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Leslie Hogben

Iowa State University and American Institute of Mathematics, hogben@iastate.edu

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Keywords

Zero forcing, hypergraph, maximum nullity, hypermatrix, infection number, power domination

Disciplines

Discrete Mathematics and Combinatorics

Comments

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Abstract

The concept of zero forcing is extended from graphs to uniform hypergraphs in analogy with the way zero forcing was defined as an upper bound for the maximum nullity of the family of symmetric matrices whose nonzero pattern of entries is described by a given graph: A family of symmetric hypermatrices is associated with a uniform hypergraph and zeros are forced in a null vector. The value of the hypergraph zero forcing number and maximum nullity are determined for various families of uniform hypergraphs and the effects of several graph operations on the hypergraph zero forcing number are determined. The hypergraph zero forcing number is compared to the infection number of a hypergraph and the iteration process in hypergraph power domination.

Keywords. Zero forcing; hypergraph; maximum nullity; hypermatrix; infection number; power domination.

AMS subject classifications. 05C50, 05C15, 05C65, 15A69, 15A03

The edges of a (simple) graph G describe the nonzero off-diagonal pattern of symmetric matrices associated with G . The study of the maximum nullity of these matrices is an active area of research (see [8] and [9] for surveys). The zero forcing number was introduced in [1] as an upper bound for maximum nullity, and independently in control of quantum systems [5] (precise definitions of these and related terms are given in Section 1). As noted in [2], zero forcing is also part of the power domination process on graphs that is used to model the optimal placement of monitoring units in electric networks [11] (see Section 2 for more detail).

A natural way to extend the idea of maximum nullity from graphs to hypergraphs is to associate a family of symmetric hypermatrices with a uniform hypergraph. When moving from matrices to hypermatrices there are several possible definitions of rank, nullity, etc. Likewise there are several reasonable choices for how to generalize zero forcing to hypergraphs. In Section 1 we present one standard definition of nullity for hypermatrices and the resulting definition of maximum nullity $M_0(H)$ of a hypergraph H that generalizes the definition of zero-diagonal maximum nullity $M_0(G)$ of a graph G introduced in [10]. We

*Department of Mathematics, Iowa State University, Ames, IA 50011, USA and American Institute of Mathematics, 600 E. Brokaw Road, San Jose, CA 95112, USA (hogben@aimath.org).

then develop a naturally related definition of the zero forcing number $Z_0(H)$ of a hypergraph H that generalizes the skew zero forcing number $Z_0(G)^1$ of a graph G , extending the graph bound $M_0(G) \leq Z_0(G)$ to hypergraphs, that is, $M_0(H) \leq Z_0(H)$. In Section 2 we show that $Z_0(H) \leq I(H) \leq Z_{\text{pd}}(H)$, where $I(H)$ is the infection number of H introduced in [3] and $Z_{\text{pd}}(H)$ is the iterated step in hypergraph power domination defined by Chang and Roussel in [6]. In Section 3 we establish additional properties of the hypergraph zero forcing number and maximum nullity of hypergraphs, including determining $Z_0(H)$ and $M_0(H)$ for several families of hypergraphs and the effects of several graph operations on the zero forcing number.

1 Maximum nullity and zero forcing number for uniform hypergraphs

Throughout F denotes a field. An *order- d hypermatrix* or a *d -hypermatrix* $A \in F^{n_1 \times \dots \times n_d}$ is specified by a d -dimensional table of values and is denoted by $A = [a_{i_1 \dots i_d}]$ where $a_{i_1 \dots i_d}$ is the entry in position (i_1, \dots, i_d) . A d -hypermatrix $A \in F^{n_1 \times \dots \times n_d}$ is *hypercubical* if $n_1 = \dots = n_d$, and this common value is the *dimension*. A hypercubical d -hypermatrix $A = [a_{i_1 \dots i_d}]$ is *symmetric* if $a_{i_{\pi(1)} \dots i_{\pi(d)}} = a_{i_1 \dots i_d}$ for all $\pi \in S_d$ (where S_d denotes the group of permutations of $\{1, \dots, d\}$). More information on hypermatrices can be found in [14].

A *d -uniform hypergraph* or a *d -hypergraph* $H = (V, E)$ has a set of vertices V (also denoted by $V(H)$) and a set of hyperedges E (also denoted by $E(H)$) with each hyperedge being a set of d distinct vertices. Since d -hypermatrices associate naturally with d -hypergraphs, most hypergraphs discussed here are uniform. The *adjacency matrix* $A(H) \in F^{n \times \dots \times n}$ of a d -hypergraph H on n vertices is the symmetric d -hypermatrix that has $a_{i_1 \dots i_d} = 1$ if $\{i_1, \dots, i_d\} \in E(H)$ and $a_{i_1 \dots i_d} = 0$ if $\{i_1, \dots, i_d\} \notin E(H)$ (the latter case includes all subscripts with a repeated index). More information on hypergraphs can be found in [4].

In this section we define a family of symmetric hypermatrices described by a hypergraph, the nullity of a hypermatrix, and the maximum nullity of hypermatrices described by a hypergraph. We define the zero forcing number for a hypergraph and show that it is an upper bound for maximum nullity.

1.1 Hypermatrix nullity and maximum nullity for uniform hypergraphs

We begin with a review of the definitions of sets of symmetric matrices associated with a graph (all graphs discussed are simple). Let $A = [a_{ij}] \in F^{n \times n}$ be a symmetric matrix. The *graph* $\mathcal{G}(A)$ of A has $V(\mathcal{G}(A)) = [n]$ (where $[n] = \{1, \dots, n\}$) and $\{i, j\} \in E(\mathcal{G}(A))$ if and only if $i \neq j$ and $a_{ij} \neq 0$. Let $G = (V, E)$ be a graph. The *set of symmetric matrices described by G* is $\mathcal{S}(G) = \{A \in F^{n \times n} : A \text{ is symmetric and } \mathcal{G}(A) = G\}$. The *set of zero-diagonal symmetric matrices described by a graph G* is

$$\mathcal{S}_0(G) = \{A \in F^{n \times n} : A \text{ is symmetric, } a_{ii} = 0 \text{ for } i = 1, \dots, n, \text{ and } \mathcal{G}(A) = G\}.$$

¹The skew zero forcing number has traditionally been denoted by Z^- but we use Z_0 to emphasize the connection with zero-diagonal matrices and hypermatrices.

Zero-diagonal symmetric matrices described by a graph G can be thought of as weighted adjacency matrices of G and were studied in [10].

A hypercubical d -hypermatrix $A = [a_{i_1 \dots i_d}] \in F^{n \times \dots \times n}$ is *graphical* if A is symmetric and $a_{i_1 \dots i_d} = 0$ whenever i_1, \dots, i_d are not all distinct. The *hypergraph* $\mathcal{H}(A)$ of a *graphical d -hypermatrix* A of dimension n has $V(\mathcal{H}(A)) = [n]$ and $\{i_1, \dots, i_d\} \in E(\mathcal{H}(A))$ if and only if $a_{i_1 \dots i_d} \neq 0$. The *set of graphical matrices described by a d -hypergraph H* is

$$\mathcal{S}_0(H) = \{A \in F^{n \times \dots \times n} : A \text{ is graphical and } \mathcal{H}(A) = H\}.$$

We choose to require the diagonal to be zero in the definition of a graphical hypermatrix for a variety of reasons: It is more natural to have $a_{ii \dots i} = 0$ given that $a_{i_1 \dots i_d} = 0$ whenever $\{i_1, \dots, i_d\}$ contains any repetition (which is necessary to obtain a uniform hypergraph). It means that a graphical hypermatrix can be viewed as a weighted adjacency matrix of a hypergraph. And it is also related to our definition of null vector below; this is discussed further after defining a null vector. The choice to require diagonal elements to be zero means that we are generalizing $\mathcal{S}_0(G)$ rather than $\mathcal{S}(G)$.

A vector $\mathbf{x} \in F^n$ is a *null vector* of a matrix $A \in F^{n \times n}$ if $A\mathbf{x} = \mathbf{0}$, i.e., $\sum_{j=1}^n a_{ij}x_j = 0$ for every $i = 1, \dots, n$. There are various possible ways to extend this definition to hypermatrices, and we choose the next definition.

Definition 1.1. Let $A = [a_{i_1 \dots i_d}] \in F^{n \times \dots \times n}$ be a symmetric d -hypermatrix. A vector $\mathbf{x} \in F^n$ is a *null vector* of A if

$$\sum_{j=1}^n a_{i_1 \dots i_{d-1}j}x_j = 0 \text{ for every submultiset } \{i_1, \dots, i_{d-1}\} \subset [n] \quad (1)$$

(it is not assumed the values of i_1, \dots, i_{d-1} in (1) are distinct). The *kernel* of A , denoted by $\ker A$, is the vector space of null vectors of A , and the *nullity* of A , denoted by $\text{null } A$, is the dimension of $\ker A$. The *maximum nullity* of a d -uniform hypergraph H is

$$M_0(H) = \max\{\text{null } A : A \in \mathcal{S}_0(H)\}.$$

The definition of null vector could have been stated for a nonsymmetric hypercubical hypermatrix. However the question then arises as to why the sum is on the last index (the d th flattening as defined below). For a symmetric hypermatrix A , $\sum_{j=1}^n a_{i_1 \dots i_{d-1}j}x_j = 0 \forall i_1, \dots, i_{d-1}$ is equivalent to $\sum_{j=1}^n a_{i_1 \dots i_{d-1}ji_{d-1}}x_j = 0 \forall i_1, \dots, i_{d-1}$, so that question is moot.

