in \mathbb{R}$. The period quadratic adjustment cost function $\tau \times (q_n - q_{n-1})^2, \tau > 0$, is in *CS*.

There are no externalities in our model, and we seek only to better understand market cycles under period heterogeneities and adjustment costs. The solution the market supports is the same as that supported by the central planner. Market welfare is given by

$$(2.2) \quad W(q) = \sum_{n=0}^{N-1} \gamma_n q_n - \sum_{n=0}^{N-1} Y(q_n) - \sum_{n=0}^{N-1} \int_0^{q_n} A(q)dq - \sum_{n=0}^{N-1} B(q_n - q_{n-1}),$$

where $q = (q_0, q_2, \ldots, q_{N-1}) \in \bar{\mathbb{R}}_+^N$ and $\gamma_n = \alpha_n - \beta_n$. The market will "choose" $q = q^*$ to maximize $W(q)$. We assume throughout that q^* is interior on $\bar{\mathbb{R}}_+^N$, and prices are $p_n(q_n^*) = \alpha_n - A(q_n^*)$.

Our interest is in cycles, and a digression is warranted on the determination of N. The cycle

[3] See, e.g., Sargent (1987) and papers referenced in chapters XIV and XV of that book. All of these references assume the particular case of quadratic adjustment costs.

must replicate and the only heterogeneities are in vectors α and β. If sequence $\alpha_0, \alpha_1, \ldots,$ $\alpha_n, \ldots$ replicates every N_α entries and $\beta_0, \beta_1, \ldots, \beta_n, \ldots$ replicates every N_β entries then choose *N* as the least common multiple of the two. This will be the periodicity of sequence $\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots$.

3. SYMMETRY ISSUES: CYCLIC AND DIHEDRAL GROUPS

Expression (2.2) possesses interesting symmetries. The sum $\sum_{n=0}^{N-1} Y(q_n) + \sum_{n=0}^{N-1} \int_0^{q_n} A(q)dq$ is permutation invariant; the value is the same when any $q_i, i \in \Omega_N$ is interchanged with any q_j, $j \in \Omega_N$. There are less apparent symmetries in expression $\sum_{n=0}^{N-1} B(q_n - q_{n-1})$. To understand these we introduce the *modulo N* function on the integer set $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$. The function, written as $\text{modN}(n)$, maps *n* to its remainder upon division by *N*. If, for example, $n = 27$ and $N = 12$ then $\text{mod12}(27) = 27 - 2 \times 12 = 3$. Observe that

$$\text{(3.1)} \quad \sum_{n=0}^{N-1} B(q_n - q_{n-1}) \equiv \sum_{n=0}^{N-1} B(q_{\text{modN}(n+i)} - q_{\text{modN}(n+i-1)}) \quad \forall i \in \mathbb{Z}.$$

Mathematically, the function is invariant under a cyclic group. In general, a group is a set of operations, or group elements, that satisfy four properties under composition.[4] Label the group itself as $\tilde{G}$, comprised of operation set *G* together with the composition operation * on ordered pairs of the operation set. The cardinality of *G*, $|G|$, is called the group's *order*. The set must be closed so that if $g_1, g_2 \in G$ then $g_1 * g_2 \in G$. The set must also have an identity, *e*; for any $g \in G$ then $e \in G$ satisfies $e * g = g * e = g$. The set's elements must satisfy associativity so that for any $g_1, g_2, g_3 \in G$ the order of association does not matter, $(g_1 * g_2) * g_3$ $= g_1 * (g_2 * g_3)$. Finally, for each $g \in G$ there must be an unique inverse element, call it g^{-1},

[4] Barnard and Neill provide an introduction to elementary group theory. See Hennessy and Lapan (2003a, 2003b) for further examples and applications in economic contexts.

such that $g^{-1} * g \equiv g * g^{-1} \equiv e$.

Our interest is entirely in groups where $|G|$ is finite, and we have specific interest in three types of groups. For function $F(x_0, x_1, \dots, x_{N-1})$, there is the group that leaves $F(\cdot)$ invariant under any permutation of arguments, i.e., $F(\dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{j-1}, x_j, x_{j+1}, \dots) \equiv$ $F(\dots, x_{i-1}, x_j, x_{i+1}, \dots, x_{j-1}, x_i, x_{j+1}, \dots) \; \forall i, j \in \Omega_N$. This is called the *symmetric group*, $\tilde{S}_N$, on set Ω_N. There are $N!$ ways (i.e., elements of *G*) to permute *N* objects, and so $|\tilde{S}_N| = N!$. The group is of passing interest in the paper, and it was already encountered when asserting that $\sum_{n=0}^{N-1} Y(q_n) + \sum_{n=0}^{N-1} \int_0^{q_n} A(q)dq$ is permutation invariant.

Two other groups are of more direct interest as they are derived from the concept of cyclic patterns in the state of the market. One is the cyclic group on *N* elements, labeled $\tilde{C}_N$ with element set C_N, and the other is the dihedral group on *N* elements, labeled $\tilde{D}_N$ with element set D_N. Both are most easily described using Cayley tables, to be explained shortly.

Consider function $F(x_0, x_1, x_2, x_3, x_4)$ with invariance $F(x_0, x_1, x_2, x_3, x_4) \equiv$ $F(x_4, x_0, x_1, x_2, x_3)$, which is labeled g_1 so that g_1 substitutes $x_0 \to x_1 \to x_2 \to x_3 \to x_4 \to x_0$. Now, by the group closure requirement, $g_1 * g_1 \in G$ and label it as g_2. This operation represents the invariance $F(x_0, x_1, x_2, x_3, x_4) \equiv F(x_3, x_4, x_0, x_1, x_2)$. Continuing, $g_3 = g_1 * g_2 \in G$, $g_4 =$ $g_1 * g_3 \in G$ and $g_5 = g_1 * g_4 = e \in G$. Thus, five-fold repetition of g_1 cycles the function back to its initial state and it is the group identity element. Cayley table 1, in appendix A, captures this cycling behavior.

Notice how the table's second row is a backward cycling of the first. This is precisely the invariance discerned in (3.1) above. Cyclic groups have a generating operation (generator) the repetition of which generates the whole group. In the case of $\tilde{C}_5$ above, it happens that any element other than e will do but let us assign g_1 to be the generator. In general, $\tilde{C}_N$ is the underlying group that represents invariance under the cyclic shuffle of a function's arguments $F(x_0, x_1, \ldots, x_{N-2}, x_{N-1}) \equiv F(x_{N-1}, x_0, \ldots, x_{N-3}, x_{N-2}) \equiv \ldots \equiv F(x_1, x_2, \ldots, x_{N-1}, x_0)$.

A more subtle invariance arises in expression $\sum_{n=0}^{N-1} B(q_n - q_{n-1})$ when $B(\cdot) \in CS$. With $N = 5$, aggregate adjustment costs are $B(q_0 - q_4) + B(q_1 - q_0) + B(q_2 - q_1) + B(q_3 - q_2) + B(q_4 - q_3)$. To be sure it has the symmetries of $\tilde{C}_N$. But under central symmetry we may also change $q_0 \leftrightarrow q_0$, $q_1 \leftrightarrow q_4$, and $q_2 \leftrightarrow q_3$ where subscripts always sum to 0, modulo 5. Then the summation of adjustment costs is invariant, i.e.,

$$\begin{aligned} &B(q_0 - q_4) + B(q_1 - q_0) + B(q_2 - q_1) + B(q_3 - q_2) + B(q_4 - q_3) \rightarrow \\ &B(q_1 - q_0) + B(q_0 - q_4) + B(q_4 - q_3) + B(q_3 - q_2) + B(q_2 - q_1). \end{aligned} \tag{3.2}$$

Labeling as h_1 the operation of simultaneously mapping $q_0 \leftrightarrow q_0$, $q_1 \leftrightarrow q_4$, and $q_2 \leftrightarrow q_3$ then the group of invariances on aggregate adjustment costs must be extended. The extension, dihedral group $\tilde{D}_5$, has 10 elements and may be represented as Cayley table 2 in appendix A.

Operation h_1 is a second generator, generating four new operations to ensure closure as a group including the elements of C_5. The additional group elements are $h_2 = g_1 * h_1$, $h_3 = g_2 * h_1$, $h_4 = g_3 * h_1$, and $h_5 = g_4 * h_1$. More generally, $\tilde{D}_N$ may be obtained from $\tilde{C}_N$ by including the additional invariances on some $F(x): \mathbb{R}^N \rightarrow \mathbb{R}$ that are generated by a permutation $x_i \leftrightarrow x_j$ $\forall$ $i, j \in \Omega_N$ such that $i + j = 0$, modulo *N*.

A comparison of tables 1 and 2 suggests a property that is more generally true. The upper left-hand quarter of table 2 is an exact replica of table 1. Defining a subgroup of $\tilde{G}$ as a subset of *G* that is a group in its own right, it is seen that $\tilde{C}_5$ is a subgroup of $\tilde{D}_5$ and the relation is written as $\tilde{C}_5 \subseteq \tilde{D}_5$. It is always true that $\tilde{C}_N \subseteq \tilde{D}_N$. Furthermore, since $\tilde{C}_N$ and $\tilde{D}_N$ are restrictions on the set of all permutation operations on *N* objects, $\tilde{S}_N$, inclusion extends to $\tilde{C}_N \subseteq \tilde{D}_N \subseteq \tilde{S}_N$. If $F(x)$ is invariant when $\tilde{G}$ acts on its arguments and if $\tilde{H} \subseteq \tilde{G}$ then $F(x)$ is invariant when $\tilde{H}$ acts on its arguments.

One further concept is needed, that of a group acting on a vector. For $x \in \mathbb{R}^N$, write x_g as the map of x under $g \in G$. For instance, g_1 has been previously labeled as the cyclic map $x_0 \to x_1 \to x_2 \to x_3 \to x_4 \to x_0$. Then $x_{g_1} = (x_4, x_0, x_1, x_2, x_3)^T$ where superscripted T represents the transpose operation. When a description of how g_1 acts on the subscripts is needed then write $x_{g_1(0)} = x_4$, meaning that x_0 is *replaced* by x_4. Given the symmetries in (2.2), i.e., $B(\cdot)$ is "$\tilde{C}_N$-symmetric" or "$\tilde{C}_N$-invariant" while it is $\tilde{D}_N$-invariant under $B(\cdot) \in CS$, we are in a position to analyze market equilibrium.

4. EQUILIBRIUM CHOICES

Expression $H(q) = \sum_{n=0}^{N-1} Y(q_n) + \sum_{n=0}^{N-1} \int_0^{q_n} A(q)dq + \sum_{n=0}^{N-1} B(q_n - q_{n-1})$ is $\tilde{C}_N$-invariant. Due to the absence of externalities, $W(q^*)$ cannot be improved upon. Given the $\tilde{C}_N$ invariances in $H(q)$, we have $H(q^*) = H(q_g^*)\ \forall g \in C_N$. Similarly, if $B(\cdot) \in CS$ then $H(q^*) = H(q_g^*)\ \forall g \in D_N$. From $H(q^*) = H(q_g^*)$, inequality $W(q^*) \geq W(q_g^*)$ implies

PROPOSITION 1: *For market surplus as laid out in (2.2), then*

(4.1) $\sum_{n=0}^{N-1} \gamma_n q_n^* \geq \sum_{n=0}^{N-1} \gamma_n q_{g(n)}^*\ \forall g \in C_N$.

