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Nordhaus-Gaddum Problems for Colin de Verdière Type Parameters, Variants of Tree-width, and Related Parameters

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Abstract A Nordhaus-Gaddum problem for a graph parameter is to determine a tight lower or upper bound for the sum or product of the parameter evaluated on a graph and on its complement. This article surveys Nordhaus-Gaddum results for the Colin de Verdière type parameters μ, ν, and ξ; tree-width and its variants largeur d'arborescence, path-width, and proper path-width; and minor monotone ceilings of vertex connectivity and minimum degree.

1 Introduction

For a graph parameter ζ, the following are *Nordhaus-Gaddum* problems:

- Determine a (sharp) lower or upper bound on $\zeta(G)+\zeta(\overline{G})$.
- Determine a (sharp) lower or upper bound on $\zeta(G)\cdot\zeta(\overline{G})$.

The name comes from the next theorem (see Section 2 for the definition of the chromatic number $\chi(G)$ of G).

Theorem 1. [31] *For any graph G,*

$$2\sqrt{|G|} \leq \chi(G)+\chi(\overline{G}) \leq |G|+1$$

and

$$|G| \leq \chi(G)\cdot\chi(\overline{G}) \leq \left(\frac{|G|+1}{2}\right)^2.$$

Each bound is assumed for infinitely many values of n.

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An extensive survey of Nordhaus-Gaddum problems in general can be found in [1]. The focus of this survey is on Nordhaus-Gaddum problems for the following parameters: the Colin de Verdière type parameters μ, ν, and ξ; tree-width and its variants largeur d'arborescence la, path-width pw, and proper path-width ppw; and minor monotone ceilings of vertex connectivity $\lceil\kappa\rceil$ and minimum degree $\lceil\delta\rceil$.

These parameters are defined in Section 2, where their background and connections are discussed. Section 3 discusses the Nordhaus-Gaddum sum upper bounds for these parameters, which take the form of a constant times the order of the graph. The Nordhaus-Gaddum sum lower bounds are discussed in Section 4, which for μ, ν, and ξ is known in linear algebra as a Graph Complement Conjecture (GCC), and Section 5 discusses Nordhaus-Gaddum product upper and lower bounds. All of the parameters μ, ν, ξ, $\mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}, \lceil\kappa\rceil, \lceil\delta\rceil$ are minor monotone, which makes the Nordhaus-Gaddum bounds more interesting (see Sections 3 and 4 for comments on the trivial values of the related non-minor monotone parameters).

2 Parameters

In this section we define some graph terminology and then define and discuss parameters. Colin de Verdière type parameters μ, ν, ξ, are discussed in Section 2.1, tree-width tw and variants largeur d'arborescence la, path-width pw, and proper-path-width ppw are discussed in Section 2.2, and the minor monotone ceilings of vertex connectivity $\lceil\kappa\rceil$ and minimum degree $\lceil\delta\rceil$ are discussed in in Section 2.4. Section 2.3 reviews the less well known Halin S-functions, which include $\mathrm{tw}+1$ and $\lceil\kappa\rceil+1$. Relationships between the parameters discussed here are shown in a diagram in Figure 5 in Section 4. A more extensive discussion of the connections between the parameters $\mu, \nu, \xi, \mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}, \lceil\kappa\rceil, \lceil\delta\rceil$ and the problem of determining maximum nullity among real symmetric matrices described by a graph can be found in [2].

All graphs are simple, undirected, and finite. The *complement* of a graph $G = (V,E)$ is the graph $\overline{G} = (V,\overline{E})$, where $\overline{E}$ consists of all two element sets from V that are not in E. For a graph $G = (V,E)$ and $W \subseteq V$, the *induced subgraph* $G[W]$ is the graph with vertex set W and edge set $\{wu \in E : w,u \in W\}$ (wu and uw denote the same edge $\{w,u\}$). The subgraph induced by $\overline{W} = V \setminus W$ is usually denoted by $G-W$, or in the case W is a single vertex $\{v\}$, by $G-v$. If $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are disjoint graphs, the *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph having vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup E$, where E consists of all the edges v_1v_2 with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. A graph is a *cograph* or *decomposable* if it can be expressed as a sequence of disjoint unions and joins of isolated vertices. If graphs G and H each contain a clique K_t, then a graph obtained by identifying these two specific copies of K_t is a *clique-sum of G and H on K_t* and is denoted by $G\langle K_t\rangle H$. The graph $\emptyset$ has no vertices, whereas $\overline{K_n}$ has $n \geq 1$ vertices but no edges.

The *vertex connectivity* $\kappa(G)$ of G is the smallest number k such that there is a set S of k vertices such that $G-S$ is disconnected, provided G is not complete. By convention, $\kappa(K_n)=n-1$. The *minimum degree* of G is $\delta(G)=\min_{v\in V}\deg_G(v)$.

The *contraction* of an edge $e=uv$ of G, denoted by G/e, is obtained by identifying the vertices u and v, deleting any loops that arise in this process, and replacing any multiple edges by a single edge. A *minor* of G arises by performing a (possibly empty) sequence of deletions of edges, deletions of isolated vertices, and/or contractions of edges. A minor H of G obtained only by contractions can also be constructed by partitioning the vertices of G as $W_1\dot{\cup}\cdots\dot{\cup}W_h=V(G)$ such that the induced subgraph $G[W_i]$ is connected for all $i=1,\ldots,h$. Then $V(H)=\{W_1,\ldots,W_h\}$ and $W_iW_j\in E(H)$ if and only if $i\neq j$ and there exist $w_i\in W_i, w_j\in W_j$ such that $w_iw_j\in E(G)$. The notation $H\preceq G$ means that H is a minor of G, and $H\prec G$ means that H is a *proper minor* of G, i.e., $H\preceq G$ and $H\neq G$. A graph parameter ζ is *minor monotone* if $H\preceq G$ implies $\zeta(H)\le\zeta(G)$.

For a graph parameter ζ that takes nonnegative integer values, the *minor monotone ceiling* $\lceil\zeta\rceil$ is defined by

$$\lceil\zeta\rceil(G)=\max\{\zeta(H):H\preceq G\}$$

(see, for example, [2]). Observe that $\lceil\zeta\rceil$ is minor monotone and for all G, $\zeta(G)\le\lceil\zeta\rceil(G)$. Minor monotone ceilings have been studied for several parameters by various authors, e.g., [2, 14, 15].