This definition of a null vector also provides another reason to restrict to zero-diagonal for d -hypermatrices with $d \geq 3$: Consider a symmetric d -hypermatrix $A \in F^{n \times \dots \times n}$ in which $a_{i_1 \dots i_d} = 0$ whenever $2 \leq |\{i_1, \dots, i_d\}| \leq d-1$ but $a_{ii \dots i} \neq 0$ is allowed, aligning with the case of graphs where the diagonal is ignored in the definition of $\mathcal{S}(G)$. Let $\mathbf{x} \in \ker A$. Then $a_{kk \dots k} \neq 0$ implies $x_k = 0$, so the values of $a_{i_1 \dots i_{d-1}k}$ for $(i_1, \dots, i_{d-1}) \neq (k, \dots, k)$ are irrelevant when determining the null vector.

This definition of null vector of a symmetric d -hypermatrix A is equivalent to taking a null vector of the transpose of the d th flattening of A : The *d th flattening* of A , denoted by $b_d(A)$, is the $n \times n^{d-1}$ matrix whose j, i -entry is $a_{i_1 \dots i_{d-1}j}$ where (i_1, \dots, i_{d-1}) is the i th entry

in the lexicographically ordered list of elements in $[n] \times \dots \times [n]$ (with $d - 1$ copies of $[n]$); see [14] for more information. For a symmetric d -hypermatrix $A \in F^{n \times \dots \times n}$ and $\mathbf{x} \in F^n$, \mathbf{x} is a null vector of A (as just defined) if and only if \mathbf{x} is a null vector of the matrix $\flat_d(A)^T$ in the usual sense, so $\text{null } A = \text{null } \flat_d(A)^T$. Thus the next result is an immediate consequence of [1, Prop 2.2] (which is Proposition 1.2 for matrices, i.e., 2-hypermatrices).

Proposition 1.2. *Let F be a field and let $A \in F^{n \times \dots \times n}$ be a d -hypermatrix. If $\text{null } A > k$, then for any set $\alpha \subset [n]$ with $|\alpha| = k$ there is a nonzero vector $\mathbf{x} = [x_i] \in \ker A$ such that $x_j = 0$ for every $j \in \alpha$.*

For a d -hypergraph H and a matrix $A = [a_{i_1 \dots i_d}] \in \mathcal{S}_0(H)$, Definition 1.1 can be restated using edges: If $\{i_1, \dots, i_{d-1}, j\} \notin E(H)$, then $a_{i_1 \dots i_{d-1} j} = 0$, so $a_{i_1 \dots i_{d-1} j} x_j = 0$. Thus a vector $\mathbf{x} \in F^n$ is a null vector of $A \in \mathcal{S}_0(H)$ if and only if

$$\sum_{\{i_1, \dots, i_{d-1}, j\} \in E(H)} a_{i_1 \dots i_{d-1} j} x_j = 0.$$

For H a d -hypergraph and $A \in \mathcal{S}_0(H)$, this observation about edges and the symmetry of A suggests using a submatrix of the transpose-flattening of A to test for null vectors. Let B be an $m \times n$ matrix over F . For $\alpha \subseteq [m]$ and $\beta \subseteq [n]$, the submatrix of B with rows indexed by α and columns indexed by β is denoted by $B[\alpha, \beta]$. Define $A^b = \flat_d(A)^T[\alpha, [n]]$ where α is the set of rows indexed by $\{i_1, \dots, i_{d-1}\}$ such that $i_1 < \dots < i_{d-1}$ and there exists an edge e of H containing $\{i_1, \dots, i_{d-1}\}$. Then \mathbf{x} is a null vector of $A \in \mathcal{S}_0(H)$ if and only if

$$A^b \mathbf{x} = \mathbf{0}. \tag{2}$$

The use of this definition is illustrated in the next two examples.

Example 1.3. Let H_1 be the 3-hypergraph with vertices $\{1, 2, 3, 4, 5\}$ and edges $\{\{1, 2, 3\},$

$$\{3, 4, 5\}\}$$
 (see Figure 1.1(a)). For $A = [a_{i_1 i_2 i_3}] \in \mathcal{S}_0(H_1)$, $A^b = \begin{bmatrix} 0 & 0 & a_{123} & 0 & 0 \\ 0 & a_{123} & 0 & 0 & 0 \\ a_{123} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{345} \\ 0 & 0 & 0 & a_{345} & 0 \\ 0 & 0 & a_{345} & 0 & 0 \end{bmatrix}$

with the rows indexed by $\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}$. Then $M_0(H_1) = 0$ because $\ker A^b = \{\mathbf{0}\}$.

Example 1.4. Let H_2 be the 3-hypergraph with vertices $\{1, 2, 3, 4\}$ and edges $\{\{1, 2, 3\},$

$$\{2, 3, 4\}\}$$
 (see Figure 1.1(b)). For $A = [a_{i_1 i_2 i_3}] \in \mathcal{S}_0(H_2)$, $A^b = \begin{bmatrix} 0 & 0 & a_{123} & 0 \\ 0 & a_{123} & 0 & 0 \\ a_{123} & 0 & 0 & a_{234} \\ 0 & 0 & a_{234} & 0 \\ 0 & a_{234} & 0 & 0 \end{bmatrix}$

with the rows indexed by $\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$. Then $M_0(H_2) = 1$ because $[a_{234}, 0, 0, -a_{123}]^T$ is a basis for $\ker A^b$.

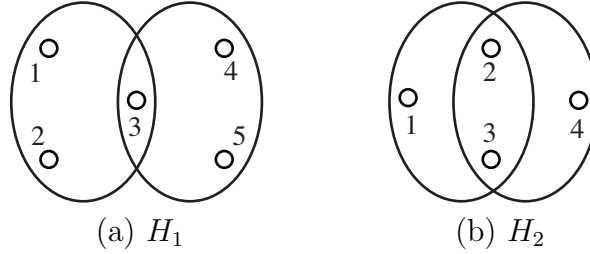


Figure 1.1: Hypergraphs H_1 and H_2 discussed in several examples

1.2 Zero forcing on uniform hypergraphs

We begin with a review of the definitions of standard and skew zero forcing on a (simple) graph G as defined in [1] and [13]. Zero forcing definitions use blue and white vertices, with a blue vertex representing a zero in the null vector of a matrix (in the older literature black is sometimes used instead of blue). A color change rule allows a white vertex to change color to blue (but once blue a vertex always remains blue); such a rule is designed by applying information about the matrix and existing zeros in the null vector to conclude that this entry of the null vector must also be zero. The *standard color change rule*, which is based on matrices in $\mathcal{S}(G)$, is: A blue vertex $v \in V(G)$ can change the color of a white vertex w to blue if (1) $\{v, w\}$ is an edge of G , and (2) u is white and $\{v, u\} \in E(G)$ implies $u = w$. The *skew color change rule*, which is based on matrices in $\mathcal{S}_0(G)$, is: A vertex $v \in V(G)$ can change the color of a white vertex w to blue if (i) $\{v, w\}$ is an edge of G , and (ii) u is white and $\{v, u\} \in E(G)$ implies $u = w$. We say v forces w and write $v \rightarrow w$ to indicate the color change rule is applied to w by using v to color w blue. The difference between the two color change rules is that for skew zero forcing, a vertex need not be blue to force.

A *standard zero forcing set* (respectively, *skew zero forcing set*) for G is a set $B \subseteq V$ such that if initially the vertices in B are blue and the vertices in $V \setminus B$ are white, then every vertex can be colored blue by repeated applications of the color change rule. The *standard zero forcing number* $Z(G)$ (respectively, *skew zero forcing number* $Z_0(G)$) of G is the minimum cardinality of a standard (skew) zero forcing set of G . The skew zero forcing number takes its name from its introduction as an upper bound for maximum nullity of skew symmetric matrices described by a graph in [13], but it also serves as an upper bound for the maximum nullity of zero-diagonal symmetric matrices described by a graph (or any family of matrices with zero-diagonal and off-diagonal nonzero entries described by the edges of the graph) [10].

Since graphical hypermatrices have all diagonal elements equal to zero, it is the skew color change rule rather than the standard color change rule that we extend to uniform hypergraphs. Note that v forcing w in a graph can be interpreted as the elements of edge $\{v, w\}$ other than w forcing w ; this is the viewpoint we adopt.

Definition 1.5. Suppose H is a d -hypergraph with $d \geq 2$, $B \subseteq V(H)$, every vertex in B is colored blue, and every vertex in $V(H) \setminus B$ is colored white. A set $S \subset V(H)$ of $d - 1$ distinct vertices can change the color of white vertex w to blue if

- (i) $S \cup \{w\}$ is an edge of H , and
- (ii) if u is a white vertex and $S \cup \{u\}$ is an edge of H , then $u = w$.

This is called the *hypergraph color change rule*. We say S forces w and write $S \rightarrow w$ to indicate the color change rule is applied to color w blue by using S . A *hypergraph zero forcing set* is a set B such that if the initial set of blue vertices is B , then every vertex can be colored blue by repeated applications of the hypergraph color change rule. The *hypergraph zero forcing number* $Z_0(H)$ of a hypergraph H is the minimum cardinality of a hypergraph zero forcing set of H .

We illustrate hypergraph zero forcing in the next example.

Example 1.6. Let H_1 be the 3-hypergraph in Example 1.3 (see Figure 1.1(a)). Then, $Z_0(H_1) = 0$ because \emptyset is a zero forcing set for H_1 : $\{1, 3\} \rightarrow 2$, $\{2, 3\} \rightarrow 1$, $\{1, 2\} \rightarrow 3$, $\{3, 4\} \rightarrow 5$, and $\{3, 5\} \rightarrow 4$.