If, in addition, $B(\cdot) \in CS$ *then*

(4.2) $\sum_{n=0}^{N-1} \gamma_n q_n^* \geq \sum_{n=0}^{N-1} \gamma_n q_{g(n)}^*\ \forall g \in D_N$.

Two especially simple cases arise. When the group is $\tilde{C}_2$ in the two-period setting then (4.1) may be written as $(\gamma_1 - \gamma_0)(q_1^* - q_0^*) \geq 0$.[5] The second simple case is when the group is $\tilde{D}_3$

[5] In a 2-period model, adjustment costs cannot be distinguished from costs of varying production.

and (4.2) provides six inequalities under $B(\cdot)\in CS$, each corresponding to a permutation of (q_0,q_1,q_2). To see this, consider cumulated adjustment costs $B(q_1-q_0)+B(q_2-q_1)+B(q_0-q_1)$ and permute any pair of production choices, say $q_1\leftrightarrow q_2$, $B(q_1-q_0)+B(q_2-q_1)+B(q_0-q_2)\equiv$ $B(q_0-q_2)+B(q_2-q_1)+B(q_1-q_0)$. Regardless of which of the six permutations on (q_0,q_1,q_2) is enacted, the sum of adjustment costs is invariant because $\tilde{D}_3$ and $\tilde{S}_3$ are the same. So if (q_0^*,q_1^*,q_2^*) is optimum then $(q_i^*-q_j^*)(\gamma_i-\gamma_j)\ge 0\ \forall i,j\in\Omega_3$. Otherwise, some rearrangement of (q_0^*,q_1^*,q_2^*) would increase the value of $W(q)$ so that the central planner would not maximize social surplus.

When $N\ge 4$ or under $\tilde{C}_3$ then the constraints as laid out in proposition 1 cannot be interpreted quite as cleanly. A graph for the case of $\tilde{C}_3$ does demonstrate much of what the constraint sets can and cannot relate.

EXAMPLE 1: For $(\gamma_0,\gamma_1,\gamma_2)$, write period outputs as share deviations $\mu_n^*=[q_n^*/(\sum_{n=0}^{2}q_n^*)$ $-1/3]$ so as to develop expressions that capture correlation attributes. Similarly, write λ_n = $3\gamma_n/(\sum_{n=0}^{2}\gamma_n)-1$ so that $\sum_{n=0}^{2}\mu_n^*=0=\sum_{n=0}^{2}\lambda_n$ and (4.1) becomes

(4.3) $\lambda_0\mu_0^*+\lambda_1\mu_1^*+\lambda_0\mu_1^*\ge 0, \qquad \lambda_0\mu_0^*+\lambda_1\mu_1^*+\lambda_1\mu_0^*\ge 0.$

Both relations, when specified with equality, pass through the origin in (μ_0^*,μ_1^*) space. If $\lambda_1\le 0$ and $\lambda_0\le 0$ then both relations, when specified with equality, have negative slopes. An example of the set of admissible points, $\mu_0^*\ge -1/3$ and $\mu_1^*\ge -1/3$ such that (4.3) holds, is described as the shaded quadrilateral (including the contained blackened triangle) in figure 1.[6]

Included also in the figure is the absolute bound under $\mu_2^*=-1/3$, the lower limit on μ_2^*.

Hennessy and Roosen (2003) study symmetries of choices in a 2-period seasonal cost model.
[6] Figure 1 is a temporal production analog to figure 1 in the Hennessy and Lapan (2003b) analysis of portfolio decisions.

This bound is represented by $\mu_0^* + \mu_1^* = 1/3$, and the shaded quadrilateral is strictly interior to the hyperplane the bound defines. Notice that the production shares over the cycle are broadly consistent with the relative preference intensity patterns $\lambda_1 \le 0$, $\lambda_0 \le 0$. Notice too that the diagram assumes $0 > \lambda_1 > \lambda_0$ because $(\lambda_0 + \lambda_1)/(3\lambda_1) > 2/3$. If $B(\cdot) \in CS$, so that the dihedral group applies, then the area can be further refined. To be specific, the diagram already assumes $\lambda_2 > 0 > \lambda_1 > \lambda_0$ and straightforward calculations confirm the admissible area under $\tilde{D}_3$ to be the blackened subset.

The invariances also have consequences for equilibrium prices over the cycle. Write $p_n(q_n^*)$ as the n+1st period equilibrium price under price-taking. Then optimize over the social planner's objective function (2.2) to obtain

(4.4) $$p_n(q_n^*) - \beta_n - Y_q(q_n^*) - B_q(q_n^* - q_{n-1}^*) + B_q(q_{n+1}^* - q_n^*) = 0 \quad \forall n \in \Omega_N,$$

where the subscripted q indicates a differentiation and the modulo arithmetic wrap-around at $n = 0$ and $n = N-1$ is understood.[7] Analysis, provided in appendix B, supports

PROPOSITION 2: *For market surplus as laid out in (2.2), then*

(4.5) $$\sum_{n=0}^{N-1}\left[p_n(q_n^*) - \beta_n\right]q_n^* \ge \sum_{n=0}^{N-1}\left[p_n(q_n^*) - \beta_n\right]q_{g(n)}^* \ \forall g \in C_N.$$

If, in addition, $B(\cdot) \in CS$ *then*

(4.6) $$\sum_{n=0}^{N-1}\left[p_n(q_n^*) - \beta_n\right]q_n^* \ge \sum_{n=0}^{N-1}\left[p_n(q_n^*) - \beta_n\right]q_{g(n)}^* \ \forall g \in D_N.$$

Neither part of the proposition asserts that high price months, net of period-specific unit costs, are matched with high output months. Both parts do suggest a more positive alignment

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[7] For a price-taking firm, the solution to (4.4) is not unique in the special case where $Y(q_n)$ is linear. In that case, if q^* solves (4.4) then so does $q^* + \theta \times (1, 1, \ldots, 1)^T$. Decreasing demand closes the model.

relative to other group maps on q^*, i.e., relative to q_g^*. Together, propositions 1 and 2 suggest a relationship between net prices $p_n(q_n^*) - \beta_n$ and net demand parameters γ_n. This is because net demand parameters tend to align with outputs and outputs tend to align with net prices. The relationship does indeed follow for certain small order groups.

COROLLARY 2.1: A) *When* $N = 2$ *then* $[p_1(q_1^*) - \beta_1 - p_0(q_0^*) + \beta_0](q_1^* - q_0^*) \geq 0$. *Furthermore,* $[p_1(q_1^*) - \beta_1 - p_0(q_0^*) + \beta_0](\gamma_1 - \gamma_0) \geq 0$.

B) *When* $N = 3$ *and* $B(\cdot) \in CS$, *arbitrarily assume* $\gamma_2 \geq \gamma_1 \geq \gamma_0$. *Then* $q_2^* \geq q_1^* \geq q_0^*$ *and furthermore* $p_2(q_2^*) - \beta_2 \geq p_1(q_1^*) - \beta_1 \geq p_0(q_0^*) - \beta_0$.

Upon identifying an invariance that has not been discussed thus far, a more general correlation result can be obtained.

PROPOSITION 3: A) *Average period unit price equals average period marginal cost, i.e.,*

$$N^{-1}\sum_{n=0}^{N-1} p_n(q_n^*) = N^{-1}\sum_{n=0}^{N-1} \beta_n + N^{-1}\sum_{n=0}^{N-1} Y_q(q_n^*). \tag{4.7}$$

In addition, if $p_n(q_n) = \alpha_n - \kappa q_n, \kappa > 0$ *while* $Y(q_n) = 0.5\chi \times (q_n)^2, \chi \geq 0$ *then aggregate output over the cycle is* $\sum_{n=0}^{N-1} q_n^* = (\kappa + \chi)^{-1} \sum_{n=0}^{N-1} \gamma_n$.

B) *Output is positively correlated with prices net of period cost parameters, i.e.,* $\sum_{n=0}^{N-1} [p_n(q_n^*) - \beta_n] q_n^* \geq \bar{q}^* \sum_{n=0}^{N-1} [p_n(q_n^*) - \beta_n)]$ *where* $\bar{q}^* = N^{-1}\sum_{n=0}^{N-1} q_n^*$. *Output is positively correlated with net demand parameters, i.e.,* $\sum_{n=0}^{N-1} \gamma_n q_n^* \geq \bar{q}^* \sum_{n=0}^{N-1} \gamma_n$.

Equation (4.7) asserts that equilibrium marginal cost pricing applies when the whole cycle is considered. Part A) also provides conditions such that mean output over the cycle, $\bar{q}^*$, must increase in response to an increase in any γ_n. Under these linearity conditions, statistic $\sum_{n=0}^{N-1} \gamma_n$

is sufficient to determine mean output.[8] Part B) describes a positive correlation concerning how production adjustments occur to take advantage of external circumstances, and so there should be surplus remaining to more than cover adjustment costs. The existence of adjustment costs does not affect the unweighted mean of prices over the cycle, only the extent of price dispersion.

A second consequence that (4.4) brings to light is the relationship between equilibrium prices and the rate of adjustment in production. Suppose $p_n(q_n^*) \geq (\leq) \beta_n + Y_q(q_n^*)$ so that $B_q(q_n^* - q_{n-1}^*) \geq (\leq) B_q(q_{n+1}^* - q_n^*)$. Given our assumption of convexity on $B(\cdot)$, $B_q(q_n^* - q_{n-1}^*) \geq$ $(\leq) B_q(q_{n+1}^* - q_n^*)$ implies $q_n^* - q_{n-1}^* \geq (\leq) q_{n+1}^* - q_n^*$. Now suppose there is a sequence of periods such that $p_1^* \geq \beta_1 + Y_q(q_1^*), p_2^* \geq \beta_2 + Y_q(q_2^*), \ldots, p_m^* \geq \beta_m + Y_q(q_m^*)$, where the choice of $n = 1$ as the initiation is without loss of generality. Write $T(q_n) = A(q_n) + Y_q(q_n)$, a strictly increasing function with inverse function $T^{-1}(\cdot)$. The sequence of inequalities may then be read as $q_1^* \leq T^{-1}(\gamma_1), \ldots, q_m^* \leq T^{-1}(\gamma_m)$, and it implies $q_1^* \geq (q_2^* + q_0^*)/2, \ldots, q_m^* \geq (q_{m+1}^* + q_{m-1}^*)/2$. Such a sequence is said to be concave because it possesses as a sequence the properties that a concave function possesses. The sequence with inequality signs reversed is said to be convex, see Pečarić, Proschan, and Tong (1992).

PROPOSITION 4: *Suppose there is a sequence of periods* $\{1, 2, \ldots, m\}$ *such that optimal output is less than output would be if there were no adjustment costs, i.e.,* $q_1^* \leq T^{-1}(\gamma_1), \ldots, q_m^* \leq T^{-1}(\gamma_m)$. *Then* $q_1^* \geq (q_2^* + q_0^*)/2, \ldots, q_m^* \geq (q_{m+1}^* + q_{m-1}^*)/2$. *Suppose instead that optimal output is greater than output would be if there were no adjustment costs over that sequence, i.e.,* $q_1^* \geq T^{-1}(\gamma_1), \ldots,$ $q_m^* \geq T^{-1}(\gamma_m)$. *Then* $q_1^* \leq (q_2^* + q_0^*)/2, \ldots, q_m^* \leq (q_{m+1}^* + q_{m-1}^*)/2$.

