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Variants of asymptotic extremes

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

This thesis is focused on the asymptotic distributions of two discrete maximum (or minimum) processes, which are closely related to Classical Extreme Value Theory — the asymptotic distributional theory for maxima (or minima) of independent, identically distributed (i.i.d.) random variables. However, the processes that we study here are not maxima (or minima) of i.i.d. random variables. These processes originate from two problems that we consider below. One problem is concerned about the asymptotic distribution of minimax point-to-point discrepancy for minimax fitting of manufactured parts, the other is about certain asymptotic distributions related to game values, for $k$ by $n$ two-person zero-sum games with i.i.d. payoff random variables, for $n$ tending to infinity with $k$ fixed. Although there is no obvious connection between these two different problems, it turns out that they both can be handled by the idea of random covering circle, suggested by Daniels (1952).

The organization of this thesis is as follows:

In Chapter 1, we first review some relevant results of Classical Extreme Value Theory, and then give some examples to illustrate its applications. Next, we outline some definitions and results of Game Theory.
In Chapter 2, we study statistical approaches to manufacturing accuracy problems under the minimax fitting criterion. Previous works related to this area include Jeon (1994), Yu (1992), McCann (1992, 1988), Jayaraman and Srinivasan (1989), Srinivasan and Jayaraman (1989), David and McCann (1988, 1987), and Spotts (1983). Mostly this chapter is first inspired by the work of Yu (1992). Yu considered multiple tolerancing problems for multi-featured manufactured planar parts under two different fitting criteria: one is that the sum of squares of point-to-point discrepancy of matching hole/peg components is minimized under translation and rotation of manufactured parts, the other is that the maximum point-to-point discrepancy of hole/peg matching pairs is minimized through only translation of manufactured parts. The least squares point-to-point fitting problem studies the way how manufacturing tolerances affect the sum of squares error penalties for failing to meet certain design specifications. In fact, the least squares fitting problem is the statistical analysis of size-and-shape by using procrustes methods (Gower, 1975). A closely related topic is the statistical procrustes analysis of shape (Goodall, 1991). The minimax point-to-point fitting problem studies the ways how manufacturing tolerances affect the likelihood of meeting composite positional tolerancing, and acceptable fit for the matching pair. It will be clear later on that the minimax point-to-point fitting problem is essentially a random covering circle problem.

Now consider an ideal planar template with n points, and a companion template with n corresponding points subject to position error. Suppose that the two templates are matched by translation through minimizing maximum point-to-point discrepancy. The resulting minimax point-to-point discrepancy is relevant to composite positional tolerancing, for gauging the quality of either the assembly of two manufactured planar parts, or the conformance of a single such planar part to its
jig. This minimax point-to-point discrepancy and the corresponding plate misalignment are studied asymptotically in $n$, using the notion of random covering circle, in the case of circular normal and circular uniform error.

In Chapter 3, we study the asymptotic value distribution of a $k$ by $n$ two-person zero-sum game with i.i.d. payoff random variables, for $n$ tending to infinity with $k$ fixed. Prior work in this area of game value distributions includes Thomas (1965, 1967), Thomas and David (1967), Soults (1968), and El-Houbi (1994). Thomas studied distributionally relevant aspects of a matrix game for any fixed $k$ and $n$; Soults studied the asymptotic game value distributions under the cases of normal and uniform payoff random variables with $k$ equal to 2. El-Houbi derived the distributions of minorant and majorant game values in the case of 2 by $n$ normal payoff distribution, and also performed certain simulations to compare empirical distributions to theoretical ones. Since there is a correspondence between the value of any matrix game and a linear programming problem, Soults dealt with this problem from the viewpoint of linear programming and obtained a rather complex expression for the limiting distributions; for example, in the normal payoff case, the limiting distribution is in the form of an integral with an integrand involving the normal cumulative distribution function.

Here, we shall attack the asymptotic value distribution problem by a variant of the random covering circle idea (Daniels, 1952) which yields a somewhat simpler expression. In particular, with the payoff distribution normal and $k$ equal to 2, the joint probability density element of the game value and player I’s optimal strategy, pertaining to the case of the non-existence of a pure saddle point is given, together with their corresponding joint asymptotic distribution. Lower and upper $k$ by $n$ game values also are considered. It is found that these generally are asymptotically in-
dependent in the i.i.d. case, and their corresponding asymptotic distributions also
are derived via Classical Extreme Value Theory. Further asymptotic analysis is de­
voted to the lower value, for certain payoff location family models, which is
relevant to sensitivity analysis in linear programming.

1.2 Some Results of Classical Extreme Value Theory

Extreme Value Theory is an elegant and mathematically fascinating theory with
an enormous variety of applications. One important property of extreme order statistics is that if the limit distribution exists, it is non-normal and depends on the
distribution only through its tail behavior. Statistical applications of the extreme
value theory can be found in, for example, Gumbel (1958) and Castillo (1988). In the
following, we summarize some relevant results from Classical Extreme Value Theory.
More detail account in this topics can be found in, for example, de Haan (1970),

Let $X_1, X_2, ..., X_n$ be a sequence of i.i.d. random variables with a cumulative dis­
tribution function $F$, and $M_n$ be the maximum of the first $n$ random variables, that is,

$$M_n = \max(X_1, X_2, ..., X_n).$$

The Classical Extreme Value Theory is concerned about the distributional properties
of $M_n$ as $n$ becomes large; or more specifically, under which conditions can we obtain

$$P[M_n \leq a_n + b_n x] \longrightarrow G(x),$$

(1.1)

for some suitable linear normalizing constants $a_n \in \mathbb{R}$ and $b_n > 0$, and non-degenerated
distribution function $G$. In particular, we are interested in which distribution functions $G$ may appear as such limits. It turned out that the possible non-
degenerate distribution function $G$, forming precisely the class of max-stable
distributions, have only the following three parametric forms (except up to location and scale changes), and are commonly called the three *Extreme Value Distributions*.

**Theorem 1.2.1 (Gnedenko, 1943)** Suppose that there exists \( a_n \in \mathbb{R} \) and \( b_n > 0; \ n \geq 1 \), such that (1.1) holds for each \( x \in C_G \) as \( n \longrightarrow \infty \), where \( G \) is a non-degenerate distribution function and \( C_G \) is the set of continuity points of a function \( G \). Then \( G \) is of the type of one of the following three classes:

**TYPE I** : \( G(x) = \exp[-e^{-x}], \ x \in \mathbb{R} \);

**TYPE II** : \( G(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \exp[-x^{-\alpha}] & \text{for some } \alpha > 0 \text{ if } x > 0 \end{cases} \);

**TYPE III** : \( G(x) = \begin{cases} \exp[-(-x)^\alpha] & \text{for some } \alpha > 0 \text{ if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases} \).

**Remark:**

(1). Note that (1.1) can be written as

\[ [F(a_n + b_n x)]^n \longrightarrow G(x), \]

and we say that \( F \) belongs to the domain of attraction of \( G \) and write \( F \in D(G) \).

(2). For a given \( F \), it may turn out that there is no extreme value distribution \( G \) such that \( F \in D(G) \). This simply means that \( M_n \) does not have a limiting distribution under any linear normalization.

(3). Since the minimum

\[ m_n = \min(X_1, X_2, \ldots, X_n) \]

is given as

\[ m_n = -\max(-X_1, -X_2, \ldots, -X_n) \],

the limiting results for minima can be obtained from those for maxima.
Theorem 1.2.2 (Khintchine) Let \( \{F_n\} \) be a sequence of distribution functions and \( G \) be a non-degenerate distribution function. Let \( \{a_n\} \) and \( \{b_n > 0\} \) be sequences of constants such that

\[
F_n(a_n + b_n x) \longrightarrow G(x) .
\]

Then for some non-degenerate distribution function \( G_\star \) and constants \( \alpha_n \) and \( \beta_n > 0 \),

\[
F_n(\alpha_n + \beta_n x) \longrightarrow G_\star(x)
\]

if and only if

\[
\frac{\alpha_n - a_n}{b_n} \longrightarrow a \text{ and } \frac{\beta_n}{b_n} \longrightarrow b
\]

for some \( a \) and \( b > 0 \), and then

\[
G_\star(x) = G(a + b x).
\]

Lemma 1.2.1 Let \( \{X_i\} \) be a sequence of i.i.d. random variables with a distribution function \( F \). Let \( 0 \leq \tau \leq \infty \) and suppose \( \{t_n\} \) is an increasing sequence of real numbers such that

\[
n[1 - F(t_n)] \longrightarrow \tau \text{ as } n \longrightarrow \infty . \tag{1.2}
\]

Then

\[
P[M_n \leq t_n] \longrightarrow \exp[-\tau] \text{ as } n \longrightarrow \infty . \tag{1.3}
\]

Conversely, if (1.3) holds for some \( \tau \) where \( 0 \leq \tau \leq \infty \), then so does (1.2).

Similarly, let \( 0 \leq \eta \leq \infty \) and suppose \( \{s_n\} \) is a sequence of real numbers such that

\[
n[F(s_n)] \longrightarrow \eta \text{ as } n \longrightarrow \infty . \tag{1.4}
\]

Then

\[
P[m_n > s_n] \longrightarrow \exp[-\eta] \text{ as } n \longrightarrow \infty . \tag{1.5}
\]

Conversely, if (1.5) holds for some \( \eta \) where \( 0 \leq \eta \leq \infty \), then so does (1.4).
Let $x_F = \sup\{x: F(x) < 1\}$ be the upper end-point of $F$.

**Theorem 1.2.3** (Gnedenko, 1943) Necessary and sufficient conditions for the distribution function $F$ of the random variables of i.i.d. sequence $\{X_n\}$ to belong to each of the three types are

**Type I:** $\exists$ some strictly positive function $g(t)$ such that

$$\lim_{t \to x_F} \frac{1 - F(t + xg(t))}{1 - F(t)} = \exp[-x] \quad \forall \ x \in \mathbb{R}.$$  

**Type II:** $x_F = \infty$ and

$$\lim_{t \to x_F} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \text{ for some } \alpha > 0 \quad \forall \ x > 0.$$  

**Type III:** $x_F < \infty$ and

$$\lim_{h \to 0} \frac{1 - F(x_F - xh)}{1 - F(x_F - h)} = x^{\alpha}, \text{ for some } \alpha > 0 \quad \forall \ x > 0.$$  

In fact, it may be shown that $\int_0^\infty (1 - F(u)) \, du < \infty$ when a Type I limit holds and one appropriate choice of $g$ is given by

$$g(t) = \frac{t}{1 - F(t)}, \text{ for } t < x_F.$$  

**Theorem 1.2.4** (Gnedenko, 1943; de Haan, 1970) The normalizing constants $a_n$ and $b_n$ in the convergence (1.1) may be taken in each case above to be

**Type I:** $b_n = g(\gamma_n), \ a_n = \gamma_n$  

**Type II:** $b_n = \gamma_n, \ a_n = 0$  

**Type III:** $b_n = x_F - \gamma_n, \ a_n = x_F$  

with $\gamma_n = F^{-1}(1 - \frac{1}{n}) = \inf\{x: F(x) \geq 1 - \frac{1}{n}\}$.  


Theorem 1.2.5 (von Mises conditions) Suppose that the distribution $F$ of the i.i.d. sequence of random variables $\{X_i\}$ is absolutely continuous with density $f$. Then sufficient conditions for $F$ to belong to each of the three possible domains of attraction are:

Type I: $f$ has a negative derivative $f'$ for all $x$ in some interval $(x_0,x_F)$, $(x_F \leq \infty)$, $f(x) = 0$ for $x \geq x_F$, and

$$\lim_{t \to x_F} \frac{f'(t)(1 - F(t))}{f^2(t)} = -1;$$

Type II: $f(x) > 0$ for all $x \geq x_0$ finite, and

$$\lim_{t \to x_F} \frac{t f(t)}{1 - F(t)} = \alpha > 0;$$

Type III: $f(x) > 0$ for all $x$ in some finite interval $(x_0,x_F)$, $f(x) = 0$ for $x > x_F$, and

$$\lim_{t \to x_F} \frac{(x_F - t)f(t)}{1 - F(t)} = \alpha > 0.$$

1.3. Illustrations

Example 1.3.1 (Normal distribution) If $\{X_i\}$ is an i.i.d. standard normal sequence of random variables, then it belongs to the Type I domain of attraction with constants

$$b_n = \frac{1}{\sqrt{2 \log(n)}},$$

$$a_n = \frac{1}{2 \log(n)} - \frac{\log \log(n) + \log(4\pi)}{2 \sqrt{2 \log(n)}}.$$

Proof: Please refer to the proof of Theorem 1.5.3 of Leadbetter et al. (1983). □
Example 1.3.2 ($\chi^2(2)$ distribution) Let $\{X_j\}$ be a sequence of i.i.d. random variables with common distribution function

$$F(x) = 1 - \exp\left[-\frac{x^2}{2}\right],$$

where $t = 0$. Define $g(t) = 2$ for all $t \geq 0$. It is easy to check that

$$\frac{1 - F(t + 2x)}{1 - F(t)} = \exp[-x],$$

for all $x$. Therefore, by Theorem 1.2.3, we obtain

$$P[M_{a_n + b_n x}] \longrightarrow \exp[-x].$$

And one choice of $a_n$ and $b_n$ can be obtained by Theorem 1.2.4. In this case, we obtain

$$\begin{cases} a_n = 2 \log(n) \\ b_n = 2 \end{cases}.$$

Example 1.3.3 (Rayleigh($\sqrt{2}$) distribution) Let $\{X_j\}$ be a sequence of i.i.d. random variables with common distribution function

$$F(x) = 1 - \exp\left[-\frac{x^2}{2}\right],$$

where $t = 0$. Define $g(t) = \frac{1}{t}$ for all $t \geq 0$. Then

$$\frac{1 - F(t + \frac{x}{t})}{1 - F(t)} \longrightarrow \exp[-x],$$

for all $x$. Therefore, by Theorem 1.2.3 and 1.2.4, we obtain

$$P[M_{a_n + b_n x}] \longrightarrow \exp[-x].$$

where

$$a_n = \inf\{x: \exp\left[\frac{x^2}{2}\right] \leq \frac{1}{n}\} = \sqrt[4]{2 \log(n)}$$

and

$$b_n = g(a_n) = \frac{1}{\sqrt[4]{2 \log(n)}}.$$
Example 1.3.4 (Uniform(0,1) distribution) Let \( \{X_i\} \) be an i.i.d. sequence of Uniform(0,1) random variables, that is, \( F(x) = x, 0 \leq x \leq 1 \). For \( \tau > 0 \) and \( t_n = 1 - \frac{\tau}{n} \), we see \( 1 - F(t_n) = \frac{\tau}{n} \) for \( n \geq \tau \). Therefore, by Lemma 1.2.1, we obtain
\[
P[M_n \leq 1 - \frac{\tau}{n}] \longrightarrow \exp(-\tau) ,
\]
Now let \( x = -\tau \), we obtain
\[
P[M_n \leq a_n + b_n x] \longrightarrow \exp[x] ,
\]
which is a Type III limit with \( \alpha = 1, b_n = \frac{1}{n}, \) and \( a_n = 1 \).

Example 1.3.5 (Uniform Disk distribution) Let \( \{(X_i, Y_i)\} \) be independently and identically distributed on a unit disk with probability density function
\[
P[X=x, Y=y] = \frac{1}{\pi} I_{\{x^2 + y^2 \leq 1\}} ,
\]
where \( I \) is the indicator function. Then the marginal density function of \( X \) is
\[
\int_{-1}^{1-x^2} \frac{1}{\pi} dy = \frac{2\sqrt{1-x^2}}{\pi} ,
\]
and hence the distribution function \( F \) of \( X \) is
\[
F(x) = \frac{1}{\pi} \{x\sqrt{1-x^2} + \sin^{-1}[x] - \sin^{-1}[1]\}I_{\{-1 \leq x \leq 1\}} .
\]
First, notice that \( F \) does not belong to Type I, since \( F^{-1}(1) = 1 < \infty \). Therefore, if the asymptotic distribution of \( M_n = \max(X_1, X_2, \ldots, X_n) \) is non-degenerate, it is either Type II or Type III. Next by Theorem 1.2.5, we obtain
\[
\lim_{t \to 1} \frac{(x_f - t)f(t)}{1 - F(t)} = \frac{3}{2} > 0 ,
\]
therefore \( F \) belongs to Type III.

Let \( n[1 - F(x)] = \tau \), that is,
\[
n[1 - \frac{1}{\pi} \{x\sqrt{1-x^2} + \sin^{-1}[x] - \sin^{-1}[1]\}I_{\{-1 \leq x \leq 1\}}] = \tau ,
\]
which implies
\[ \{x-1-x^2 + \sin^{-1}[x]\}_{-1 \leq x \leq 1} = \frac{1}{2} - \pi. \]
Let \( y = \pi - 2 \sin^{-1}[x] \), then we have
\[ \sin(y) + (\pi - y) = \pi - \frac{2\pi}{n}. \]
Assume
\[ f_1(y) = \sin(y) + (\pi - y), \]
\[ f_2(y) = y - \frac{y^3}{3!} + (\pi - y). \]
Let \( y_1 \) and \( y_2 \) such that
\[ f_1(y_1) = \pi - \frac{2\pi}{n}, \]
\[ f_2(y_2) = \pi - \frac{2\pi}{n}. \]
Then we have
\[ - \frac{y^3}{3!} + \pi = \pi - \frac{2\pi}{n}, \]
which implies
\[ y_2 = \left( \frac{2\pi}{n} \right)^{1/3}. \]
Since
\[ 0 = f_1(y_1) - f_2(y_2) = f_1(y_1) - f_2(y_1) + f_2(y_1) - f_2(y_2) = O(y_1^5) + f_2'(\xi) (y_1 - y_2), \]
we obtain
\[ \frac{y_1 - y_2}{y_1} = \frac{O(y_1^4)}{\xi^{3/2}} \sim O(y_1^2). \]
Hence we have
\[ y_1 - y_2 \sim O(y_1^2) - O(y_2^3) \text{ if } n \text{ is sufficiently large,} \]
which implies \( y_1 = y_2 + O(n^{-1}) \), that is,
Notice that

\[ x = \sin\left(\frac{\pi}{2} - \frac{y_1}{2}\right) = \cos\left(\frac{y_1}{2}\right) = 1 - \frac{1}{2!}\left(\frac{y_1}{2}\right)^2 + O(y_1^4) = 1 - \frac{1}{8}\left(\frac{12\pi}{n}\right)^{2/3} + O(n^{-4/3}). \]

Therefore, we obtain

\[ P[M_n \leq 1 + \left[\frac{3\sqrt{2\pi}}{8n}\right]^{2/3} x] \rightarrow \begin{cases} 1 & \text{if } x \geq 0 \\ \exp[-(-x)^{3/2}] & \text{if } x < 0 \end{cases}. \]

1.4 Some Definitions and Results of Game Theory

The game theory might be loosely defined as "a theory of mathematical models of optimal decision making in conflict situations (Vorob'ev, 1994)". For future reference, a summary of relevant definitions and results is given below; for more details, see, for example, Owen (1982) or Vorob'ev (1994).