For a hypergraph H and initial set B of blue vertices the *derived set of B* is the set of vertices that are blue after applying the color change rule until no more color changes are possible.

Remark 1.7. As noted in [1] for a graph G , the derived set of an initial set B is unique. The same reasoning applies to hypergraph zero forcing (and hypergraph infection, and hypergraph power domination zero forcing): Any vertex that turns blue under one sequence of applications of the color change rule can always be turned blue regardless of the order of color changes: Suppose H is a d -hypergraph, B is a set of blue vertices, and there is a sequence of forces that results in a derived set D_1 , and there is another forcing process that colors D_2 blue. If $D_1 \not\subseteq D_2$, then it is possible to continue forcing: Among vertices in $D_1 \setminus D_2$, let u be the first vertex colored blue in the forcing process that produces D_1 , with $S \rightarrow u$. When u is colored blue, all the vertices that were blue before u are in D_2 . Then after the second forcing process produces D_2 , it is still possible to perform the force $S \rightarrow u$ (since coloring additional vertices makes it easier to force). Note that the set of forces used to produce the derived set is usually not unique.

Theorem 1.8. *Suppose H is a d -hypergraph on $n \geq d \geq 2$ vertices. Then*

$$M_0(H) \leq Z_0(H).$$

Proof. We prove the following statement:

$$A \in \mathcal{S}_0(H), \mathbf{x} \in \ker A, B \text{ a zero forcing set for } H, \text{ and } x_i = 0 \forall i \in B \implies \mathbf{x} = \mathbf{0}. \quad (3)$$

Once (3) is established, we can choose arbitrarily at most $Z_0(H) - 1$ zeros in a nonzero vector in $\ker A$ for any $A \in \mathcal{S}_0(H)$. Then, $M_0(H) \leq Z_0(H)$ by Proposition 1.2.

Assume that $A \in \mathcal{S}_0(H)$, $\mathbf{x} \in \ker A$, $x_i = 0$ for all $i \in B' \subseteq V(H)$, and the color change rule allows the force $S \rightarrow w$ with the vertices in B' blue. Denote the vertices in S by i_1, \dots, i_{d-1} . Then

$$\begin{aligned} 0 &= \sum_{j=1}^n a_{i_1 \dots i_{d-1} j} x_j \\ &= \sum_{\{i_1 \dots i_{d-1} j\} \in E(H)} a_{i_1 \dots i_{d-1} j} x_j \\ &= \sum_{\{i_1 \dots i_{d-1} j\} \in E(H), j \neq w} (a_{i_1 \dots i_{d-1} j} \cdot 0) + a_{i_1 \dots i_{d-1} w} x_w \end{aligned}$$

Since $\{i_1 \cdots i_{d-1} w\} \in E(G)$ implies that $a_{i_1 \cdots i_{d-1} w} \neq 0$, necessarily $x_w = 0$. If B is a zero forcing set for H , then by the zero forcing process $\mathbf{x} = 0$. Thus, (3) is established. \square

Theorem 1.8 is applied in the next example.

Example 1.9. Let H_2 be the 3-hypergraph in Example 1.4 (see Figure 1.1(b)). Then, $Z_0(H_2) = 1$ because $\{1\}$ is a zero forcing set for H_2 with forces $\{1, 3\} \rightarrow 2$, $\{1, 2\} \rightarrow 3$, and $\{2, 3\} \rightarrow 4$, which implies $Z_0(H_2) \leq 1$. Note that $Z_0(H_2) \geq 1$ since it was shown that $M_0(H_2) = 1$ in Example 1.3.

When H is defined as a hypergraph, the only color change and zero forcing definitions that apply are those in Definition 1.5, so “hypergraph” may be omitted from the terminology. We use the symbol Z_0 that is associated with skew zero forcing, so for a graph G it does not matter whether we view G as a graph or a 2-hypergraph when writing $Z_0(G)$, but for a graph the correct term is “skew zero forcing.” The case $d = 2$ in Remark 1.10 coincides with the known result $Z_0(G) \leq n - 2$ for a graph G that has an edge [13].

Remark 1.10. Let $n \geq d \geq 2$ and H be a d -hypergraph H on n vertices that has an edge. Then, $Z_0(H) \leq n - d$, because we can choose any one edge $e = \{w_1, \dots, w_d\}$ and color the remaining $n - d$ vertices blue. Define $S_i = \{w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_d\}$. Then $S_i \rightarrow w_i$ for $i = 1, \dots, d$, so $Z_0(H) \leq n - d$.

The *degree* of a vertex v of a hypergraph H is the number of edges that contain v and is denoted by $\deg(v)$ (or $\deg_H(v)$ if the hypergraph is not clear).

Remark 1.11. Let $n \geq d \geq 2$ and H be a d -hypergraph H on n vertices. Suppose $\deg(v) = 1$ and let $e = \{v, w_2, \dots, w_d\}$ be the edge that contains v . Then every vertex in e except v can be colored blue by the empty set, because $\{v, w_2, \dots, w_{i-1}, w_{i+1}, \dots, w_d\} \rightarrow w_i$. If edge e has two or more vertices of degree one, then all vertices in e can be colored blue by the empty set.

Remark 1.11 illustrates a feature of hypergraph zero forcing that is significantly different from skew zero forcing on graphs: In a d -hypergraph with $d \geq 3$, a vertex may participate in any number of forces (by combining it with distinct sets of other vertices), whereas in a graph a vertex acts alone to force and thus may perform at most one force.

2 Comparison of zero forcing and infection and power domination for uniform hypergraphs

In this section we compare the extension of zero forcing to uniform hypergraphs discussed in Section 1 to other extensions of zero forcing to hypergraphs and show our definition is the best upper bound for maximum nullity among these definitions.

2.1 Infection for hypergraphs

Bergen et al. defined the infection number of a hypergraph as a generalization of the zero forcing number of a graph [3]. In this section we show that the infection number of a uniform hypergraph H is at least as large as the zero forcing number of H . Suppose H is a hypergraph with a set B of infected vertices (and vertices in $V(H) \setminus B$ are uninfected). The *infection rule* [3] allows a non-empty set $S \subseteq B$ of infected vertices to infect all the other vertices in an edge $e \in E$ if

- (1) $S \subset e$, and
- (2) if u is an uninfected vertex and $u \notin e$, then $S \cup \{u\} \not\subseteq e'$ for every edge e' .

An *infection set* is a set B such that if initially the set of infected vertices is B , then every vertex can be infected by repeated applications of the infection rule. The *infection number* of H , denoted by $I(H)$, is the minimum cardinality of an infection set.

Theorem 2.1. *Suppose H is a uniform hypergraph. Then any infection set for H is a zero forcing set for H and*

$$Z_0(H) \leq I(H).$$

Proof. Observe that condition (ii) in the hypergraph color change rule could be restated as (2') if u is a white vertex and $u \notin e := S \cup \{w\}$, then $S \cup \{u\} \not\subseteq e'$ for every edge e' of H , which more clearly parallels the infection rule condition (2). Differences include that

- (a) a set S of vertices need not be blue/infected to apply the hypergraph color change rule, and
- (b) a set of maximum cardinality is used for the hypergraph color change rule.

The first of these properties makes it easier to perform a force, and given that vertices need not be blue to apply the hypergraph color change rule, choosing a maximal set makes it easier to perform a force. Thus every infection set is a zero forcing set and $Z_0(H) \leq I(H)$. \square

The next example shows equality is possible in Theorem 2.1.

Example 2.2. Let H be the 3-hypergraph shown in Figure 1.1(b). Then $I(H) = 1$ because $I(G) \geq 1$ for every hypergraph G and $\{1\}$ is an infection set: $\{1\}$ infects $\{2,3\}$ and $\{2,3\}$ infects $\{4\}$. It was shown in Example 1.9 that $Z_0(H) = 1$.

It is also possible to have an arbitrarily large separation between $Z_0(H)$ and $I(H)$ for any $d \geq 3$. For $d \geq 3$ and $p \geq 2$, define a d -hypergraph $S_p^{(d)} = (V, E)$ by $V = \{0, 1, 2, \dots, p(d-1)\}$, and $E = \{e_i := \{0, (i-1)(d-1) + 1, (i-1)(d-1) + 2, \dots, i(d-1)\} : i = 1, \dots, p\}$. The hypergraph in Example 1.6 is $S_2^{(3)}$, and $S_p^{(d)}$ is called a *star* because $|\cap_{e \in E} e| = 1$.

Example 2.3. Remark 1.11 implies that $Z_0(S_p^{(d)}) = 0$, since every edge has at least two vertices of degree one (because $d \geq 3$). It is shown that $I(S_p^{(d)}) = p - 1$ in [3] ($S_p^{(d)}$ is one of the graphs there called a flower).

2.2 Power domination zero forcing for hypergraphs

For graphs it was shown in [2] that power domination as defined in [11] can be viewed as a domination step followed by a zero forcing process. Chang and Roussel extended power domination (and more generally k -power domination) to hypergraphs in [6]. In this section we identify the hypergraph zero forcing process in [6] and show that for uniform hypergraphs the associated zero forcing number is at least as large as the infection number.

Vertex w is a *neighbor* of vertex v if there is an edge that contains both v and w . A *power dominating set* of a hypergraph $H = (V, E)$ is a set D of vertices that observes all vertices according to the *observation rules* [6], where the first rule is applied once and the second rule is applied repeatedly:

- (1) A vertex in D observes itself and all its neighbors.
- (2) If v is observed and all the unobserved neighbors of v are in one edge that contains v , then all these unobserved neighbors of v become observed as well.