The finding points to solution harmonics, and further intuition in this regard will be provided

[8] An analog exists with Cournot oligopoly in a homogeneous good industry; see, e.g., Salant and

when we linearize the model.

5. SURPLUSES

Specify social surplus as

(5.1) $\mathfrak{L}(\gamma) = \max_q W(q)$.

This function is quite structured, as the following definition will help to illustrate.

DEFINITION 1: Vector $u \in \mathbb{R}^N$ is *cyclic majorized* by $v \in \mathbb{R}^N$ with respect to group $\tilde{C}_N$ (written as $u \leq_{\tilde{C}_N} v$) if u lies in the convex hull of the points $v_g, g \in C_N$. If the group is instead $\tilde{D}_N$ then u is said to be *dihedral majorized* by v with respect to $\tilde{D}_N$, written as $u \leq_{\tilde{D}_N} v$. If the group is instead $\tilde{S}_N$, then u is said to be *majorized* by v, written as $u \leq_{\tilde{S}_N} v$.[9]

A simple illustration of $u \leq_{\tilde{S}_N} v$ should facilitate an understanding of the economics underlying the relation. In $\mathbb{R}^2$, let $u_1 + u_2 = v_1 + v_2$ and $\min[u_1, u_2] \geq \min[v_1, v_2]$. Then $u \leq_{\tilde{S}_2} v$. Because the sums of vector components have been constrained to be equal, the relation concerns dispersion among vector components. Cyclic majorization is less direct, and is best described using matrices. If $u \leq_{\tilde{C}_N} v$ then there exists a vector $\lambda \in \overline{\mathbb{R}}^N_+$ such that $\sum_{n=0}^{N-1} \lambda_n = 1$ and[10]

$$(5.2)\quad \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} \lambda_0 & \lambda_1 & \vdots & \lambda_{N-1} \\ \lambda_{N-1} & \lambda_0 & \vdots & \lambda_{N-2} \\ \cdots & \cdots & \ddots & \cdots \\ \lambda_1 & \lambda_2 & \vdots & \lambda_0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{pmatrix}.$$

A square matrix with rows generated by cyclic action such that the diagonal elements are

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Shaffer (1999). There the sum of marginal cost parameters determines industry output.

[9] Majorization as it is usually understood, $\leq_{\tilde{S}_N}$, has been used widely in the welfare economics of income inequality. See Marshall and Olkin (1979) for more on linear transformation group majorization, which is sufficiently general to include all three of the defined relations.

the same, as in (5.2), is referred to as a *circulant*. A circulant is generated by the transpose of the first row as a *reference vector*, $(\lambda_0, \lambda_1, \ldots, \lambda_{N-1})^T$ in (5.2). Condition $u \leq_{\tilde{C}_N} v$ is more restrictive than $u \leq_{\tilde{S}_N} v$, which may be written as the requirement that there exists an $N \times N$ matrix with non-negative entries $\lambda_{i,j}$, $\sum_{i=0}^{N-1} \lambda_{i,j} = 1 \; \forall j \in \Omega_N, \sum_{j=0}^{N-1} \lambda_{i,j} = 1 \; \forall i \in \Omega_N$, such that[11]

$$(5.3) \quad \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} \lambda_{0,0} & \lambda_{0,1} & \vdots & \lambda_{0,N-1} \\ \lambda_{1,0} & \lambda_{1,1} & \vdots & \lambda_{1,N-1} \\ \cdots & \cdots & \ddots & \cdots \\ \lambda_{N-1,0} & \lambda_{N-1,1} & \vdots & \lambda_{N-1,N-1} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{pmatrix}.$$

Intuitively, more dispersion in γ should increase welfare in our model because welfare surplus triangles are convex under location shifts of the defining curves. How the α values are matched with the β values should matter too, but this will not be an issue when only the net parameter values, $\gamma_n = \alpha_n - \beta_n$, are considered. Also, how contiguous γ_n parameter values relate in the cycle should matter because sudden reversals in preference and cost parameters will require large adjustments. The $\leq_{\tilde{C}_N}$ and $\leq_{\tilde{D}_N}$ pre-orderings are tailored to accommodate this concern about non-smooth cycles.

PROPOSITION 5: A) $\mathfrak{L}(\gamma') \leq \mathfrak{L}(\gamma'')$ *whenever* $\gamma' \leq_{\tilde{C}_N} \gamma''$.

B) *If* $B(\cdot) \in CS$, *then* $\mathfrak{L}(\gamma') \leq \mathfrak{L}(\gamma'')$ *whenever* $\gamma' \leq_{\tilde{D}_N} \gamma''$.

EXAMPLE 2: In contrast to two and three period contexts, four period contexts do not generally lend themselves to graphical analysis. Nonetheless it is possible to graph the dominated convex hull when the reference vector is amenable. Let the parameter vector, when written in deviation form as was done in Example 1, be chosen as realizations of the cosine

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[10] See Giovagnoli and Wynn (1996, p. 219).
[11] All three relations in definition 1 have the property of bistochasticity. This means that the matrix representation of each convex hull relation has all rows and all columns summing to zero

function at $\pi / 2$ radian intervals on the real domain with initial observation at $-\pi / 2$. Thus, $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (0,1,0,-1)$. With simplex weightings $\omega = (\omega_0, \omega_1, \omega_2, \omega_3) \geq 0$, $\sum_{k=0}^{3} \omega_k = 1$, the $\tilde{C}_4$ group acts on the vector to generate convex hull $(\omega_3 - \omega_1, \omega_0 - \omega_2, \omega_1 - \omega_3, \omega_2 - \omega_0)$. Given that the first and third coordinates differ only in sign while the same is true of the second and fourth coordinates, this convex hull can be compressed into two dimensions. These will have to be either the first and second coordinates, the first and fourth coordinates, the second and third coordinates, or the third and fourth coordinates. In any of these cases the convex hull will be the shaded, oriented square given in figure 2, where the axis labels used are λ_0 and λ_1. Formally, the set may be represented as $\{(\lambda_0', \lambda_1', \lambda_2', \lambda_3') = (\lambda_0', \lambda_1', -\lambda_0', -\lambda_1') : |\lambda_0'| + |\lambda_1'| \leq 1\}$.

Proposition 5 makes the important distinction between the roles of dispersion and smoothing on the production side. Firms should like more dispersion in unit period cost parameters because gains from expanding production in lower cost periods should more than offset losses on contracted production in higher cost periods. But that dispersion cannot be arbitrary, for surely in a five period cycle the sequence $\gamma_0' \leq \gamma_1' \leq \gamma_2' \leq \gamma_3' \geq \gamma_4' \geq \gamma_0'$ should be preferred over $\gamma_0'' \leq \gamma_1'' \geq \gamma_2'' \leq \gamma_3'' \geq \gamma_4'' \leq \gamma_0''$ when the sequence $\gamma'' \in \overline{\mathbb{R}}^5$ is a re-arrangement of γ'. Sequence γ'' is too disrupted to allow smooth adjustments, and adjustment costs will reflect that. Parts A) and B) control for this, so that convexity is availed of and yet adjustment costs are modest.

6. LINEAR MODEL

A linear system allows for a precise characterization of production dynamics. With linear demand, period cost $\beta_n q_n + 0.5\chi \times (q_n)^2$, and adjustment cost $0.5\tau \times (q_n - q_{n-1})^2, \tau \in \overline{\mathbb{R}}_+$, then equilibrium output is given as:

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while all matrix entries are non-negative. See Marshall and Olkin (1979, pp. 18-23).

PROPOSITION 6: *In the linear system with N periods, define* $\vartheta_n = \gamma_n /(\kappa + \chi)$, $\xi = \tau /(\kappa + \chi)$, $\rho = (2\xi + 1 + \Delta)/(2\xi)$, *and* $\Delta = \sqrt{1 + 4\xi}$. *Then output choices may be written in the form*

$$(6.1)\quad q_n^* = \sum\nolimits_{m=0}^{N-1} u_{m-n} \vartheta_m, \qquad u_j = \frac{\rho^j + \rho^{N-j}}{\Delta \times (\rho^N - 1)}, \qquad \sum\nolimits_{n=0}^{N-1} u_n = 1, \qquad u_n \geq 0 \; \forall n \in \Omega_N.$$

The equilibrium effects of parameter dispersion on consumer and producer surplus are not as readily obtained as for the welfare effect in proposition 5 because the optimality of markets cannot be appealed to. Knowledge from proposition 6 is useful in this regard. Write producer surplus as $\Pi(\gamma)$ and consumer surplus as $\mathbb{C}(\gamma)$ when viewed as functions of the primitives.

PROPOSITION 7: *In the linear system,*

A) $\Pi(\gamma') \leq \Pi(\gamma'')$ *whenever* $\gamma' \leq_{\tilde{D}_N} \gamma''$, *and*

B) $\mathbb{C}(\gamma') \leq \mathbb{C}(\gamma'')$ *whenever* $\gamma' \leq_{\tilde{D}_N} \gamma''$. *Furthermore, consumption is more seasonal under* γ'' *in the sense that the value of* $\sum\nolimits_{n=0}^{N-1} (s_n^*)^2$ *increases under* $\gamma' \to \gamma''$, *where* $s_m^* = q_m^* / \sum\nolimits_{n=0}^{N-1} q_n^*$.

Proposition 6 conveys much more about the structure of vector $u = (u_0, u_1, \dots, u_{N-1})^T$ besides its location on the unit simplex.

PROPOSITION 8: A) *The circulant with reference vector* $(u_0, u_1, \dots, u_{N-1})^T$ *is bistochastic, i.e., all entries are non-negative, all rows and columns sum to 1.*

B) *The weightings are centrally symmetric, i.e.,* $u_j = u_{N-j} \; \forall n \in \Omega_N$.

C) $\text{Lim}_{\xi \to 0} u_j = 0 \; \forall j \in \{1, 2, \dots, N-1\}$, $\text{Lim}_{\xi \to 0} u_0 = 1$, *and* $\text{Lim}_{\xi \to \infty} u_j = N^{-1} \; \forall j \in \Omega_N$.

D) *If* $(N-1)/2 > n$ *then* $u_n \geq u_{n+1}$ *and* u_{n+1}/u_n *is increasing in* ξ. *If* $(N-1)/2 < n$ *then* $u_n \geq u_{n-1}$ *and* u_n / u_{n-1} *is decreasing in* ξ.

E) *For the given sum value* $\sum\nolimits_{n=0}^{N-1} \vartheta'_n = \sum\nolimits_{n=0}^{N-1} \vartheta''_n$, *suppose* $\sum\nolimits_{n=0}^{m} \vartheta'_n + \sum\nolimits_{n=N-1-m}^{N-1} \vartheta'_n \geq \sum\nolimits_{n=0}^{m} \vartheta''_n +$

$\sum_{n=N-1-m}^{N-1} \vartheta''_n \; \forall m \in \{0,1, \dots ,(N-3)/2\}$ *whenever* $(N-1)/2$ *is even and* $\forall m \in$ $\{0,1, \dots ,(N-2)/2\}$ *whenever* $(N-1)/2$ *is odd. Then* $q'_0 \geq q''_0$.