The *clique number* $\omega(G)$ is the maximum order of a clique in G, and the *Hadwiger number* $\eta(G)$ is the maximum order of a clique minor of G, i.e., $\eta=\lceil\omega\rceil$. The *chromatic number* $\chi(G)$ is the minimum number of colors necessary to color the vertices of G so that adjacent vertices have different colors. Since $\chi(K_n)=n$, $\omega(G)\le\chi(G)$ and thus $\eta(G)\le\lceil\chi\rceil(G)$ for all graphs G. The *Hadwiger Conjecture* is that for all graphs G, $\chi(G)\le\eta(G)$, or equivalently, $\lceil\chi\rceil=\eta$.

2.1 Colin de Verdière Type Parameters

In 1990 Colin de Verdière ([8] in English) introduced the graph parameter μ to characterize planarity; $\mu(G)$ is equal to the maximum nullity among all matrices satisfying several conditions including the Strong Arnold Hypothesis (defined below). The Colin de Verdière number μ is the first of several parameters (called *Colin de Verdière type parameters*) that take the maximum nullity among real symmetric matrices that are described by the graph and satisfy the Strong Arnold Hypothesis, and possibly other conditions. All the Colin de Verdière type parameters we discuss have been shown to be minor monotone [4, 8, 9].

All matrices discussed are real and symmetric; the set of $n\times n$ real symmetric matrices is denoted by $S_n(\mathbb{R})$. Define $[n]=\{1,\ldots,n\}$. For $A=[a_{ij}]\in S_n(\mathbb{R})$, the *graph* of A is $\mathscr{G}(A)=(V,E)$ where $V=[n]$ and $E=\{ij:i,j\in[n], i\neq j,\text{ and } a_{ij}\neq 0\}$; the diagonal of A is ignored in determining $\mathscr{G}(A)$. The *set of symmetric matrices* *described by* G is $\mathscr{S}(G) = \{A \in S_n(\mathbb{R}) : \mathscr{G}(A) = G\}$. The *maximum nullity* of a graph is $\mathrm{M}(G) = \max\{\mathrm{null} A : A \in \mathscr{S}(G)\}$. We note that this parameter is not a Colin de Verdière type parameter and is not minor monotone.

A real symmetric matrix A satisfies the *Strong Arnold Hypothesis* provided there does not exist a nonzero real symmetric matrix X satisfying

- $AX = 0$,
- $A \circ X = 0$, and
- $I \circ X = 0$,

where $\circ$ denotes the entry-wise product, i.e., $(A \circ B)_{ij} = a_{ij}b_{ij}$, and I is the identity matrix. The Strong Arnold Hypothesis is equivalent to the requirement that certain manifolds intersect transversally (see [22]).

The Colin de Verdière number $\mu(G)$ is defined to be the maximum nullity among symmetric matrices $A = [a_{ij}]$ such that:

- $A \in \mathscr{S}(G)$;
- A satisfies the Strong Arnold Hypothesis;
- A has exactly one negative eigenvalue (counting multiplicity); and
- A is a generalized Laplacian (i.e., for all $i \neq j$, $a_{ij} \leq 0$).

Colin de Verdière introduced the parameter ν in [9]; the definition utilizes positive semidefinite matrices. A matrix $A \in S_n(\mathbb{R})$ is *positive semidefinite* if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, or equivalently all eigenvalues of A are nonnegative. Then $\nu(G)$ is defined to be the maximum nullity among matrices A such that:

- $A \in \mathscr{S}(G)$;
- A satisfies the Strong Arnold Hypothesis; and
- A is positive semidefinite.

The parameter $\xi(G)$ was introduced in [4]; $\xi(G)$ is the maximum nullity among real symmetric matrices such that:

- $A \in \mathscr{S}(G)$ and
- A satisfies the Strong Arnold Hypothesis.

Remark 1. For every graph G, $\nu(G) \leq \xi(G)$ and $\mu(G) \leq \xi(G)$, but μ and ν are noncomparable (see [2] for examples).

As noted in [19], the following result can be derived from results in [28, 29].

Theorem 2. [19, 28, 29] *For every graph G, $\kappa(G) \leq \nu(G)$.*

Since all Colin de Verdière parameters are minor monotone, the next corollary is immediate.

Corollary 1. [2] *For every graph G, $\lceil\kappa\rceil(G) \leq \nu(G)$.*

2.2 Tree-width and Variants

Tree-width is a well known parameter with many equivalent characterizations, any one of which may be chosen as the definition; for more information see a standard reference such as [10]. Here we use the k-tree characterization as the definition.

A *k-tree* is constructed inductively by starting with a complete graph on $k+1$ vertices and connecting each new vertex to the vertices of an existing clique on k vertices. A tree is a 1-tree. A k-tree is a k-connected chordal graph with the maximum order of a clique equal to $k+1$ (G is *chordal* if G has no induced cycle of length greater than three). A *partial k-tree* is a subgraph of a k-tree. The maximal cliques of a k-tree are of order $k+1$, and the *facets* of a maximal clique are its k-clique subgraphs. Two maximal cliques are *adjacent* if they share a facet. For a graph G, $\mathrm{tw}(G)$ is the minimum k such that G is a partial k-tree. Note that $\omega(G)-1 \leq \mathrm{tw}(G)$ and an equivalent way to define tree-width is $\mathrm{tw}(G) = \min\{\omega(H)-1 : H \text{ is chordal supergraph of } G\}$.

Several related parameters are defined as the minimum k for which the graph is a subgraph of a certain type of k-tree. Examples with $k=2$ are shown in Figure 1.

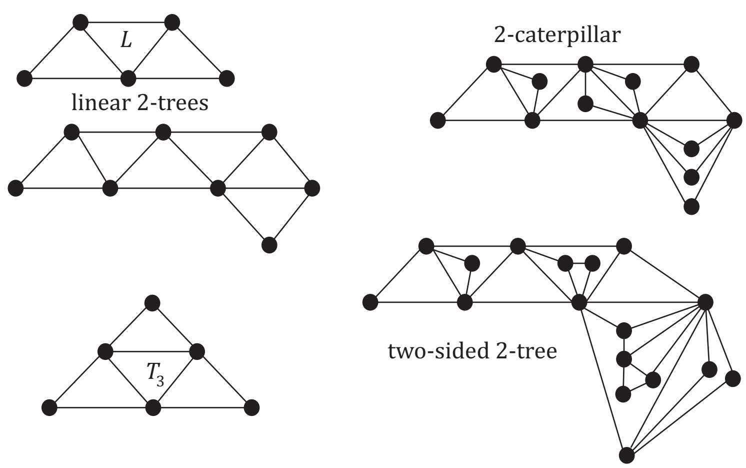

Fig. 1 Examples of 2-trees. Clockwise from top left: two linear 2-trees including the specific 2-tree L; a 2-caterpillar that is not a linear 2-tree; a two-sided 2-tree that is not a 2-caterpillar; the supertriangle T_3, which is not a two-sided 2-tree.