The *power domination number* $\gamma_p(H)$ is the minimum cardinality of a power dominating set of H . The second rule can be interpreted as a color change rule, here called the *power domination color change rule*:

- If all the white neighbors of a blue vertex v are in one edge that contains v , then all these white neighbors of v change color to blue.

A (*hypergraph*) *power domination zero forcing set* is a set $B \subseteq V(H)$ such that if initially the set of blue vertices is B , then every vertex can be colored blue by repeated applications of the power domination color change rule. The *power domination zero forcing number* $Z_{\text{pd}}(H)$ of a hypergraph H is the minimum cardinality of a power domination zero forcing set of H .

Theorem 2.4. *For a hypergraph H , any power domination zero forcing set is an infection set and $I(H) \leq Z_{\text{pd}}(H)$.*

Proof. The power domination hypergraph color change rule is the same as the infection rule with the restriction that S be a single vertex. Thus, whenever the power domination hypergraph color change rule can be applied, so can the infection rule. \square

Corollary 2.5. *For a uniform hypergraph H ,*

$$M_0(H) \leq Z_0(H) \leq I(H) \leq Z_{\text{pd}}(H).$$

The next two examples show that it is possible to have all three parameters Z_0 , I , and Z_{pd} equal or distinct.

Example 2.6. Let H be the 3-hypergraph shown in Figure 1.1(b). It was shown in Examples 1.9 and 2.2 that $Z_0(H) = I(H) = 1$. Also, $Z_{\text{pd}}(H) = 1$ because $I(G) \leq Z_{\text{pd}}(G)$ for every hypergraph G and $\{1\}$ is a power domination zero forcing set: 1 colors 2 and 3 blue, and 2 colors 4 blue.

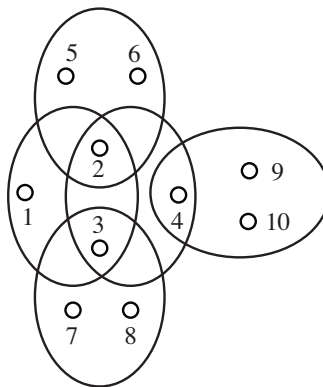


Figure 2.1: The hypergraph H in Example 2.7

Example 2.7. Let H be the 3-hypergraph shown in Figure 2.1. We show that $Z_0(H) = 0$, $I(H) = 1$, and $Z_{\text{pd}}(H) = 2$.

- $Z_0(H) = 0$ because edges $\{2, 5, 6\}$, $\{3, 7, 8\}$, and $\{4, 9, 10\}$ each have two degree-one vertices so all vertices except 1 can be colored blue, and then $\{2, 3\} \rightarrow 1$.
- $I(H) = 1$ because $I(H) \geq 1$ for every hypergraph H and $\{1\}$ is an infection set: $\{1\}$ infects $\{2,3\}$, $\{2,3\}$ infects $\{4\}$, $\{2\}$ infects $\{5,6\}$, $\{3\}$ infects $\{7,8\}$, and $\{4\}$ infects $\{9,10\}$.
- $Z_{\text{pd}}(H) = 2$ because we show no one vertex of H is a power domination hypergraph zero forcing set and $\{1,5\}$ is a power domination hypergraph zero forcing set.
 - The only one vertex sets we need to consider as possible power domination hypergraph zero forcing sets are those vertices that in exactly one edge. We consider each in turn.
 - * $D = \{1\}$: 1 colors 2 and 3. 2 has white neighbors in $\{2,5,6\}$ and $\{2,3,4\}$. 3 has white neighbors in $\{3,7,8\}$ and $\{2,3,4\}$.
 - * $D = \{5\}$ ($D = \{6\}$, $\{7\}$ and $\{8\}$ are similar): 5 colors 2 and 6. 2 has white neighbors in $\{1,2,3\}$ and $\{2,3,4\}$. 6 has no white neighbors.
 - * $D = \{9\}$ ($D = \{10\}$ is similar): 9 colors 4 and 10. 4 colors 2 and 3. 2 has white neighbors in $\{2,5,6\}$ and $\{2,3,4\}$. 3 has white neighbors in $\{3,7,8\}$ and $\{2,3,4\}$. 10 has no white neighbors.
 - $D = \{1,5\}$: 1 colors 2 and 3 blue, 5 colors 6 blue, 2 colors 4 blue, 3 colors 7 and 8 blue, and 4 colors 9 and 10 blue.

Example 2.3 shows that the difference between the infection number and the hypergraph zero forcing number is unbounded for a d -hypergraph (independent of d), Example 3.17 in the next section shows that the difference between the power domination zero forcing number and the infection number is unbounded if d is allowed to go to infinity.

3 Families and further results

In this section we study hypergraph zero forcing and maximum nullity further. In particular, we determine the values of these parameters for several families of hypergraphs and study the effect of certain hypergraph operations on the zero forcing number. We begin with some definitions.

Let H be a d -hypergraph. A d -subhypergraph of H is a d -hypergraph \tilde{H} such that $V(\tilde{H}) \subseteq V(H)$ and $E(\tilde{H}) \subseteq E(H)$; in this case H is a d -superhypergraph of \tilde{H} . A d -subhypergraph \tilde{H} of H is *induced* if $E(\tilde{H}) = \binom{V(\tilde{H})}{d} \cap E(H)$ where $\binom{S}{d}$ denotes the set of all d -element subsets of the set S ; $H[U]$ denotes the induced subhypergraph with vertex set $U \subseteq V(H)$.

A *path* in a hypergraph H is a vertex-hyperedge alternating sequence

$$v_1, e_1, v_2, e_2, \dots, v_s, e_s, v_{s+1}$$

such that v_1, \dots, v_{s+1} are distinct vertices, e_1, \dots, e_s are distinct hyperedges, and $v_i, v_{i+1} \in e_i$ for $i = 1, \dots, s$ [4]; such a path is also called a *path from v_1 to v_{s+1}* . A hypergraph H is *connected* if for any two vertices u and v of H there is a path in H from u to v . A *connected component* of a hypergraph $H = (V, E)$ is a maximal set of vertices U such that the hypergraph $H[U]$ is connected; in this case, the hypergraph $H[U]$ is called a *connected component hypergraph*. A vertex v is *isolated* in a hypergraph H if it is not in any edge of H ; a connected hypergraph with more than one vertex has no isolated vertices.

Suppose that the d -hypergraph H has connected component hypergraphs H_1, \dots, H_c . It is noted in [3] that $I(H) = \sum_{i=1}^c I(H_i)$. Similarly, it is immediate from the color change rule and the power domination color change rule that $Z_0(H) = \sum_{i=1}^c Z_0(H_i)$ and $Z_{\text{pd}}(H) = \sum_{i=1}^c Z_{\text{pd}}(H_i)$. Let A be a graphical d -hypermatrix, and suppose that $\mathcal{H}(A)$ has c connected components V_1, \dots, V_c with connected component hypergraphs $H_i = \mathcal{H}(A)[V_i]$, $i = 1, \dots, c$. Then $b_d(A)^T$ is a block diagonal matrix with the i th diagonal block C_i associated with H_i , so $\text{null } A = \sum_{i=1}^c \text{null } C_i$. Thus, $M_0(H) = \sum_{i=1}^c M_0(H_i)$. Since all of the parameters sum across connected components, it is common to focus on connected hypergraphs.

3.1 Zero forcing number and maximum nullity for families of hypergraphs

In this section we study the zero forcing number and maximum nullity of complete hypergraphs, linear hypergraphs, interval hypergraphs (analogous to path graphs), and circular-arc hypergraphs (analogous to cycle graphs). The complete d -hypergraph on $n \geq d$ vertices has all possible edges, i.e., $K_n^{(d)} = \left([n], \binom{[n]}{d} \right)$.

Proposition 3.1. *For $n \geq d$, $Z_0(K_n^{(d)}) = n - d$.*

Proof. By Remark 1.10, $Z_0(K_n^{(d)}) \leq n - d$. Consider any set B of $n - d - 1$ blue vertices, so there are $d + 1$ white vertices. Then any set S of $d - 1$ vertices omits at least two white vertices, say w and u . Thus S cannot force w because u violates condition (ii). Therefore, $Z_0(K_n^{(d)}) \geq n - d$. \square

Next we show that $M_0(K_n^{(d)}) < Z_0(K_n^{(d)})$ for $n \geq d + 1$. This parallels the situation for zero-diagonal maximum nullity of complete graphs: $M_0(K_n) = n - 3 < n - 2 = Z_0(K_n)$ [10].