F) $dq^*_m / d\vartheta_{m-n} = dq^*_m / d\vartheta_{m+n} \; \forall n,m \in \Omega_N$ *where modular arithmetic on indices is understood.*

The proposition's findings may be depicted as a figure rather similar to a string between parallel walls, see figure 3 where we have assumed a continuous weighting function on $[0, N)$ for convenience. A cyclical sequence is said to be unimodal if it has a single local peak over its period. Due to parts B) and D), the weightings are inverted symmetric unimodal, i.e., lowest value is at the middle. When adjustment costs are small then the preponderant weight is on the own-parameter, the string is quite lax. As adjustment costs increase then the weighting shifts toward other parameters. At the limit of infinite adjustment costs, the weighting is uniform, $u_n = N^{-1} \; \forall n \in \Omega_N$. Further analysis of these observations supports

PROPOSITION 9: *If* $\vartheta = (\vartheta_0, \vartheta_1, \dots , \vartheta_{N-1})^T$ *is centrally symmetric unimodal, then* q^* *is centrally symmetric unimodal and peak output occurs at the time of the largest coordinate in* ϑ.

The central symmetry and unimodality properties are inherited by q^* through convolution $q^*_n = \sum_{m=0}^{N-1} u_{m-n} \vartheta_m$. It is readily shown, though, that neither unimodality only on ϑ nor central symmetry only on ϑ are sufficient for q^* to be either unimodal or centrally symmetric. Figure 4 depicts the proposition's content. Consistent with proposition 8, vector u (given as a continuous approximation) is unimodal and centrally symmetric around $n = 0$. One example of the weighting vector is the sequence with entries N^{-1} that arises at the limit as $\xi \to \infty$, see part C). More typically, the curve will look like the other curve in figure 4.

Suppose market parameter vector $\vartheta = (\vartheta_0, \vartheta_1, \dots , \vartheta_{N-1})^T$ happens to be centrally symmetric and unimodal around $n = 0$. In that case it can be constructed from positive weightings of discrete uniform weightings centered around $n = 0$. The blackened block is one such weighting (as a continuous approximation). Alternatively, the market parameter vector could be unimodal and centrally symmetric around $n = N/2$ rather than $n = 0$. This situation is represented as the pair of patched blocks on either side of $n = N/2$, and there are two blocks just because the circle representing the cycle was cut at $n = N/2$ in order to present it in two dimensions. In the linear model, both market parameter vectors give the same aggregate output, and in each case the output vectors are both symmetric and unimodal. The only difference is that the coordinate values in the period output vector are shifted 180 degrees around the cycle.

If the patched block is moved in either direction toward the center then the value of q_0 will be monotone non-decreasing along the trajectory until the patched block rests on the blackened block. The idea holds true more generally for the symmetric unimodal weightings because these can be constructed as limits from positive sums of such uniform blocks, see Dharmadhikari and Joag-Dev (1988, p. 10). In fact the idea behind the proposition is known in the statistics literature as the moving set inequality, as reported in Dharmadhikari and Joag-Dev (1988).

Proposition 9 suggests that the output vector inherits some structure from the exogenous parameter vector. The inheritance relationship is very clear when the exogenous parameters follow the cosine function. Specifically, let $\vartheta_m = \mathrm{Cos}(2\pi m / N)$ so that ϑ is unimodal (with mode at $m = 0$) and centrally symmetric.[12] To see that the cosine function (as with the sine function) places pertinent structure on the exogenous parameters, pair n with $n + N/2$. Then notice that $\varsigma \times \mathrm{Cos}(2\pi n / N) + (1-\varsigma) \times \mathrm{Cos}[2\pi(n + N/2)/N] \equiv (2\varsigma - 1) \times \mathrm{Cos}(2\pi n / N) \in [\mathrm{Cos}(2\pi n / N), \mathrm{Cos}[2\pi(n + N/2)/N]]\ \forall \varsigma \in [0,1]$. The convex combination represents a dihedral

[12] A constant could be added to the cosine function without affecting the analysis.

dominance relation and that relation is synonymous with a reduction in amplitude. In our case the reduction in amplitude is from 1 to $|2\varsigma-1|\le 1$.

PROPOSITION 10: A) *In the linear model, let* $\vartheta_m = \mathrm{Cos}(2\pi m/N)\ \forall m\in\Omega_N$. *Write peak output* $q_0^{*,N}$ *as the period 0 output when there are* N *outputs. Then*

$$
(6.2)\qquad
\begin{aligned}
&(6.2\mathrm{A})\quad q_n^{*,N}-\overline{q}^{*}=(q_0^{*,N}-\overline{q}^{*})\mathrm{Cos}(2\pi n/N)=(q_0^{*,N}-\overline{q}^{*})\vartheta_n,\\
&(6.2\mathrm{B})\quad q_0^{*,N}-\overline{q}^{*}=\frac{(\rho-1)(\rho+1)}{\Delta\times[\rho^2-2\rho\times\mathrm{Cos}(2\pi/N)+1]}>0,
\end{aligned}
$$

so that the periods of vectors q^{*} *and* ϑ *are the same. The peak and trough of the vectors occur at the same time points while* $\mathrm{Var}(q^{*,N})=(q_0^{*,N}-\overline{q}^{*})^2\mathrm{Var}(\vartheta)$.

B) *Peak output* $q_0^{*,N}$ *is decreasing in adjustment cost parameter* τ.

C) $q_0^{*,N+1}>q_0^{*,N}$ *and* $\mathrm{Lim}_{N\to\infty}q_0^{*,N}=\overline{q}^{*}+(\rho+1)/[\Delta\times(\rho-1)]$.

D) *The ratio of adjustment costs to output variance is given as* $\tau\sum_{n=0}^{N-1}(q_n^{*,N}-q_{n-1}^{*,N})^2/\mathrm{Var}(q^{*,N})=$ $4\tau N\times\mathrm{Sin}^2(\pi/N)$. *This is the least value the ratio can take.*

Part D) asserts that under cosine structured exogenous parameters the tradeoff between output variability to increase consumer and producer surplus and the adjustment costs of varying output is as low as it could possibly be. When exogenous parameters are of form $\vartheta_n =$ $\mathrm{Cos}(2\pi n/N)$, then the gains from convexity in objective functions are bought as cheaply as adjustment costs could possibly allow. Parts A) and B) provides some insights on Proposition 4. Production is less (more) than it would be absent adjustment costs when the production sequence is concave (convex). For the cosine function, the sequence is concave on interval $[-\pi/2,\pi/2)$

and convex on $[\pi/2, 3\pi/2)$. The production schedule is an amplitude modified version of the exogenous parameter vector.

Part C) confirms a point suggested in part B); that peak and trough outputs become less constrained by adjustment costs as the number of periods increases. This is because the adjustments in exogenous parameters become smaller from period to period as the number of periods increases. When the partition of evaluations on the cosine function becomes sufficiently small, adjustment costs become infinitesimally small and do not affect production choices.

Given that an increase in adjustment costs tends to flatten out the weighting vector, by part C) of proposition 8, it is natural to ask what impacts adjustment costs have on output variability. In order to address this question, and returning to the more general case where the ϑ_n do not necessarily follow a trigonometric function, we will first develop a characterization of how an increase in adjustment costs smooths the ξ-dependent weighting vector.

PROPOSITION 11: *For each* $\xi_1 > 0$ *there exists an* ξ_1*-dependent finite real number* $\hat{\xi}_2(\xi_1)$ *such that* $u(\xi_2) \leq_{\tilde{C}_N} u(\xi_1)$ *for all* $\xi_2 \geq \hat{\xi}_2(\xi_1)$.

While we have not shown that $u(\xi_2) \leq_{\tilde{C}_N} u(\xi_1)\ \forall \xi_2 \geq \xi_1$, the smoothing partial order applies whenever ξ_2 is sufficiently large. For $\bar{q}^*(\xi) = N^{-1}\sum_{n=0}^{N-1} q_n^*(\xi)$, $\bar{\vartheta} = N^{-1}\sum_{n=0}^{N-1}\vartheta_n$ and $U(\xi)$ as the ξ-conditioned circulant with reference vector $(u_0(\xi), u_1(\xi), \ldots, u_{N-1}(\xi))^T$, (6.1) and part B) of proposition 8 provide

(6.3) $$\mathrm{Var}[q^*(\xi)] = N^{-1}\sum_{n=0}^{N-1}[q_n^*(\xi) - \bar{q}^*(\xi)]^2 = N^{-1}(\vartheta - \bar{\vartheta})^T [U(\xi)]^2 (\vartheta - \bar{\vartheta}).$$

Proposition 11 may be applied to conclude:

PROPOSITION 12: $\mathrm{Var}[q^*(\xi_1)] \geq \mathrm{Var}[q^*(\xi_2)]\ \forall \xi_2 \geq \hat{\xi}_2(\xi_1)$, $\hat{\xi}_2(\xi_1)$ *defined in Proposition 11.*

While an increase in the adjustment cost parameter does not affect aggregate output over the cycle in the linear model, we have found conditions under which it reduces output variance. Linearization has simplified the system in a mathematical way. It will be shown that external trade opportunities provide an economic simplification.

7. EXTERNAL TRADE

In the presence of an external, i.e., world, market then it is not necessary for production and consumption to match period-by-period even when product is non-storable. All that matters is that the external market can be accessed to make up the difference. For simplicity it is assumed that the period varying world price, p_n^w, is exogenous. Write q_n and x_n as the n+1st period production and consumption levels, respectively. With linear-in-money preferences, then market welfare becomes

$$\text{(7.1)}\quad \begin{aligned} &W(q,x)=\sum_{n=0}^{N-1}(\alpha_n-p_n^w)x_n-\sum_{n=0}^{N-1}\int_0^{x_n}A(x)dx \\ &-\sum_{n=0}^{N-1}(\beta_n-p_n^w)q_n-\sum_{n=0}^{N-1}Y(q_n)-\sum_{n=0}^{N-1}B(q_n-q_{n-1}). \end{aligned}$$

Conditional on the p_n^w values, external trade separates consumption from production. The optimality conditions reflect this separation,

$$\text{(7.2)}\quad \begin{aligned} &\text{(7.2A)}\quad p_n^w=\alpha_n-A(x_n)\quad \forall n\in\Omega_N; \\ &\text{(7.2B)}\quad p_n^w=\beta_n+Y_q(q_n)+B_q(q_n-q_{n-1})-B_q(q_{n+1}-q_n)\quad \forall n\in\Omega_N; \end{aligned}$$

with solution vectors x^*,q^*. When looking at welfare effects it is convenient to note that the optimization problem presented in (7.1) may be separated into consumption and production problems. With $\hat{\alpha}=\alpha-p^w$ and $\hat{\beta}=\beta-p^w$, consumer and producer surpluses are respectively

$$\text{(7.3)}\quad \begin{aligned} &\text{(7.3A)}\quad \mathfrak{L}^{\mathbb{C}}(\hat{\alpha})=\max_{\{x_0,x_1,\dots,x_{n-1}\}}\sum_{n=0}^{N-1}\hat{\alpha}_n x_n-\sum_{n=0}^{N-1}\int_0^{x_n}A(x)dx, \\ &\text{(7.3B)}\quad \mathfrak{L}^{\Pi}(\hat{\beta})=\min_{\{q_0,q_1,\dots,q_{n-1}\}}\sum_{n=0}^{N-1}\hat{\beta}_n q_n+\sum_{n=0}^{N-1}Y(q_n)+\sum_{n=0}^{N-1}B(q_n-q_{n-1}). \end{aligned}$$

It is straightforward to show the following regularities for equilibrium decisions:

PROPOSITION 13: A) *Under external trade, both of the following are true*

(7.4) $\sum_{n=0}^{N-1} \hat{\alpha}_n x_n^* \geq \sum_{n=0}^{N-1} \hat{\alpha}_n x_{g(n)}^* \; \forall g \in S_N$, $\quad \sum_{n=0}^{N-1} \hat{\beta}_n q_n^* \leq \sum_{n=0}^{N-1} \hat{\beta}_n q_{g(n)}^* \; \forall g \in C_N$,

If, in addition, $B(\cdot) \in CS$ *then*

(7.5) $\sum_{n=0}^{N-1} \hat{\beta}_n q_n^* \leq \sum_{n=0}^{N-1} \hat{\beta}_n q_{g(n)}^* \; \forall g \in D_N$.