**Definition 1.4.1** A two-person zero-sum game is a system of the form

\[ \Gamma = <\xi, \eta, X>, \]

where \( \xi \) and \( \eta \) are arbitrary disjoint sets, called sets of strategies of players I and II, together with \( X: \xi \times \eta \rightarrow \mathbb{R}, \) the payoff function. Here the pairs \( (\xi, \eta) \in \{\xi\} \times \{\eta\} \) are called situations in \( \Gamma, \) and the number \( X(\xi, \eta) \) is the payoff to player I or the loss to player II in the situation \( (\xi, \eta). \)
Definition 1.4.2 A matrix game is defined as a finite two-person zero-sum game, that is, as a game \( \Gamma = \langle \xi, \eta, X \rangle \) where the sets \( \xi \) and \( \eta \) of the players' (pure) strategies are finite. Unless the contrary is stated, we shall always suppose that \( \xi = \{1, \ldots, k\} \) and \( \eta = \{1, \ldots, n\} \).

Definition 1.4.3 A mixed strategy for a player is a probability distribution on the set of his pure strategies.

In case the player has only a finite number, \( k \), of pure strategies \( \{1, \ldots, k\} \), a mixed strategy reduces to a \( k \)-vector, \( \xi = (\xi_1, \ldots, \xi_k) \), satisfying

\[
\xi_i \geq 0, \quad 1 \leq i \leq k,
\]

and

\[
\sum_{i=1}^{k} \xi_i = 1.
\]

Now let \( X = [X_{ij}] \) where \( 1 \leq i \leq k, \quad 1 \leq j \leq n \) be a \( k \) by \( n \) payoff matrix. Suppose that players I and II are playing the matrix game \( X \). If player I chooses the mixed strategy \( \xi \) and player II chooses the mixed strategy \( \eta \), then the expected payoff will be

\[
X(\xi, \eta) = \xi^T X \eta.
\]

Naturally, player I must fear that player II will discover his/her choice of a strategy. Should this situation happen, then player II will certainly choose strategy \( \eta \) to minimize the payoff \( X(\xi, \eta) \); that is, the expected minimum gain of player I will be

\[
v(\xi) = \min_\eta \xi^T X \eta.
\]
Now, $\xi^T X \eta$ can be thought of as a weighted average of the expected payoffs for player I if he/she uses $\xi$ against the pure strategies of player II. Therefore, the minimum will be obtained by a pure strategy; that is
\[ v(\xi) = \min_j \xi^T X_{j,} \]
where $X_{j,}$ is the jth column of the matrix $X$. Thus, player I should choose $\xi$ to maximize $v(\xi)$, and obtain
\[ v_1 = \max_\xi \min_j \xi^T X_{j,} \]
Such an $\xi$ is called maximin strategy of player I.

Similarly, if player II chooses strategy $\eta$, then he/she will obtain the maximum loss
\[ v(\eta) = \max_i X_{i,} \eta \]
where $X_{i,}$ is the ith row of $X$. Hence, player II should choose strategy $\eta$ to minimize $v(\eta)$ and obtain
\[ v_2 = \min_\eta \max_i X_{i,} \eta \]
Such an $\eta$ is called minimax strategy of player II. The two numbers $v_1$ and $v_2$ defined above are called the values of the matrix game $\Gamma$ to play I and II, respectively.

Next, let's state one of the most important theorems in game theory, which guarantees the existence of game value for a two-person zero-sum matrix game,

**Theorem 1.4.1 (The Minimax Theorem)**
\[ v_1 = v_2 . \]
Definition 1.4.4 The value $V_n$ of a (two-person zero-sum) matrix game $\Gamma = <\xi, \eta, X>$ is defined as

$$V_n^{(k)} = \max_{\xi} \min_{\eta} \xi^T X \eta$$

or equivalently

$$V_n^{(k)} = \min_{\eta} \max_{\xi} \xi^T X \eta$$

where

$$\xi^T = (\xi_1, \ldots, \xi_k)$$

such that $\sum_{i=1}^{k} \xi_i = 1$, $\xi_i \geq 0$, $1 \leq i \leq k$,

and

$$\eta^T = (\eta_1, \ldots, \eta_n)$$

such that $\sum_{j=1}^{n} \eta_j = 1$, $\eta_j \geq 0$, $1 \leq j \leq n$.

Note that the existence of the value $V_n^{(k)}$ for any matrix game $\Gamma$ is guaranteed by the minimax theorem.

There is a correspondence between the value of any matrix game and a linear programming problem which is stated in the following lemma.

Lemma 1.4.1 Let $V_n^{(k)}$ be the value of a two-person zero-sum matrix game $\Gamma = <\xi, \eta, X>$, then

$$V_n^{(k)} = \max_{\lambda} \lambda \text{ such that } \sum_{i=1}^{k} \xi_i X_{ij} \geq \lambda \\forall j = 1, \ldots, n$$

where $\sum_{i=1}^{k} \xi_i = 1$, $\xi_i \geq 0$, $1 \leq i \leq k$. 
From Lemma 1.4.1, we know that $V_n^{(k)}$ is the value of a matrix game $\Gamma$ if and only if $V_n^{(k)}$ is the optimal value of a linear programming problem induced by $\Gamma$. 
CHAPTER 2
MINIMAX FITTING OF MANUFACTURED PARTS
AND RANDOM COVERING CIRCLES

2.1 Introduction

One aspect of the "quality" of a complex product is that the parts, of which it is made, fit together acceptably. Two such parts may be, for example, a rigid plate of "holes" and a rigid plate of "pegs". We consider perfectly manufactured pegs and holes, but faulty peg and hole positioning. We suppose as well an ideal amount of clearance between pegs and holes. That ideal clearance will be compromised to the extent that peg and hole centers fail to coincide after alignment. In the present instance, this alignment is assumed to be achieved by minimizing the maximum discrepancy between corresponding faultily positioned peg and hole centers. The resulting minimax point-to-point discrepancy may be used to assess the quality of the assembly. Such minimax point-to-point discrepancy, now for the case of a single planar multi-featured manufactured part, with respect to an ideal jig, also is relevant to so-called "composite positional tolerancing" (cf. Foster (1986)). Misalignment of the plates induced by minimizing maximum point-to-point discrepancy is of use in the case when the peg-hole pairs act as fasteners.
A statistical view of faulty positioning in the plate involves two-dimensional error distributions. These distributions, together with an agreed-upon alignment criterion, say the above mentioned maximum discrepancy between corresponding features minimized under translation, produce a joint distribution of the minimized maximum discrepancy $r_n$ and the misalignment $\rho_n$, under the translation achieving $r_n$, of corresponding ideal features in the two plates.

Previous work in this area of statistical approaches to manufacturing accuracy includes Yu (1992), McCann (1992), Jayaraman and Srinivasan (1989), Srinivasan and Jayaraman (1989), and Spotts (1983). Our primary concern here is to study the joint asymptotic behavior of $r_n$ and $\rho_n$, as the number $n$ of features becomes large, in the case of the circular normal and circular uniform position error distributions. We note that, under translation, the positions of the ideal features are irrelevant to the statistical analysis.

In Section 2.2, we give a brief description of the minimax point-to-point fitting problem, and summarize some relevant results. In Section 2.3, we identify the equivalence between our minimax point-to-point fitting problem and the random covering circle problem first described and studied by Daniels (1952) in the normal case. It turns out that the minimax point-to-point discrepancy $r_n$ corresponds to the radius of a corresponding covering circle, and the magnitude of misalignment $\rho_n$ corresponds to the distance from the center of the covering circle to the origin. In Section 2.4, we describe the construction of the joint density of $r_n$ and $\rho_n$ for an arbitrary circular symmetric error distribution, along the lines of Daniels' argument. In Sections 2.5 and 2.6, we derive the joint asymptotic distributions of $r_n$ and $\rho_n$ in the case of circular normal and circular uniform error, where our
analysis of the latter case appears to call for a certain "approximate Scheffe's Lemma". Concluding remarks are given in Section 2.7.

2.2 Minimax Point-to-Point Fitting

Consider two rigid plates designated here, for purposes of discussion, as a "hole-plate" and a corresponding "peg-plate". Suppose for either plates, a coordinate system assigns \{\((u_i,v_i), i = 1, \ldots, n\)\} to the common ideal locations of the pegs and holes centers. Suppose further that manufacturing errors locate the actual holes and pegs centers respectively at \{\((u_i+\varepsilon_{1i},v_i+\eta_{1i}), i = 1, \ldots, n\)\} and \{\((u_i+\varepsilon_{2i},v_i+\eta_{2i}), i = 1, \ldots, n\)\}. The point-to-point discrepancy of a peg-center and its corresponding hole-center is defined to be the Euclidean distances between them.

Suppose that we superimpose the two plates in such a way that the two coordinate systems differ only in that the hole-plate origin is displaced by \((a,b)\) with respect to the peg-plate origin. Then the point-to-point discrepancy for the \(i\)th peg/hole pair, \(i = 1, \ldots, n\), is

\[
\frac{\sqrt{[(u_i+\varepsilon_{1i}) - (u_i+\varepsilon_{2i}+a)]^2 + [v_i+\eta_{1i}) - (v_i+\eta_{2i}+b)]^2}}{\sqrt{[(\varepsilon_{1i} - \varepsilon_{2i})]^2 + (\eta_{1i} - \eta_{2i})^2}}
\]

From (2.1), it is clear that, in this pure translation formulation, the ideal feature positions \{\((u_i,v_i), i = 1, \ldots, n\)\} are irrelevant to the statistical analysis. Let \(I_n\) be the minimax point-to-point discrepancy; that is,

\[
I_n = \min_{(a,b) \in \mathbb{R}^2} \max_{1 \leq i \leq n} \sqrt{[(\varepsilon_{1i} - \varepsilon_{2i})]^2 + (\eta_{1i} - \eta_{2i})^2}.
\]

Denoting \(X_i = \varepsilon_{1i} - \varepsilon_{2i}, Y_i = \eta_{1i} - \eta_{2i}\) for \(i = 1, \ldots, n\), the minimax point-to-point fitting problem is that of minimizing the maximum distance between \((a,b)\) and the points \{\((X_i,Y_i), i = 1, \ldots, n\)\}; that is,
Yu (1992) investigated the above problem under the assumption that \{(X_i, Y_j); i = 1, \ldots, n\} are i.i.d standard bivariate normal random vectors. She gave upper and lower bounds for \(I_n\), and studied their asymptotic behaviors. Yu's bounds \(I_{1,n}\) and \(I_{2,n}\) are as follows, with \((a_0, b_0)\) any point of \(\mathbb{R}^2\)

\[
I_{1,n} = \max_{1 \leq i \leq n} \sqrt{(X_i-a_0)^2 + (Y_i-b_0)^2} \geq \min_{(a,b) \in \mathbb{R}^2} \max_{1 \leq i \leq n} \sqrt{(X_i-a)^2 + (Y_i-b)^2} = I_n
\]

\[
= \min_{(a,b) \in \mathbb{R}^2} \max_{1 \leq i \leq n} \max_{1 \leq j \leq n} \sqrt{(X_i-a)^2 + (Y_j-b)^2} = I_{2,n}.
\]

Note that if we choose \((a_0, b_0) = (0,0)\), then \(I_{1,n}\) is the maximum of \(n\) i.i.d Rayleigh\((\sqrt{2})\) random variables. Hence, from classical extreme value theory, the asymptotic distribution of the upper bound \(I_{1,n}\) is given by

\[
\lim_{n \to \infty} P[I_{1,n} \leq \sqrt{2 \log(n)} + \frac{t}{\sqrt{2 \log(n)}}] = \exp[-e^{-t^2}], \text{ for all } t \in \mathbb{R}.
\] 

Next notice (McCann (1992), Yu (1992)), with regard to the next-to-last line of (2.4), that

\[
\min_{(a,b) \in \mathbb{R}^2} \max_{1 \leq i \leq n} \max_{1 \leq j \leq n} \sqrt{(X_i-a)^2 + (Y_j-b)^2} = \sqrt{(X_i-a^*)^2 + (Y_j-b^*)^2}
\]
\[ = \sqrt{(X_j-a^*)^2 + (Y_j-b^*)^2} \]
\[ = I_{ij} \]

where \(a^* = \frac{1}{2}(X_j+X_j)\), and \(b^* = \frac{1}{2}(Y_j+Y_j)\). Hence \(I_{2,n}\) of (2.4) may be rewritten as

\[ I_{2,n} = \max_{1 \leq i < j \leq n} I_{ij} . \] (2.6)

From (2.6), it is clear that \(2I_{2,n}\) is the two-dimensional sample range \(R_n\), defined as the largest interpoint Euclidean distance among the \(n\) points \((X_i,Y_i), i = 1, ..., n\), that is,

\[ R_n = \max_{1 \leq i < j \leq n} \sqrt{(X_i-X_j)^2 + (Y_i-Y_j)^2} = 2I_{2,n} . \]

Now, if we define the "directional maximum" \(M(n,\theta)\) and the "directional minimum" \(m(n,\theta)\) by

\[
\begin{align*}
M(n,\theta) &= \max_{1 \leq i \leq n} \{ \varepsilon_i \cos(\theta) + \eta_i \sin(\theta) \} \\
m(n,\theta) &= \min_{1 \leq i \leq n} \{ \varepsilon_i \cos(\theta) + \eta_i \sin(\theta) \}
\end{align*}
\]

and the "directional range" \(R(n,\theta)\) by

\[ R(n,\theta) = M(n,\theta) - m(n,\theta) , \]

then we obtain

\[ I_{2,n} = \frac{1}{2} \max_{0 \leq \theta < \pi} R(n,\theta) . \] (2.7)

From (2.7), it is easy to see that a further lower bound \(I_{3,n,k}\) of \(I_{2,n}\) can be easily given by

\[ I_{3,n,k} = \frac{1}{2} \max_{1 \leq i \leq k} R(n,\theta_i) , \]

where \(0 \leq \theta_1 < \theta_2 < \cdots < \theta_k < \pi\) are \(k\) different directions. Yu (1992) showed that \(\{R(n,\theta_i): 1 \leq i \leq k, k \in \mathbb{N}\}\) are asymptotically independent, and
for all \( t \in \mathbb{R} \), where

\[
\begin{cases}
  a_n = \frac{\sqrt[2]{\log(n)} - \frac{\log\log(n) + \log(4\pi)}{2\sqrt[2]{\log(n)}}}{2}\log(n)
  \\
  b_n = \frac{1}{2\sqrt[2]{\log(n)}}
\end{cases}
\]

Yu further showed that if \( \theta_i = \frac{\pi}{k_n} \), \( i = 1, \ldots, 2k_n \), where \( k_n \) is chosen such that

\[
\frac{k_n}{\log n} \rightarrow 0, \text{ as } n \rightarrow 0.
\]

Then for all \( t \in \mathbb{R} \),

\[
\lim_{n \rightarrow \infty} P[\max_{1 \leq i \leq k_n} I_{3,n,k_i} \leq a_n + b_n t] = \exp(-e^{-t})
\]

where

\[
\begin{cases}
  a_n = \sqrt[2]{\log(n)} - \frac{\log\log(n) - \log\log\log(n) + \log(4\pi)}{4\sqrt[2]{\log(n)}}
  \\
  b_n = \frac{1}{2\sqrt[2]{\log(n)}}
\end{cases}
\]

Since \( I_{2,n} = \lim_{n \rightarrow \infty} \max_{1 \leq i \leq k_n} I_{3,n,k_i} \), Yu conjectured that the normalizing constants obtained in (2.8) could be the normalizing constants of \( I_{2,n} \). However, the asymptotic distribution of \( k \)-dimensional sample range in the normal case was obtained by Matthews and Rukhin (1993). They showed that the number of interpoint distances exceeding an increasing high level follows a limiting Poisson distribution, which implies

\[
\lim_{n \rightarrow \infty} P[I_{2,n} \leq a_n + b_n t] = \exp(-e^{-t}), \text{ for all } t \in \mathbb{R},
\]

where
\[
\begin{align*}
\alpha_n &= \frac{\frac{1}{2} \log \log(n) - \log \log \log(n) + \frac{1}{2} \log(32\pi)}{4\sqrt{2 \log(n)}} \\
\beta_n &= \frac{1}{2 \sqrt{2 \log(n)}}.
\end{align*}
\] (2.10)

2.3 Random Covering Circles

Consider \( n \geq 3 \) sample points \( \{(X_i, Y_i) : i = 1, \ldots, n\} \). The covering circle of the sample is defined to be the smallest circle containing every sample point on or within it (Daniels (1952)). While the random covering circle problem is interesting in itself, it was first originated to be a measure of precision in ballistic quality control. For a discussion of other statistical measures of precision in bivariate samples, we refer to Grubbs (1964), Cacoullos and DeCicco (1967), and Eckler (1969).

Some facts about covering circles are as follows:

**Fact 0.** The covering circle of a given sample is uniquely determined.

**Fact 1.** The covering circle contains on it at least two sample points.

**Fact 2.** If the covering circle contains on it exactly two sample points, then these two points lie diametrically opposite on the circle.

**Fact 3.** If the covering circle contains on it exactly three sample points, then these three points form either an acute-angled triangle or a right-angled triangle.
Fact 4. If the covering circle contains on it more than three sample points, then either some two of them lie diametrically opposite on the circle, or some three of them form an acute-angled triangle.