Proposition 3.2. *For $n \geq d + 1$, $M_0(K_n^{(d)}) \leq n - d - 1$. In particular, $M_0(K_{d+1}^{(d)}) = 0$.*

Proof. Define $r_1 = [d + 1] \setminus \{d, d + 1\}$, $r_2 = [d + 1] \setminus \{d - 1, d + 1\}$, $r_3 = [d + 1] \setminus \{d - 1, d\}$, and $r_i = [d + 1] \setminus \{d - i + 2, d + 1\}$ for $i = 4, \dots, d + 1$. Let $\alpha = \{r_1, \dots, r_{d+1}\}$. Then

$$A^b[\alpha, [d + 1]] = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 0 & b_{d+1} & b_d \\ 0 & 0 & \dots & 0 & 0 & b_{d+1} & 0 & b_{d-1} \\ 0 & 0 & \dots & 0 & 0 & b_d & b_{d-1} & 0 \\ 0 & 0 & \dots & 0 & b_{d+1} & 0 & 0 & b_{d-2} \\ 0 & 0 & \dots & b_{d+1} & 0 & 0 & 0 & b_{d-3} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & b_{d+1} & \dots & 0 & 0 & 0 & 0 & b_2 \\ b_{d+1} & 0 & \dots & 0 & 0 & 0 & 0 & b_1 \end{bmatrix}$$

where $b_i = a_{1, \dots, i-1, i-1, \dots, d+1}$. From the form of $A^b[\alpha, [d + 1]]$ and successive Laplace expansions on the first column,

$$\begin{aligned} \det(A^b[\alpha, [d + 1]]) &= \pm (b_{d+1})^{d-2} \det \begin{bmatrix} 0 & b_{d+1} & b_d \\ b_{d+1} & 0 & b_{d-1} \\ b_d & b_{d-1} & 0 \end{bmatrix} \\ &= \pm 2(a_{1, \dots, d})^{d-1} a_{1, \dots, d-2, d-1, d+1} a_{1, \dots, d-2, d, d+1} \neq 0. \end{aligned}$$

Since $\text{rank } A^b \geq \text{rank } A^b[\alpha, [d + 1]] = d + 1$, $\text{null } A = \text{null } A^b \leq n - d - 1$. \square

Remark 3.3. For infection, it is shown in [3] that $I(K_n^{(d)}) = n - d + 1$. The infection set given there (color blue all the vertices outside one edge e , and also color one vertex of e blue) also works for power domination zero forcing, and the infection number is a lower bound for power domination zero forcing number, so $Z_{\text{pd}}(K_n^{(d)}) = n - d + 1$.

A hypergraph is *linear* if distinct edges intersect in at most one vertex. For example, the star $S_p^{(d)}$ is linear. Every graph is linear when viewed as a 2-hypergraph; however, Corollary 3.5 below shows that the hypergraph zero forcing numbers of linear d -hypergraphs with $d \geq 3$ do not behave like those of linear 2-hypergraphs, which are graphs (see [13] for more information about the skew zero forcing numbers of graphs). The next, more general, result explains why.

Proposition 3.4. *Let H be a d -hypergraph with no isolated vertices such that $|e \cap e'| \leq d - 2$ for every pair of distinct edges e and e' of H . Then, $M_0(H) = Z_0(H) = 0$.*

Proof. Since H does not have isolated vertices, for each $v \in V(H)$ there is some edge e_v such that $v \in e_v$; let $S_v = e_v \setminus \{v\}$. Then $|S_v| = d - 1 > |e_v \cap e'|$ for every edge $e' \neq e$. Thus, $S_v \not\subseteq e'$ for $e' \neq e$ and $S_v \rightarrow v$. So H can be forced by the empty set. That $M_0(H) = 0$ follows from $0 \leq M_0(H) \leq Z(H)$ by Theorem 1.8. \square

Since $|e \cap e'| \leq 1$ for a linear d -hypergraph, the next result is immediate.

Corollary 3.5. *If $d \geq 3$ and H is a linear d -hypergraph with no isolated vertices, then $M_0(H) = Z_0(H) = 0$.*

It was established in [3] that $I(H) \leq 2$ for a connected linear d -hypergraph H in which all vertices have degree at most two and $I(H) = 1$ if and only if in addition H has a vertex of degree one. For the zero forcing number of a linear d -hypergraph with $d \geq 3$, the degree restriction is unnecessary, and in place of connected we need require only no isolated vertices. However, the restriction on the degree is essential for infection number: The infection number (and therefore also the power domination zero forcing number) is unbounded for linear d -hypergraphs as illustrated by stars, for which $I(S_p^{(d)}) = p - 1$ (see Example 2.3).

Recall that a *path graph* P_n is a graph for which there is a linear order of vertices v_1, \dots, v_n such that $E(P_n) = \{\{v_i v_{i+1}\} : i = 1, \dots, n-1\}$ (this is equivalent to defining a path as already done for hypergraphs and defining a path graph to be a graph whose vertices and edges are those of a path). Path graphs can be generalized to hypergraphs as interval hypergraphs.

An *interval d -hypergraph*² is a d -hypergraph H for which there is a linear order of vertices v_1, \dots, v_n such that every edge is of the form $\{v_\ell, v_{\ell+1}, \dots, v_{\ell+d-1}\}$ for some $\ell \in \{1, \dots, n - d + 1\}$. A graph is a connected interval 2-hypergraph if and only if it is a path graph. It was established in [3] that $I(H) = 1$ for a connected interval d -hypergraph H . Thus the zero forcing number for an interval d -hypergraph is zero or one. Example 3.6 below provides for each $d \geq 2$ an infinite family of interval d -hypergraphs H with $M_0(H) = Z_0(H) = 1$, and we show in Theorem 3.10 that all other d -hypergraphs have $M_0(H) = Z_0(H) = 0$.

It is convenient to assume that the vertices of an interval hypergraph are denoted by $1, \dots, n$, and we make this assumption. An interval d -hypergraph is determined by its number of vertices n and its *left endpoint set* $L(H) = \{\ell_i : e_i = \{\ell_i, \ell_i + 1, \dots, \ell_i + d - 1\}, i = 1, \dots, m\}$ where the edges are $e_i, i = 1, \dots, m$ in order, meaning that $i < j$ implies $\ell_i < \ell_j$. If H has no isolated vertices, then $\ell_1 = 1$, and $\ell_m = n - d + 1$. Since we are interested in generalizing the idea of path, we are interested in connected interval d -hypergraphs. This is equivalent to requiring that $\ell_{i+1} \leq \ell_i + d - 1$. If H is a connected interval d -hypergraph with an edge, then $1, e_1, \ell_2, e_2, \dots, \ell_m, e_m, n$ is a path that includes every edge of H . The 3-hypergraphs shown in Figure 1.1 are both connected interval 3-hypergraphs on 5 and 4 vertices with left endpoint sets $\{1, 3\}$ and $\{1, 2\}$, respectively. Note that for any interval hypergraph on n vertices, $\deg(1) = 1 = \deg(n)$.

The 3-hypergraph H_2 shown in Figure 1.1(b) and discussed in Examples 1.4 and 1.9 is actually the smallest member of the unique family of interval 3-hypergraphs that have zero forcing number equal to one, as discussed in the next example and theorem.

Example 3.6. For integers $d \geq 2$ and $s \geq 1$, let $SI^{(d)}(s)$ denote the *special interval d -hypergraph* defined by $n = sd + 1$ and $L(H) = \{(i - 1)d + 1, (i - 1)d + 2 : i = 1, \dots, s\}$; $SI^{(3)}(1)$ is shown in Figure 3.1. Examination of Equation (2) shows that the vector $\mathbf{x} = [x_j]$ defined by $x_{(i-1)d+1} = (-1)^i$ for $i = 1, \dots, s$ and all other entries equal to 0 is a null vector of the adjacency matrix of $SI^{(d)}(s)$. Since $Z_0(SI^{(d)}(s)) \leq I(SI^{(d)}(s)) = 1$, $M_0(SI^{(d)}(s)) =$

²The reader is warned that the term *interval graph* often means a graph that can be modeled by associating vertices to intervals of the real line with two vertices adjacent if and only if the corresponding intervals overlap.

$Z_0(SI^{(d)}(s)) = 1$. Note that $SI^{(d)}(s)$ has $2s$ edges and d divides $n-1$. Since $SI^{(2)}(s) = P_{2s+1}$, for $d = 2$ this is consistent with the known result that maximum nullity and skew zero forcing number of an odd path are one.

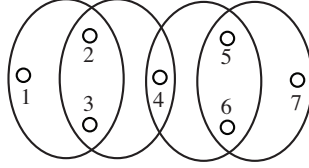


Figure 3.1: The special interval hypergraph $SI^{(3)}(2)$

We use a series of lemmas to show that a connected interval d -hypergraph has zero forcing number and maximum nullity equal to zero unless it is a special interval d -hypergraph.

Lemma 3.7. *For any connected interval uniform hypergraph H on n vertices, $\{1\}$ and $\{n\}$ are zero forcing sets for H .*

Proof. It is shown in the proof of [3, Lemma 4.2] that $\{1\}$ is an infection set for any such H , and a similar argument applies to n . The result then follows from Theorem 2.1. \square

Lemma 3.8. *Let H be a connected interval d -hypergraph determined by n and $L(H) \subset [n]$ with m edges. For $2 \leq k \leq n-1$, the empty set can force k unless there exists i such that $k = \ell_i = \ell_{i-1} + d - 1$ and $\ell_{i+1} = k + 1$. The empty set can force 1 unless $\ell_2 = 2$, and the empty set can force n unless $\ell_{m-1} = n - d$.*

Proof. Assume $2 \leq k \leq n-1$. For $k \in \{\ell_i + 1, \dots, \ell_i + d - 2\}$ no edge except e_i contains $e_i \setminus \{k\}$. Thus $e_i \setminus \{k\} \rightarrow k$ for $k \in \{\ell_i + 1, \dots, \ell_i + d - 2\}$. Thus k can be colored blue by the empty set unless k is in exactly two consecutive edges e_{i-1} and e_i and $k = \ell_i = \ell_{i-1} + d - 1$. Furthermore, if $\ell_{i+1} \neq \ell_i + 1$, then $\deg(\ell_i + 1) = 1$ so $e_i \setminus \{\ell_i\} \rightarrow \ell_i$. The cases $k = 1$ and $k = n$ are simpler. \square

Lemma 3.9. *Let H be a connected interval d -hypergraph determined by n and $L(H) \subset [n]$. Any set of d consecutive vertices is a zero forcing set.*

Proof. Suppose $B = \{j_1, j_1 + 1, \dots, j_2 - 1, j_2\}$ is a maximal set of consecutive blue vertices with $j_2 - j_1 \geq d - 1$. If $j_2 = n$, then B is a zero forcing set by Lemma 3.7. So assume $j_2 + 1$ exists; $j_2 + 1$ is white by the maximality of B . We show $j_2 + 1$ can be colored blue, and then repeat this procedure until n is colored blue.