B) *Consumption and demand parameters, net of period world prices, are matched, i.e.,*

$(\hat{\alpha}_i - \hat{\alpha}_j)(x_i^* - x_j^*) \geq 0 \; \forall i, j \in \Omega_N$.

C) *Outputs correlate negatively with period unit costs net of period world prices, i.e.,*

$\sum_{n=0}^{N-1} \hat{\beta}_n q_n^* \leq \bar{q}^* \sum_{n=0}^{N-1} \hat{\beta}_n$.

Part A) clarifies that only demand heterogeneities matter in determining the share allocation of cycle consumption over periods, while only supply heterogeneities matter for production share allocations over periods. Part B) writes the consumption implications in A) in a more convenient form. This form does not apply on the production side because $\tilde{C}_N$ and $\tilde{D}_N$ invariances are too restrictive, but part C) presents a statistical interpretation of supply implications.

The advent of an external market cannot reduce welfare. Less obvious is how it affects the determinants of welfare. An important insight for understanding how is to appreciate that $u \leq_{\tilde{S}_N} v$ implies $u \leq_{\tilde{D}_N} v$, which in turn implies $u \leq_{\tilde{C}_N} v$.[13]

PROPOSITION 14: A) $\forall \hat{\beta}, \mathfrak{L}^{\mathbb{C}}(\hat{\alpha}') \leq \mathfrak{L}^{\mathbb{C}}(\hat{\alpha}'')$ whenever $\alpha' \leq_{\tilde{S}_N} \alpha''$.

B) $\forall \hat{\alpha}, \mathfrak{L}^{\Pi}(\hat{\beta}') \leq \mathfrak{L}^{\Pi}(\hat{\beta}'')$ *whenever* $\beta' \leq_{\tilde{C}_N} \beta''$.

C) *If* $B(\cdot) \in CS$ *then* $\forall \hat{\alpha}, \mathfrak{L}^{\Pi}(\hat{\beta}') \leq \mathfrak{L}^{\Pi}(\hat{\beta}'')$ *whenever* $\beta' \leq_{\tilde{D}_N} \beta''$.

Part A) asserts that more dispersion among demand parameters is always beneficial, albeit having adjusted for period world prices. Heretofore how α related to β mattered too, but external trade breaks that link. Furthermore, the sort of dispersion for α in part A) is $\leq_{\tilde{S}_N}$ and not the more restrictive $\leq_{\tilde{C}_N}$ or $\leq_{\tilde{D}_N}$ as in proposition 5. The symmetry generalization does not occur on the production side, see part B) of proposition 14, because adjustment costs remain.

8. DISCUSSION

This paper has provided a deterministic framework for understanding markets exposed to adjustment costs and cyclic heterogeneities. We have found that the number of periods in a cycle matters, and in rather subtle ways. Notwithstanding the existence of adjustment costs, output does correlate positively with strength of demand over the cycle while price and strength of demand also correlate positively. Despite the presence of adjustment costs, we also identify conditions under which more dispersion in market heterogeneities across periods is beneficial for welfare. A linearized model provides insights on symmetries in equilibrium production levels and responses. External trade affects the results, but the model adapts readily. The linear model should allow an extension to endogenize world prices. Concerning the realism of posited invariances, statistical methods exist to test for symmetry structures and it may be possible to adapt these to provide evidence on the plausibility of invariance assumptions.[14] A further challenge would be to account for random demand and supply innovations, so as to better reflect the noisy cycles that firms, markets, and economies actually face.

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[13] This is because $\tilde{C}_N \subseteq \tilde{D}_N \subseteq \tilde{S}_N$.

[14] Serfling (2003) presents an overview of statistical tests for symmetries.

APPENDIX A

TABLE 1

CYCLIC 5 GROUP, $\tilde{C}_5$

*=after[a]	e	g_1	g_2	g_3	g_4
e	e	g_1	g_2	g_3	g_4
g_1	g_1	g_2	g_3	g_4	e
g_2	g_2	g_3	g_4	e	g_1
g_3	g_3	g_4	e	g_1	g_2
g_4	g_4	e	g_1	g_2	g_3

[a] Column operation is after row operation

TABLE 2

DIHEDRAL 5 GROUP, $\tilde{D}_5$

*=after[a]	e	g_1	g_2	g_3	g_4	h_1	h_2	h_3	h_4	h_5
e	e	g_1	g_2	g_3	g_4	h_1	h_2	h_3	h_4	h_5
g_1	g_1	g_2	g_3	g_4	e	h_2	h_3	h_4	h_5	h_1
g_2	g_2	g_3	g_4	e	g_1	h_3	h_4	h_5	h_1	h_2
g_3	g_3	g_4	e	g_1	g_2	h_4	h_5	h_1	h_2	h_3
g_4	g_4	e	g_1	g_2	g_3	h_5	h_1	h_2	h_3	h_4
h_1	h_1	h_5	h_4	h_3	h_2	e	g_4	g_3	g_2	g_1
h_2	h_2	h_1	h_5	h_4	h_3	g_1	e	g_4	g_3	g_2
h_3	h_3	h_2	h_1	h_5	h_4	g_2	g_1	e	g_4	g_3
h_4	h_4	h_3	h_2	h_1	h_5	g_3	g_2	g_1	e	g_4
h_5	h_5	h_4	h_3	h_2	h_1	g_4	g_3	g_2	g_1	e

[a] Column operation is after row operation

APPENDIX B

Proof of Proposition 2: For the cyclic group, the sum of adjustment costs is $\tilde{C}_N$-invariant. It is also convex. A consequence of proposition 2.2, p. 834, in Eaton and Perlman (1977) is that if $f(x): \mathbb{R}^M \to \mathbb{R}$ is $\tilde{G}$-invariant and convex then $\sum_{m=0}^{M-1}(x_m - x_{g(m)})\partial f(x)/\partial x_m \geq 0 \; \forall g \in G$, i.e., $\sum_{m=0}^{M-1}\left[\partial f(x)/\partial x_m - \partial f(x)/\partial x_{g(m)}\right]x_m \geq 0 \; \forall g \in G$. Applying the inequality to aggregate adjustment costs under group $\tilde{C}_N$ provides $\sum_{n=0}^{N-1}\left[B_q(q_n^* - q_{n-1}^*) - B_q(q_{n+1}^* - q_n^*)\right](q_n^* - q_{g(n)}^*) \geq 0$ $\forall g \in C_N$. In addition, $\sum_{n=0}^{N-1} Y_q(q_n^*)(q_n^* - q_{g(n)}^*) \geq 0 \; \forall g \in C_N$ because $\sum_{n=0}^{N-1} Y(q_n)$ is $\tilde{S}_N$-invariant and convex while $\tilde{C}_N \subseteq \tilde{S}_N$. Thus,

(B1) $$\sum_{n=0}^{N-1}\left[Y_q(q_n^*) + B_q(q_n^* - q_{n-1}^*) - B_q(q_{n+1}^* - q_n^*)\right](q_n^* - q_{g(n)}^*) \geq 0 \quad \forall g \in C_N.$$

Substitute in (4.4) to obtain (4.5). Relation (4.6) follows similarly. Q.E.D.

Proof of Proposition 3: For part A), sum optimality conditions (4.4) over all periods to obtain (4.7). The remainder of the part follows immediately. For part B), define $\delta_n \equiv q_n - q_{n-1}$ and $q_N \equiv q_0$ where all subscripts are interpreted mod *N*. Hence $\sum_{n=0}^{N-1}[B_q(\delta_n) - B_q(\delta_{n+1})] = 0$ and

(B2) $$\sum_{n=0}^{N-1}\left[B_q\left(\delta_n^*\right) - B_q\left(\delta_{n+1}^*\right)\right]q_n^* = \sum_{n=0}^{N-1} B_q\left(\delta_n^*\right)q_n^* - \sum_{n=0}^{N-1} B_q\left(\delta_n^*\right)q_{n-1}^* = \sum_{n=0}^{N-1} B_q\left(\delta_n^*\right)\delta_n^*,$$

because $\delta_N = \delta_0$. But $\sum_{n=0}^{N-1} B_q(\delta_n^*)\delta_n^* \geq 0$ as $B(\cdot)$ is convex and $\sum_{n=0}^{N-1}\delta_n = 0$. Continuing,

(B3) $$\begin{aligned}&\sum_{n=0}^{N-1}\left[p_n\left(q_n^*\right) - \beta_n\right]\left(q_n^* - \bar{q}^*\right) = \sum_{n=0}^{N-1}\left[Y_q\left(q_n^*\right) + B_q\left(\delta_n^*\right) - B_q\left(\delta_{n+1}^*\right)\right]\left(q_n^* - \bar{q}^*\right)\\ &= \sum_{n=0}^{N-1} B_q\left(\delta_n^*\right)\delta_n^* + \sum_{n=0}^{N-1} Y_q\left(q_n^*\right)\left[q_n^* - \bar{q}^*\right] \geq 0,\end{aligned}$$

where optimality conditions (4.4) and the property $Y_{qq}(q_n) \geq 0$ have been used. As for the second correlation in part B), substitute the demand function into (B3) to obtain

$$\text{(B4)}\quad \begin{aligned} &0 \le \sum_{n=0}^{N-1} \gamma_n \left(q_n^* - \overline{q}^*\right) - \sum_{n=0}^{N-1} A(q_n^*)\left(q_n^* - \overline{q}^*\right) \\ &\le \sum_{n=0}^{N-1} \gamma_n \left(q_n^* - \overline{q}^*\right) - \sum_{n=0}^{N-1}\left(q_n^* - \overline{q}^*\right)\sum_{n=0}^{N-1} A(q_n^*) = \sum_{n=0}^{N-1} \gamma_n q_n^* - \overline{q}^* \sum_{n=0}^{N-1} \gamma_n. \end{aligned}$$

The correlation inequality has been used in relation (B4). Q.E.D.