A *linear k-tree* is a k-tree constructed with the restriction that at each stage the new vertex is connected to the vertices of an existing K_k that includes a vertex of degree k (without loss of generality it can be assumed that the new vertex is adjacent to the vertex just added). Equivalently, a linear k-tree is a k-tree in which exactly two vertices have degree k (or is the k-tree K_{k+1}). A *partial linear k-tree* is a subgraph of a linear k-tree. The *proper path-width* of a graph G, denoted by $\mathrm{ppw}(G)$, is the minimum k for which G is a partial linear k-tree. Proper path-width was introduced in [35].

A *k-caterpillar* is a k-tree constructed with the restriction that at each stage the new vertex is adjacent to the k vertices of some facet of the maximal clique that was added at the previous stage; an example with $k = 2$ is shown in Figure 1. A k-caterpillar can also be constructed by first constructing a linear k-tree and then possibly adding extra vertices, with each new vertex adjacent to all the vertices in a facet that is shared by two maximal cliques of the linear k-tree used to construct the k-caterpillar. The *path-width* of a graph G, denoted by $\mathrm{pw}(G)$, is the minimum k for which G is a subgraph of a k-caterpillar.

A *two-sided k-tree* (or *straight k-tree*) is a k-tree constructed with the restriction that at each stage the new vertex is adjacent to the k vertices of an existing K_k that either includes a vertex of degree k or is the same as the K_k to which some previous vertex was connected. Every tree is a two-sided 1-tree. The *largeur d'arborescence* of a graph G is the minimum k for which G is a subgraph of a two-sided k-tree, and is denoted by $\mathrm{la}(G)$ [9, 20].

Any linear k-tree is a k-caterpillar, any k-caterpillar is a two-sided k-tree, and any two-sided k-tree is a k-tree, but not vice versa, as illustrated in Figure 1. Thus $\mathrm{tw}(G) \le \mathrm{la}(G) \le \mathrm{pw}(G) \le \mathrm{ppw}(G)$. There are known close relationships between tree-width and largeur d'arborescence, and between path-width and proper path-width, as described in the next theorem.

Theorem 3. *For every graph G,*

1. [9] $\mathrm{tw}(G) \le \mathrm{la}(G) \le \mathrm{tw}(G) + 1$, *and*
2. [36] $\mathrm{pw}(G) \le \mathrm{ppw}(G) \le \mathrm{pw}(G) + 1$.

Tree-width, largeur d'arborescence, path-width, and proper path-width are all minor monotone. Alternative characterizations of largeur d'arborescence, path-width, and proper path-width in terms of a two-coloring with color change rule (called zero forcing) were given in [2] and we summarize these here.[1] Let G be a graph.

- A subset $B \subseteq V(G)$ defines a *coloring* by coloring all vertices in B blue and all the vertices not in B white.
- A *color change rule* (CCR) is a rule describing conditions on a vertex u and its neighbors under which u can cause a white vertex w to change color to blue; in this case we say *u forces w* and write $u \to w$.
- Given a coloring of G and a color change rule CCR-X, a CCR-X *final coloring* is a set of blue vertices obtained by applying CCR-X until no more color changes are possible.
- A CCR-X *zero forcing set* for G is a subset B of vertices such that if initially the vertices in B are colored blue and the remaining vertices are colored white, $V(G)$ is a CCR-X final coloring.

[1] There is also a characterization for tree-width by color change rule but it is more complicated and there are many other useful characterizations for tree-width, so we omit it here.

- The CCR-X *zero forcing number* is the minimum cardinality of B over all CCR-X zero forcing sets $B \subseteq V(G)$. The CCR-X zero forcing number is often denoted by X.

The study of zero forcing arose independently in mathematical physics in the study of control of quantum systems, and in combinatorial matrix theory in the study of maximum nullity/minimum rank of a graph, where it is associated with forcing zeros in a null vector of a matrix. Since our interest is in minor monotone parameters, we skip over the original parameters (see [2] for more information) and proceed directly to the minor monotone floors. The color change rules for $\lfloor \mathrm{Z} \rfloor =$ ppw, $\lfloor \mathrm{Z}_\ell \rfloor =$ pw, and $\lfloor \mathrm{Z}_+ \rfloor =$ la are:[2]

CCR-$\lfloor \mathrm{Z} \rfloor$
: If u is blue and w is the only white neighbor of u, then change the color of w to blue. If u is blue, all neighbors of u are blue, and u has not yet performed a force, then change the color of any white vertex w to blue; in this case we say that u *hops* to w. It is not required that u have any neighbors for u to hop.

CCR-$\lfloor \mathrm{Z}_\ell \rfloor$
: If u is blue and exactly one neighbor w of u is white, then change the color of w to blue. If w is white, w has a neighbor, and every neighbor of w is blue, then change the color of w to blue. If u is blue, all neighbors of u are blue, and u has not yet performed a force, then change the color of any white vertex w to blue (this does not require that u have any neighbors); in this case we say that u hops to w.

CCR-$\lfloor \mathrm{Z}_+ \rfloor$
: Let B be the set consisting of all the blue vertices. Let $W_1, \ldots, W_k$ be the sets of vertices of the k components of $G - B$ (note that it is possible that $k = 1$). For each component $1 \leq i \leq k$, let $C_i \subseteq B$ be the subset of blue vertices that are considered to be "active" with regard to that component. (Initially, each $C_i = B$.) If $u \in C_i$, $w \in W_i$, and u has no white neighbors in $G[W_i \cup B] - w$, then change the color of w to blue (this allows for either a normal forcing move, or a hop from u to w). To each connected component of $G[W_i] - w$, associate a new active set equal to $(C_i \setminus \{u\}) \cup \{w\}$.

2.3 Halin's S-functions and Tree-width

Halin [15] defined an S-function to be an integer valued graph parameter ζ such that

1. ζ is minor monotone,
2. $\zeta(\emptyset) = 0$ [or equivalently, $\zeta(K_1) = 1$],
3. $\zeta(G \vee K_1) = \zeta(G) + 1$, and
4. $\zeta(G \langle K_t \rangle H) = \max\{\zeta(G), \zeta(H)\}$.

[2] All of these equalities of parameters require the graph to have at least one edge.