Let \( r_n \) and \( c_n \) be, respectively, the radius and the center of the covering circle of the sample points \( \{(X_i, Y_i), i = 1, \ldots, n\} \). We first observe that the strict convexity of the function \( \max\{(X_i-a)^2 + (Y_i-b)^2\} \) in \((a,b)\) guarantees the existence of a unique \((\alpha_n, \beta_n)\) satisfying

\[
I_n = \min_{(a,b) \in \mathbb{R} \times \mathbb{R}} \max_{1 \leq i \leq n} (X_i-a)^2 + (Y_i-b)^2
= \max_{1 \leq i \leq n} (X_i-\alpha_n)^2 + (Y_i-\beta_n)^2. \tag{2.11}
\]

Lemma 2.3.1 There exists uniquely a point \((\alpha_n, \beta_n) \in \mathbb{R} \times \mathbb{R}\) such that the equality (2.11) holds.

Proof. Denote \( h_i(a,b) = \sqrt{(X_i-a)^2 + (Y_i-b)^2} \), \( 1 \leq i \leq n \). Then, by the triangle inequality of Euclidean norm, we know that \( h_i(a,b) \) is a convex function for all \( i \), which implies \( \max_{1 \leq i \leq n} h_i(a,b) \) is also a convex function. Therefore, these exists a point \((\alpha_n, \beta_n) = d_n\) such that (2.11) holds. To prove the uniqueness, suppose that there also exist a point \( e_n \neq d_n \) such that (2.11) holds. Let \( B_r(c) \) be the circle with center \( c \) and radius \( r \). Note that by the definition of \( I_n \), \( B_n(d_n) \) and \( B_n(e_n) \) both contain all the sample points. Thus, \( B_n(d_n) \cap B_n(e_n) \) also contains all the sample points. Let \( m_n = \frac{1}{2} (d_n + e_n) \), then there exists \( \delta > 0 \) such that
which contradicts the definition of $I_n$. Therefore, we are done. \( \Box \)

Next, let $B_r(c)$ be the circle with center $c$ and radius $r$. From the definition of covering circles and (2.11), we know

**Lemma 2.3.2** $B_n((\alpha_n, \beta_n))$ is the covering circle $B_{r_n}(c_n)$ of the sample.

**Proof:** First, from (2.11) it is obvious that $B_n((\alpha_n, \beta_n))$ contains all the sample points. Therefore, by the definition of covering circles, $\text{Area}(B_n((\alpha_n, \beta_n))) \geq \text{Area}(B_{r_n}(c_n))$, which implies $I_n \geq r_n$. If $I_n > r_n$, then there exists a constant $\delta > 0$ such that $r_n < I_n - \delta < I_n$. Since $\text{Area}(B_{r_n}(c_n)) < \text{Area}(B_{r_n}(c_n))$, we know that $B_{r_n}(c_n)$ also covers all the sample points. Denote $c_n = (\alpha, \beta)$, then we have

$$I_n - \delta > \max_{1 \leq i \leq n} \left( (X_i - \alpha)^2 + (Y_i - \beta)^2 \right)$$

$$= \min_{(a, b) \in \mathbb{R} \times \mathbb{R}} \max_{1 \leq i \leq n} \left( (X_i - a)^2 + (Y_i - b)^2 \right)$$

$$= I_n$$

which is a contradiction. Therefore $I_n = r_n$.

Next, since $I_n = r_n$, we have $\text{Area}(B_n((\alpha_n, \beta_n))) = \text{Area}(B_{r_n}(c_n))$. But $B_n((\alpha_n, \beta_n))$ contains all the sample points and $B_{r_n}(c_n)$ is the covering circle; hence, by the uniqueness property of covering circles, it must be that $(\alpha_n, \beta_n) = c_n$.

\( \Box \)

From Lemma 2.3.2, we know that $I_n = r_n$ and $(\alpha_n, \beta_n) = c_n$. Therefore, for the circularly symmetric random sample, studying the joint asymptotic behavior of $I_n$ and
\[ \alpha_n^2 + \beta_n^2 \] is equivalent to studying the joint asymptotic distribution of \( r_n \) and \( \|c_n\| = \rho_n \), where \( \| \cdot \| \) denotes the Euclidean norm. In other words, our minimax point-to-point fitting problem is equivalent to the random covering circle problem.

### 2.4 Joint Density of Radius and Center Norm, of a Random Covering Circle

Suppose that \((X_i, Y_i); 1 \leq i \leq n\), are i.i.d. bivariate random vectors with density circularly symmetric about the origin. Now, let \( \rho_n \) be the center norm (that is, distance from the origin to the center \( c_n \) of the random covering circle). Daniels (1952) derives the joint density of the covering circle radius \( r_n \) and \( \rho_n \) in the normal case.

In accordance with Facts 1 - 4 given above, Daniels deals with the following four mutually exclusive and exhaustive cases:

(C1): The covering circle passes through exactly two sample points, and these are diametrically opposite on the circle.

(C2): The covering circle passes through exactly three sample points, and these form an acute-angled triangle.

(C3): The covering circle passes through exactly three sample points, and these form a right-angled triangle.

(C4): The covering circle passes through more than three sample points.
Daniels further observes

**Lemma 2.4.1** The probability of the cases (C3) and (C4) is zero.

**Proof:** First consider case (C4). Then by Fact 4, there exist either some two sample points diametrically opposite each other on the covering circle, or some three of them forming an acute-angled triangle. Thus, the covering circle is determined by either those two or those three sample points. Since the probability of any other sample point falling on a determined circle is zero, therefore the probability that the covering circle passes through more than three of the samples points is zero. For case (C3), notice that if a circle passes through exactly three points and these form a right-angled triangle, then some two of them are diametrically opposite on the circle. Hence, similar argument can be applied in case (C3). •

Since cases (C1) and (C2) are mutually exclusive, we have, by Lemma 4.1, that the probability element $dF_n$ for the desired density is obtained by adding $dF_2$ for (C1) and $dF_3$ for (C2) as follows:

$dF_2$ is the joint probability element under case (C1), for $r_n$ lying between $r$ and $r+dr$, and $\rho_n$ lying between $\rho$ and $\rho+dp$, that is,

$$dF_2 = P[\text{case (C1)}; r \leq r_n < r+dr, \rho \leq \rho_n < \rho+dp]$$

$$= \frac{n(n-1)}{2} dQ_2 P^{n-2}(r,\rho) ,$$

(2.12)

where $dQ_2$ is the probability element for the two diametrically opposite "labelled" (says, 1 and 2) points on the circle, and $P(r,\rho)$ is the probability that any particular one of the remaining (n-2) sample points falls within the circle. The factor $n(n-1)$ is the number of ways of selecting two points among the sample, and the divisor 2 accounts for the identity of orders 12, 21 on the circle.
Similarly, dF_3 is the joint probability element under case (C2), for \( r_n \) lying between \( r \) and \( r + dr \), and \( \rho_n \) lying between \( \rho \) and \( \rho + d\rho \), that is,

\[
dF_3 = P[\text{case (C2)}; \, r_n < r + dr, \, \rho_n < \rho + d\rho] = \frac{n(n-1)(n-2)}{3} dQ_3 \, P^{(n-3)}(r, \rho), \tag{2.13}
\]

where \( dQ_3 \) is the probability element of three "labelled" (says, 1, 2, and 3) points lying in a specified order around the circle and forming an acute-angled triangle, and \( P(r, \rho) \) is the probability that any particular one of the remaining \( (n-3) \) sample points falls within the circle. The factor \( n(n-1)(n-2) \) is the number of ways of selecting the three points, and the divisor 3 accounts for the identity of orders 123, 231, and 312 on the circle.

The addition of these two kind of density elements then yields

\[
dF_n(r, \rho) = dF_2 + dF_3 = \frac{1}{2} n(n-1) dQ_2 \, P^{(n-2)}(r, \rho) + \frac{1}{3} n(n-1)(n-2) dQ_3 \, P^{(n-3)}(r, \rho). \tag{2.13}
\]

2.5 Circular Normal Case

Daniels (1952) assumed that \((X_i, Y_i); \, 1 \leq i \leq n\) are i.i.d. circular normal random vectors with joint density

\[
f(x, y) = \frac{1}{2\pi\sigma^2} e^{-(x^2 + y^2)/2\sigma^2}, \tag{2.14}
\]

and calculated the joint probability element \( dF_n(r, \rho) \) of \( r_n \) and \( \rho_n \). We summarize some relevant ones of his results below. And, without loss of generality, we assume \( \sigma = 1 \).

To derive \( dF_n(r, \rho) \), we need to compute \( dF_2 \) and \( dF_3 \), and hence \( dQ_2 \), \( dQ_3 \) and \( P(r, \rho) \). First under (2.14), the probability \( P(r, \rho) \) for any sample point falling within a circle \( B_r(\rho) \) of radius \( r \) and center norm \( \rho \) is
\[
P(r, \rho) = \int \int \frac{1}{2\pi} e^{-\left(x^2+y^2\right)/2} \, dx \, dy
\]

\[
B_r(\rho) = \exp\left[-\frac{\rho^2}{2}\right] \int_0^r s \exp\left[-\frac{s^2}{2}\right] I_0(sp) \, ds ,
\]  
(2.15)

where \( I_k(z) \) is the \textit{modified Bessel function of the first kind} also called the Bessel function of imaginary argument, of order \( k \), defined by

\[
I_k(z) = \frac{1}{2\pi} \int_0^{2\pi} \exp[z \cos(\theta)] \cos(k\theta) \, d\theta .
\]  
(2.16)

Next, by the formula

\[\frac{d}{dz}[z^k I_k(z)] = z^k I_{k-1}(z)\]  
(2.17)

we obtain

\[
\frac{\partial}{\partial \rho} P(r, \rho) = -\rho \, P(r, \rho) + \exp\left[-\frac{\rho^2}{2}\right] \int_0^r t \exp\left[-\frac{t^2}{2}\right] I_1(\rho t) \, dt
\]

\[
= -r \, I_1(rp) \exp\left[-\frac{r^2+\rho^2}{2}\right] .
\]  
(2.18)

Furthermore, from (2.18), we obtain

\[
P(r, \rho) = r \exp\left[-\frac{r^2}{2}\right] \int_0^{\infty} \exp\left[-\frac{u^2}{2}\right] I_1(\rho u) \, du .
\]  
(2.19)

But from (2.15) and the fact that \( P(\omega, \rho) = 1 \), we can rewrite \( P(r, \rho) \) as

\[
P(r, \rho) = 1 - \exp\left[-\frac{\rho^2}{2}\right] \int_r^\infty \exp\left[-\frac{s^2}{2}\right] I_1(sp) \, ds
\]

\[
= 1 - \exp\left[-\frac{r^2+\rho^2}{2}\right] I_0(rp) - P(\rho, r) ,
\]  
(2.20)
where the second equality follows from (2.19) and integration by parts. Further partial integration gives

\[ P(r,p) = 1 - \exp\left[-\frac{r^2+p^2}{2}\right] \{I_0(rp) + \frac{p^2}{r} I_1(rp) + \frac{p^4}{r^2} I_2(rp) + \cdots \} , \quad (2.21) \]

and similarly, from (2.15),

\[ P(r,p) = \exp\left[-\frac{r^2+p^2}{2}\right] \{\frac{r^2}{p} I_1(rp) + \cdots \} . \]

Both series converge for all \( r \) and \( p \), and their sum yields the \textit{Laurent development}, that is,

\[ \exp\left[-\frac{r^2+p^2}{2}\right] = \sum_{m=-\infty}^{\infty} \left( \frac{r}{p} \right)^m I_m(rp) . \]

For more detailed calculation and other properties of \( P(r,p) \), we refer to Quenouille (1949) and Daniels (1952).

Next, to calculate \( dQ_2 \), let the two diametrically opposite points be given by

\[
\begin{align*}
  x_1 &= p \cos(\phi) + r \cos(\theta_1) \\
  y_1 &= p \sin(\phi) + r \sin(\theta_1) \\
  x_2 &= p \cos(\phi) + r \cos(\theta_2) \\
  y_2 &= p \sin(\phi) + r \sin(\theta_2) ,
\end{align*}
\]

(2.22)

where \( 0 \leq \theta_1 < 2\pi \), \( \theta_2 = \theta_1 + \pi \), and \( 0 \leq \phi < 2\pi \). Because the Jacobian is

\[ \frac{\partial(x_1,y_1,x_2,y_2)}{\partial(p,r,\phi,\theta_1)} = \begin{vmatrix}
\cos(\phi), \cos(\theta_1), -p \sin(\phi), -r \sin(\theta_1) \\
\sin(\phi), \sin(\theta_1), p \cos(\phi), r \cos(\theta_1) \\
\cos(\phi), -\cos(\theta_1), -p \sin(\phi), r \sin(\theta_1) \\
\sin(\phi), -\sin(\theta_1), p \cos(\phi), -r \cos(\theta_1)
\end{vmatrix} = 4rp , \quad (2.23) \]

the probability element for the two points transforms to
\[
\begin{align*}
\left\{ f(p \cos(\phi) + r \cos(\theta_1), p \sin(\phi) + r \sin(\theta_1)) \right. \\
\left. f(p \cos(\phi) + r \cos(\theta_2), p \sin(\phi) + r \sin(\theta_2)) \right. \nonumber \\
\left. 4 rp \right. \nonumber \\
& = 4 \exp[-(r^2+p^2)] \exp[-rp\{\cos(\phi-\theta_1) - \sin(\theta_1)\}] d\theta_1 d\phi dr dp. 
\end{align*}
\]

Therefore, \( dQ_2 \) is obtained by integrating \( \theta_1 \) and \( \phi \) from 0 to \( 2\pi \), that is,

\[
\begin{align*}
dQ_2 &= \int_{0}^{2\pi} \int_{0}^{2\pi} 4 \exp[-(r^2+p^2)] \exp[-rp\{\cos(\phi-\theta_1) - \sin(\theta_1)\}] d\theta_1 d\phi dr dp. \\
& = 4 \exp[-(r^2+p^2)] \exp[r \cos(\phi_1) - \sin(\phi_1)] d\phi dr dp.
\end{align*}
\]

To calculate \( dQ_3 \), let the three points have coordinates \( \{(x_i,y_i), i = 1, 2, 3\} \), and write

\[
\begin{align*}
x_i &= p \cos(\phi) + r \cos(\theta_i) \\
y_i &= p \sin(\phi) + r \sin(\theta_i)
\end{align*}
\]

where \( p \) and \( \phi \) are the polar coordinates of the center, and \( 0 \leq \theta_1 \leq \theta_2 \leq \theta_3 < 2\pi \).

Then the Jacobian is obtained by

\[
\frac{\partial(x_1,y_1,x_2,y_2,x_3,y_3)}{\partial(p,r,\phi,\theta_1,\theta_2,\theta_3)}
\]

\[
= \begin{vmatrix}
\cos(\phi), \cos(\theta_1), -p \sin(\phi), -r \sin(\theta_1), 0, 0 \\
\sin(\phi), \sin(\theta_1), p \cos(\phi), r \cos(\theta_1), 0, 0 \\
\cos(\phi), \cos(\theta_2), -p \sin(\phi), 0, -r \sin(\theta_2), 0 \\
\sin(\phi), \sin(\theta_2), p \cos(\phi), 0, r \cos(\theta_2), 0 \\
\cos(\phi), \cos(\theta_3), -p \sin(\phi), 0, 0, -r \sin(\theta_3) \\
\sin(\phi), \sin(\theta_3), p \cos(\phi), 0, 0, r \cos(\theta_3)
\end{vmatrix}
\]

\[
= r^3 \rho \{\sin(\theta_2-\theta_1) + \sin(\theta_3-\theta_2) + \sin(\theta_1-\theta_3)\}.
\]

Notice that these three points will form an acute-angled triangle if and only if

\[
\begin{align*}
0 &\leq \theta_1 < 2\pi \\
\theta_1 &\leq \theta_2 < \theta_1 + \pi \\
\theta_1 + \pi &\leq \theta_3 < \theta_2 + \pi
\end{align*}
\]

(2.26)

And the probability element for the three points transforms to
\[
\begin{align*}
&\{ f(\rho \cos(\phi) + r \cos(\theta_1), \rho \sin(\phi) + r \sin(\theta_1)) \\
&f(\rho \cos(\phi) + r \cos(\theta_2), \rho \sin(\phi) + r \sin(\theta_2)) \\
&f(\rho \cos(\phi) + r \cos(\theta_3), \rho \sin(\phi) + r \sin(\theta_3)) \\
&\frac{r^3}{(2\pi)^3} \{ \sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_2) + \sin(\theta_1 - \theta_3) \} \right\} d\theta_3 d\theta_2 d\theta_1 d\phi dr d\rho \\
&= \left\{ \exp\left[\frac{-3(r^2 + \rho^2)}{2}\right] \frac{1}{(2\pi)^3} \exp[-rp(\cos(\theta_1 - \phi) + \cos(\theta_2 - \phi) + \cos(\theta_3 - \phi))] \\
&\frac{r^3}{(2\pi)^3} \{ \sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_2) + \sin(\theta_1 - \theta_3) \} \right\} d\theta_3 d\theta_2 d\theta_1 d\phi dr d\rho.
\end{align*}
\]

Since, by symmetry of circular normal random vector, integration of \( \theta_1, \theta_2, \) and \( \theta_3 \) over (2.26) must give a result independent of \( \phi \), the integration of \( \phi \) only produces a factor of \( 2\pi \). The integrations of \( \theta_1, \theta_2, \) and \( \theta_3 \) can be effected by putting \( \phi = 0 \), expanding \( \sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_2) + \sin(\theta_1 - \theta_3) \), and integrating the six terms separately. Therefore, we obtain
\[
\begin{align*}
2\pi \quad 2\pi \\
\theta_1 + \pi \quad \theta_2 + \pi \\
dQ_3 = \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{\pi} \exp\left[\frac{-3(r^2 + \rho^2)}{2}\right] \exp[-rp(\cos(\theta_1 - \phi) + \cos(\theta_2 - \phi) + \cos(\theta_3 - \phi))] \\
&\frac{r^3}{(2\pi)^3} \{ \sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_2) + \sin(\theta_1 - \theta_3) \} d\theta_3 d\theta_2 d\theta_1 d\phi dr d\rho \\
&= 3 \exp\left[\frac{3}{2}(r^2 + \rho^2)\right] I_1(rp) \quad r^2 \quad dr d\rho.
\end{align*}
\]

Finally, combining all the results above, the joint probability element of \( r_n \) and \( \rho_n \) is obtained as
\[
\begin{align*}
dF_n(r, \rho) &= \left\{ 2n(n-1) \exp[-(r^2 + \rho^2)] \right\} r^2 P^{n-2}(r, \rho) \quad dr d\rho + \right. \\
&n(n-1)(n-2) \exp\left[\frac{3}{2}(r^2 + \rho^2)\right] I_1(rp) \quad r^2 P^{n-3}(r, \rho) \right\} dr d\rho \\
&= f_2(r, \rho) \quad dr d\rho + f_3(r, \rho) \quad dr d\rho. \quad (2.27)
\end{align*}
\]

From (2.27), we have
Proposition 2.5.1 Let \( r_n \) be the radius and \( \rho_n \) be the center norm, of the covering circle of i.i.d. standard bivariate normal random samples \( \{(X_i, Y_i): 1 \leq i \leq n\} \). Then, the joint asymptotic distribution of \( r_n \) and \( \rho_n \) is given by

\[
P[r_n \leq \frac{1}{2} \log(n) + \frac{s}{\sqrt{2 \log(n)}}, \rho_n \leq \frac{t}{\sqrt{2 \log(n)}}] \quad (2.28)
\]

\[
\int_{-\infty}^{\infty} \int_{0}^{\infty} \exp[-3u-I_0(v)e^{-u}] I_1(v) \, du \, dv , \text{ for all } s \in \mathbb{R} \text{ and } t \geq 0
\]

\[
= \exp[-e^{-s}] \{1 + e^{-s}\} - \exp[-I_0(t)e^{-s}] \left\{ \frac{1}{I_0^2(t)} + \frac{e^{-s}}{I_0(t)} \right\},
\]

where \( I_k(t) \) is the modified Bessel function of the first kind of order \( k \) defined by (2.16).