Proof of Proposition 5: In part A), by standard arguments

$$\text{(B5)}\quad \mathfrak{L}(\gamma) = \max_q \sum_{n=0}^{N-1} \gamma_n q_n - \sum_{n=0}^{N-1} \int_0^{q_n} A(q)dq - \sum_{n=0}^{N-1} Y(q_n) - \sum_{n=0}^{N-1} B(q_n - q_{n-1})$$

is convex in γ. It is also $\tilde{C}_N$ -invariant because

$$\text{(B6)}\quad \begin{aligned} &\mathfrak{L}(\gamma) \equiv \mathfrak{L}(\gamma_g) = \max_q \sum_{n=0}^{N-1} \gamma_{g(n)} q_n - \sum_{n=0}^{N-1} \int_0^{q_n} A(q)dq - \sum_{n=0}^{N-1} Y(q_n) \\ &- \sum_{n=0}^{N-1} B(q_n - q_{n-1}) \quad \forall g \in C_N, \end{aligned}$$

where the welfare function's cyclic symmetry has allowed us to re-label arguments. Vector $q^*(\gamma)$ will change as $\gamma \to \gamma_g$, it must cycle too for function invariance would be violated otherwise. But $\mathfrak{L}(\gamma)$ will not change as $\gamma \to \gamma_g$. Then apply Jensen's inequality: $\mathfrak{L}(\gamma) = \lambda_0 \mathfrak{L}(\gamma) + \lambda_1 \mathfrak{L}(\gamma_{g_1}) + \ldots + \lambda_{N-1} \mathfrak{L}(\gamma_{g_{N-1}}) \ge \mathfrak{L}(\lambda_0 \gamma + \lambda_1 \gamma_{g_1} + \ldots + \lambda_{N-1} \gamma_{g_{N-1}})$, whenever $\lambda \in \overline{\mathbb{R}}_+^N$ and $\sum_{n=0}^{N-1} \lambda_n = 1$. By construction, $\lambda_0 \gamma + \lambda_1 \gamma_{g_1} + \ldots + \lambda_{N-1} \gamma_{g_{N-1}} \le_{\tilde{C}_N} \gamma$. Part B) follows similarly. Q.E.D.

Proof of Proposition 6: Write the optimality conditions for the linear system as

$$\text{(B7)}\quad \begin{pmatrix} 1+2\xi & -\xi & 0 & \vdots & 0 & -\xi \\ -\xi & 1+2\xi & -\xi & \vdots & 0 & 0 \\ 0 & -\xi & 1+2\xi & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 1+2\xi & -\xi \\ -\xi & 0 & 0 & \vdots & -\xi & 1+2\xi \end{pmatrix} \begin{pmatrix} q_0^* \\ q_1^* \\ q_2^* \\ \vdots \\ q_{N-2}^* \\ q_{N-1}^* \end{pmatrix} = \begin{pmatrix} \vartheta_0 \\ \vartheta_1 \\ \vartheta_2 \\ \vdots \\ \vartheta_{N-2} \\ \vartheta_{N-1} \end{pmatrix},$$

or $Aq^* = \vartheta$ where the symbols have the obvious interpretations. Solve to obtain $q^* = U\vartheta$ where

(B8) $$U = \frac{1}{\Delta \times (\rho^N - 1)} \begin{pmatrix} 1+\rho^N & \rho+\rho^{N-1} & \rho^2+\rho^{N-2} & \vdots & \rho^2+\rho^{N-2} & \rho+\rho^{N-1} \\ \rho+\rho^{N-1} & 1+\rho^N & \rho+\rho^{N-1} & \vdots & \rho^3+\rho^{N-3} & \rho^2+\rho^{N-2} \\ \rho^2+\rho^{N-2} & \rho+\rho^{N-1} & 1+\rho^N & \vdots & \rho^4+\rho^{N-4} & \rho^3+\rho^{N-3} \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ \rho+\rho^{N-1} & \rho^2+\rho^{N-2} & \rho^3+\rho^{N-3} & \vdots & \rho+\rho^{N-1} & 1+\rho^N \end{pmatrix}.$$

Weighting positivity is obvious. Algebra confirms $\sum_{n=0}^{N-1}(\rho^n + \rho^{N-n}) \equiv \Delta \times (\rho^N - 1)$. Q.E.D.

Proof of Proposition 7: **Part A):** With adjustment costs, profits are

(B9) $$\Pi(\gamma) = \sum_{n=0}^{N-1}\left[(p_n - \beta_n)q_n - 0.5\chi q_n^2 - \tau(q_n^2 - q_n q_{n-1})\right],$$

where we have used cyclic property $\sum_{n=0}^{N-1} q_n^2 = \sum_{n=0}^{N-1} q_{n-1}^2$ under modular arithmetic. Define

(B10) $$P \equiv \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{N-1} \end{pmatrix}; \quad \beta \equiv \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{N-1} \end{pmatrix}; \quad E_{-1} \equiv \begin{pmatrix} 0 & 0 & 0 & \vdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \vdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \vdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 0 & 1 & 0 \end{pmatrix}; \quad q \equiv \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ \vdots \\ q_{N-1} \end{pmatrix},$$

where circulant E_{-1} is obtained from the identity matrix by a backward cycling of each row. Observe that

(B11) $$E_{-1}^T = \left(E_{-1}\right)^{-1} = \begin{pmatrix} 0 & 1 & 0 & \vdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \vdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \vdots & 0 & 0 & 0 \end{pmatrix},$$

so that the transpose of E_{-1} equals the inverse of E_{-1} and both equal the matrix representing the forward cycling of each row in the identity matrix.

From (B9), profits and the profit-maximizing condition are

$$\text{(B12)}\quad \begin{aligned} \Pi &= \left[\left\{(P^T-\beta^T)-0.5(\chi+2\tau)\right\}q^T+\tau q^T E_{-1}\right]q, \\ \frac{d\Pi}{dq} &= P-\beta-\left\{(\chi+2\tau)-\tau(E_{-1}+E_{-1}^T)\right\}q=0. \end{aligned}$$

Solving generates optimal choice vector

$$\text{(B13)}\quad \chi G q^* = P-\beta; \quad G \equiv \left[I_N+\upsilon\left(2I_N-(E_{-1}+E_{-1}^T)\right)\right]; \quad \upsilon \equiv \tau/\chi;$$

where I_N is the rank N identity. Insert back into (B12) to obtain optimal producer surplus as

$$\text{(B14)}\quad \Pi^* = 0.5\chi\left(q^*\right)^T G q^*.$$

Notice that G is a circulant whose rows and columns sum to one. It is readily shown that G^{-1} is bistochastic, see footnote 11. From proposition 6, equilibrium output is

$$\text{(B15)}\quad Aq^* = \vartheta \quad \text{where} \quad \vartheta_n = \frac{\gamma_n}{\kappa+\chi}; \quad A = I_N+\xi\left(2I_N-(E_{-1}+E_{-1}^T)\right); \quad \xi = \frac{\tau}{\kappa+\chi}.$$

Re-write matrix G as

$$\text{(B16)}\quad G \equiv \frac{\kappa+\chi}{\chi}\left[A-\frac{\kappa}{\kappa+\chi}I_N\right].$$

Using (B13)-(B16), optimal producer surplus can be written as

$$\text{(B17)}\quad \Pi^* = 0.5(\kappa+\chi)\vartheta^T M\vartheta; \quad M \equiv \left(A^{-1}\right)^T\left[A-\frac{\kappa}{\kappa+\chi}I_N\right]A^{-1} = A^{-1}-\frac{\kappa}{\kappa+\chi}A^{-2},$$

where the fact that A^{-1} is symmetric has been used.

If λ_n is an eigenvalue of A^{-1} then it is readily shown that $\eta_n = \lambda_n - \kappa\lambda_n^2/(\kappa+\chi)$ is an eigenvalue of M and that every eigenvector of A^{-1} is an eigenvector of M. Since A^{-1} is a bistochastic, symmetric circulant, all its eigenvalues are real, non-negative and no larger than one. Hence, all eigenvalues of M are non-negative (and strictly less than one). Furthermore, since M is real and symmetric, it has a full set of eigenvectors, call them $(\vec{v}_0, \ldots, \vec{v}_{N-1})$, that span $\mathbb{R}^N$ and that can be chosen to be orthonormal. Hence, any vector $\vec{u} \in \mathbb{R}^N$ can be written as a linear combination of these eigenvectors; $\vec{u} = \sum_{n=0}^{N-1} \omega_n \vec{v}_n$. By the choice of an orthonormal base, this implies

(B18) $\vec{u}^T M \vec{u} = \left(\sum_{n=0}^{N-1} \omega_n \vec{v}_n^T\right) M \left(\sum_{n=0}^{N-1} \omega_n \vec{v}_n\right) = \sum_{n=0}^{N-1} \omega_n^2 \eta_n \geq 0$

since $\vec{v}_n^T \vec{v}_m = \delta_{nm}$, the Kronecker delta function with value 1 whenever $n = m$ and zero otherwise. Thus, M is positive semi-definite. It is also a circulant. Part A) follows.

Part B): Consumer surplus is $\mathbb{C}(\gamma) = \sum_{n=0}^{N-1} \int_0^{q_n^*} p_n(q)dq - \sum_{n=0}^{N-1} p_n(q_n^*) q_n^* = 0.5\kappa \sum_{n=0}^{N-1} (q_n^*)^2$. Expression $\sum_{n=0}^{N-1} (q_n^*)^2$ is symmetric and convex in period equilibrium quantities. Given the linear system, each $q_n^*(\gamma)$ is linear in γ. Further, proposition 3 has shown that aggregate output $\sum_{n=0}^{N-1} q_n^*$ is fixed on simplex $\sum_{n=0}^{N-1} \gamma_n$ so we need not be concerned that the shift $\gamma' \to \gamma''$, with $\gamma' \leq_{\tilde{C}_N} \gamma''$, induces a change in the value of $\sum_{n=0}^{N-1} q_n^*$. From (6.1), $\sum_{n=0}^{N-1} [q_n^*(\gamma)]^2$ is also $\tilde{D}_N$-invariant, $\sum_{n=0}^{N-1} [q_n^*(\gamma)]^2 = \sum_{n=0}^{N-1} [q_n^*(\gamma_g)]^2 \; \forall g \in D_N$. Then apply Jensen's inequality: $\mathbb{C}(\gamma) = \sum_{g \in D_N} \lambda_g \mathbb{C}(\gamma_g) \geq \mathbb{C}(\sum_{g \in D_N} \lambda_g \gamma_g)$ whenever $\lambda_g \in \overline{\mathbb{R}}_+ \; \forall g \in D_N$ and $\sum_{g \in D_N} \lambda_g = 1$.[15] To demonstrate that the value of $\sum_{n=0}^{N-1} (s_n^*)^2$ increases under $\gamma' \to \gamma''$, write $\sum_{n=0}^{N-1} (s_n^*)^2 = 2\kappa^{-1} \mathbb{C}(\gamma) / (\sum_{n=0}^{N-1} q_n^*)^2$ where the denominator is invariant to the shift $\gamma' \to \gamma''$. Q.E.D.

Proof of Proposition 8: Parts A) and B) follow from inspection of (B8). The first two limits in part C) are due to the facts that $\text{Lim}_{\xi \to 0} \rho = \infty$ and the dominant power wins out in limits of polynomial ratios as a variable goes to infinity. For the final limit in part C), note $\text{Lim}_{\xi \to \infty} \rho = 1$

[15] In this notation it is important to recognize that, in contrast to the subscripted g in $\gamma_g \in \mathbb{R}^N$, the subscripted g in $\lambda_g \in \overline{\mathbb{R}}_+$ is not a group operation. Rather it is the identifier on simplex weighting coordinate λ_g in the summation over group operation set $G = D_N$ acting on vector γ.

and $\text{Lim}_{\xi\to\infty} u_m / u_n = \text{Lim}_{\xi\to\infty}(\rho^m + \rho^{N-m})/(\rho^n + \rho^{N-n}) = 1$ so that all weightings converge to the uniform weighting as adjustment costs become arbitrarily large. For assertion $u_n > u_{n+1}$ in part D), the inequality may be written as $\rho^n + \rho^{N-n} > \rho^{n+1} + \rho^{N-n-1}$, or $(N-1)/2 > 2$.