If there does not exist an i such that $j_2 + 1 = \ell_i = \ell_{i-1} + d - 1$, then Lemma 3.8 shows that $j_2 + 1$ can be colored blue by the empty set. So assume $j_2 + 1 = \ell_i = \ell_{i-1} + d - 1$ for some i . Since $j_2 - j_1 \geq d - 1$, $\ell_{i-1} - 1 \geq j_1$. The only edges that can possibly contain $\{\ell_{i-1}, \dots, \ell_{i-1} + d - 2\} = e_{i-1} \setminus \{j_2 + 1\}$ are e_{i-1} and e_{i-2} . If e_{i-2} contains $e_{i-1} \setminus \{j_2 + 1\}$, then $\ell_{i-2} = \ell_{i-1} - 1$, which is blue. So $e_{i-1} \setminus \{j_2 + 1\} \rightarrow j_2 + 1$. \square

Theorem 3.10. *Let $d \geq 3$ and let H be a connected interval d -hypergraph determined by n and $L(H) \subset [n]$.*

$$M_0(H) = Z_0(H) = \begin{cases} 1 & \text{if } d \text{ divides } n - 1 \text{ and } H = SI^{(d)}\left(\frac{n-1}{d}\right); \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It was shown in Example 3.6 that $M_0(SI^{(d)}(s)) = Z_0(SI^{(d)}(s)) = 1$. Since $Z_0(H) \leq I(H) \leq 1$ by Theorem 2.1 and [3], it suffices to show that $Z_0(H) \neq 0$ implies $H = SI^{(d)}(s)$. So assume the empty set has failed to color all vertices of H blue and no additional forces are possible. Then $\ell_1 = 1$ is white by Lemma 3.7 and $\ell_2 = 2$ by Lemma 3.8. Then $e_1 \setminus \{k\} \rightarrow k$ for $k = 2, \dots, d-1$ and $e_2 \setminus \{d\} \rightarrow d$. Thus $2, \dots, d$ are blue. Since there cannot be d consecutive blue vertices by Lemma 3.9, $d+1$ is white, so $\ell_3 = d+1$ and $\ell_4 = d+2$. Proceeding in order we construct $SI^{(n)}(d)$. \square

The conclusion of Theorem 3.10 is valid for 2-hypergraphs also, since it is known that $Z_0(P_n) = 0$ for n even and $Z_0(P_n) = 1$ for n odd, but the proof given here needs $d \geq 3$.

Interval hypergraphs also behave nicely for power domination zero forcing.

Proposition 3.11. *Let H be a connected interval d -hypergraph. Then $Z_{\text{pd}}(H) = 1$.*

Proof. Note first that $1 = I(H) \leq Z_{\text{pd}}(H)$. Let $B = \{1\}$. Then 1 can color blue any white vertices in e_1 . Let $j \geq 2$ and assume that all the vertices in e_i are blue for $i \leq j-1$; this implies ℓ_j is blue. Observe that $\ell_j \notin e_i$ for $i > j$. So ℓ_j can color blue any white vertices of e_j . By repeating this process, all the vertices of H can be colored blue. Thus $Z_{\text{pd}}(H) = 1$. \square

It is noted in [3] that $I(H) = c$ for an interval d -hypergraph H with c connected components, and $Z_{\text{pd}}(H) = c$ by similar reasoning. The characterization of hypergraph zero forcing number also extends to interval d -hypergraphs that are not connected, but we need to consider the types of connected component hypergraphs.

Corollary 3.12. *Let H be an interval d -hypergraph, let c_1 denote the number of isolated vertices in H , and let c_2 denote the number of connected components H_i such that $H_i = SI^{(d)}(s_i)$ for some s_i . Then $M_0(H) = Z_0(H) = c_1 + c_2$.*

A *cycle* in a hypergraph H is a vertex-hyperedge alternating sequence

$$v_1, e_1, v_2, e_2, \dots, v_s, e_s, v_1$$

such that v_1, \dots, v_s are distinct vertices, e_1, \dots, e_s are distinct hyperedges, and $v_i, v_{i+1} \in e_i$ for $i = 1, \dots, s$ (with $s+1$ interpreted as 1) [4]. Recall that a *cycle graph* C_n is a graph for which there is a cyclic order of vertices v_1, \dots, v_n such that $E(C_n) = \{\{v_i v_{i+1}\} : i = 1, \dots, n\}$ (with $n+1$ interpreted as 1). The definitions of cycle and interval hypergraph lead naturally to the idea of a circular-arc hypergraph, which generalizes the idea of cycle graphs to hypergraphs [3]. A *circular-arc d -hypergraph* is a d -hypergraph H on $n \geq d+1$ vertices for which there is a cyclic order of vertices v_1, \dots, v_n such that every edge is of the form $\{v_\ell, v_{\ell+1}, \dots, v_{\ell+d-1}\}$ for some $\ell \in \{1, \dots, n\}$ (with $n+k$ interpreted as k); examples are shown in Figures 3.2 and 3.3.

As with an interval d -hypergraph, we assume $V = [n]$, so a circular-arc d -hypergraph is determined by n and the *first endpoint set* $L(H) = \{\ell_i : e_i = \{\ell_i, \ell_i + 1, \dots, \ell_i + d - 1\}, i = 1, \dots, m\}$ where the edges are $e_i, i = 1, \dots, m$ and $i < j$ implies $\ell_i < \ell_j$; by convention, $\ell_1 = 1$. We are interested in generalizing the idea of a cycle graph and want $1, e_1, \ell_2, e_2, \dots, \ell_m, e_m, 1$ to be a cycle that includes every edge of H . Thus we also assume that $\ell_{i+1} \leq \ell_i + d - 1$ for $i = 1, \dots, m - 1$ and $\ell_m \geq n - d + 2$. For $n \geq d + 2$, we obtain a characterization of zero forcing number and maximum nullity that parallels the characterization of these parameters for interval d -hypergraphs in Theorem 3.10.

Lemma 3.13. *Let H be a circular-arc d -hypergraph determined by $n \geq d + 2$ and $L(H) \subset [n]$ with $\ell_{i+1} \leq \ell_i + d - 1$ for $i = 1, \dots, m - 1$ and $\ell_m \geq n - d + 2$. The empty set can force k unless there exists i such that $k = \ell_i = \ell_{i-1} + d - 1$ and $\ell_{i+1} = k + 1$.*

Proof. Note that for $k \in \{\ell_i + 1, \dots, \ell_i + d - 2\}$ no edge except e_i contains $e_i \setminus \{k\}$ because each edge proceeds in cyclic order and the distance going the reverse order is longer since $n \geq d + 2$. Thus $e_i \setminus \{k\} \rightarrow k$ for $k \in \{\ell_i + 1, \dots, \ell_i + d - 2\}$. Furthermore, if $\ell_{i+1} \neq \ell_i + 1$, then $\deg(\ell_i + 1) = 1$ so $e_i \setminus \{\ell_i\} \rightarrow \ell_i$. Thus the empty set can force k unless there exists i such that $k = \ell_i = \ell_{i-1} + d - 1$ and $\ell_{i+1} = k + 1$. \square

The proof of the next lemma is analogous to that of Lemma 3.9.

Lemma 3.14. *Let H be a circular-arc d -hypergraph determined by $n \geq d + 2$ and $L(H) \subset [n]$ with $\ell_{i+1} \leq \ell_i + d - 1$ for $i = 1, \dots, m - 1$ and $\ell_m \geq n - d + 2$. Any set of d consecutive vertices is a zero forcing set.*

For integers $d \geq 2$ and $s \geq 2$, $SCA^{(d)}(s)$ denotes the *special circular arc d -hypergraph* defined by $n = sd$ and $L(H) = \{(i - 1)d + 1, (i - 1)d + 2 : i = 1, \dots, s\}$; $SCA^{(3)}(3)$ is shown in Figure 3.2. Note that $SCA^{(d)}(s)$ can be constructed from $SI^{(d)}(s)$ by identifying vertices 1 and $sd + 1$.