Proof: First, let

\[
r = \frac{1}{2} \log(n) + \frac{s}{\sqrt{2 \log(n)}} \equiv a_n + b_n s,
\]

and

\[
\rho = \frac{t}{\sqrt{2 \log(n)}} \equiv d_n t.
\]

Then by (2.21) and (2.27), it is easy to check that

\[
f_2(a_n + b_n s, d_n t) |b_n d_n| = 2n(n-1) \exp[-(a_n + b_n s)^2 + (d_n t)^2] (a_n + b_n s)(d_n t) \, P^{n-2}(a_n + b_n s, d_n t) \, b_n d_n,
\]

where

\[
2n(n-1) \exp[-(a_n + b_n s)^2 + (d_n t)^2] \longrightarrow 2 \exp[-2s],
\]

\[
(a_n + b_n s)(d_n t) \longrightarrow t,
\]

\[
P^{n-2}(a_n + b_n s, d_n t) \longrightarrow \exp[-I_0(t)e^{-s}],
\]

and

\[
b_n d_n \longrightarrow 0.
\]
Therefore, we have
\[ f_2(a_n + b_n s, d_n t) \mid b_n d_n \mid \longrightarrow 0. \]

Similarly,
\[
\begin{align*}
  f_3(a_n + b_n s, d_n t) \mid b_n d_n \\
  &= n(n-1)(n-2) \exp\left[ \frac{3}{2}(a_n + b_n s)^2 + (d_n t)^2 \right] (a_n + b_n s)^2 \ I_1((a_n + b_n s)(d_n t)) \\
  &P^{n-3}(a_n + b_n s, d_n t) \ b_n d_n ,
\end{align*}
\]
where
\[
\begin{align*}
  n(n-1)(n-2) \exp\left[ \frac{3}{2}(a_n + b_n s)^2 + (d_n t)^2 \right] \longrightarrow \exp[-3s] , \\
  (a_n + b_n s)^2 \ b_n d_n \longrightarrow 1 , \\
  I_1((a_n + b_n s)(d_n t)) \longrightarrow I_1(t),
\end{align*}
\]
and
\[
P^{n-3}(a_n + b_n s, d_n t) \longrightarrow \exp[-e^{I_0(t)}e^{s}];
\]
hence
\[
f_3(a_n + b_n s, d_n t) \mid b_n d_n \mid \longrightarrow \exp[-3s-I_0(t)e^{s}] I_1(t).
\]
Next, denote \( E(s,t) = \exp[-3s-I_0(t)e^{s}] I_1(t) \), then by formula (2.27) and integration by parts, we observe
\[
\begin{align*}
  \int_{-\infty}^{\infty} \int_{0}^{\infty} E(s,t) \ dt \ ds \\
  &\int_{-\infty}^{\infty} \int_{0}^{\infty} \exp[-3s-I_0(t)e^{s}] I_1(t) \ dt \ ds \\
  &\int_{-\infty}^{\infty} \int_{0}^{\infty} \exp[-3s-I_0(t)e^{s}] \ dI_0(t) \ ds \\
  &= \int_{-\infty}^{\infty} \exp[-3s] \int_{0}^{\infty} \exp[-I_0(t)e^{s}] \ dI_0(t) \ ds
\end{align*}
\]
\begin{align*}
&= \int_{-\infty}^{+\infty} \exp[-3s] \left( \exp[-I_0(\infty)e^{-s}] - \exp[-I_0(0)e^{-s}] \right) \frac{ds}{-\exp[-s]} \\
&= \int_{-\infty}^{+\infty} \exp[-2s-e^{-s}] \, ds \\
&= 1.
\end{align*}

That is, $E(s,t)$ is a joint density function. Therefore, (2.28) follows by Scheffe's Lemma. □

Next, by letting $s = \infty$ or $t = \infty$ in (2.28), we obtain

**Corollary 2.5.1** The asymptotic distribution of $r^n$ is given by

\[ P[r_n \leq \frac{1}{2} \log(n) + \frac{s}{42 \log(n)} \exp[\log(1+e^{-s}) - e^{-s}] , \]

for all $s \in \mathbb{R}$. And the asymptotic distribution of $\rho_n$ is given by

\[ P[\rho_n \leq \frac{t}{42 \log(n)} \exp[\log(1+e^{-s}) - e^{-s}] , \]

for all $t \geq 0$.

Note that let $G(s) = \exp[\log(1+e^{-s}) - e^{-s}]$ and $H(t) = 1 - \frac{1}{I_0(t)^2}$. It is obvious

that $G(s) \rightarrow 1$ as $s \rightarrow \infty$, $G(s) \rightarrow 0$ as $s \rightarrow -\infty$, and $G(s)$ is continuous, hence

$G(s)$ is a distribution function. Similarly, $I_0(t) \rightarrow 1$ as $t \rightarrow 0$, $I_0(t) \rightarrow \infty$ as $t \rightarrow \infty$, and $I_0(t)$ is continuous, therefore $H(t)$ is also a distribution function.

Furthermore, it is clear that

\[ E(s,t) \neq \frac{d}{ds}G(s) \frac{d}{dt}H(t) . \]

Therefore, we have
Corollary 2.5.2 Under the circular normal case, $r_n$ and $\rho_n$ are not asymptotically independent.

2.6 Circular Uniform Case

Now suppose that $(X_i, Y_i); 1 \leq i \leq n$, are i.i.d. circular uniform random vectors with joint density function

$$f(x, y) = \frac{1}{\pi} I_{\{x^2 + y^2 \leq 1\}}, \quad (2.29)$$

where $I$ is the indicator function.

A special feature in the circular uniform case, different from the circular normal case, is that the range of the random sample is bounded by a unit disk now. Hence the covering circle of a given sample is either fully contained in the unit disk or partially contained in the unit disk.

Notice that under (2.29), the probability $P(r, \rho)$ of any sample point lying within a circle of radius $r$ at a given distance $\rho$ from the origin is

$$P(r, \rho) = r^2 I_{\{0 \leq r + \rho \leq 1\}} + \frac{A(r, \rho)}{\pi} I_{\{1 \leq r + \rho\}}; \quad 0 \leq r, \rho \leq 1, \quad (2.30)$$

where

$$(r, \rho) = \cos^{-1}\left[1 + \frac{\rho^2 - r^2}{2r^2}\right] - \frac{4r^2 \rho^2 - (1 - r^2 - \rho^2)^2}{2} + r^2 \cos^{-1}\left[\frac{\rho^2 + r^2 - 1}{2r\rho}\right]. \quad (2.31)$$

Therefore, the probability of $n$ sample points lying within the circle is

$$P^n(r, \rho) = r^{2n} I_{\{0 \leq r + \rho < 1\}} + \left\{\frac{A(r, \rho)}{\pi}\right\}^n I_{\{1 \leq r + \rho\}}. \quad (2.31)$$

Next, $dQ_2$ and $dQ_3$ defined in (2.12) and (2.13) are partitioned as

$$dQ_2 = dQ_2 I_{\{0 \leq r + \rho < 1\}} + dQ_2 I_{\{1 \leq r + \rho\}}$$

$$= dQ_{21} + dQ_{22},$$
and

\[ dQ_3 = dQ_3 \{ 0 \leq r + \rho < 1 \} + dQ_3 \{ 1 \leq r + \rho \} \]

\[ = dQ_{31} + dQ_{32} \, . \]

To calculate \( dQ_{21} \) and \( dQ_{22} \), let the two diametrically opposite points \((x_1, y_1)\) and \((x_2, y_2)\) be given as (2.22); then the Jacobian is obtained by (2.23). It follows that under the condition \( 0 \leq r + \rho < 1 \), we have

\[
\frac{2\pi 
}{2\pi 
} = \int_0^\pi \int_0^{2\pi} \left( \frac{1}{\pi} \right)^2 4\rho \, d\theta_1 d\phi drdp \, I_{\{0 \leq r + \rho < 1\}}
\]

\[ = 16\rho \, drdp \, I_{\{0 \leq r + \rho < 1\}} \, . \]

Next, under the condition \( r + \rho \geq 1 \), \( \theta_1 \) and \( \theta_2 \) must satisfy

\[
(A1) \left\{ \begin{array}{l}
0 \leq \theta_1 < \Psi - \pi \\
\theta_2 = \theta_1 + \pi
\end{array} \right.
\]
or

\[
(A2) \left\{ \begin{array}{l}
\pi \leq \theta_1 < \Psi \\
\theta_2 = \theta_1 + \pi
\end{array} \right.
\]

where

\[
\Psi = \Psi(r, \rho) = 2 \, \cos^{-1} \left[ \frac{\rho^2 + r^2 - 1}{2\rho} \right] \, . \quad (2.32)
\]

Hence, we obtain

\[
dQ_{22} = \left\{ \int_0^\pi \int_0^{2\pi} \left( \frac{1}{\pi} \right)^2 4\rho \, d\theta_1 d\phi drdp + \int_0^\pi \int_0^{2\pi} \left( \frac{1}{\pi} \right)^2 4\rho \, d\theta_1 d\phi drdp \right\} I_{\{1 \leq r + \rho\}}
\]

\[ = \frac{16\rho}{\pi} \, (\Psi - \pi) \, drdp \, I_{\{1 \leq r + \rho\}} \, . \]

To calculate \( dQ_{31} \) and \( dQ_{32} \), let the three points \( \{(x_i, y_i)\}: i = 1, 2, 3 \) be given as (2.24); then the Jacobian is obtained by (2.25). First, under the condition \( 0 \leq
$r + \rho < 1$, these three points will form an acute-angled triangle if and only if (2.26) is satisfied. Therefore, we obtain

$$dQ_{31} = \int \int \int (\frac{1}{\pi})^3 r^3 \rho \{\sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_2) + \sin(\theta_1 - \theta_3)\}$$

$$d\theta_3 d\theta_2 d\theta_1 d\phi dp dp I_{\{0 < r + \rho < 1\}}$$

$$= 12 r^3 \rho \int dp I_{\{0 < r + \rho < 1\}} .$$

Next, under the condition $r + \rho \approx 1$, these three points will form an acute-angled triangle if and only if:

(B1) \[
\begin{cases}
0 \leq \theta_1 < \Psi - \pi \\
\theta_1 \leq \theta_2 < \Psi - \pi \\
\theta_1 + \pi \leq \theta_3 < \theta_2 + \pi
\end{cases}
\]

or

(B2) \[
\begin{cases}
0 \leq \theta_1 < \Psi - \pi \\
\Psi - \pi \leq \theta_2 < \pi \\
\theta_1 + \pi \leq \theta_3 < \Psi
\end{cases}
\]

or

(B3) \[
\begin{cases}
0 \leq \theta_1 < \Psi - \pi \\
\pi \leq \theta_2 < \theta_1 + \pi \\
\theta_1 + \pi \leq \theta_3 < \Psi \text{ or } 2\pi \leq \theta_3 < \theta_2 + \pi
\end{cases}
\]

or

(B4) \[
\begin{cases}
\Psi - \pi \leq \theta_1 < \pi \\
\pi \leq \theta_2 < \Psi \\
2\pi \leq \theta_3 < \theta_2 + \pi
\end{cases}
\]

or

(B5) \[
\begin{cases}
\pi \leq \theta_1 < \Psi \\
\theta_1 \leq \theta_2 < \Psi \text{ or } 2\pi \leq \theta < \theta_1 + \pi \\
\theta_1 + \pi \leq \theta_3 < \theta_2 + \pi
\end{cases}
\]
where $\Psi = \Psi(r,\rho)$ is defined by (2.32). Therefore,

$$
2\pi
\int_0^{2\pi} \int_0^\pi \int_{\{r+p\}} \rho \left\{ \sin(\theta_2-\theta_1) + \sin(\theta_3-\theta_2) + \sin(\theta_1-\theta_3) \right\}
\rho^2 d\theta_2 d\theta_3 d\phi dr dp \cdot
$$

Finally, combining all the results above, the joint probability element of $r_n$ and $\rho_n$ is therefore

$$
dF_n(r,\rho) = dF_2 + dF_3
= \{f_1(n,\rho) + f_2(n,\rho) + f_3(n,\rho) + f_4(n,\rho)\} dr dp
$$

where

$$
\begin{align*}
f_1(n,\rho) &= 4n(n-1)(n-2)\rho^{2n-3} \pi^{n-3} \int_{\{0\leq r+p\leq 1\}} \\
f_2(n,\rho) &= 8n(n-1)\rho^{2n-3} \pi^{n-3} \int_{\{0\leq r+p\leq 1\}} \\
f_3(n,\rho) &= \frac{2}{\pi^2} n(n-1)(n-2)\rho^{2n-3} \left\{ \Psi^2 - \Psi\pi - \Psi\sin(\Psi) + 2\pi\sin(\Psi) \right\} \int_{\{r+p\leq 1\}} \\
f_4(n,\rho) &= \frac{8}{\pi^2} n(n-1)\rho^{2n-3} \left(\Psi - \pi\right) \int_{\{r+p\leq 1\}}
\end{align*}
$$

with $A = A(r,\rho)$ defined by (2.31) and $\Psi = \Psi(r,\rho)$ defined by (2.32).

Intuitively, it is clear that $r_n \rightarrow 1$ in probability and $\rho_n \rightarrow 0$ in probability, as $n \rightarrow \infty$. Suppose that $r_n = 1 - b_n u_n$ and $\rho_n = d_n v_n$, where $\{b_n\}$ and $\{d_n\}$ are sequences of positive constants such that $b_n \rightarrow 0$, $d_n \rightarrow 0$, and $u_n$ and $v_n$ are normalized random variables, converging in distribution to non-degenerated random variables. Then the joint density $f_n(s,t)$ of $(u_n,v_n)$ is given by

$$
\begin{align*}
f_{1n}(1-b_n s,d_n t)b_n d_n + f_{2n}(1-b_n s,d_n t)b_n d_n + \\
f_{3n}(1-b_n s,d_n t)b_n d_n + f_{4n}(1-b_n s,d_n t)b_n d_n \\
= g_{1n}(s,t) + g_{2n}(s,t) + g_{3n}(s,t) + g_{4n}(s,t)
\end{align*}
$$
Now let \( b_n = d_n = \frac{1}{n} \). We first note that the indicator function

\[ I\{0 \leq 1 - (s/n) + (t/n) < 1\} = I\{s \geq 0\} \]

is constant in \( n \) for any fixed \( s \geq 0 \) and \( t \geq 0 \), and it is clear that

\[
g_{1n}(s,t) = 4n(n-1)(n-2)(1 - \frac{s}{n})^{2n-3} \frac{t}{n^2} \frac{1}{n} I\{0 \leq 1 - (s/n) + (t/n) < 1\} \rightarrow 4te^{2s} I\{s \geq 0\}
\]

and

\[
g_{2n}(s,t) = 8n(n-1)(1 - \frac{s}{n})^{2n-3} \frac{t}{n^2} \frac{1}{n} I\{0 \leq 1 - (s/n) + (t/n) < 1\} \rightarrow 0,
\]

for all \( s \geq 0 \) and \( t \geq 0 \).