For the monotone derivative in part D), assume (the other case follows similarly) that $(N-1)/2 > n$. Clearly, $d\rho / d\xi \le 0$. Differentiate $u_{n+1}/u_n = (\rho + \rho^{N-2n-1})/(1+\rho^{N-2n})$,

$$\text{(B19)} \quad \frac{d(u_{n+1}/u_n)}{d\xi} = \frac{1-\rho^{4[0.5(N-1)-n]} - 2[0.5(N-1)-n](\rho^{N-2n} - \rho^{N-2n-2})}{(1+\rho^{N-2n})^2}\frac{d\rho}{d\xi} \ge 0.$$

Turning to part E), let $\sum_{n=0}^{N-1} \vartheta'_n = \sum_{n=0}^{N-1} \vartheta''_n = \Phi$. Then

$$\text{(B20)} \quad \begin{aligned} q'_0 &= u_0\vartheta'_0 + u_1(\vartheta'_1 + \vartheta'_{N-1}) + u_2(\vartheta'_2 + \vartheta'_{N-2}) + \ \dots\ + u_{(N-1)/2}(\vartheta'_{(N-1)/2} + \vartheta'_{(N+1)/2}) \\ &= (u_0 - u_1)\vartheta'_0 + (u_1 - u_2)(\vartheta'_0 + \vartheta'_1 + \vartheta'_{N-1}) + (u_2 - u_3)(\vartheta'_0 + \vartheta'_1 + \vartheta'_{N-1} + \vartheta'_2 + \vartheta'_{N-2}) \\ &+ \ \dots\ + (u_{(N-3)/2} - u_{(N-1)/2})(\Phi - \vartheta'_{(N-1)/2} - \vartheta'_{(N+1)/2}) + u_{(N-1)/2}\Phi, \end{aligned}$$

for $N-1$ even and

$$\text{(B21)} \quad \begin{aligned} q'_0 &= u_0\vartheta'_0 + u_1(\vartheta'_1 + \vartheta'_{N-1}) + u_2(\vartheta'_2 + \vartheta'_{N-2}) + \ \dots\ + u_{(N-2)/2}(\vartheta'_{(N-2)/2} + \vartheta'_{(N+2)/2}) + u_{N/2}\vartheta'_{N/2} \\ &= (u_0 - u_1)\vartheta'_0 + (u_1 - u_2)(\vartheta'_0 + \vartheta'_1 + \vartheta'_{N-1}) + (u_2 - u_3)(\vartheta'_0 + \vartheta'_1 + \vartheta'_{N-1} + \vartheta'_2 + \vartheta'_{N-2}) \\ &+ \ \dots\ + (u_{(N-2)/2} - u_{N/2})(\Phi - \vartheta'_{N/2}) + u_{N/2}\Phi, \end{aligned}$$

for $N-1$ odd. The result follows from applying the set of dominance summation relations in part E) to (B20) and (B21). As to part F), see part B) and equation (6.1). Q.E.D.

Proof of Proposition 9: Let market parameter vector ϑ be unimodal and centrally symmetric around $n = 0$. Without loss of generality we may characterize the vector as of uniform weighting on a connected subset J_m of Ω_N, $\vartheta_{-m} = \vartheta_{1-m} = \dots = \vartheta_{-1} = \vartheta_0 = \vartheta_1 = \dots = \vartheta_{m-1} = \vartheta_m = 1$ and $\vartheta_n = 0$ otherwise. All other vectors that are centrally symmetric and unimodal around period 0 can be constructed by taking positive combinations and limits of this weighting set, e.g., $2J_2 + J_0$. Thus, under modular arithmetic,

(B22) $q_0 = \sum_{n=-m}^{m} u_n, \qquad q_1 = q_{-1} = \sum_{n=1-m}^{m+1} u_n = q_0 + u_{m+1} - u_{-m} = q_0 + u_{m+1} - u_m \le q_0,$
$q_2 = q_{-2} = q_1 + u_{m+2} - u_{m+1} \le q_1, \qquad \ldots\ldots\ldots \qquad , q_r = q_{-r} = q_{r-1} + u_{m+r} - u_{m+r-1} \le q_{r-1},$

and the decline continues as r increases until the output at issue is phased halfway around the circle from q_0. Then the added term is larger than the dropped term and q_r begins to increase toward peak output q_0. Q.E.D.

Proof of Proposition 10: **Part A):** To demonstrate (6.2A), insert coefficients in (6.1) to obtain

(B23) $$q_n^{*,N} - \bar{q}^* = \frac{\sum_{j=0}^{N\text{-}1} (\rho^j + \rho^{N-j}) \mathrm{Cos}[2\pi(n+j)/N]}{\Delta \times (\rho^N - 1)}.$$

We seek to show

(B24) $$q_n^{*,N} - \bar{q}^* = \frac{\mathrm{Cos}(2\pi n/N) \sum_{j=0}^{N\text{-}1} (\rho^j + \rho^{N-j}) \mathrm{Cos}(2\pi j/N)}{\Delta \times (\rho^N - 1)}.$$

A readily demonstrated trigonometric relationship is $\mathrm{Cos}(2\pi n/N)\mathrm{Cos}(2\pi j/N) =$ $0.5 \times \mathrm{Cos}[2\pi(n+j)/N] + 0.5 \times \mathrm{Cos}[2\pi(n-j)/N]$, so our claim in part A) can be written as

(B25) $$q_n^{*,N} - \bar{q}^* = \frac{0.5 \times \sum_{j=0}^{N\text{-}1} (\rho^j + \rho^{N-j})(\mathrm{Cos}[2\pi(n+j)/N] + \mathrm{Cos}[2\pi(n-j)/N])}{\Delta \times (\rho^N - 1)}.$$

To reconcile (B25) with (B23), it is necessary and sufficient to establish

(B26) $$\sum_{j=0}^{N\text{-}1} (\rho^j + \rho^{N-j}) \mathrm{Cos}[2\pi(n-j)/N] = \sum_{j=0}^{N\text{-}1} (\rho^j + \rho^{N-j}) \mathrm{Cos}[2\pi(n+j)/N].$$

Central symmetry assures $\sum_{j=0}^{N\text{-}1} \rho^{N-j} \mathrm{Cos}[2\pi(n-j)/N] = \sum_{i=0}^{N\text{-}1} \rho^i \mathrm{Cos}[2\pi(n+i-N)/N] =$ $\sum_{i=0}^{N\text{-}1} \rho^i \mathrm{Cos}[2\pi(n+i)/N] = \sum_{j=0}^{N\text{-}1} \rho^j \mathrm{Cos}[2\pi(n+j)/N]$. Likewise, $\sum_{j=0}^{N\text{-}1} \rho^j \mathrm{Cos}[2\pi(n-j)/N]$ $= \sum_{j=0}^{N\text{-}1} \rho^{N-j} \mathrm{Cos}[2\pi(n+j)/N]$ and (B26) is validated.

To verify (6.2B), we need to show that

(B27) $$q_0^{*,N} - \overline{q}^* = \frac{\sum_{j=0}^{N\text{-}1}(\rho^j + \rho^{N-j})\text{Cos}(2\pi j/N)}{\Delta\times(\rho^N - 1)} = \frac{(\rho-1)(\rho+1)}{\Delta\times[\rho^2 - 2\rho\text{Cos}(2\pi/N)+1]},$$

i.e., that

(B28) $$\sum_{j=0}^{N\text{-}1}(\rho^j + \rho^{N-j})\text{Cos}(2\pi j/N) = \frac{(\rho^N - 1)(\rho-1)(\rho+1)}{[\rho^2 - 2\rho\text{Cos}(2\pi/N)+1]}.$$

Break (B28) into two parts, L_1 and L_2, considering first $L_1 = \sum_{j=0}^{N\text{-}1}\rho^j\text{Cos}(2\pi j/N)$. Write $z = \rho\times[\text{Cos}(2\pi/N) + i\times\text{Sin}(2\pi/N)] = \rho\times e^{i2\pi/N}$ in the complex plane. Then, by De Moivre's theorem, L_1 is the real part of $\sum_{j=0}^{N\text{-}1}z^j$ because $\text{Re}(\sum_{j=0}^{N\text{-}1}z^j) = \sum_{j=0}^{N\text{-}1}\text{Re}(z^j) = \sum_{j=0}^{N\text{-}1}\rho^j\text{Cos}(2\pi j/N)$. By the truncated summation formula

(B29) $$\text{Re}\left(\sum_{j=0}^{N\text{-}1}z^j\right) = \text{Re}\left(\frac{z^N - 1}{z-1}\right) = \text{Re}\left(\frac{\rho^N - 1}{\rho\times[\text{Cos}(2\pi/N) + i\times\text{Sin}(2\pi/N)] - 1}\right),$$

because $z^N = \rho^N e^{i2\pi} = \rho^N$. Using complex conjugates, we also have

(B30) $$\frac{\rho^N - 1}{\rho\times[\text{Cos}(2\pi/N) + i\times\text{Sin}(2\pi/N)] - 1} = \frac{(\rho^N - 1)(\rho\times[\text{Cos}(2\pi/N) - i\times\text{Sin}(2\pi/N)] - 1)}{\rho^2 - 2\rho\times\text{Cos}(2\pi/N)+1}$$

so that

(B31) $$L_1 = \text{Re}\left(\frac{\rho^N - 1}{\rho\times[\text{Cos}(2\pi/N) + i\times\text{Sin}(2\pi/N)] - 1}\right) = \frac{(\rho^N - 1)[\rho\times\text{Cos}(2\pi/N) - 1]}{\rho^2 - 2\rho\times\text{Cos}(2\pi/N)+1}.$$

And for $L_2 = \sum_{j=0}^{N\text{-}1}\rho^{N-j}\text{Cos}(2\pi j/N) = \rho^N\sum_{j=0}^{N\text{-}1}\rho^{-j}\text{Cos}(2\pi j/N)$, substituting in ρ^{-1} for ρ in (B29), with $z = \rho^{-1}\times[\text{Cos}(2\pi/N) + i\times\text{Sin}(2\pi/N)] = \rho^{-1}\times e^{i2\pi/N}$ and following through confirms the value of L_2 as

(B32)
$$\mathrm{Re}\left(\rho^N \sum_{j=0}^{N\text{-}1} z^j\right) = \mathrm{Re}\left(\frac{1-\rho^N}{z-1}\right) = \mathrm{Re}\left(\frac{1-\rho^N}{\rho^{-1}\times[\mathrm{Cos}(2\pi/N)+i\times\mathrm{Sin}(2\pi/N)]-1}\right)$$
$$= \frac{(1-\rho^N)[\rho^{-1}\times\mathrm{Cos}(2\pi/N)-1]}{\rho^{-2}-2\rho^{-1}\times\mathrm{Cos}(2\pi/N)+1} = \frac{(1-\rho^N)\rho\times[\mathrm{Cos}(2\pi/N)-\rho]}{1-2\rho\times\mathrm{Cos}(2\pi/N)+\rho^2}.$$

Upon summing, $L_1 + L_2 = (\rho^N - 1)(\rho - 1)(\rho + 1)/[\rho^2 - 2\rho\mathrm{Cos}(2\pi/N)+1]$ as sought. Finally observe that $\rho > 1$ when τ is finite and that N finite implies both $|\mathrm{Cos}(2\pi/N)| < 1$ and $\rho^2 - 2\rho\times\mathrm{Cos}(2\pi/N)+1 > 0$.