He studied the complete lattice of S-functions and observed that η is the minimum element of the lattice. The first definition of tree-width was given by Halin for the parameter $\mathrm{tw}+1$, which he called $s(G)$ [15]; tree-width was redefined later by Robertson and Seymour using tree-decompositions. Halin defined $s(G)$ as the maximum element of the lattice of S-functions, and characterized it as follows: A graph H is an *S-graph* if it has a simplicial decomposition in which each member is a simplex, which is another way of saying that H has a decomposition as clique-sums of cliques. As Halin notes, this is a well-known characterization of chordal graphs. An *S-covering of order k* of a graph G is an S-graph H that contains G and such that the maximum order of a member of the simplicial decomposition of H is k, i.e., H is a chordal supergraph of G with $\omega(H)=k$. Then $s(G)$ is the minimum order of an S-covering of G, i.e., $s(G)=\min\{\omega(H):H \text{ is chordal supergraph of } G\}=\mathrm{tw}(G)+1$.

Halin [15] also studied other parameters such as $\lceil\chi\rceil(G)$, which he called the modified chromatic number and denoted by $c(G)$; he showed that $\lceil\chi\rceil$ is an S-function (recall that Hadwiger's Conjecture is equivalent to $\lceil\chi\rceil=\eta$). Halin [15] studied $\lceil\kappa\rceil+1$, which he called the *modified vertex connectivity* and denoted by $d(G)$; he showed that $\lceil\kappa\rceil+1$ is an S-function.

The next two examples show that $\mu+1$ and $\nu+1$ are not S-functions (even ignoring issues of the correct starting value).

Example 1. Observe that $K_{1,3}=P_3\,\langle K_1\rangle\,P_2$ but $\mu(P_3)=\mu(P_2)=1$ and $\mu(K_{1,3})=2$.

Example 2. Observe that $T_3=L\,\langle K_2\rangle\,K_3$ but $\nu(L)=\nu(K_3)=2$ and $\nu(T_3)=3$ (see Figure 1 for L and T_3 and [2] for values of ν).

2.4 Minor Monotone Ceilings of Vertex Connectivity and Minimum Degree

In this section we discuss the minor monotone ceilings $\lceil\kappa\rceil$ and $\lceil\delta\rceil$ in more detail. For all G, $\kappa(G)\le\delta(G)$, so $\lceil\kappa\rceil(G)\le\lceil\delta\rceil(G)$. It is clear from the definition of a k-tree that if G a subgraph of a k-tree, then $\delta(G)\le k$. Thus for every graph G, $\delta(G)\le\mathrm{tw}(G)$. and since tree-width is minor monotone, $\lceil\delta\rceil(G)\le\mathrm{tw}(G)$. If $K_r\preceq G$, then $r-1=\kappa(K_r)\le\lceil\kappa\rceil(G)$, so $\eta(G)-1\le\lceil\kappa\rceil(G)$ for all G. Fijavž [13] and Fijavž and Wood [14] studied $\lceil\kappa\rceil$ and $\lceil\delta\rceil$ via minimal forbidden minors.

Example 3. The graph D_3 shown in Figure 2 is defined in [13, 14], where it is shown that $\lceil\kappa\rceil(D_3)=4$, $\lceil\delta\rceil(D_3)=5$, and $\lceil\delta\rceil(G_{5,4})=4$ for $G_{5,4}\simeq D_3[\{1,2,3,4,5,6,13\}]$ $\simeq D_3[\{7,8,9,10,11,12,13\}]$. Observe that $D_3=G_{5,4}\,\langle K_1\rangle\,G_{5,4}$, but $\lceil\delta\rceil(G_{5,4})=4$ and $\lceil\delta\rceil(D_3)=5$. This shows that $\lceil\delta\rceil$ is not a Halin S-function.

In our study of Nordhaus-Gaddum sum upper bounds (see Section 3) we use a lower bound on the number of edges in G in terms of ξ (see Theorem 7 below). This

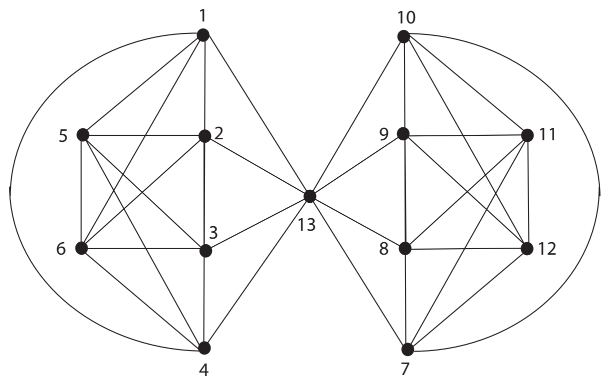

Fig. 2 The graph D_3.

bound carries over to $\lceil\kappa\rceil \leq \nu \leq \xi$ (and also to μ, although a similar bound for μ was known previously [32]). Here we prove an analogous bound for $\lceil\delta\rceil$.

Proposition 1. *For any graph $G = (V,E)$, $|E| \geq \frac{\lceil\delta\rceil(G)(\lceil\delta\rceil(G)+1)}{2}$.*

Proof. Let $G' = (V',E')$ be a minor of G having $\delta(G') = \lceil\delta\rceil(G)$. Observe that the operations used to produce minors (edge contraction, edge deletion, isolated vertex deletion) cannot increase the number of edges, so $|E| \geq |E'|$. Since each vertex of G' must have degree at least $\delta(G')$ and there must be at least $\delta(G')+1$ vertices in G', $|E'| \geq \frac{\delta(G')(\delta(G')+1)}{2} = \frac{\lceil\delta\rceil(G)(\lceil\delta\rceil(G)+1)}{2}$. □

3 Nordhaus-Gaddum Sum Upper Bounds

Let $\beta \in \{\eta, \lceil\kappa\rceil, \lceil\delta\rceil, \underline{\mu}, \nu, \xi, \mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}\}$. What can be said about an upper bound for $\beta(G) + \beta(\overline{G})$? In [6], the *NG sum upper multiplier for* β, denoted by b_β, is defined to be the least value of b making $\beta(G) + \beta(\overline{G}) \leq b|G|$ true for all G with $|G| \geq 2$. In the spirit of Nordhaus and Gaddum's requirement that the tight bound be realized for arbitrarily large order graphs, here we modify the definition slightly. For $n \geq 2$,

$$b(n,\beta) := \min\{b : \beta(G) + \beta(\overline{G}) \leq b|G| \text{ for all } G \text{ with } |G| \geq n\}$$

and

$$b_\beta := \lim_{n\to\infty} b(n,\beta);$$

since $b(n,\beta)$ is nonincreasing and $b(n,\beta) \geq 0$, the limit exists. In fact, K_n immediately shows that $b_\beta \geq 1$.

Since $\beta(G) \le |G|-1$ (except $\eta(G) \le |G|$), $b_\beta \le 2$. It is known that the value of b_β is less than 2 for some of the parameters and equal to 2 for others; some are known precisely whereas for others we have bounds and open questions. We begin with the parameters β for which b_β is known exactly.