Next, we have

\[
g_{3n}(s,t) = (\frac{2}{\pi^2})n(n-1)(n-2)(1 - \frac{s}{n})^{3} \frac{t}{n^3} (\frac{B_n}{\pi})^{n-3} \{\Phi_n^2 - \Phi_n \pi - \Phi_n \sin(\Phi_n) + 2\pi \sin(\Phi_n)\}
\]

\[
\frac{1}{n^2} I\{1 \leq 1 - (s/n) + (t/n)\},
\]

where \( B_n = B_n(s,t) = A(1-s/n,t/n) \) and \( \Phi_n = \Phi_n(s,t) = \Psi(1-s/n,t/n) \). Similarly, the indicator function \( I\{1 \leq 1 - (s/n) + (t/n)\} = I\{0 \leq s \leq t\} \) is constant in \( n \) for any fixed \( s \geq 0 \) and \( t \geq 0 \), and

\[
B_n(s,t) = \cos^{-1}\left[\frac{1 + (t/n)^2 - (1-s/n)^2}{2u/n}\right] - \frac{1}{2} \sqrt{4\left(\frac{1-s^2}{n}\right)^2 - \left[1 - (\frac{t}{n})^2 - (1-s/n)^2\right]^2} +
\]

\[
(\frac{1}{n})^2 \cos^{-1}\left[\frac{(t/n)^2 + (1-s/n)^2 - 1}{2(1-s/n)(t/n)}\right]
\]

\[ = H_{1n} - H_{2n} + H_{3n}.\]

Since
\[ H_{1n} = \cos^{-1}\left[1 + \left(\frac{t}{n}\right)^2 - \left(1-\frac{s}{n}\right)^2\right] \]

\[ = \cos^{-1}\left[\frac{s}{t}\right] - \frac{1}{n} \frac{t^2-s^2}{2} + o(\frac{1}{n}) , \]

\[ H_{2n} = \frac{1}{2} \sqrt{4(1-\frac{s}{n})^2 - \left[1-(\frac{t}{n})^2 - (1-\frac{s}{n})^2\right]} \]

\[ = \frac{\sqrt{t^2-s^2}}{n} + o(\frac{1}{n}) , \]

and

\[ H_{3n} = (1-\frac{s}{n})^2 \cos^{-1}\left[\frac{(t/n)^2 + (1-s/n)^2 - 1}{2(1-s/n)(t/n)}\right] \]

\[ = (1-\frac{s}{n})^2 \left\{ \cos^{-1}\left[\frac{s}{t}\right] - \frac{1}{n} \frac{t^2-s^2}{2} + o(\frac{1}{n}) \right\} , \]

we have

\[ B_n(s,t) = \pi - \frac{1}{n} \left\{ 2\sqrt{t^2-s^2} + 2s \cos^{-1}\left[\frac{s}{t}\right] \right\} + o(\frac{1}{n}) \longrightarrow \pi I_{\{t=s=0\}} , \]

and

\[ \Phi_n(s,t) = 2 \cos^{-1}\left[\frac{(t/n)^2 + (1-s/n)^2 - 1}{2(1-s/n)(t/n)}\right] \]

\[ = 2\left\{ \cos^{-1}\left[\frac{s}{t}\right] - \frac{1}{n} \frac{t^2-s^2}{2} + o(\frac{1}{n}) \right\} \longrightarrow 2 \cos^{-1}\left[\frac{s}{t}\right] I_{\{t=s=0\}} . \]

Therefore,

\[ \left\{ \frac{B_n(s,t)}{\pi} \right\}^n = \left\{ 1 - \frac{1}{n} \left[2\sqrt{t^2-s^2} + 2s \cos^{-1}\left[\frac{s}{t}\right] \right] + o(\frac{1}{n}) \right\}^n \longrightarrow e^{-\frac{\alpha(s,t)}{\pi}} \]

and

\[ \Phi_n^2 - \Phi_n \pi - \Phi_n \sin(\Phi_n) + 2\pi \sin(\Phi_n) \longrightarrow \beta(s,t) , \]

where

\[ \alpha(s,t) = \left\{ 2\sqrt{t^2-s^2} + 2s \cos^{-1}\left[\frac{s}{t}\right] \right\} I_{\{t=s=0\}} . \] (2.33)

and
\[ \beta(s,t) = \left\{ (2 \cos^{-1}\left(\frac{s}{t}\right))^2 - \pi(2s \cos^{-1}\left(\frac{s}{t}\right)) \right\} - \left(2 \cos^{-1}\left(\frac{s}{t}\right)\right) \sin(2 \cos^{-1}\left(\frac{s}{t}\right)) + 2\pi \sin(2 \cos^{-1}\left(\frac{s}{t}\right)) \right\} I\{t \geq s \geq 0\}. \] 

(2.34)

Thus,

\[ g_{3n}(s,t) \rightarrow \frac{2t}{\pi^2} e^{-\frac{\alpha(s,t)}{\pi}} \beta(s,t) I\{t \geq s \geq 0\}. \]

Furthermore, it is clear that

\[ g_{4n}(s,t) \rightarrow 0 \text{ as } n \rightarrow \infty. \]

Finally, combining all the results above, we obtain

\[ f_n(s,t) \rightarrow 4t e^{-2s} I\{s > t \geq 0\} + \frac{2t}{\pi^2} e^{-\frac{\alpha(s,t)}{\pi}} \beta(s,t) I\{t \geq s \geq 0\}. \] 

(2.35)

for all \( s \geq 0 \) and \( t \geq 0 \), where \( \alpha(s,t) \) and \( \beta(s,t) \) are defined by (2.33) and (2.34) respectively.

Note that the integration of the first part in (2.35) is

\[
\int_0^t \int_s^t \frac{4ve^{-2u}}{\pi} I\{u > v \geq 0\} du dv \\
= \left\{ \int_0^t \int_s^t 4ve^{-2u} dv du \right\} I\{s > t \geq 0\} + \\
\left\{ \int_s^t \int_0^t 4ve^{-2u} dv du + \int_s^t \int_0^t 4ve^{-2u} dv du \right\} I\{t \geq s \geq 0\}
\]

\[
= \frac{t^2}{e^{2s}} I\{s > t \geq 0\} + \left\{ \frac{1/2 + s + s^2}{e^{2s}} - \frac{1/2 + t}{e^{2t}} \right\} I\{t \geq s \geq 0\},
\]

for all \( s \geq 0 \) and \( t \geq 0 \); therefore

\[
\int_0^\infty \int_0^s 4te^{-2s} I\{s > t \geq 0\} dt ds = \frac{1}{2}.
\]
Next, it is not easy to do the second integration symbolically in (2.35). But, by numerical approximation, it is shown that

\[ \int_0^\infty \int_0^1 \frac{2t}{\pi^2} e^{-\frac{\alpha(s,t)}{2t}} \beta(s,t) I_{\{t_s=0\}} \, ds \, dt = \frac{1}{2} \quad \text{(up to machine error)}; \]

so \( 4te^{-2s} I_{\{s>t_s=0\}} \) may well be a density function. We now note that, even if it is not, it can still be used for approximate large-sample approximations, by resort to Lemma 2.6.1 below, a kind of "approximate Scheffe's Lemma".

Let \( \varepsilon > 0 \) be the machine error, so that

\[ \left| \int_0^\infty \int_0^1 \left( 4te^{-2s} I_{\{s>t_s=0\}} + \frac{2t}{\pi^2} e^{-\frac{\alpha(s,t)}{2t}} \beta(s,t) I_{\{t_s=0\}} \right) \, ds \, dt - 1 \right| \leq \varepsilon . \] (2.37)

Then we observe, for any sequence \( g_n \) of densities

**Lemma 2.6.1** If \( g_n \) is a sequence of densities with respect to a measure \( \nu \), and \( g_n \rightarrow g \) almost everywhere where \( \left| \int g \, d\nu - 1 \right| \leq \varepsilon \) for some non-negative constant \( \varepsilon \), then

\[ \lim_{n \to \infty} \left| \int_D g_n \, d\nu - \int_D g \, d\nu \right| \leq \varepsilon , \]

where \( D \) is any measurable set.

**Proof:** We first have
\[-\epsilon \leq \int (g - g_n) \, dv \]

\[= \int (g - g_n)^+ \, dv - \int (g - g_n)^- \, dv ,\]

where \((g - g_n)^+ = \max(g - g_n, 0)\), and \((g - g_n)^- = \max(g_n - g, 0)\).

Next,

\[\int |g - g_n| \, dv = \int (g - g_n)^+ \, dv + \int (g - g_n)^- \, dv ,\]

which implies

\[\int |g - g_n| \, dv \leq 2 \int (g - g_n)^+ \, dv + \epsilon .\]

Since \((g - g_n)^+ \leq g\) and \(g\) is integrable, by LDCT we have

\[\int_{D} (g - g_n)^+ \, dv \longrightarrow 0 ,\]

where \(D\) is any measurable set. Therefore

\[\lim_{D} \int g_n \, dv - \int g \, dv \]

\[\leq \lim_{D} \int |g - g_n| \, dv \]

\[\leq \epsilon .\]

\[\Box\]

By (2.37) and Lemma 6.1, we have
Proposition 2.6.1 Let $r_n$ be the radius and $\rho_n$ be the center norm of the covering circle of i.i.d. standard circular uniform random samples $\{(X_i, Y_i); 1 \leq i \leq n\}$. Then, the joint asymptotic distribution of $r_n$ and $\rho_n$ satisfies

$$
\lim_{t \to \infty} \left| P[r_n \leq 1 - \frac{s}{n}, \rho_n \leq \frac{t}{n}] - \left\{ \int_0^t \int_0^\infty 4ve^{2u} I\{u > 0\} \, dudv \right. \right.
+ \int_0^t \int_0^\infty \frac{e^{-\alpha(u,v)}}{\pi} \beta(u,v) I\{v = 0\} \, dudv \left. \right\} \leq \epsilon
$$

for all $s \geq 0$ and $t \geq 0$, where $\epsilon$ is defined in (2.37), and $\alpha(s,t)$ and $\beta(s,t)$ are defined by (2.33) and (2.34) respectively.

Now let $s \geq 0$ be fixed and $t$ tend to $\infty$, or let $t \geq 0$ be fixed and choose $s = 0$, then we have

Corollary 2.6.1 The the asymptotic distribution of $r_n$ satisfies

$$
\lim_{t \to \infty} \left| P[r_n \leq 1 - \frac{s}{n}] - \left\{ \frac{1}{2} \frac{1}{e^{2s}} + \int_0^\infty \frac{2v}{\pi^2} e^{-\alpha(u,v)} \beta(u,v) I\{v = 0\} \, dudv \right\} \right| \leq \epsilon
$$

for all $s \geq 0$, and the asymptotic distribution of $\rho_n$ satisfies

$$
\lim_{t \to \infty} \left| P[\rho_n \leq \frac{1}{n}] - \left\{ \frac{1}{2} \frac{1}{e^{2t}} + \int_0^t \int_0^\infty \frac{2v}{\pi^2} e^{-\alpha(u,v)} \beta(u,v) I\{v = 0\} \, dudv \right\} \right| \leq \epsilon
$$

for all $t \geq 0$. 


Finally, if we assume that (2.35) yields a density function, then as in the circular normal case, $r_n$ and $\rho_n$ are not asymptotically independent.

2.7 Some Conclusions

In this chapter, we study the minimax point-to-point fitting problem under two important cases: the circular normal and circular uniform cases. We identify the equivalence between our minimax point-to-point fitting problem and the random covering circle problem, and therefore focus on the joint asymptotic behavior of the center norm $\rho_n$ and the radius $r_n$, of a random covering circle. Essentially, $2r_n$ and $\rho_n$ are two-dimensional generalizations of the one-dimensional range and midrange, defined by maximum minus minimum and half of maximum plus minimum respectively, for any given one-dimensional sample. The same concept can be easily extended to any finite dimensional case, but the calculation becomes extremely tedious; for example, in the three-dimensional covering sphere problem, by the same construction, one would need to perform a symbolical calculation of the determinant of a twelve by twelve Jacobian matrix. Daniels (1952) described distributionally relevant properties of random covering circles, and gave exact small-sample joint and marginal densities of $r_n$ and $\rho_n$ in the normal case. However, as shown in previous sections, although the general description of the joint density of $r_n$ and $\rho_n$, for any circularly symmetric finite sample, was implied by Daniels (1952), the corresponding joint asymptotic distribution seems not subject to a unified formulation. Some hypotheses for general circularly symmetric error distributions, suggested by our two special analyses, are

(H1) $r_n$ and $\rho_n$ are not asymptotically independent.

(H2) The scale normalization for $r_n$ and $\rho_n$ are the same.

These two hypotheses, (H1) and (H2), do hold in the general one-dimensional case.
For the assembly of two planar parts, the distribution of \( r_n \) is relevant to the statistical analysis of the process of verifying composite positional tolerancing, and the distribution of \( \rho_n \) is relevant to the displacement of two plates with respect to one another, required for meeting such composite positional tolerancing. One open problem is to allow not only the translation but also the rotation of "peg-plate" and "hole-plate" with respect to one another. Again, this problem can be reduced to some special covering problem, but now the location of features becomes relevant.
CHAPTER 3

ASYMPTOTIC VALUE DISTRIBUTIONS FOR MATRIX GAMES
AND RELATED TOPICS

3.1 Introduction

One of the most important quantities in game theory is the value of a game. The Minimax Theorem, which proves the existence of a game value, also assures us that every two-person zero-sum game will have optimal strategies. Given any $k$ by $n$ two-person zero-sum matrix game with i.i.d. payoff random variables $X_{ij}$'s where $1 \leq i \leq k$, $1 \leq j \leq n$, the concern of this chapter is with the asymptotic value distribution of the game as $n$ tends to infinity with $k$ being fixed, and also with the asymptotic distribution of player I's optimal strategy. Prior work in this area of game value distributions includes Thomas (1965), Soults (1968), and El-Houbi (1994). Thomas studied distributionally relevant aspects of a matrix game for any fixed $k$ and $n$; Soults studied the asymptotic game value distributions under the cases of normal and uniform pay-off random variables with $k$ equal to 2. El-Houbi derived the distributions of minorant and majorant game values in the case of 2 by $n$ normal payoff distribution, and also performed certain simulations to compare empirical distributions to theoretical ones. Soults' approach yielded a rather complex expression for the limit distribution, in the form of an integral with an integrand involving the normal cumula-
tive distribution function, in the normal payoff case. Here, we shall attack the asymptotic value distribution problem by a variant of the random covering circle idea (Daniels, 1952) which yields a somewhat simpler expression. Section 3.2 is devoted to game value bounds. In Section 3.3, we derive the joint probability density element for the distribution of the value of a 2 by n matrix game and player I's optimal strategy pertaining to the case of the non-existence of a pure value, essentially using the random covering circle idea, and then specialize in Section 3.4 to the case of i.i.d. normal payoff random variables, for which we derive the conditional joint and marginal asymptotic distributions of the value and player I's optimum strategy. Some miscellaneous results are given in Section 3.5 concerning perturbations of the payoff matrix of a matrix game. Finally, concluding remarks are given in Section 3.6.

3.2 Some Upper Bounds and Lower Bounds of a Game Value

Consider a two-person zero-sum matrix game with a payoff matrix $X = [x_{ij}]$ where $1 \leq i \leq k$, $1 \leq j \leq n$. Without loss of generality, we label the pure strategies of player I by the ordinal numbers of corresponding rows and the pure strategies of player II by the ordinal numbers of the corresponding columns. Player I chooses a distribution $\xi$ over the rows of this matrix, and player II selects a distribution $\eta$ over the columns. These choices are made independently by player I and player II. After choices are made, player I obtains the expected pay-off $\xi^T X \eta$ from player II, with the interpretation that, if the expectation $\xi^T X \eta$ is negative, then player I loses the absolute value of that amount to player II. The value $V_n^{(k)}$ of the game is then defined as
\[ V_n^{(k)} = \max \min_{\xi} \xi^T X \eta \]
or, equivalently, by the Minimax Theorem,
\[ V_n^{(k)} = \min \max_{\xi} \xi^T X \eta . \]
where \( \xi^T = (\xi_1, \ldots, \xi_k) \) with \( \sum_{i=1}^k \xi_i = 1, \xi_i \geq 0, 1 \leq i \leq k, \) and \( \eta^T = (\eta_1, \ldots, \eta_n), \)
with \( \sum_{j=1}^n \eta_j = 1, \eta_j \geq 0, 1 \leq j \leq n. \)

Lower and upper bounds of \( V_n^{(k)} \) can easily be obtained as follows:

\[
V_n^{(k)} = \max \min_{\xi} \left[ \sum_{i=1}^k \sum_{j=1}^n \xi_i \eta_j X_{ij} \right] = \max \min_{\xi} \left[ \sum_{i=1}^k \left( \sum_{j=1}^n \eta_j X_{ij} \right) \right] = \max_{\xi} \left[ \sum_{i=1}^k \left( \min_{\eta} \sum_{j=1}^n \eta_j X_{ij} \right) \right] = \max_{\xi} \left[ \sum_{i=1}^k \min_{1 \leq j \leq n} X_{ij} \right] = \max_{1 \leq i \leq k} \min_{1 \leq j \leq n} X_{ij} \equiv L_n^{(k)}. \]

Similarly, we obtain
\[ V_n^{(k)} \leq \min \max_{1 \leq i \leq k} X_{ij} \equiv U_n^{(k)}. \]

Here, \( L_n^{(k)} \) and \( U_n^{(k)} \) are the values of minorant and majorant games, also often called the lower and the upper values of the game respectively. From the definition of upper and lower game values, it is clear that the asymptotic distributions of \( L_n^{(k)} \) and \( U_n^{(k)} \) can be obtained from Classical Extreme Value Theory.
Next, the three mutually exclusive and exhaustive cases of Classical Extreme Value Theory lead to the asymptotic distribution of \( L_n^{(k)} \) and \( U_n^{(k)} \) as follows, in a manner indicated in El-Houbi (1992) in the normal 2 by n case.

First, the asymptotic distribution of \( L_n^{(k)} \) is provided by

**Proposition 3.2.1** Let \( X_{ij}, 1 \leq i \leq k, 1 \leq j \leq n, \) be i.i.d. random variables with distribution function \( F. \) Let \( 0 \leq \tau(x) \leq \omega, \) and \( L_n^{(k)} = \max_{1 \leq i \leq k} \min_{1 \leq j \leq n} X_{ij}. \) Suppose that there exists constants \( a_n \) and \( b_n > 0 \) such that

\[
n \{ F(a_n + b_n X) \} \rightarrow \tau(x), \quad \text{for all } x \in \mathbb{R}, \tag{3.1}
\]

where \( \tau(x) \) has one of the following parametric forms (up to location and scale changes).

- **TYPE I:** \( \tau(x) = e^x, \) \( x \in \mathbb{R}; \)

- **TYPE II:** \( \tau(x) = \begin{cases} \infty & \text{if } x \geq 0 \\ (-x)^\alpha & \text{for some } \alpha > 0 \text{ if } x < 0 \end{cases}; \)

- **TYPE III:** \( \tau(x) = \begin{cases} x^\alpha & \text{for some } \alpha > 0 \text{ if } x = 0 \\ 0 & \text{if } x < 0 \end{cases}. \)

Then,

\[
\text{P}[L_n^{(k)} \leq a_n + b_n X] \rightarrow \{1 - \exp[-\tau(x)]\}^k, \quad \text{for all } x \in \mathbb{R}. \tag{3.2}
\]

Conversely, if (3.2) holds for some constants \( a_n \) and \( b_n > 0, \) and one of the \( \tau(x) \) given above, then so does (3.1).

**Proof:** Note that

\[
\text{P}[L_n^{(k)} \leq a_n + b_n X] \\
= \{ \text{P}[\min_{1 \leq j \leq n} X_{ij} \leq a_n + b_n X] \}^k \\
= \left\{ 1 - \text{P}[\min_{1 \leq j \leq n} X_{ij} \geq a_n + b_n X] \right\}^k
\]
\[
\left\{ 1 - \left[ 1 - \left( 1 - \frac{1}{n} \{ F(a_n + b_n x) \} \right) \right]^n \right\}^k
\]

Therefore, the result follows immediately from Lemma 1.2.1 and Theorem 1.2.1. □

Next, the asymptotic distribution of \( U_{n}^{(k)} \) is provided by

**Proposition 3.2.2** Let \( X_{ij} \), \( 1 \leq i \leq k \), \( 1 \leq j \leq n \), be i.i.d. random variables with distribution function \( F \), and denote \( U_{n}^{(k)} = \min \max X_{ij} \).