Part B): With no adjustment costs, then $q_0^{*,N} = \overline{q}^* + 1$. We claim that, $\forall \rho \in (1,\infty)$, $(\rho-1)(\rho+1)/(\Delta\times[\rho^2 - 2\rho\times\mathrm{Cos}(2\pi/N)+1]) \in (0,1)$ where $\rho = 1+(1+\Delta)/(2\xi)$, $\Delta = \sqrt{1+4\xi}$, and $\xi > 0$, as previously defined. To demonstrate this claim, note that $2\xi = (\Delta^2 - 1)/2$ so that $\rho = 1 + 2/(\Delta - 1)$. Substitution provides

(B33)
$$\frac{(\rho-1)(\rho+1)}{\Delta\times[\rho^2-2\rho\times\mathrm{Cos}(2\pi/N)+1]} = \frac{4}{(\Delta+1)^2 - 2(\Delta^2-1)\mathrm{Cos}(2\pi/N)+(\Delta-1)^2}$$
$$= \frac{2}{\Delta^2[1-\mathrm{Cos}(2\pi/N)]+[1+\mathrm{Cos}(2\pi/N)]} \in (0,1),$$

where positivity is immediate and upper limit 1 is due to the following logic. The upper limit is true if relation $4 < (\Delta+1)^2 - 2(\Delta^2-1)\mathrm{Cos}(2\pi/N)+(\Delta-1)^2$ applies, i.e., (from repackaging terms) if $(\Delta^2-1)[1-\mathrm{Cos}(2\pi/N)] > 0$. This latter condition is clearly true for N finite. In addition, note that expression $(\rho-1)(\rho+1)/(\Delta\times[\rho^2-2\rho\times\mathrm{Cos}(2\pi/N)+1])$ is decreasing in ξ (and so in τ) so that higher adjustment costs unambiguously lower peak output.

Part C): Monotonicity of (6.2B) in N is immediate for $N \geq 2$. Also, take the limit in (6.2B) noting that $\mathrm{Cos}(0) = 1$.

Part D): A variant on the Wirtinger inequality, as given in Block (1957), asserts that if $x_{-1} = x_{N-1}$ and $\sum_{n=0}^{N-1} x_n = 0$ then $\sum_{n=0}^{N-1} (x_n - x_{n-1})^2 / \sum_{n=0}^{N-1} x_n^2 \geq 4\text{Sin}^2(\pi / N)$. The inequality is an equality if and only if each x_n takes form $x_n = A_1\text{Cos}(2\pi n / N) + A_2\text{Sin}(2\pi n / N)$. But part A) has demonstrated that this form is taken. Q.E.D.

Proof of Proposition 11: Set $\xi_2 > \xi_1$ with adjustment cost conditioned weighting vectors $u(\xi) = (u_0(\xi), u_1(\xi), \dots, u_{N-1}(\xi))^T$. If $u(\xi_2) \leq_{\tilde{C}_N} u(\xi_1)$ then, from definition 1, there exists a weighting vector, $\{\lambda(\xi_2;\xi_1) \in [0,1]^N : \lambda_n(\xi_2;\xi_1) \geq 0 \ \forall n \in \Omega_N, \sum_{n=0}^{N-1} \lambda_n(\xi_2;\xi_1) = 1\}$, such that

$$\text{(B34)} \quad \begin{pmatrix} u_0(\xi_2) \\ u_1(\xi_2) \\ \vdots \\ u_{N-1}(\xi_2) \end{pmatrix} = \begin{pmatrix} u_0(\xi_1) \\ u_1(\xi_1) \\ \vdots \\ u_{N-1}(\xi_1) \end{pmatrix} \lambda_0(\xi_2;\xi_1) + \begin{pmatrix} u_{N-1}(\xi_1) \\ u_0(\xi_1) \\ \vdots \\ u_{N-2}(\xi_1) \end{pmatrix} \lambda_1(\xi_2;\xi_1) + \dots + \begin{pmatrix} u_1(\xi_1) \\ u_2(\xi_1) \\ \vdots \\ u_0(\xi_1) \end{pmatrix} \lambda_{N-1}(\xi_2;\xi_1).$$

Re-arrange the right-hand side as the product of the λ weighting vector and a matrix previously labeled as U. The inverse is given in (B7) above so that

$$\text{(B35)} \quad \begin{pmatrix} 1+2\xi_1 & -\xi_1 & \vdots & 0 & -\xi_1 \\ -\xi_1 & 1+2\xi_1 & \vdots & 0 & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ -\xi_1 & 0 & \vdots & -\xi_1 & 1+2\xi_1 \end{pmatrix} \begin{pmatrix} u_0(\xi_2) \\ u_1(\xi_2) \\ \vdots \\ u_{N-2}(\xi_2) \\ u_{N-1}(\xi_2) \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{N-2} \\ \lambda_{N-1} \end{pmatrix}.$$

By part C) of proposition 8, $\text{Lim}_{\xi_2 \to \infty} u(\xi_2) = N^{-1}(1,1, \dots, 1)^T$ so that $\text{Lim}_{\xi_2 \to \infty} \lambda(\xi_2;\xi_1) = N^{-1}(1,1, \dots, 1)^T$. By continuity of each $\lambda_n(\xi_2;\xi_1)$, there exists a $\hat{\xi}_2(\xi_1)$ such that $\min\{\lambda_0(\xi_2;\xi_1), \lambda_1(\xi_2;\xi_1), \dots, \lambda_{N-1}(\xi_2;\xi_1)\} \geq 0 \ \forall \xi_2 \geq \hat{\xi}_2(\xi_1)$. Therefore, $u(\xi_2) \leq_{\tilde{C}_N} u(\xi_1) \ \forall \xi_2 \geq \hat{\xi}_2(\xi_1)$. Q.E.D.

Proof of Proposition 12: Write $\mathrm{Var}[q(\xi_1)] - \mathrm{Var}[q(\xi_2)]$ as

$$\text{(B36)}\quad \begin{aligned} & N^{-1}(\vartheta - \bar{\vartheta})^T [U(\xi_1)]^T U(\xi_1)(\vartheta - \bar{\vartheta}) - N^{-1}(\vartheta - \bar{\vartheta})^T [U(\xi_2)]^T U(\xi_2)(\vartheta - \bar{\vartheta}) \\ & = N^{-1}(\vartheta - \bar{\vartheta})^T \left([U(\xi_1)]^2 - [U(\xi_2)]^2 \right)(\vartheta - \bar{\vartheta}). \end{aligned}$$

A property of circulant matrices with defining vector $(u_0, u_1, \ldots, u_{N-1})$ on the unit simplex is that one eigenvalue is $\sigma = 1$ while any distinct eigenvalue, real or complex, satisfies the radius of convergence criterion $|\sigma| < 1$ [Davis (1979 p. 132)]. Proposition 11 showed conditions under which $u(\xi_2) \leq_{\hat{C}_N} u(\xi_1)$. But from Giovagnoli and Wynn (1996, p. 219) this means there exists a circulant $\hat{S}$ with defining vector $(s_0, s_1, \ldots, s_{N-1})^T$ on the unit simplex such that $u(\xi_2) = \hat{S}u(\xi_1)$, $U(\xi_2) = \hat{S}U(\xi_1)$, and $[U(\xi_2)]^2 = [\hat{S}U(\xi_1)]^2$. We seek to understand when $(\vartheta - \bar{\vartheta})^T [U(\xi_1)]^2 (\vartheta - \bar{\vartheta}) - (\vartheta - \bar{\vartheta})^T [\hat{S}U(\xi_1)]^2 (\vartheta - \bar{\vartheta}) \geq 0$, and so we require information on the eigenvalues of the difference.

The $n \times n$ Fourier matrix F with complex number entries is defined as follows. Set ω as the n^{th} root of unity, i.e., $\omega^n = 1$ and write

$$\text{(B37)}\quad F^* = \begin{pmatrix} 1 & 1 & 1 & \vdots & 1 & 1 \\ 1 & \omega & \omega^2 & \vdots & \omega^{n-2} & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \vdots & \omega^{2(n-2)} & \omega^{2(n-1)} \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \vdots & \omega^{(n-2)(n-1)} & \omega^{(n-1)(n-1)} \end{pmatrix}.$$

Fourier matrix F is the complex conjugate of F^*, i.e., $F = \bar{F}^*$ where the bar represents the conjugation operation. Thus, $FF^* = F^*F = I_n$. The matrices are useful because if S is a circulant then $S = F^*\Lambda F$ where Λ is diagonal. From Davis (1979, pp. 72-73) the diagonal entries in Λ are an ordering of the eigenvalues for S.

If S_1 and S_2 are circulants then S_1S_2 is also. A second property is that if S_1 and S_2 are circulants with respective ordered eigenvalue sets (from the top along the diagonal) established by the Fourier matrix diagonalization as $\sigma_1 = (\sigma_{1,0}, \sigma_{1,1}, \ldots, \sigma_{1,n-1})^T$ and $\sigma_2 =$

$(\sigma_{2,0}, \sigma_{2,1}, \ldots, \sigma_{2,n-1})^T$, then the eigenvalue vector of $S_3 = S_1 S_2$ is the component-wise product of σ_1 and σ_2.[16] So $\sigma_3 = (\sigma_{1,0}\sigma_{2,0}, \sigma_{1,1}\sigma_{2,1}, \ldots, \sigma_{1,n-1}\sigma_{2,n-1})^T$. To see this, suppose $S_1 = F^* \Lambda_1 F$ and $S_2 = F^* \Lambda_2 F$. Then $S_1 S_2 = F^* \Lambda_1 F F^* \Lambda_2 F = F^* \Lambda_1 \Lambda_2 F$ where $\Lambda_1 \Lambda_2$ is diagonal. If S_1 and S_2 are circulants with defining vectors on the unit simplex, then $S_1 S_2$ is a circulant with defining vector on the unit simplex. Since the modulus of a product is the product of the moduli, the eigenvalues of $S_1 S_2$ have moduli as follows; $(|\sigma_{1,0}\sigma_{2,0}|, |\sigma_{1,1}\sigma_{2,1}|, \ldots, |\sigma_{1,n-1}\sigma_{2,n-1}|) =$ $(|\sigma_{1,0}||\sigma_{2,0}|, |\sigma_{1,1}||\sigma_{2,1}|, \ldots, |\sigma_{1,n-1}||\sigma_{2,n-1}|)$. Let $S_1 = \widehat{S}$, the circulant that smooths $U(\xi_1)$ to $U(\xi_2)$ in proposition 11, and let $S_2 = U(\xi_1)$. So $|\sigma_{1,i}| \le 1 \; \forall i \in \Omega_N$, $|\sigma_{1,i}|^2 \le 1 \; \forall i \in \Omega_N$ and $(|\sigma_{1,0}|^2 |\sigma_{2,0}|^2, |\sigma_{1,1}|^2 |\sigma_{2,1}|^2, \ldots, |\sigma_{1,n-1}|^2 |\sigma_{2,n-1}|^2) \le (|\sigma_{2,0}|^2, |\sigma_{2,1}|^2, \ldots, |\sigma_{2,n-1}|^2)$ where the comparison is entry-wise. By Theorem 4.5.5. in Davis, this means $N^{-1}(\vartheta - \bar{\vartheta})^T \left([U(\xi_2)]^2 - [U(\xi_1)]^2 \right) (\vartheta - \bar{\vartheta}) \le 0$. Q.E.D.

[16] See Davis (1979, pp. 91-92).

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