Theorem 4. [24] *For all G*

$$\eta(G)+\eta(\overline{G}) \le \frac{6}{5}|G|$$

and there exist graphs of arbitrarily large order achieving $\eta(G)+\eta(\overline{G}) = \frac{6}{5}|G|$, *so*

$$b_{\eta-1} = b_\eta = \frac{6}{5} = 1.2.$$

The lower bound on b_η is established by the next example.

Example 4. [24] Construct a graph G on $n=5s$ vertices partitioned into 5 sets $V_1,\dots,V_5$ of s vertices each. Include all edges between pairs of vertices u and v with $u,v \in V_i$ for $i \in \{1,3\}$, and all edges between pairs of vertices u and v with $u \in V_i$ and $v \in V_{i+1}$ (where $V_6 := V_1$). This is illustrated in Figure 3, where the gray areas indicate all edges present. Contracting a perfect matching between edges in V_1 and V_2 and contracting perfect matching between V_4 and V_5 gives a K_{3s} minor of G, so $\eta(G) \ge \frac{3}{5}n$. Since $G \prec \overline{G}$, $\eta(\overline{G}) \ge \frac{3}{5}n$. Thus $\eta(G)+\eta(\overline{G}) \ge \frac{6}{5}|G|$.

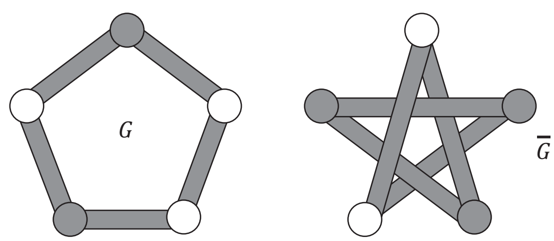

Fig. 3 Schematic diagram of the edges in the graph G and its complement $\overline{G}$.

The next result, due to Joret and Wood, follows from a recent result of Perarnau and Serra [33] that $\mathrm{tw}(G(n,\frac{1}{2})) = n-o(n)$ as $n \to \infty$.

Theorem 5. [23] *Asymptotically almost surely as* $n \to \infty$,

$$\mathrm{tw}\left(G\left(n,\frac{1}{2}\right)\right)+\mathrm{tw}\left(\overline{G\left(n,\frac{1}{2}\right)}\right) = 2n-o(n).$$

and so $b_{\mathrm{tw}} = 2$.

Corollary 2. $b_{\mathrm{la}} = b_{\mathrm{pw}} = b_{\mathrm{ppw}} = 2$.

Although it is known that for almost all large graphs $\mathrm{tw}(G)+\mathrm{tw}(\overline{G}) \approx 2|G|$, we do not know explicit examples showing $b_{\mathrm{tw}} = 2$.

Question 1. Can we construct explicit examples showing $b_{\mathrm{tw}} = 2$?

Theorem 6. [6]

$$1.333 < \frac{4}{3} \le b_{\lceil\kappa\rceil} \le b_{\nu} \le b_{\xi} \le \sqrt{2} < 1.415.$$

As for Hadwiger number, the lower bound on $b_{\lceil\kappa\rceil}$ comes from the construction of an example, but the example is considerably more complicated, and we refer the reader to [6]. The upper bound comes from the next theorem.

Theorem 7. [17] *For a connected graph $G = (V,E)$,*

$$|E| \ge \frac{\xi(G)(\xi(G)+1)}{2} - 1$$

(where the -1 is unnecessary unless G is bipartite and every optimal matrix for $\xi(G)$ has zero diagonal).

As noted in [6], $b_{\lceil\kappa\rceil} \le b_{\lceil\delta\rceil}$, so $\frac{4}{3} \le b_{\lceil\delta\rceil}$. Then $b_{\lceil\delta\rceil} \le \sqrt{2}$ follows from Proposition 1 by a simplification of the argument that deduces $b_{\xi} \le \sqrt{2}$ from Theorem 7. Here we reproduce the brief proof for the convenience of the reader.

Theorem 8.

$$1.333 < \frac{4}{3} \le b_{\lceil\delta\rceil} \le \sqrt{2} < 1.415.$$

Proof. Suppose $|G| = n$. By Proposition 1, $2E(G) \ge \lceil\delta\rceil(G)(\lceil\delta\rceil(G)+1) \ge (\lceil\delta\rceil(G))^2$ and similarly for $\overline{G}$. Then $n^2 \ge n(n-1) = 2E(G) + 2E(\overline{G}) \ge \lceil\delta\rceil(G)^2 + \lceil\delta\rceil(\overline{G})^2$. Thus

$$b_{\beta} \le \max_{\theta}(\cos\theta + \sin\theta) = \sqrt{2}.$$ □

Question 2. Is there an example that increases the lower bound $\frac{4}{3}$ for $b_{\lceil\kappa\rceil}, b_{\lceil\delta\rceil}, b_{\nu}$ or b_{ξ}? Or a way to reduce the upper bound $\sqrt{2}$ for b_{ξ}, b_{ν}, $b_{\lceil\delta\rceil}$, or $b_{\lceil\kappa\rceil}$?

From prior results for η and ξ,

$$1.2 = \frac{6}{5} = b_{\eta} \le b_{\mu} \le b_{\xi} \le \sqrt{2} < 1.415.$$

Question 3. Is there an example that increases the (Hadwiger) lower bound $\frac{6}{5}$ for b_{μ}? Or a way to reduce the upper bound $\sqrt{2}$?

Since $\mathrm{M}(K_n) = n-1$ and $\mathrm{M}(\overline{K_n}) = n$, trivially the Nordhaus-Gaddum sum upper bound for M is $2|G|-1$. (Note that $\mathrm{M}(G)+\mathrm{M}(\overline{G}) \le 2n-1$ requires $n \ge 2$, as we have assumed throughout this section.)

4 Nordhaus-Gaddum Sum Lower Bound

Within the combinatorial matrix community, the assertion that $|G|-2$ is the Nordhaus-Gaddum sum lower bound for β, i.e., for all graphs G

$$|G|-2 \leq \beta(G)+\beta(\overline{G}) \tag{1}$$

is referred to as the Graph Complement Conjecture for β or GCC$_\beta$, because it has been conjectured by various authors for the Colin de Verdière type parameters (and for other parameters related to the maximum nullity of matrices associated with a graph). Here we discuss Nordhaus-Gaddum sum lower bounds for the parameters $\beta \in \{\lceil\kappa\rceil, \lceil\delta\rceil, \mu, \nu, \xi, \mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}\}$ in terms of parameters for which GCC$_\beta$ is true, conjectured, and false. In Section 4.3 we show that when it is true this bound is tight.