1. If there are sequences \( a_n \) and \( b_n > 0 \) such that

\[
n [F(a_n + b_n x)] \longrightarrow \exp(x) \text{ for all } x \in \mathbb{R},
\]

then

\[
P[U_{n}^{(k)} \leq -c_n - d_n x] \longrightarrow 1 - \exp(-e^{-x}), \text{ for all } x \in \mathbb{R},
\]

where \( c_n = -a_k \int_{n}^{1} \) and \( d_n = \frac{1}{k} b_k \int_{n}^{1} \).

2. If there are sequences \( a_n \) and \( b_n > 0 \) such that

\[
n [F(a_n + b_n x)] \longrightarrow \begin{cases} 
\infty & \text{if } x \geq 0 \\
(-x)^{-\alpha} & \text{for some } \alpha > 0 \text{ if } x < 0,
\end{cases}
\]

then

\[
P[U_{n}^{(k)} \leq -c_n - d_n x] \longrightarrow \begin{cases} 
1 & \text{if } x \leq 0 \\
1 - \exp(-x^{-\alpha}) & \text{if } x > 0
\end{cases}
\]

where \( c_n = -a_k \int_{n}^{1} \) and \( d_n = b_k \int_{n}^{1} \).

3. If there are sequences \( a_n \) and \( b_n > 0 \) such that

\[
n F(a_n + b_n x) \longrightarrow \begin{cases} 
x^\alpha & \text{for some } \alpha > 0 \text{ if } x \geq 0 \\
0 & \text{if } x < 0,
\end{cases}
\]

then
\[ P[U_n^{(k)} \leq -c_n - d_n x] = \begin{cases} 
1 - \exp[-(-x)^{k\alpha}] & \text{if } x \leq 0 \\
0 & \text{if } x > 0 
\end{cases} \]

where \( c_n = -\frac{a_k}{k^\alpha} \) and \( d_n = \frac{b_k}{k^\alpha} \).

**Proof:** (1). First, define \( C_j = \min_{1 \leq i \leq k} (-X_{ij}); 1 \leq j \leq n \), then \( C_j \)'s are i.i.d. random variables with the common distribution

\[ G(t) = 1 - [F(-t)]^k. \]

If \( n [F(a_n + b_n x)] \to \exp[x] \) for all \( x \in \mathbb{R} \), then by Lemma 1.2.1 we know that

\[ P[\min_{1 \leq i \leq n} X_{ij} \leq a_n + b_n x] \to 1 - \exp[-\exp[x]], \quad \text{as } n \to \infty. \]

Therefore, by Theorem 1.2.2, there is a strictly positive function \( g(t) \) such that

\[ \lim_{t \to x_F} \frac{F(t+\frac{x}{k} g(t))}{F(t)} = \exp[x], \quad \text{for all } x \in \mathbb{R}, \]

where \( x_F = \inf\{x; F(x) > 0\} \). Therefore

\[
\lim_{t \to x^G} \frac{1 - G(t+\frac{x}{k} g(t))}{1 - G(t)} = \left( \frac{F(-t+\frac{x}{k} g(t))}{F(-t)} \right)^k \\
= \lim_{t \to x^G} \left( \frac{F(t+\frac{x}{k} g(-t))}{F(t)} \right)^k \\
= \exp[-x]
\]

where \( x^G = \sup\{x; G(x) < 1\} \), that is, \( G \) belongs to the Type I domain of attraction.

Hence, there are constants \( c_n \) and \( d_n > 0 \) such that

\[ n[1 - G(c_n + d_n x)] \to \exp[-x], \quad \text{for all } x \in \mathbb{R}. \]

Now, let \( y = -\frac{x}{k} \). Note that
\[
\begin{align*}
\text{n} \ [1 - G(c_n + d_n x)] & \longrightarrow \exp[-x] \\
\Rightarrow \text{n} \ [F(-c_n - d_n x)]^k & \longrightarrow \exp[-x] \\
\Rightarrow \text{n} \ [F(-c_n + d_n x)]^k & \longrightarrow \exp[ky] \\
\Rightarrow \text{k} \left[ F(-c_n + d_n x) \right] & \longrightarrow \exp[y] \\
\Rightarrow \text{m} \ [F(-c_m + d_m x)] & \longrightarrow \exp[y].
\end{align*}
\]

Denote \( c_m = -a_m \) and \( d_m k = b_m \), then we get \( c_n = -a_k \) and \( d_n = \frac{1}{k} b_k \).

Next, note that

\[
P[U_{n}^{(k)} \leq -c_n - d_n x]
\]

\[
= 1 - P[\min_{1 \leq j \leq n} \max_{1 \leq i \leq k} X_{ij} > -c_n - d_n x]
\]

\[
= 1 - \left\{ 1 - [1 - G(c_n + d_n x)] \right\}^n
\]

\[
= 1 - \left\{ 1 - \left[1 - G(c_n + d_n x)\right] \right\}^n
\]

Therefore, the result follows immediately. Finally, (2) and (3) can easily be proved by similar methods. □

A lower bound \( L_{n,m}^{(k)} \) for \( V_n^{(k)} \), better than \( L_n^{(k)} \), also can be obtained. For purposes of illustration, we detail here the case of \( k = 2 \). The argument is as follows:

\[
V_n^{(2)} = \max_{\xi_1, \xi_2} \min_{1 \leq j \leq n} (\xi_1 X_{1j} + \xi_2 X_{2j})
\]

\[
= \max_{\xi_1} \min_{1 \leq j \leq n} (\xi_1 X_{1j} + (1-\xi_1) X_{2j})
\]
\[ \begin{align*}
\max_{\xi_t \in S_{2m}} \min_{1 \leq j \leq n} [\xi_t X_{ij} + (1-\xi_t) X_{2j}] \\
\equiv L_{n,m}^{(2)},
\end{align*} \]

where \( S_{2m} = \{0, \frac{1}{2m}, \frac{2}{2m}, \ldots, \frac{2m-1}{2m}, 1\} \). Furthermore, if \( X_{ij} \)'s are i.i.d. normal random variables, then the asymptotic distribution of \( L_{n,m}^{(2)} \) can easily be obtained.

First, we need the following lemma, which can be found in Resnick (1987).

**Lemma 3.2.1** Let \((Z_{ij}, \ldots, Z_{nj}, \ldots, Z_{dj}), 1 \leq j \leq n, \) be i.i.d. \( d \)-dimensional multivariate normal random vectors. If all \( \frac{d(d-1)}{2} \) correlations are less than 1, then \( \max_{1 \leq j \leq n} Z_{ij}, \ldots, \max_{1 \leq j \leq n} Z_{nj}, \ldots, \max_{1 \leq j \leq n} Z_{dj} \) are jointly asymptotically independent.

Next, we have, for fixed \( m \),
\[ P[L_{n,m}^{(2)} \leq t] = P[\max_{0 \leq h \leq 2m} \min_{1 \leq j \leq n} \{\frac{h}{2m} X_{ij} + (1 - \frac{h}{2m}) X_{2j}\} \leq t] \]
\[ = P[\min_{1 \leq j \leq n} \{\frac{h}{2m} X_{ij} + (1 - \frac{h}{2m}) X_{2j}\} \leq t, h = 0, 1, \ldots, 2m] \]

Note that
\[ \frac{h}{2m} X_{ij} + (1 - \frac{h}{2m}) X_{2j} \sim N(0, \frac{h^2 + (2m-h)^2}{4m^2}), h = 0, 1, \ldots, 2m, \]
and
\[ \text{Cov}[\frac{h}{2m} X_{ij} + (1 - \frac{h}{2m}) X_{2j}, \frac{j}{2m} X_{ij} + (1 - \frac{j}{2m}) X_{2j}] \]
\[ = \frac{hj}{4m^2} + \frac{(2m-h)(2m-j)}{4m^2} \]
\[ = \frac{4m^2 - 2(h+j)m + hj}{4m^2}, \]

which implies
\[ \text{Corr}[\frac{h}{2m} X_{ij} + (1 - \frac{h}{2m}) X_{2j}, \frac{j}{2m} X_{ij} + (1 - \frac{j}{2m}) X_{2j}] < 1 \quad \text{if} \quad h \neq j. \]
Therefore, by Lemma 3.2.1, with \( d = 2m + 1 \) and \( Z_{bij} = \frac{h}{2m}X_{ij} + (1-\frac{h}{2m})X_{2j} \), we know that 
\[
\min_{1 \leq j \leq n} \left\{ \frac{h}{2m}X_{ij} + (1-\frac{h}{2m})X_{2j}; \ h = 0, 1, \ldots, 2m \right\}
\]
are asymptotically independent.

Now denote \( \sigma_{mh}^2 = \frac{h^2 + (2m-h)^2}{4m^2} \); \( h = 0, \ldots, 2m \), then from Classical Extreme Value Theory, we observe, for all \( h, 0 \leq h \leq 2m \), that
\[
\lim_{n \to \infty} P\left[ \min_{1 \leq j \leq n} \left\{ \frac{h}{2m}X_{ij} + (1-\frac{h}{2m})X_{2j} \right\} \leq \sigma_{mh}(-a_n + b_nt) \right] = 1 - \exp\left[-e^\psi\right], \quad -\infty < t < \infty .
\]

where
\[
\begin{align*}
a_n &= \frac{1}{2} \log(n) - \frac{\log \log(n) + \log(4\pi)}{2} \\
b_n &= \frac{1}{\sqrt{2} \log(n)}
\end{align*}
\]

Next, by Cauchy-Schwartz inequality, we obtain \( \sigma_{mh}^2 > \sigma_{mm}^2 \) for all \( h \neq m \), and it is easy to check that
\[
\frac{\sigma_{mm} b_n}{\sigma_{mh} b_n} \to \frac{\sigma_{mm}}{\sigma_{mh}}
\]
and
\[
\frac{(-\sigma_{mm} a_n) - (-\sigma_{mh} a_n)}{\sigma_{mh} b_n} \to \infty, \text{ for all } h \neq m .
\]

Therefore, by Khintchine’s Theorem, we obtain
\[
\lim_{n \to \infty} P\left[ \min_{1 \leq j \leq n} \left\{ \frac{h}{2m}X_{ij} + (1-\frac{h}{2m})X_{2j} \right\} \leq \sigma_{mm}(-a_n + b_nt) \right] = 1, \text{ for all } h \neq m .
\]

In other words, since \( \sigma_{mm} = \frac{1}{4\sqrt{2}} \), for any \( m \), and by asymptotic independence of
\[
\min_{1 \leq j \leq n} \left\{ \frac{h}{2m}X_{ij} + (1-\frac{h}{2m})X_{2j}; \ h = 0, 1, \ldots, 2m \right\},
\]
\[
\lim_{n \to \infty} P[L_{n,m}^{(2)} \leq \frac{1}{\sqrt{2}}(-a^t_n + b^t_n t)] \\
= \lim_{n \to \infty} P[\min_{1 \leq i \leq k} \{\frac{h}{2m}X_{ij} + (1 - \frac{h}{2m})X_{2j}\} \leq \frac{1}{\sqrt{2}}(-a^t_n + b^t_n t), h = 0, 1, \ldots, 2m] \\
= 1 - \exp[-e^t], \ -\infty < t < \infty,
\]

Thus, the asymptotic distributions of all \(L_{n,m}^{(2)}; m = 1, 2, \ldots\) agree. Similarly, for arbitrary \(k\), we have

**Proposition 3.2.3** Let \(X_{ij}\)'s be i.i.d. standard normal random variables where \(1 \leq i \leq k\) and \(1 \leq j \leq n\). Denote

\[
L_{n,m}^{(k)} = \max_{(\xi_1, \ldots, \xi_{k-1}) \in S_{2m}} \min_{1 \leq i \leq k-1} \left[ \sum_{i=1}^{k-1} \xi_i X_{ij} + (1 - \xi_{i-1} - \cdots - \xi_{k-1}) X_{kj} \right]
\]

where \(S_{2m} = \{0, \frac{1}{2m}, \frac{2}{2m}, \ldots, \frac{2m-1}{2m}, 1\}\). Then, the asymptotic distributions of \(L_{n,m}^{(k)}; m = 1, 2, \ldots\) all agree, with

\[
\lim_{n \to \infty} P[L_{n,m}^{(k)} \leq \frac{1}{\sqrt{k}}(-a^t_n + b^t_n t)] = 1 - \exp[-e^t], \ -\infty < t < \infty,
\]

where \(a^t_n\) and \(b^t_n\) are given by (3.3).

By comparing the location normalization constants for \(L_{n,m}^{(k)}\) and \(L_{n,m}^{(k)}\), we note that the superiority of the latter over the former is maintained in an asymptotic stochastic comparison sense.

We next consider the asymptotic independence of the lower value \(L_n^{(k)}\) and upper value \(U_n^{(k)}\). As before, let \(X_{ij}\), \(1 \leq i \leq k\) and \(1 \leq j \leq n\), be i.i.d. random variables, and denote that \(A_n^{(k)} = \max_{1 \leq i \leq k} \min_{1 \leq j \leq n} Y_{ij}\), \(B_n^{(k)} = \min_{1 \leq i \leq k} \max_{1 \leq j \leq n} Y_{ij}\), and \(W_n = \max_{1 \leq i \leq n} Y_{ij}; 1 \leq i \leq k\), where \(Y_{ij} = -X_{ij}\). Notice that \(L_n^{(k)}\) and \(U_n^{(k)}\) are asymptotically indepen-
dent if and only if $A_n^{(k)}$ and $B_n^{(k)}$ are asymptotically independent. Then we have the following observation.

**Proposition 3.2.4** Let $Y_{ij}$, $1 \leq i \leq k$ and $1 \leq j \leq n$, be i.i.d. random variables.

Define $A_n^{(k)} = \max_{1 \leq j \leq n} \min_{1 \leq i \leq k} Y_{ij}$, $B_n^{(k)} = \min_{1 \leq i \leq k} \max_{1 \leq j \leq n} Y_{ij}$, and $W_i = \max_{1 \leq j \leq n} Y_{ij}$; $1 \leq i \leq k$. Suppose that there are constants $\alpha_n$, $\beta_n$, $\alpha_{in}$, and $\beta_{in}$, such that for all $s$ and all $t_i$, $1 \leq i \leq k$,

$$P[A_n^{(k)} \leq \alpha_n + \beta_n s] \longrightarrow \exp[-\tau(s)] \text{ as } n \longrightarrow \infty$$

and

$$P[W_i \leq \alpha_{in} + \beta_{in} t_i] \longrightarrow \exp[-\tau_i(t_i)] \text{ as } n \longrightarrow \infty, \ 1 \leq i \leq k,$$

where $\tau(\cdot)$ and $\tau_i(\cdot)$; $1 \leq i \leq k$, are one of the possible three types of functions discussed in Theorem 1.2.1. Then $A_n^{(k)}$ and $B_n^{(k)}$ are asymptotically independent.

**Proof:** Note that if we can show that $A_n^{(k)}$ and $\{W_i; 1 \leq i \leq k\}$ are asymptotically independent, then $A_n^{(k)}$ and $B_n^{(k)}$ are asymptotically independent.

Denote $u_n(s) = \alpha_n + \beta_n s$ and $w_{in}(t_i) = \alpha_{in} + \beta_{in} t_i$; $1 \leq i \leq k$. By Lemma 1.2.1, we have

$$P[A_n^{(k)} \leq u_n(s)] \longrightarrow \exp[-\tau(s)] \text{ as } n \longrightarrow \infty$$

if and only if

$$nP[\min_{1 \leq i \leq k} Y_{ij} > u_n(s)] \longrightarrow \tau(s) \text{ as } n \longrightarrow \infty,$$

and

$$P[W_i \leq w_{in}(t_i)] \longrightarrow \exp[-\tau_i(t_i)] \text{ as } n \longrightarrow \infty$$

if and only if

$$nP[Y_{ij} > w_{in}(t_i)] \longrightarrow \tau_i(t_i) \text{ as } n \longrightarrow \infty, \ 1 \leq i \leq k.$$
Now denote \( S_0 = \{ \max Y_{ii} \leq u_n(s) \} \) and \( S_i = \{ Y_{ii} \leq w_{in}(t_i) \}; \ 1 \leq i \leq k \). Consider

\[
P[A_n^{(k)} \leq u_n(s), W_{in} \leq w_{in}(t_i); \ 1 \leq i \leq k]
\]

\[
= \{ P[ \max_{1 \leq i \leq k} Y_{ii} \leq u_n(s), Y_{ii} \leq w_{in}(t_i); \ 1 \leq i \leq k] \}^n
\]

\[
= \{ P[S_0 \cap S_1 \cap \ldots \cap S_k] \}^n
\]

\[
= \{ 1 - P[S_0^c \cup S_1^c \cup \ldots \cup S_k^c] \}^n
\]

\[
= \{ 1 - \frac{1}{n} n \left( \sum_{i=0}^{k} P[S_i^c] - \sum_{0 \leq i \neq j \leq k} P[S_i^c \cap S_j^c] + \ldots + (-1)^k P[S_0^c \cap S_1^c \cap \ldots \cap S_k^c] \right) \}^n.
\]

Notice that \( n P[S_0^c] \longrightarrow \tau(s) \) and \( n P[S_i^c] \longrightarrow \tau_i(t_i); \ 1 \leq i \leq k \). Hence, in view of inclusion, if

\[
n P[S_i^c \cap S_j^c] \longrightarrow 0, \text{ for all } 0 \leq i \neq j \leq k,
\]

then all other terms involving more two event \( S_i^c; \ 0 \leq i \leq k \), will tend to zero, and

\[
P[A_n^{(k)} \leq u_n(s), W_{in} \leq w_{in}(t_i); \ 1 \leq i \leq k] \longrightarrow \exp[-\tau(s)] \prod_{i=1}^{k} \exp[-\tau_i(t_i)],
\]

which implies the independence of \( A_n^{(k)} \) and \( B_n^{(k)} \).