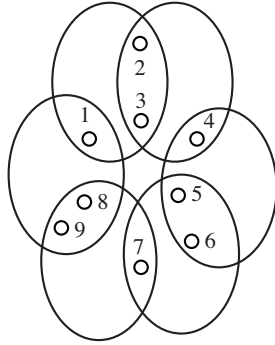


Figure 3.2: The special circular arc hypergraph $SCA^{(3)}(3)$

Theorem 3.15. *Let $d \geq 3$ and let H be a circular-arc d -hypergraph determined by $n \geq d + 2$ and $L(H) \subset [n]$ with $\ell_{i+1} \leq \ell_i + d - 1$ for $i = 1, \dots, m - 1$ and $\ell_m \geq n - d + 2$.*

$$M_0(H) = Z_0(H) = \begin{cases} 1 & \text{if } d \text{ divides } n \text{ and } H = SCA^{(d)}\left(\frac{n}{d}\right); \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First consider $H = SCA^{(d)}\left(\frac{n}{d}\right)$. For edge $e = \{i_1, \dots, i_d\}$ let a_e denote the value of $a_{i_1 \dots i_d}$ (with entries in all possible orders). Define $A \in \mathcal{S}_0(SCA^{(d)}\left(\frac{n}{d}\right))$ by $a_{e_{2(i-1)+1}} = -1$ and $a_{e_{2i}} = 1$ for $i = 1, \dots, s$. Examination of Equation (2) shows that the vector $\mathbf{x} = [x_j]$ defined by $x_{(i-1)d+1} = 1$ for $i = 1, \dots, s$ and all other entries equal to 0 is a null vector of A . Thus $M_0(SCA^{(d)}(s)) \geq 1$. Since $d \geq 3$, the empty set can color blue every vertex except those of the form $d(i-1) + 1, i = 0, \dots, s-1$ by Lemma 3.13. Let $B = \{1\}$. Then the d consecutive vertices $1, \dots, d$ are blue, so all vertices can be colored blue by Lemma 3.14. Thus $Z_0(SCA^{(d)}(s)) \leq 1$, so $M_0(SCA^{(d)}(s)) = Z_0(SCA^{(d)}(s)) = 1$.

The proof that $Z_0(H) \neq 0$ implies $H = SCA^{(d)}\left(\frac{n}{d}\right)$ is analogous to the proof in Theorem that $H = SI^{(d)}\left(\frac{n-1}{d}\right)$ if we begin by choosing a white vertex to label as $\ell_1 = 1$ (after performing all possible forces using the empty set). \square

The behavior of the zero forcing number of a circular arc d -hypergraph graph H with $d \geq 3$ and $n \geq d+2$ differs somewhat from that of the skew zero forcing number of a cycle, because $Z_0(H) \leq 1$ and $Z_0(C_n) = 2$ for n even (whether viewed as a 2-hypergraph or a graph).

The situation is much more complicated for the infection number, and results in [3] are obtained primarily for a t -tight circular-arc d hypergraph, which is defined (in our notation) by choosing positive integers $d \geq 3$, $t \leq d-1$, and $s \geq \frac{d+1}{d-t}$, and setting $n = s(d-t)$ and $\ell_i = (i-1)(d-t) + 1$ for $i = 1, \dots, s$; this hypergraph is denoted by $C_n^{(d)}(t)$. Figure 3.3 shows $C_{12}^{(5)}(2)$. Since $SCA^{(d)}(s)$ is not tight for $n \geq d+2$, we know that $Z_0(C_n^{(d)}(t)) = 0$ for $n \geq d+2$.

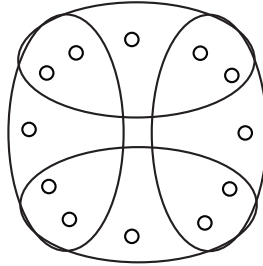


Figure 3.3: The 2-tight circular arc 5-hypergraph on 12 vertices

Proposition 3.16. For positive integers $d \geq 3$ and $t \leq d-1$ such that $d-t$ divides $d+1$,

$$0 \leq M_0(C_{d+1}^{(d)}(t)) \leq Z_0(C_{d+1}^{(d)}(t)) = 1.$$

Proof. Note that $Z_0(C_{d+1}^{(d)}(t)) \leq (d+1) - d = 1$ by Remark 1.10. To see that \emptyset is not a zero forcing set, observe that $v_i := (i-1)(d-t)$ is not in edge e_i for $i = 1, \dots, m := \frac{d+1}{d-t}$ (with 0 interpreted as n) but is in every other edge, and every other vertex is in every edge. Let $S \subset e_i$ with $|S| = d-1$, so S omits vertex v_i and one other vertex. If the other vertex is $w \neq v_j$ for $j = 1, \dots, m$, then S is a subset only of e_i and $S \rightarrow w$. Thus all vertices $w \neq v_j$ for $j = 1, \dots, m$ can be colored blue. Now assume S omits both v_i and v_j for some $j \neq i$. Then $S = e_i \cap e_j$ and each of e_i and e_j has a white vertex not in S , so S cannot perform a force. \square

The bounds on $M_0(C_{d+1}^{(d)}(t))$ are both tight. If we renumber so that the two vertices that are in both edges are 1 and 3, then Example 1.4 is $C_4^{(3)}(1)$, and $M_0(C_4^{(3)}(1)) = 1$. Note that $C_{d+1}^{(d)}(d-1) = K_{d+1}^{(d)}$ so $M_0(C_{d+1}^{(d)}) = 0$ by Proposition 3.2. That $Z_0(C_{d+1}^{(d)}(t)) = 1$ is significantly different from the complicated situation for infection number described in [3], where it is shown that $I(C_{d+1}^{(d)}(d-2)) = \frac{d-1}{2}$ when d is odd, implying that the infection number is unbounded as $d \rightarrow \infty$. The next example shows that $Z_{\text{pd}}(H) - I(H)$ can also be unbounded for tight circular arc d -hypergraphs as $d \rightarrow \infty$.

Example 3.17. Let $n \geq 2d - 1$. It is shown in [3] that $I(C_n^{(d)}(d-1)) = 2$. We show that $Z_{\text{pd}}(C_n^{(d)}(d-1)) = d$. Let $B = \{1, \dots, d\}$. The neighbors of d are $1, \dots, d-1, d+1, \dots, 2d-1$. Since $1, \dots, d-1$ are all blue and $d+1, \dots, 2d-1 \subset e_2$, d colors $d+1, \dots, 2d-1$ blue. We can proceed to color the entire hypergraph one edge at a time. Now suppose $B' \subset V(H)$ contains fewer than d vertices. If $v \in B'$, then at most $d-2$ other vertices are blue. Since every vertex has $2d-2$ neighbors, v has d white neighbors, which is too many to include in one edge together with v . So v cannot color any vertex blue.

3.2 Hypergraph operations and zero forcing number

Many results about the effect of operations on the infection number for hypergraphs [3], zero forcing number for graphs [1, 7], and skew zero forcing number for graphs [12, 13] have related versions for the zero forcing number for hypergraphs. In this section we exhibit several.

Let H be a d -hypergraph with $d \geq 2$. The d -hypergraph $H - e$ is obtained from H by deleting e from the set of edges of G . The d -hypergraph $H - v$ is obtained from H by deleting v from the set of vertices of H and deleting from the set of edges every edge that contains v . (For $d = 2$, these operations are the same as deleting an edge or a vertex from a graph.) For standard zero forcing on a graph G , there are nice bounds relating the zero forcing number of G and the graph resulting from deleting a vertex or an edge: $Z(G) - 1 \leq Z(G - v) \leq Z(G) + 1$ and $Z(G) - 1 \leq Z(G - e) \leq Z(G) + 1$ [7]. Things are not as simple for hypergraph zero forcing with $d \geq 3$. Since $Z_0(S_p^{(d)}) = 0$ for $d \geq 3$ and deleting the center vertex of $S_p^{(d)}$ leaves $p(d-1)$ isolated vertices, there is no useful upper bound on $Z_0(H - v)$ in terms of $Z_0(H)$ for $d \geq 3$.

Proposition 3.18. *Let H be a d -hypergraph with $d \geq 2$.*

1. $Z_0(H) - d \leq Z_0(H - e) \leq Z_0(H) + d$.
2. $Z_0(H) - 1 \leq Z_0(H - v)$.
3. For $d = 2$ (i.e., when H is a graph), $Z_0(H - v) \leq Z_0(H) + 1$.

All of these bounds are tight.

Proof. (1): If B is a zero forcing set for H , then $B \cup e$ is a zero forcing set for $H - e$, so $Z_0(H - e) \leq Z_0(H) + d$. If \hat{B} is a zero forcing set for $H - e$, then $\hat{B} \cup e$ is a zero forcing set for H , so $Z_0(H) \leq Z_0(H - e) + d$. A d -hypergraph with d vertices and one edge shows the upper

bound is tight. A cycle graph on an even number of vertices shows the lower bound is tight for $d = 2$, because deleting an edge produces a path graph, $Z_0(C_{2s}) = 2$, and $Z_0(P_{2s}) = 0$.

(2): If \hat{B} is a zero forcing set for $H - v$, then $B' \cup \{v\}$ is a zero forcing set for H , so $Z_0(H) - 1 \leq Z_0(H - v)$. The complete d -hypergraph $K_n^{(d)}$ shows the bound is tight.

(3): Let B be a zero forcing set for H . Since H is 2-hypergraph, v is involved in at most one force. If $\{v\} \rightarrow w$, then $B \cup \{w\}$ is a zero forcing set for $H - v$. If v does not perform a force, then B is a zero forcing set for $H - v$. Thus $Z_0(H - v) \leq Z_0(H) + 1$. A path graph on an even number of vertices with v adjacent to a vertex of degree one shows the bound is tight. \square

The proof of the next result is adapted from that of Proposition 3.2 in [3], where it is shown that for any d -hypergraph H with $d \geq 3$, there exists a superhypergraph H' such that $I(H') = 1$. However, our result applies to 2-hypergraphs also and matches the behavior of the skew zero forcing number on graphs, whereas it is noted in [3] that this behavior is very different from the behavior of the zero forcing number on graphs.