4.1 Parameters β for which GCC$_\beta$ is True

For the four parameters $\mathrm{tw}, \mathrm{la}, \mathrm{pw}$, and ppw, (1) is known to be true.

Theorem 9. [23], [11] *For all graphs G,*

$$|G|-2 \leq \mathrm{tw}(G)+\mathrm{tw}(\overline{G}).$$

Corollary 3. [11] *For $\beta \in \{\mathrm{la}, \mathrm{pw}, \mathrm{ppw}\}$, $|G|-2 \leq \beta(G)+\beta(\overline{G})$ for all graphs G.*

It is worth noting that for large random graphs (1) is immediate from Perarnau and Serra's result on the expected value of tree-width [33]. However, although suggestive, large random graph results do not imply (1) is true for all graphs. Compare Kostochka's result that $\eta(G(n,\frac{1}{2})) \leq \frac{n}{\sqrt{\log n}}$ almost always as $n \to \infty$ [25] and his Nordhaus-Gaddum sum upper bound for η quoted in Theorem 4.

4.2 Parameters β for which GCC$_\beta$ is Conjectured

For the parameters $\beta \in \{\mu, \nu, \xi\}$, various authors have conjectured that GCC$_\beta$ is true:

Conjecture 1. [27] For all graphs G,

$$|G|-2 \leq \mu(G)+\mu(\overline{G}).$$

Conjecture 2. [3] For all graphs G,

$$|G|-2 \leq \nu(G)+\nu(\overline{G}).$$

Of course, Conjectures 1 and 2 each imply the following:

Conjecture 3. [3] For all graphs G,

$$|G| - 2 \leq \xi(G) + \xi(\overline{G}).$$

Note that for $\beta \in \{\nu, \xi\}$, where GCC$_\beta$ (1) is conjectured, the information on large random graphs is inconclusive: Hall et al. [17] showed that for $\beta \in \{\nu, \xi\}$,

$$\frac{1}{2} \leq \liminf_{n\to\infty} \frac{\mathbf{E}\left[\beta(G(n, \frac{1}{2}))\right]}{n} \leq \frac{1}{\sqrt{2}}.$$

However, there is some evidence for GCC$_\mu$ and GCC$_\nu$ (and of course either of those would imply GCC$_\xi$).

Theorem 10. [27] *If G is planar, then $|G| - 2 \leq \mu(G) + \mu(\overline{G})$.*

Corollary 4. *$|G| - 2 \leq \mu(G) + \mu(\overline{G})$ for a graph G whenever*

1. *$\mu(G) \geq |G| - 6$ or $\mu(\overline{G}) \geq |G| - 6$;*
2. *$|G| \leq 10$.*

Proof. That $\mu(G) \geq |G| - 6$ implies GCC$_\mu$ was noted in [12]; here we reproduce the brief explanation. Assume $\mu(G) \geq |G| - 6$. If $\mu(\overline{G}) \geq 4$ then $\mu(G) + \mu(\overline{G}) \geq |G| - 2$. If $\mu(\overline{G}) \leq 3$, then $\overline{G}$ is planar [8], so by Theorem 10, $\mu(G) + \mu(\overline{G}) \geq |G| - 2$.

If $\mu(G) \leq 3$ or $\mu(\overline{G}) \leq 3$, then G or $\overline{G}$ is planar [8], so $\mu(G) + \mu(\overline{G}) \geq |G| - 2$ by Theorem 10. Thus if $\mu(G) + \mu(\overline{G}) < |G| - 2$, necessarily $\mu(G) \geq 4$ and $\mu(\overline{G}) \geq 4$, and thus $10 < |G|$. □

Theorem 11. *$|G| - 2 \leq \nu(G) + \nu(\overline{G})$ for a graph G whenever*

1. [34] $\mathrm{tw}(G) \leq 3$ *or* $\mathrm{tw}(\overline{G}) \leq 3$*;*
2. [34] *G is a k-connected partial k-tree, in particular, if G is a k-tree;*
3. *$\nu(G) \geq \mathrm{tw}(G)$ or $\nu(\overline{G}) \geq \mathrm{tw}(\overline{G})$;*
4. *$\nu(G) \leq 2$ or $\nu(\overline{G}) \leq 2$;*
5. [12] *$\nu(G) \geq |G| - 4$ or $\nu(\overline{G}) \geq |G| - 4$;*
6. [3] *$|G| \leq 8$;*
7. [3] *G is a cograph;*
8. [30] *G is chordal.*

Furthermore, if G and H are graphs that satisfy GCC$_\nu$, then the join $G \vee H$ and disjoint union $G \dot{\cup} H$ satisfy GCC$_\nu$ [3].

Proof. Most of the statements appear in the literature. For statement (3): By [34, Theorem 5], $\nu(\overline{G}) \geq |G| - \mathrm{tw}(G) - 2$ for every graph G. So if $\nu(G) \geq \mathrm{tw}(G)$,

$$\nu(G) + \nu(\overline{G}) \geq \mathrm{tw}(G) + |G| - \mathrm{tw}(\overline{G}) - 2 = |G| - 2.$$

Statement (4) follows from Statement (3) and the fact that $\nu(G) \le 2$ implies $\mathrm{tw}(G) \le \nu(G)$, which follows from the minimal forbidden minor characterizations of tw and ν: The minimal forbidden minor for $\mathrm{tw}(G) \le 2$ is K_4 [7]. The minimal forbidden minors for $\nu(G) \le 2$ are K_4 and T_3 [19]. □

4.3 GCC$_\beta$ is Tight if True for β

We show in Proposition 3 below that for $\beta \in \{\mu, \nu, \xi, \mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}\}$, $\beta(\overline{P_n}) = n-3$, and thus for any of these parameters for which the inequality (1) is correct (in particular, tw, la, pw, and ppw), it is tight because for $n \ge 4$,

$$\beta(P_n) + \beta(\overline{P_n}) = 1 + (n-3) = n-2.$$

Proposition 2. *Let T be a forest that contains P_4 as a subgraph. Then* $\mathrm{ppw}(\overline{T}) \le n-3$.