Now, with regard to terms (3.4) where neither \( i \) nor \( j \) is zero, we note that, by the i.i.d. assumption, we have

\[
n P[S_i^c \cap S_j^c] \longrightarrow 0, \text{ for all } 1 \leq i \neq j \leq k.
\]

It remains then to show that

\[
n P[S_0^c \cap S_i^c] \longrightarrow 0, \text{ for all } 1 \leq i \leq k.
\]

Thus, without loss of generality, consider

\[
n P[S_0^c \cap S_i^c]
\]

\[
= n P[Y_{ii} > u_n(s), \text{ for all } 1 \leq i \leq k, Y_{1i} > w_{in}(t_i)]
\]
\[ = n \mathbb{P}[Y_{11} > \max\{u_n(s), w_{1n}(t_j)\}] (\mathbb{P}[Y_{11} > u_n(s)])^{k-1} \]

\[ \leq n \mathbb{P}[Y_{11} > w_{1n}(t_j)] \{\mathbb{P}[Y_{11} > u_n(s)]\}^{k-1}, \text{ for sufficiently large } n \]

\[ \longrightarrow 0 \text{ as } n \longrightarrow \infty . \]

Therefore, we are done. \[ \square \]

### 3.3 Probability Density Elements of the Value and Optimal Strategy

Consider a 2 by \( n \) matrix game with i.i.d. payoff random variables \( \{X_{ij}: 1 \leq i \leq 2, 1 \leq j \leq n\} \). For convenience, let us write \( X_j \equiv X_{ij}, Y_j \equiv X_{2j} \). By an argument similar to Daniels' approach to the random covering circle problem (Daniels, 1952), the asymptotic game value distribution can be easily derived. First, we need the following lemma, which can be found, for example, in Luce and Raiffa (1964).

**Lemma 3.3.1** In a \( k \) by \( n \) matrix game with absolutely continuous i.i.d. payoff random variables \( \{X_{ij}: 1 \leq i \leq k, 1 \leq j \leq n\} \), the probability that the value \( V_n^{(k)} \) is a pure value, that is, the game possesses a pure saddle point, does not depend on the distribution of \( X_{ij} \), and is given by

\[ \mathbb{P}[V_n^{(k)} \text{ is a pure value}] = \frac{k! \cdot n!}{(k + n - 1)!} . \quad (3.5) \]

According to so-called S-game theory (for examples, Blackwell and Girshick (1954)), the following considerations yield the value \( V_n^{(2)} \), and optimal strategies \((1 - \lambda_n, \lambda_n)\) for player I, of a 2 by \( n \) game: The columns of the matrix are plotted as points (that is, "sample points") in the plane, and their convex hull \( S \), whose extreme points will consist of all or some of the points so plotted, is constructed. Then a right angular wedge \( W \) is constructed, with apex on the equi-angular line, that just touches (that is, osculates) \( S \). The apex \( w \) of this osculating wedge will be
(V_n^{(2)}, V_n^{(2)}), and, if $s$ is the slope of any line separating $W$ and $S$, called a separating line, then $(1-\lambda_n, \lambda_n) = \left(\frac{|s|}{1 + |s|}, \frac{1}{1 + |s|}\right)$ is a good strategy for player I.

In the absolutely continuous situation that we are considering, the intersection $S \cap W$ will consist of only one point; that point either (C1) is an extreme point of $S$ not equal to $w$, implying the existence of a pure saddle point, or (C2) is $w$, with $w$ interior to an edge of $S$ connecting two extreme points of $S$, implying the non-existence of a pure saddle point. Thus, in view of Lemma 3.3.1,

$$P[\text{case (C2)}] \rightarrow 1$$

in the absolutely continuous case.

In case (C2), there will be a unique separating line for $S$ and $W$, with slope $s$ satisfying $-\infty < s < 0$, so that there will be a unique optimal strategy $(1-\lambda_n, \lambda_n)$, with $\lambda_n = \frac{1}{1 + |s|}$, for player I, that is positive. Moreover, all the "sample points" will fall either above or on the separating line. Let $\theta$, with $\tan(\theta) = |s|$, be the (clockwise) angle between the x axis and the separating line, which, as indicated above, corresponds to the optimal strategy of player I, and let $v$ be the value of $V_n^{(2)}$. Then the separating line can be written as

$$y - v = \tan(\pi - \theta) (x - v)$$

or

$$y = -x \tan(\theta) + v(1 + \tan(\theta)),$$

where $0 \leq \theta \leq \frac{\pi}{2}$.

Next, the probability of $n$ sample points lying on or above a given supporting line, characterized by $v$ and $\theta$, is

$$[P(v,\theta)]^n,$$

where
\[ P(v, \theta) = P[Y \geq -X \tan(\theta) + v(1 + \tan(\theta))] . \]  

(3.7)

Now the probability element for the game value \( V_n^{(2)} \) lying between \( v \) and \( v + dv \), and the angle \( \theta_n \) falling between \( \theta \) and \( \theta + d\theta \), and for case (C2) pertaining, is

\[ dF_2(v, \theta) = P[\text{Case(C2)}; \ v = V_n^{(2)} < v + dv, \ \theta \leq \theta_n < \theta + d\theta] \]

\[ = n(n-1) \ dQ_2(v, \theta) [P(v, \theta)]^{n-2} , \]

where the factor \( n(n-1) \) is the number of ways of selecting the two extreme points from among the \( n \) sample points, \( dQ_2(v, \theta) \) is the probability element that these two extreme points are positioned in a specified manner on the line compatible with case (C2), and \( [P(v, \theta)]^{n-2} \) is the probability that all remaining \( (n-2) \) sample points fall above or on the separating line. Hence, the probability density element of the value \( V_n^{(2)} \) of a 2 by \( n \) matrix game, conditioned on the non-existence of a pure saddle point, is

\[ n(n-1) \int_{\theta=0}^{\pi/2} [P(v, \theta)]^{n-2} dQ_2(v, \theta) \frac{1}{1 - P[V_n^{(2)} \text{ is a pure value}]} . \]

Since \( \theta_n = \tan^{-1}\left(\frac{1-\lambda_n}{\lambda_n}\right) \), the conditional probability density element of \( \lambda_n \) is therefore

\[ n(n-1) \int_{v=-\infty}^{+\infty} [P(v, \tan^{-1}\left(\frac{1-\lambda_n}{\lambda_n}\right))]^{n-2} dQ_2(v, \tan^{-1}\left(\frac{1-\lambda_n}{\lambda_n}\right)) \frac{1}{1 - P[V_n^{(2)} \text{ is a pure value}]} . \]

### 3.4 Asymptotic Value Distribution of a 2 by \( n \) Normal Matrix Game

Now assume that the above absolutely continuous payoff random variable is the standard normal distribution. Let \( \Phi \) be the cumulative distribution function and \( \phi \) be the probability density function of the standard normal random variable \( N(0,1) \). To
derive the asymptotic game value distribution, first we need to calculate \( P(v, \theta) \) and \( dQ_2(v, \theta) \).

With regard to \( P(v, \theta) \), we have from (3.7) that, for all \( v \in \mathbb{R} \) and \( 0 < \theta < \frac{\pi}{2} \),

\[
P(v, \theta) = P[Y \geq -X \tan(\theta) + v(1 + \tan(\theta))]
\]

\[
= P[\cos(\theta) Y + \sin(\theta) X \geq \cos(\theta) v(1 + \tan(\theta))]
\]

\[
= P[N(0,1) \geq \cos(\theta) v(1 + \tan(\theta))]
\]

\[
= 1 - \Phi(\cos(\theta) v(1 + \tan(\theta))).
\]

Next, to calculate \( dQ_2(v, \theta) \), let the two extreme points have coordinates \((x_1, y_1)\) and \((x_2, y_2)\), and write

\[
\begin{align*}
x_1 &= v + r_1 \cos(\theta) \\
y_1 &= v - r_1 \sin(\theta) \\
x_2 &= v + r_2 \cos(\theta + \pi) \\
y_2 &= v - r_2 \sin(\theta + \pi)
\end{align*}
\]

where \((v, v)\) is the coordinates of the apex \( w \), and \( r_1 \) and \( r_2 \) are the distance from the two extreme points to the apex \( w \). The Jacobian is then

\[
\frac{\partial(x_1, y_1, x_2, y_2)}{\partial(r_1, r_2, v, \theta)} = \begin{vmatrix} 
\cos(\theta) & 0 & 1 & -r_1 \sin(\theta) \\
-sin(\theta) & 0 & 1 & -r_1 \cos(\theta) \\
0 & -\cos(\theta) & 1 & r_2 \sin(\theta) \\
0 & \sin(\theta) & 1 & r_2 \cos(\theta) \\
\end{vmatrix}
\]

\[
= (r_1 + r_2)[\cos(\theta) + \sin(\theta)].
\]

And the probability element \( \phi(x_1, y_1) \Phi(x_2, y_2) \) \( dx_1 dy_1 dx_2 dy_2 \) for the two points transforms to

\[
\left\{ \phi(v + r_1 \cos(\theta), v - r_1 \cos(\theta)) \phi(v + r_2 \cos(\theta + \pi), v - r_2 \cos(\theta + \pi)) \\
(r_1 + r_2)[\cos(\theta) + \sin(\theta)] \right\} dr_1 dr_2 d\theta dv
\]
\[ dQ_2(v, \theta) = \left\{ \frac{1}{4\pi^2} e^{-\frac{(r_1^2 + r_2^2 + 2r_1v(\cos(\theta) - \sin(\theta)) + 2r_2v(\sin(\theta) - \cos(\theta))}{2}} \cdot (r_1 + r_2)[\cos(\theta) + \sin(\theta)] \right\} \, dr_1 dr_2 d\theta dv. \]

Hence, \( dQ_2(v, \theta) \) is obtained by integrating \( r_1 \) and \( r_2 \) from 0 to \( \infty \), that is,

\[
dQ_2(v, \theta) = \int_0^\infty \int_0^\infty \phi(v + r_1 \cos(\theta), v - r_1 \cos(\theta)) \phi(v + r_2 \cos(\theta + \pi), v - r_2 \cos(\theta + \pi)) (r_1 + r_2)[\cos(\theta) + \sin(\theta)] \, dr_1 dr_2 d\theta dv.
\]

\[
= \frac{1}{4\pi^2} \left[ \cos(\theta) + \sin(\theta) \right] e^{-2v^2} H(v, \theta) \, d\theta dv,
\]

where

\[
H(v, \theta) = \int_0^\infty \int_0^\infty (r_1 + r_2)e^{-\frac{(r_1^2 + r_2^2 + 2r_1v(\cos(\theta) - \sin(\theta)) + 2r_2v(\sin(\theta) - \cos(\theta))}{2}} \, dr_1 dr_2. \quad (3.8)
\]

It is possible to perform the double integration in (3.8) explicitly, as follows.

Let's denote \( w = \sqrt{2}v[\sin(\theta) - \cos(\theta)] \), \( a = \sqrt{2}r_1 \), and \( b = \sqrt{2}r_2 \), then we can write

\[
H(v, \theta) = \int_0^\infty \int_0^\infty (r_1 + r_2) e^{-\frac{(r_1^2 + r_2^2 + 2r_1w + 2r_2w)}{2}} \, dr_1 dr_2
\]

\[
= \frac{1}{2\sqrt{2}} \int_0^\infty \int_0^\infty (a + b) \exp\left[ -\frac{(a - w)^2 + (b + w)^2}{4} \right] \, dadb \exp\left[ \frac{w^2}{2} \right]
\]

\[
= \frac{1}{2\sqrt{2}} (I_1 + I_2) \exp\left[ \frac{w^2}{2} \right],
\]

where
\[
I_1 = \int \int_{0}^{\infty} a \exp\left[\frac{(a - w)^2 + (b + w)^2}{4}\right] \, da \, db
\]
\[
= \int_{0}^{\infty} a \exp\left[\frac{(a - w)^2}{4}\right] \, da \exp\left[\frac{(b + w)^2}{4}\right] \, db
\]
\[
= \left\{ 2 \exp\left[\frac{w^2}{4}\right] + w \left[1 - \Phi(w)\right] 2\sqrt{\pi} \right\} \Phi(w) \, 2\sqrt{\pi},
\]
and
\[
I_2 = \int \int_{0}^{\infty} b \exp\left[\frac{(a - w)^2 + (b + w)^2}{4}\right] \, da \, db
\]
\[
= \int_{0}^{\infty} b \exp\left[\frac{(b + w)^2}{4}\right] \, db \exp\left[\frac{(a - w)^2}{4}\right] \, da
\]
\[
= \left\{ 2 \exp\left[\frac{w^2}{4}\right] - w [\Phi(w)] 2\sqrt{\pi} \right\} \Phi(-w) \, 2\sqrt{\pi}.
\]
Noting that
\[
I_1 + I_2 = 4\sqrt{\pi} \exp\left[\frac{w^2}{4}\right] \left[\Phi(w) + \Phi(-w)\right] + 4\pi w \{[1 - \Phi(w)]\Phi(w) - \Phi(w)\Phi(-w)\}
\]
\[
= 4\sqrt{\pi} \exp\left[\frac{w^2}{4}\right],
\]
we obtain
\[
H(v, \theta) = \frac{1}{2\sqrt{\pi}} 4\sqrt{\pi} \exp\left[\frac{-w^2}{4}\right] \exp\left[\frac{w^2}{2}\right]
\]
\[
= \frac{1}{2\sqrt{\pi}} \exp\left[\frac{w^2}{4}\right]
\]
\[
= \frac{1}{2\sqrt{\pi}} \exp\left[\frac{v^2}{2} \sin^2(\theta) - \cos^2(\theta)\right] .
\]
Finally, combining (3.8) and (3.9) above, the joint probability density element of
\[V_n^{(2)}\] and \(\theta_n\), and for case (C2) pertaining, is given by
\[ dF(v, \theta) = \left\{ n(n-1) \left[ 1 - \Phi(\cos(\theta) v(1 + \tan(\theta))) \right]^{b_2} \right. \]

\[ = \frac{1}{4\pi^2} \left[ \cos(\theta) + \sin(\theta) \right] e^{2v^2} \sqrt{2\pi} \exp\left\{ \frac{v^2}{2} \left[ \sin(\theta) - \cos(\theta) \right]^2 \right\} \] \, d\theta \, dv 

\[ = f_n(v, \theta) \, d\theta \, dv \quad (3.10) \]

We next calculate the limit of (3.10). To that end, let

\[ v = -\frac{1}{\sqrt{2}} (a_n + b_n s) \]

and

\[ \theta = \frac{\pi}{4} - \frac{1}{4\sqrt{2}} b_n t \]

where \( a_n \) and \( b_n \) are given by (3.3). Then the joint density element of \((V_n, \theta_n)\), pertaining to case (C2), is transformed to

\[ f_n^{-1} \left( \frac{1}{2} (a_n + b_n s), \frac{\pi}{4} - \frac{1}{4\sqrt{2}} b_n t \right) \frac{1}{2} b_n^2 \]

\[ = G_{1n}(s, t) \, G_{2n}(s, t) \, G_{3n}(s, t) \, G_{4n}(s, t), \]

where

\[ G_{1n}(s, t) = \left\{ 1 - \Phi(\cos(\frac{\pi}{4} - \frac{1}{4\sqrt{2}} b_n t) \frac{1}{\sqrt{2}} (a_n + b_n s) (1 + \tan(\frac{\pi}{4} - \frac{1}{4\sqrt{2}} b_n t)) \right\}^{n-2}, \]

\[ G_{2n}(s, t) = \frac{1}{4\pi^2} \left\{ \cos(\frac{\pi}{4} - \frac{1}{4\sqrt{2}} b_n t) + \sin(\frac{\pi}{4} - \frac{1}{4\sqrt{2}} b_n t) \right\}, \]

\[ G_{3n}(s, t) = n(n-1) \frac{1}{2} b_n^2 \exp\left\{ -2\left( \frac{1}{\sqrt{2}} (a_n + b_n s) \right)^2 \right\}, \]

\[ G_{4n}(s, t) = \sqrt{2\pi} \exp\left\{ \frac{1}{2} (a_n + b_n s)^2 \left[ \sin(\frac{\pi}{4} - \frac{1}{\sqrt{2}} b_n t) - \cos(\frac{\pi}{4} - \frac{1}{\sqrt{2}} b_n t) \right]^2 \right\} \]

It is easy to check that

\[ \cos(\frac{\pi}{4} - \frac{1}{\sqrt{2}} b_n t) + \sin(\frac{\pi}{4} - \frac{1}{\sqrt{2}} b_n t) \]

\[ = \sqrt{2} \cos(\frac{1}{\sqrt{2}} b_n t) \]

\[ = \sqrt{2} \left( 1 - \frac{1}{4} b_n^2 t^2 + o(b_n^3) \right), \]
and

\[
\cos(\theta) - \sin(\theta) = \cos\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}} b_n t\right) - \sin\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}} b_n t\right)
\]

\[
= \sqrt{2} \cos\left(\frac{\pi}{2} - \frac{1}{\sqrt{2}} b_n t\right)
\]

\[
= \sqrt{2} \sin\left(\frac{1}{\sqrt{2}} b_n t\right)
\]

\[
= b_n t + o(b_n^2).
\]

Therefore, we obtain

\[
G_{2n}(s,t) \longrightarrow \frac{1}{2\sqrt{2 \pi}}, \quad (3.11)
\]

\[
G_{3n}(s,t) = n(n-1) \frac{1}{2} b_n^2 \exp\left[-2\left\{\frac{1}{\sqrt{2}} (a_n + b_n s)^2\right\}\right]
\]

\[
= n(n-1) \frac{1}{2} b_n^2 \exp\left[-\left\{2\log(n) + o(b_n) - \log\log(n + \log(4\pi)) + (2 + o(b_n))s + o(b_n)\right\}\right]
\]

\[
= n(n-1) \frac{1}{2} b_n^2 n^{-2} \log(n) 4\pi \exp[-2s] \exp[o(b_n)]
\]

\[
\longrightarrow \pi \exp[-2s], \quad (3.12)
\]

and

\[
G_{4n}(s,t) = \sqrt{2\pi} \exp\left[\frac{1}{4} \left(a_n + b_n s\right)^2 \left(b_n t + o(b_n^2)\right)^2\right]
\]

\[
\longrightarrow \frac{1}{2\sqrt{2 \pi}} \exp\left[\frac{1}{4} t^2\right] \text{ as } n \longrightarrow \infty. \quad (3.13)
\]