Proposition 3.19. *Let $H = (V, E)$ be a d -hypergraph with $d \geq 2$. There exists a d -hypergraph H' such that $H = H'[V]$ and $Z_0(H') = 0$.*

Proof. Let n denote the number of vertices of H . Partition V into $\lfloor \frac{n}{d-1} \rfloor$ sets of $d-1$ vertices, and possibly one additional set of less than $d-1$ vertices; denote these sets by W_1, \dots, W_k where $k = \lfloor \frac{n}{d-1} \rfloor$; if $|W_k| < d-1$ then add additional vertices to W_k to make its cardinality equal to $d-1$ (it does not matter which vertices are added). Define $H' = (V', E')$ by $V' = V \cup \{w_1, \dots, w_k\}$ and $E' = E \cup \{W_i \cup \{w_i\} : i = 1, \dots, k\}$. Then for $i = 1, \dots, k$ and $v \in W_i$, $(W_i \setminus \{v\}) \cup \{w_i\} \rightarrow v$, and then once every vertex of V is blue, $W_i \rightarrow w_i$ for $i = 1, \dots, k$. \square

For any set S and object v , define $S \times v = \{(x, v) : x \in S\}$; $v \times S$ is defined analogously. The *Cartesian product* of d -hypergraphs H and H' , denoted by $H \square H'$ is the d -hypergraph with vertex set $V(H) \times V(H')$ and edge set

$$\{e \times v' : e \in E(H), v' \in V(H')\} \cup \{v \times e' : v \in V(H), e' \in E(H')\}.$$

Lemma 3.20. *Let $H = (V, E)$ and $H' = (V', E')$ be d -hypergraphs with $d \geq 3$, $B', S' \subset V'$, $w' \in V' \setminus B'$, and $u \in V$. Suppose that when the vertices in B' are blue and those in $V' \setminus B'$ are white, S' can force w' in H' . Then in $H \square H'$ when the vertices in $u \times B'$ are blue and those in $u \times (V' \setminus B')$ are white, $u \times S'$ can force (u, w') .*

Proof. Since $d \geq 3$, $|S'| \geq 2$. Thus $u \times S' \not\subseteq e \times v'$ for any $e \in E$ and $v' \in V$. So if $u \times S'$ is contained in an edge of $H \square H'$, it is contained in an edge of the form $u \times e'$. Suppose $u \times S' \cup \{(u, v')\} = u \times e'$ and (u, v') is white. Then $e' = S' \cup \{v'\}$ and v' is white because all vertices in $u \times B'$ are blue, so $v' \notin B'$. Since $S' \rightarrow w'$ in H' , this implies $v' = w'$, i.e., $S' \cup \{(u, v')\} = S' \cup \{(u, w')\}$. Thus $u \times S' \rightarrow (u, w')$ in $H \square H'$. \square

Theorem 3.21. *Let $H = (V, E)$ and $H' = (V', E')$ be d -hypergraphs with $d \geq 3$. Then*

$$Z_0(H \square H') \leq Z_0(H) Z_0(H').$$

Proof. If $Z_0(H) = 0$ or $Z_0(H') = 0$, then assume without loss of generality that $Z_0(H') = 0$ and let $\widehat{B} = \emptyset$. Otherwise, let $\widehat{B} = B \times B'$ for minimum zero forcing sets B for H and B' for H' . We show that \widehat{B} is a zero forcing set for $H \square H'$. If $\widehat{B} = \emptyset$, then fix any vertex $u \in V(H)$. Otherwise, fix any vertex $u \in B$. Lemma 3.20 implies that the zero forcing process on H' can proceed in $H \square H'$ to color all the vertices in $u \times V(H')$. Thus all vertices are blue if $\widehat{B} = \emptyset$, and otherwise all vertices in $u \times V'$ are blue for every $u \in B$. In the latter case then apply the same reasoning using the zero forcing process in H to color all the remaining vertices. Therefore, $Z_0(H \square H') \leq |B||B'| = Z_0(H)Z_0(H')$. \square

In contrast to the simple upper bound on the zero forcing number of a Cartesian product in Theorem 3.21, the analogous bound for infection number requires defining the 2-infection number, in which at least two infected vertices are needed to infect (see [3]). Note also that for graphs G (i.e., 2-hypergraphs), it is not generally true that $Z_0(G \square G') \leq Z_0(G)Z_0(G')$: For example, $Z_0(K_2) = 0$ but $K_2 \square K_2' = C_4$ and $Z_0(C_4) = 2$.

For d -hypergraphs H and H' with $d \geq 3$, it is immediate that $Z_0(H) = 0$ or $Z_0(H') = 0$ implies $Z_0(H \square H') = Z_0(H)Z_0(H') = 0$, and we do not know of any examples where $Z_0(H \square H') < Z_0(H)Z_0(H')$. This naturally raises the next question.

Question 3.22. *Is it true that $Z_0(H \square H') = Z_0(H)Z_0(H')$ for all d -hypergraphs H and H' with $d \geq 3$?*

We answer Question 3.22 in the affirmative for hypergraphs with $Z_0(H') = 1$ in the next result, which of course also applies when $Z_0(H) = 1$.

Theorem 3.23. *Let $H = (V, E)$ and $H' = (V', E')$ be d -hypergraphs with $d \geq 3$ such that $Z_0(H') = 1$. Then $Z_0(H \square H') = Z_0(H)$.*

Proof. Let $\tilde{B} \subset V(H \square H')$ with $|\tilde{B}| < Z_0(H)$. Since $Z_0(H \square H') \leq Z_0(H)$ by Theorem 3.21, it suffices to show that \tilde{B} cannot color all vertices of $H \square H'$ blue. Let $B = \{u \in V(H) : \exists u' \in V(H') \text{ such that } (u, u') \in \tilde{B}\}$. Note that $B \leq |\tilde{B}| < Z_0(H)$, so B is not a zero forcing set for H . Let D denote the derived set of B (in H), and let D' denote the derived set of the empty set in H' . Observe that both $V \setminus D$ and $V' \setminus D'$ are nonempty. We show that none of the vertices in $(V \setminus D) \times (V' \setminus D')$ are in the derived set of \tilde{B} (in $H \square H'$).

By definition, every edge of $H \square H'$ is of the form $v \times e'$ or $e \times v'$ where $v \in V, e \in E, v' \in V',$ and $e' \in E'$. So if $\tilde{S} \subset V \times V'$ is a set of $d - 1$ vertices that can force (w, w') , then $\tilde{S} \subset v \times e'$ or $\tilde{S} \subset e \times v'$, but not both because $d \geq 3$ implies $|S'| \geq 2$. By Remark 1.7, the order in which we perform the forces does not matter, provided at the end we ensure all options have been tested. So first perform all possible forces using $\tilde{S} \subset v \times e'$. This will color (v, u') blue for all $v \in V$ and $u' \in D'$. Also some additional vertices of the form (u, v') with $u \in B$ and $v \in V'$ will be colored blue. Assuming additional vertices have been colored blue can not reduce the derived set of \tilde{B} . So assume (u, v') is blue for all $u \in B$ and $v' \in V'$. Observe that every vertex of the form (y, x') with $y \in V \setminus B$ and $x' \in V' \setminus D'$ is still white; this includes every vertex of the form (x, x') with $x \in V \setminus D$ and $x' \in V' \setminus D'$. Now perform all possible forces using $\tilde{S} \subset e \times v'$. After this set of forces, every vertex of the form $u \times v'$ is blue for $u \in D$ and $v' \in V'$, but no additional vertices of the form (x, v') for $x \in V \setminus D$ have been colored blue. Thus (x, x') is still white for every $x \in D$ and $x' \in D'$. Since no additional

vertices in $x \times V'$ have been colored blue, and all vertices in $u \times V'$ are blue for $u \in D$, no further forces using $\tilde{S} \subset v \times e'$ are possible, and thus no additional forces are possible. \square

Corollary 3.24. *Let $H = (V, E)$ and $H' = (V', E')$ be d -hypergraphs with $d \geq 3$ such that $Z_0(H)Z_0(H') \leq 3$. Then $Z_0(H \square H') = Z_0(H)Z_0(H')$.*

Proof. Without loss of generality, $Z_0(H') \leq Z_0(H)$. Since $Z_0(H)Z_0(H') \leq 3$, $Z_0(H') \leq 1$ and the result follows from Theorems 3.21 and 3.23. \square

4 Concluding remarks

It is not surprising that the zero forcing number defined in Section 1 is a better bound for maximum nullity than the infection number defined in [3], since the authors of [3] are explicit that they are not trying to generalize zero forcing as an upper bound for maximum nullity. They say, “For hypergraphs, there is no matrix analogous to the adjacency matrix of a graph and this notion of a set of entries in a vector forcing the other entries to be zero in a proposed null vector does not apply.” While it is true that for a hypergraph there is no *matrix* analogous to the adjacency matrix, we would argue that for a uniform hypergraph the natural analog is the adjacency hypermatrix. When using hypermatrices, the issue is not that there are no analogous concepts for nullity and forcing zeros in a null vector, but rather there are several possible choices for nullity of a hypermatrix, which lead naturally to associated concepts of zero forcing. Here we have made one standard choice for the definition of nullity that led to the concept of zero forcing that has been studied. Other choices for nullity of a hypermatrix will lead to other definitions of zero forcing in hypergraphs, and these are also worthy of study.

We have obtained a variety of results for the zero forcing number and maximum nullity as defined here, but numerous interesting questions remain. Here we highlight a few, in addition to Question 3.22. There are many additional interesting families of hypergraphs for which the zero forcing number could be determined, and additional hypergraph operations whose effect on zero forcing number could be studied. There are also many interesting questions to be studied regarding the power domination zero forcing number, including many of the types of results obtained in [3] for the infection number and here for the zero forcing number, but also the question of a relationship between the hypergraph power domination number defined in [6] and power domination zero forcing defined here, paralleling that between graph power domination number and zero forcing number [2].

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