Proof. Since $\mathrm{ppw} = \lfloor \mathrm{Z} \rfloor$, it suffices to show $\lfloor \mathrm{Z} \rfloor(\overline{T}) \le n-3$. Assume the vertices of T are numbered starting with the P_4 as 1, 2, 3, 4 in path order. Then $B = \{2, 5, 6, \ldots, n\}$ is a zero forcing set of cardinality $n-3$ for $\overline{T}$ (with $2 \to 4, 4 \to 1, 1 \to 3$). □

Proposition 3. *For $\beta \in \{\kappa, \delta, \lceil\kappa\rceil, \lceil\delta\rceil, \mu, \nu, \xi, \mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}\}$ and $n \ge 4$, $\beta(\overline{P_n}) = n-3$.*

Proof. For all β, $\beta \le \mathrm{ppw} = \lfloor \mathrm{Z} \rfloor$, and $\kappa \le \beta$ for all β except μ. By Proposition 2, $\lfloor \mathrm{Z} \rfloor(\overline{P_n}) \le n-3$. Since P_n is planar, it is known that $\mu(P_n) + \mu(\overline{P_n}) \ge n-2$ and thus that $\mu(\overline{P_n}) \ge n-3$.

To see that $\kappa(\overline{P_n}) \ge n-3$, assume $P_n = (1, 2, \ldots, n)$ and suppose $S \subset [n]$ is a cut-set of $\overline{P_n}$. Note first that this implies there must exist $x < y \in [n] \setminus S$ that are not joined by a path in $\overline{P_n} - S$. For $x \sim y$ in $\overline{P_n}$, this is impossible. So $x = k, y = k+1$ for some k with $1 \le k \le n-1$. Assume first that $2 \le k \le n-2$. Then for $z = 1, \ldots, k-2, k+3, \ldots, n$, $(k, z, k+1)$ is a path between k and $k+1$. So the $n-4$ vertices $1, \ldots, k-2, k+3, \ldots, n$ must be in S. But k and $k+1$ are still connected by the path $(k, k+2, k-1, k+1)$, so at least one of $k-1$ or $k+2$ must also in S, and thus $|S| \ge n-3$. The case $k = 1$ or $k = n-1$ is even easier, as there are $n-3$ paths on 3 vertices between k and $k+1$ in $\overline{P_n}$. Thus $\kappa(\overline{P_n}) \ge n-3$. □

4.4 Parameters β for which GCC$_\beta$ is False

In this section we show that GCC$_\beta$ is false for $\beta \in \{\lceil\delta\rceil, \lceil\kappa\rceil, \lceil\chi\rceil - 1\}$ and reproduce the known Nordhaus-Gaddum lower bound for η.

Theorem 12. [25, Corollary 5] [3]

$$\min_{|G|=n}\left(\eta(G)+\eta(\overline{G})\right)=\Theta\left(\frac{n}{\sqrt{\log n}}\right).$$

The tree T shown Figure 4 provides a counterexample to (1) for $\lceil\delta\rceil$ and thus for $\lceil\kappa\rceil$ and $\eta-1$, and also for $\lceil\chi\rceil-1$. (A smaller but less structured example was found by Berrera-Cruz and Lin [5].)

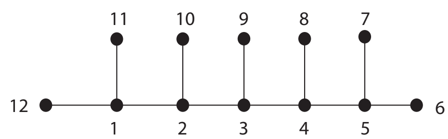

Fig. 4 A tree T for which $\mathrm{GCC}_{\lceil\delta\rceil}$ fails.

Proposition 4. *For the tree T shown in Figure 4, $\lceil\delta\rceil(\overline{T})=8$ and $\lceil\chi\rceil(\overline{T})=9$ and thus*

$$|T|-3\geq\lceil\delta\rceil(T)+\lceil\delta\rceil(\overline{T})\geq\lceil\kappa\rceil(T)+\lceil\kappa\rceil(\overline{T}),$$
$$|T|-1\geq\lceil\chi\rceil(T)+\lceil\chi\rceil(\overline{T})\geq\eta(T)+\eta(\overline{T}).$$

Proof. We first show $\lceil\delta\rceil(\overline{T})\leq 8$. Deletion of a vertex in a connected graph cannot result in a greater minimum degree than contraction of an edge incident with that vertex, and deletion of an edge cannot raise minimum degree, so there is a minor realizing $\lceil\delta\rceil(\overline{T})$ is obtained from $\overline{T}$ by contractions. That is, there exists $H\prec\overline{T}$ with $\delta(H)=\lceil\delta\rceil(\overline{T})$ and H is defined by a partition $W_1,\ldots,W_h$. For $v=1,2,3,4,5$, $\deg_T v=3$, so $\deg_{\overline{T}} v=8$. Thus to achieve $\delta(H)\geq 9$, each of these 5 vertices must be in a W_i of cardinality greater than one. But that would imply $|H|\leq 9$ and thus $\delta(H)\leq 8$.

For edges $e=\{1,5\},f=\{2,4\},g=\{3,6\}$, $\overline{T}/\{e,f,g\}=K_9$, so $\eta(\overline{T}/\{e,f,g\})=9$ and $\lceil\delta\rceil(\overline{T}/\{e,f,g\})=8$. Note that $9=\eta(\overline{T})\leq\lceil\chi\rceil(\overline{T})$. Since any order n graph $G\neq K_n$ has $\chi(G)\leq n-1$, any order 10 minor H of $\overline{T}$ has $\chi(H)\leq 9$. Since $\chi(\overline{T})=7$ and for every contraction of a single edge e, $\chi(\overline{T}/e)\leq 8$,[4] $\lceil\chi\rceil(\overline{T})=9$. □

The fact that (1) fails for η for large random graphs is clear from Kostochka's sum lower bound for η quoted in Theorem 12. But it is nice to have concrete examples, such as T shown in Figure 4 or the icosahedral graph [3].

We note that the construction in Proposition 4 can be generalized to further lower the bound; see [18] where it is shown that a Nordhaus-Gaddum sum lower bound for $\lceil\delta\rceil$ cannot exceed $n-c\sqrt{n}$ for any $c>0$.

[3] $f(n)=\Theta(g(n))$ means there exist constants K,C,k,c such that $f(n)\leq Cg(n)$ for all $n\geq K$ and $f(n)\geq cg(n)$ for all $n\geq k$.

[4] It is easy to verify this in a computer algebra system such as *Sage*.

Question 4. What is the Nordhaus-Gaddum sum lower bound for $\lceil\delta\rceil$? Or what is its form?

Question 5. What is the Nordhaus-Gaddum sum lower bound for $\lceil\kappa\rceil$? Or what is its form?

It is easy to construct a graph G of arbitrarily large order with $\delta(G) = \delta(\overline{G}) = 1$, so trivially the Nordhaus-Gaddum sum lower bound for δ and κ is 2.