Furthermore, By the well-known relation for the tail of \(\Phi(\cdot)\), that is,

\[
1 - \Phi(u) \sim \frac{\phi(u)}{u} \text{ as } u \longrightarrow \infty,
\]

we obtain

\[
n \Phi(\cos\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}} b_n t\right)^{-1} (a_n + b_n s) (1 + \tan\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}} b_n t\right))
\]

\[
= n \Phi\left(\frac{1}{\sqrt{2}} (a_n + b_n s) \left(\cos\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}} b_n t\right) + \sin\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}} b_n t\right)\right)\right)
\]

\[
= n \Phi\left(\frac{1}{\sqrt{2}} (a_n + b_n s) \left[1 - \frac{1}{4} b_n^2 t^2 + o(b_n^3)\right]\right)
\]
\[ = n \left\{ 1 - \Phi(a_n + b_n(s - \frac{1}{4} \bar{t})^2) + o(b_n) \right\} \rightarrow \exp[-(s - \frac{1}{4} \bar{t})] , \]

which implies
\[ G_{in}(s,t) = \left\{ 1 - \Phi(\cos(\frac{\pi}{4} - \frac{1}{\sqrt{2}} b_n t) - \frac{1}{\sqrt{2}} (a_n + b_n s) (1 + \tan(\frac{\pi}{4} - \frac{1}{\sqrt{2}} b_n t)) \right\}^{n^2} \rightarrow \exp[-e^{-(s - \frac{t^2}{4})}] . \]

Finally, combining the results from (3.10) to (3.14), we obtain
\[ \frac{d}{dt} \left( \frac{1}{\sqrt{2}} (a_n + b_n s), \frac{\pi}{4} - \frac{1}{\sqrt{2}} b_n t \right) \rightarrow \frac{1}{2\sqrt{\pi}} \int \exp[-2s - e^{-(s - \frac{t^2}{4})} + \frac{t^2}{4}] \ ds dt = E(s,t) \ ds dt . \]

Note that
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(s,t) \ ds dt = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \int \exp[-2s - e^{-(s - \frac{t^2}{4})} + \frac{t^2}{4}] \ ds dt \\
= \frac{1}{2\sqrt{\pi}} \lim_{u \to \infty} \lim_{v \to \infty} \int_{-\infty}^{u} [e^{-\frac{t^2}{2}} + e^{\frac{t^2}{2}} - v] e^{-\frac{t^2}{4}} - v e^{\frac{t^2}{4}} \ dt \\
= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{4}} \ dt = 1.
\]

Therefore, by Lemma 3.3.1 and Scheffe’s Lemma, we have, for all \( u \in \mathbb{R} \) and \( v \in \mathbb{R} \),
\[ p[V_n = \frac{-1}{\sqrt{2}} (a_n + b_n \nu), \theta_n = \pi - \frac{1}{\sqrt{2}} b_n \mu] \leq \frac{1}{1 - p[V_n^{(2)} \text{ is a pure value}]} \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{24\pi} \exp[-2s - e^{-(s-t^2/4) + \frac{t^2}{4}}] ds dt \]

\[ = \frac{1}{24\pi} \int \left[ e^{-t^2/2} + e^{-t^2/4} - \nu \right] e^{-t^2/4} \nu \ e^{t^2/4} dt \]

where \( a_n \) and \( b_n \) are defined by (3.3).

Since \( \theta_n = \tan^{-1}(\frac{1 - \lambda_n}{\lambda_n}) \) and \( \tan(x) = x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \cdots \), we have

\[ p[V_n = \frac{-1}{\sqrt{2}} (a_n + b_n \nu), \theta_n = \pi - \frac{1}{\sqrt{2}} b_n \mu] \]

\[ = p[V_n = \frac{-1}{\sqrt{2}} (a_n + b_n \nu), \tan^{-1}(\frac{1 - \lambda_n}{\lambda_n}) = \pi - \frac{1}{\sqrt{2}} b_n \mu] \]

\[ = p[V_n = \frac{-1}{\sqrt{2}} (a_n + b_n \nu), \lambda_n = \frac{1}{1 + \tan(\frac{\pi}{4} - \frac{1}{\sqrt{2}} b_n \mu)}] \]

\[ = p[V_n = \frac{-1}{\sqrt{2}} (a_n + b_n \nu), \lambda_n = \frac{1}{2} + \frac{1}{2} \tan(\frac{1}{\sqrt{2}} b_n \mu)] \]

\[ = p[V_n = \frac{-1}{\sqrt{2}} (a_n + b_n \nu), \lambda_n = \frac{1}{2} + \frac{1}{2} \frac{b_n \mu + o(b_n)}{\sqrt{2}}] \]

Therefore, we obtain

**Proposition 3.4.1** Let \( V_n \) be the value of a 2 by n matrix game with i.i.d. standard normal payoff distributions and \((1-\lambda_n, \lambda_n)\) be the optimal strategy of player I. Then for all \( u \in \mathbb{R} \) and \( v \in \mathbb{R} \), as \( n \) goes to infinity,

\[ p[V_n = \frac{-1}{\sqrt{2}} (a_n + b_n \nu), \lambda_n = \frac{1}{2} + \frac{1}{2 \sqrt{2}} b_n \nu] \leq \frac{1}{1 - p[V_n^{(2)} \text{ is a pure value}]} \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{24\pi} \left[ e^{-t^2/2} + e^{-t^2/4} - \nu \right] e^{-t^2/4} \nu \ e^{t^2/4} dt \]


where $a_n$ and $b_n$ are defined by (3.3).

Next, by letting $u = \infty$ or $v = \infty$ in Proposition 3.4.1, we then have

**Corollary 3.4.1** The asymptotic distribution of $V_n$, conditioned on case (C2), satisfies

$$P[V_n \leq \sum\frac{1}{\sqrt{2}} (a_n + b_n v)] \frac{1}{1 - P[V_n^{(2)} \text{is a pure value}]}$$

$$\rightarrow \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} [e^{-t^2/2} + e^{-t^2/4} - v] e^{-e^t/4} \cdot e^{t^2/4} \cdot e^{-t^2/4} dt, \text{ for all } v \in \mathbb{R},$$

and the asymptotic distribution of $\lambda_n$, conditioned on case (C2), satisfies

$$P[\lambda_n \leq \frac{1}{2} + \frac{1}{2\sqrt{2}} b_n u] \frac{1}{1 - P[V_n^{(2)} \text{is a pure value}]}$$

$$\rightarrow 1 - \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{u} e^{-t^2/4} dt, \text{ for all } u \in \mathbb{R}.$$

### 3.5 Miscellaneous Results

In this section, we consider perturbations in the payoff of a matrix game. The study of these is relevant to the sensitivity analysis of linear programming problems. First, we need the following lemma, which can be found in Theorem 6.2.1 of Galambos (1987).

**Lemma 3.5.1** Let $X_1, X_2, \ldots$ be i.i.d. random variables with the common distribution function $F(x)$. Let $N(n)$ be a positive integer valued random variable. Assume $\frac{N(n)}{n} \rightarrow \tau$ in probability, as $n \rightarrow \infty$, where $\tau$ is a positive random variable. Assume
further that there are sequences \(a_n\) and \(b_n > 0\) such that \(\frac{\max X_i - a_n}{b_n}\) converges weakly to a non-degenerate distribution function \(H(x)\). Then, as \(n \to +\infty\),

\[
\lim_{n \to +\infty} P[\max_{1 \leq i \leq n} X_i \leq a_n + b_n] = \int_{-\infty}^{+\infty} H'(x) \, dP\{\tau < y\}.
\]

Now if \(Y_n = c_n + X_n\), where the \(X_n\)'s are i.i.d. random variables belonging to some domain of attraction and the \(c_n\)'s are constants such that \(c_n \in S = \{s_1, \ldots, s_m; s_1 < s_2 < \cdots < s_m\}\). Then the extreme value distribution of the maximum of \(\{Y_1, \ldots, Y_n\}\) can be obtained as follows.

**Proposition 3.5.1** Consider \(Y_n = c_n + X_n\), where \(X_n\)'s are i.i.d. random variables, and \(c_n \in S = \{s_1, \ldots, s_m; s_1 < s_2 < \cdots < s_m\}\). Assume further that there are sequences \(\max X_i - a_n\)

\(a_n\) and \(b_n > 0\) such that \(\frac{\max X_i - a_n}{b_n}\) converges weakly to a non-degenerate distribution function \(H(x)\). Let \(\{n_{ki}\}\) be the subsequence of \(\{n\}\) such that \(c_{n_{ki}} = s_i, 1 \leq i \leq m\). Denote \(k_i(n) = \text{number of } c_j, 1 \leq j \leq n, \text{ such that } c_j = s_i\). Further, assume \(\frac{k_i(n)}{n} \to \tau_i\), where \(\tau_i\) is a positive random variables, \(1 \leq i \leq m\). Then

\[
\lim_{n \to +\infty} P[\max_{1 \leq i \leq n} Y_i \leq a_n + b_n] = \int_{-\infty}^{+\infty} H'(x) \, dP\{\tau < y\} \prod_{j=1}^{m-1} H(x + \frac{s_m - s_j}{\lim b_n}).
\]

**Proof:** It is easy to see that

\[
P[\max_{1 \leq i \leq n} Y_i \leq t_n]
= P[\max_{1 \leq i \leq n} (c_i + X_i) \leq t_n]
\]
\[ P[\max\{ \max (s_1 + X_{nk1}), \cdots, \max (s_m + X_{nk_m}) \} \leq t_n] \]

\[ = P[\max (s_1 + X_{nk1}) \leq t_n, \cdots, \max (s_m + X_{nk_m}) \leq t_n] \]

\[ = \prod_{i=1}^{m} P[\max (s_i + X_{nk}) \leq t_n] \]

\[ = \prod_{i=1}^{m} P[s_i + \max X_i \leq t_n] \]

Now, let \( t_n = s_m + a_n + b_n x \); then, by Lemma 3.5.1,

\[ \lim_{n \to \infty} P[\max Y_i \leq s_m + a_n + b_n x] \]

\[ = \int_{-\infty}^{+\infty} H'(x) dP[\tau_m < y] \prod_{j=1}^{m-1} H(x + \frac{s_m - s_j}{\lim_{n \to \infty} b_n}) . \]

\[ \Box \]

A further result on the role of additive constants in Extreme Value Theory, not directly based on Lemma 3.5.1, is as follows. Let \( Y_n = c_n + X_n \), where the \( X_n \)'s are i.i.d. random variables belonging to some domain of attraction and the \( c_n \)'s are constants such that \( \lim_{n \to \infty} c_n = c \). Then we have

**Proposition 3.5.2** Consider \( Y_n = c_n + X_n \), where \( X_n \)'s are i.i.d. random variables, and \( c_n \)'s are constants such that \( \lim_{n \to \infty} c_n = c \). Assume further that there are constants \( a_n \) and \( b_n > 0 \) such that \( \frac{\max X_i - a_n}{b_n} \) converges weakly to a non-degenerate distribution function \( H(x) \). Then

\[ \lim_{n \to \infty} P[\max Y_i < c + a_n + b_n x] = H(x) . \]
Proof: Since \( \lim_{n \to \infty} c_n = c \), for all \( \epsilon > 0 \), there exists \( N_\epsilon \in \mathbb{N} \) such that
\[
c - \epsilon < c_n < c + \epsilon, \ \forall \ n \geq N_\epsilon .
\]
In particular, choose \( \epsilon_m = o(b_m) \) and \( \epsilon_m \) decreases to 0, then
\[
c - \epsilon_m < c_n < c + \epsilon_m, \ \forall \ n \geq N_m, \ \forall \ m \in \mathbb{N}.
\]
Therefore, for all \( n \geq N_m \),
\[
c - \epsilon_m + X_n < Y_n = c_n + X_n < c + \epsilon_m + X_n, \ \forall \ n, \forall \ m ,
\]
which implies
\[
c - \epsilon_m + \max_{N_m \leq j \leq N_m + n - 1} X_j
\leq \max_{N_m \leq j \leq N_m + n - 1} Y_j
\leq c + \epsilon_m + \max_{N_m \leq j \leq N_m + n - 1} X_j, \ \forall \ n, \forall \ m .
\]
From the above, we obtain
\[
P[c - \epsilon_m + \max_{N_m \leq j \leq N_m + n - 1} X_j \leq a_n + b_n x + c - \epsilon_m]
\leq P[\max_{1 \leq j \leq n} Y_j \leq a_n + b_n x + c - \epsilon_m], \ \forall \ n, \forall \ m .
\]
And, since the \( X_n \)'s are i.i.d., we can write
\[
P[c - \epsilon_m + \max_{1 \leq j \leq n} X_j \leq a_n + b_n x + c - \epsilon_m]
\leq P[\max_{1 \leq j \leq n} Y_j \leq a_n + b_n x + c - \epsilon_m], \ \forall \ n, \forall \ m ,
\]
which implies
\[
H(x) \leq \lim_{n \to \infty} P[\max_{1 \leq j \leq n} Y_j \leq a_n + b_n x + c - \epsilon_m], \ \forall \ m .
\]
Similarly, we get
\[
H(x) \leq \lim_{n \to \infty} P[\max_{1 \leq j \leq n} Y_j \leq a_n + b_n x + c + \epsilon_m], \ \forall \ m ;
\]
that is,
\[ \lim_{n \to \infty} \mathbb{P}(\max_{1 \leq j \leq n} Y_j \leq a_n + b_n x + c + \epsilon_m) \leq H(x) \]
\[ \Rightarrow \lim_{n \to \infty} \mathbb{P}(\max_{1 \leq j \leq n} Y_j \leq a_n + b_n x + c - \epsilon_m), \quad \forall m. \]

In particular, let \( m = n \), then we obtain
\[ \lim_{n \to \infty} \mathbb{P}(\max_{1 \leq j \leq n} Y_j \leq a_n + b_n x + c + \epsilon_n) \leq H(x) \]
\[ \Rightarrow \lim_{n \to \infty} \mathbb{P}(\max_{1 \leq j \leq n} Y_j \leq a_n + b_n x + c - \epsilon_n). \]

Since \( \epsilon_n = o(b_n) \), by Khintchine’s Theorem we obtain
\[ \lim_{n \to \infty} \mathbb{P}(\max_{1 \leq j \leq n} Y_j \leq a_n + b_n x + c) \leq H(x) \]
\[ \Rightarrow \lim_{n \to \infty} \mathbb{P}(\max_{1 \leq j \leq n} Y_j \leq a_n + b_n x + c), \]
which implies
\[ \lim_{n \to \infty} \mathbb{P}(\max_{1 \leq i \leq n} Y_i < c + a_n + b_n x) = H(x). \]

\( \Box \)

Proposition 3.5.1 and 3.5.2 are now applied to models relevant to the sensitivity analysis of lower game values. First, assume that \( Y_{ij} = c_{ij} + X_{ij}, 1 \leq i \leq k, 1 \leq j \leq n \), where \( X_{ij} \)'s are i.i.d. random variables, and \( c_{ij} \)'s are constants such that \( c_{ij} \in S = \{s_1, \ldots, s_m; s_1 < s_2 < \cdots < s_m\} \). Assume further that there are constants \( a_n \) and \( b_n > 0 \) such that for any fixed \( 1 \leq i \leq k \), \( \frac{\max X_{ij} - a_n}{b_n} \) converges weakly to a non-degenerate distribution function \( H(x) \). Let \( \{n_{\alpha}\} \) be the subsequence of \( \{n\} \) such
that $c_{inh} = s_h$, $1 \leq h \leq m$. Denote $k_{nh}(n) = \text{number of } \{c_j: 1 \leq j \leq n\} \text{ such that } c_j = s_n$. Furthermore, if \( \frac{k_{nh}(n)}{n} \to \tau_h \), where $\tau_h$ is a positive random variables, $1 \leq h \leq m$, $1 \leq i \leq k$. Then

\[
P\left[ \min_{1 \leq i \leq k} \max_{1 \leq j \leq n} Y_{ij} \leq t_n \right]
= 1 - P\left[ \min_{1 \leq i \leq k} \max_{1 \leq j \leq n} Y_{ij} > t_n \right]
= 1 - \prod_{i=1}^{k} P\left[ \max_{1 \leq j \leq n} Y_{ij} > t_n \right]
= 1 - \prod_{i=1}^{k} \left( 1 - P\left[ \max_{1 \leq j \leq n} Y_{ij} \leq t_n \right] \right)
= 1 - \prod_{i=1}^{k} \left( 1 - P\left[ \max_{1 \leq j \leq n} Y_{ij} \leq t_n \right] \right)
\]

Now let $t_n = s_m + a_n + b_n x$, then from Proposition 3.5.1, we have

\[
\lim_{n \to \infty} P\left[ \min_{1 \leq i \leq k} \max_{1 \leq j \leq n} Y_{ij} \leq s_m + a_n + b_n x \right]
= 1 - \prod_{i=1}^{k} \left( 1 - \int_{-\infty}^{\infty} H^y(x) \, dP(\tau_{ih} < y) \right) \prod_{h=1}^{m-1} H(x + \frac{s_m - s_h}{\lim b_n})
\]

Next, assume that $Y_{ij} = c_{ij} + X_{ij}$; $1 \leq i \leq k$, $1 \leq j \leq n$, as above with $c_{ij}$'s being constants such that $c_{in} \to c_i$ as $n \to \infty$, $1 \leq i \leq k$. Then from Proposition 3.5.2, we have

\[
\lim_{n \to \infty} P\left[ \min_{1 \leq i \leq k} \max_{1 \leq j \leq n} Y_{ij} \leq \min_{1 \leq i \leq k} c_i + a_n + b_n x \right]
= 1 - \prod_{h=1}^{k} \left( 1 - H(x + \frac{\min_{1 \leq i \leq k} c_i - c_h}{\lim b_n}) \right)
\]

3.6 Some Conclusions

In this chapter, we study the asymptotic value distribution of a matrix game with i.i.d. payoff distributions. In particular, in the 2 by n case, we give the
probability density elements for the value distribution and the optimal strategies of player I; and with the further assumption of standard normal payoff, we derived the asymptotic game value distribution. Although the method presented can be generalized to any finite $k$, the computation becomes tedious. Finding a way to overcome this difficulty is a challenging topic. Some open problems are, for example, allowing $k$ to be a function of $n$ instead of a fixed constant, or dropping the i.i.d. assumption in the payoff distributions. As for the asymptotic perturbation analysis of a payoff matrix, it remains to consider the upper game value and the value itself.
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