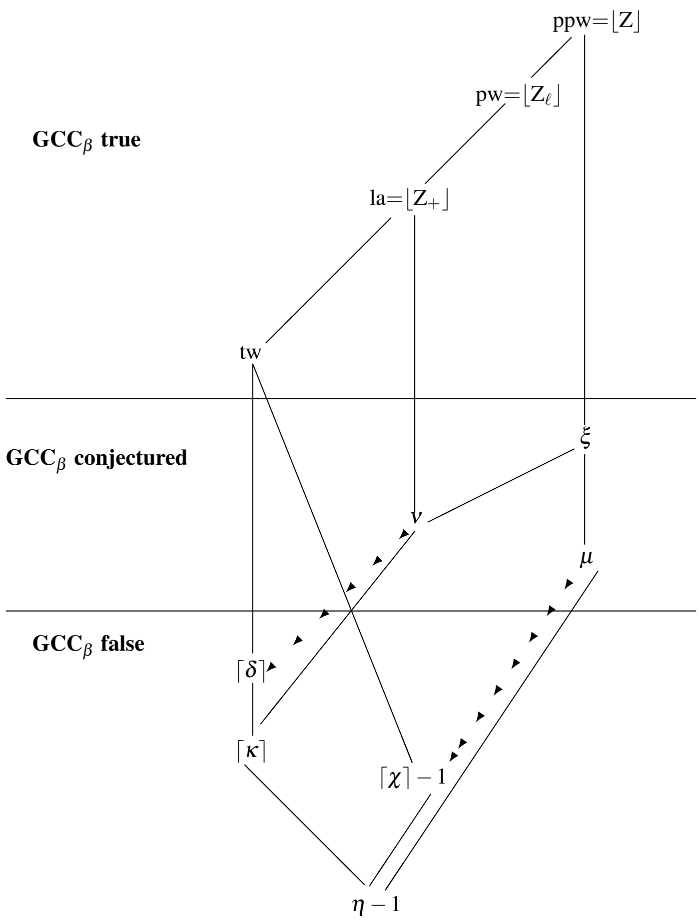

Fig. 5 Relationships between minor monotone parameters and status of GCC$_\beta$.

4.5 Relationships between Parameters and GCC

Relationships between the parameters discussed here are shown in Figure 5 together with a summary concerning the parameters for which (1) is known to be true, is conjectured, and is false. If two parameters are connected by a line segment, then the upper parameter is greater than or equal to the lower parameter for all graphs that have an edge. A dashed line of small triangles indicates a conjecture. Note that if Hadwiger's Conjecture is true then $\lceil\chi\rceil - 1 = \eta - 1$, but that is not indicated on the diagram (except by the conjecture that $\lceil\chi\rceil - 1 \leq \mu$, which it implies). The horizontal lines delineate the status of (1) as the Nordhaus-Gaddum sum lower bound. This diagram is adapted from [2, Figure 1.1][5], which provides additional information such as proofs of the claimed inequalities, examples showing all the parameters are distinct (except $\lceil\chi\rceil - 1$ and $\eta - 1$), and examples showing noncomparability.

It has been conjectured by various authors that δ is less than several parameters related to maximum nullity (see [12, Section 46.7] for more information). H. Tracy Hall has presented a proof that $\delta(G) \leq \nu(G)$ [16] but it is not yet fully written up, so is still considered a conjecture.

5 Nordhaus-Gaddum Product Bounds

For $n \geq 2$, the *NG product upper multiplier for β,* denoted by c_β, is defined by

$$c(n,\beta) := \min\{c : \beta(G)\cdot\beta(\overline{G}) \leq c\,|G|^2 \text{ for all } G \text{ with } |G| \geq n\}$$

and

$$c_\beta := \lim_{n\to\infty} c(n,\beta).$$

Since $c(n,\beta)$ is nonincreasing and $c(n,\beta) \geq 0$, the limit exists.

As in the case of Nordhaus and Gaddum's original results (cf. Theorem 1), we can often obtain good information about the NG product upper multiplier for β from the NG sum upper multiplier for β.

Remark 2. If $\beta(G)+\beta(\overline{G}) \leq bn$ for all graphs of order n, then $\beta(G)\beta(\overline{G}) \leq x(bn - x) \leq \frac{1}{4}b^2n^2$, so $c_\beta \leq \frac{1}{4}b_\beta^2$. If there exists a construction of graphs G_n of order n such that $\beta(G_n) = \beta(\overline{G_n})$ and $\lim_{n\to\infty}\beta(G_n)/n = d$, then $d^2 \leq c_\beta$.

Since Kostochka's construction in [24] for $\eta(G)+\eta(\overline{G}) = \frac{6}{5}|G|$ realizes $\eta(G) = \frac{3}{5}n = \eta(\overline{G})$, the next corollary is immediate.

Corollary 5. *The NG product upper multiplier for η (or for $\eta - 1$) is*

$$c_{\eta-1} = c_\eta = \frac{9}{25}.$$

[5] [2, Figure 1.1] does not includes $\lceil\chi\rceil - 1$ nor does it have any information about the Nordhaus-Gaddum lower bound.

The next corollary follows from the fact that the construction in [6] of $H(a)$ that realizes $\lceil\kappa\rceil(H(a)) \approx \frac{2}{3}n$ is self-complementary.

Corollary 6. *We have the following bounds on the NG product upper multiplier for* $\lceil\kappa\rceil, \lceil\delta\rceil, \nu, \xi$

$$0.444 < \frac{4}{9} \le c_{\lceil\kappa\rceil} \le c_\nu \le c_\xi \le \frac{1}{2} = 0.5 \text{ and } 0.444 < \frac{4}{9} \le c_{\lceil\delta\rceil} \le \frac{1}{2} = 0.5.$$

For a lower bound to $\beta(G)\cdot\beta(\overline{G})$, we assume neither G nor $\overline{G}$ is the empty graph (no edges), because for many of our parameters this would result in a lower bound of zero.

Remark 3. For $\beta \in \{\lceil\kappa\rceil, \lceil\delta\rceil, \nu, \mu, \xi, \mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}\}$, $\beta(P_n) = 1$ and $\beta(\overline{P_n}) = n-3$ (cf. Proposition 3), so any lower bound on $\beta(G)\beta(\overline{G})$ cannot exceed $|G|-3$. If $|G|-2 \le \beta(G)+\beta(\overline{G})$ (as is true for $\beta \in \{\mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}\}$), then

$$|G|-3 \le \beta(G)\cdot\beta(\overline{G})$$

and this bound is tight.

It should be noted that whereas subtracting 1 from a parameter (e.g., $\eta - 1$ vs. η) has no effect on the upper multiplier, it makes a dramatic difference in the NG product lower bound, which Kostochka has determined for η.

Theorem 13. [26] (cited in [1]) *For all graphs such that G and $\overline{G}$ each have an edge,*

$$\left\lceil \frac{3n-5}{2} \right\rceil \le \eta(G)\cdot\eta(\overline{G}).$$

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