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Practical feedback stabilization of nonlinear control systems and applications

Ruey-Gang Lai
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Practical feedback stabilization of nonlinear control systems and applications

by

Ruey-Gang Lai

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of the Requirements for the Degree of DOCTOR OF PHILOSOPHY

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1996

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LIST OF NOTATIONS

The following standard notations will be used throughout:

\( A_\varepsilon \) compact subset of \( A \) with \( d_H(A, A_\varepsilon) < \varepsilon \)
\( A(B) \) domain of attraction of \( B \)
\( \text{co}\{\omega_1, \ldots, \omega_k\} \) convex hull generated by \( \{\omega_1, \ldots, \omega_k\} \)
\( d_H(A, B) \) the Hausdorff distance between \( A \) and \( B \)
\( F(x) \) \( \{f(x, u)\}_{u \in \mathcal{U}(x)} \)
\( \text{int}(B), \overline{B}, \partial B \) interior, closure, boundary of \( B \)
\( \mathcal{K}(M) \) the family of all nonempty compact subsets of \( M \)
\( \mathcal{L}A(X, Y) \) the Lie algebra generated by the vector fields \( X \) and \( Y \)
\( N_\varepsilon(L) \) \( \{x \in M \mid d(x, L) < \varepsilon\} \), an open \( \varepsilon \)-neighborhood of \( L \)
\( O^+_t(B) \) \( \{y \in M \mid \text{there are } x \in B, \text{ and } u \in \mathcal{U} \text{ with } y = \varphi(t, x, u)\} \)
\( O^+_{\leq T}(B) \) \( \bigcup_{0 \leq t \leq T} O^+_t(B) \)
\( O^+(B) \) \( \bigcup_{0 \leq t} O^+_{\leq t}(B) \)
\( O^-_t(B) \) \( \{y \in M \mid \text{there are } x \in B, \text{ and } u \in \mathcal{U} \text{ with } x = \varphi(t, y, u)\} \)
\( O^-_{\leq T}(B) \) \( \bigcup_{0 \leq t \leq T} O^-_t(B) \)
\( O^-(B) \) \( \bigcup_{0 \leq t} O^-_{\leq t}(B) \)
\( O^-_{\omega_i}(B) \) \( \{y \in M \mid \text{there is } x \in B \text{ with } x = \varphi(t, y, \omega_i)\} \)
\[ \mathcal{O}_{\omega_i \leq T}(B) = \bigcup_{0 \leq t \leq T} \mathcal{O}_{\omega_i,t}(B) \]

\[ \mathcal{O}_{\omega_i}(B) = \bigcup_{0 \leq t} \mathcal{O}_{\omega_i,t}(B) \]

\[ \mathcal{O}_{\mathcal{U},t}(B) = \{ y \in M \mid \text{there are } x \in B, \text{ and } u \in \hat{U}_{pc} \text{ with } x = \varphi(t, y, u) \} \]

\[ \mathcal{O}_{\mathcal{E},\leq T}(B) = \bigcup_{0 \leq t \leq T} \mathcal{O}_{\mathcal{E},t}(B) \]

\[ \mathcal{O}_{\mathcal{F}^+}(B) = \bigcup_{0 \leq t} \mathcal{O}_{\mathcal{F}^+,t}(B) \]

\[ \mathcal{O}_{\omega_{11}^+(B)} = \mathcal{O}_{\omega_{1}^-,\leq T_{11}}(B) \]

\[ \mathcal{O}_{\omega_{j_{11},12}^+(B)} = \mathcal{O}_{\omega_{21},\leq T_{12}}(\mathcal{O}_{\omega_{1}^-,\leq T_{11}}(B)) \]

\[ \mathcal{O}_{\omega_{11},\omega_{j_{11},12}}^+(B) = \mathcal{O}_{\omega_{21},\leq T_{12}}(\mathcal{O}_{\omega_{1}^-,\leq T_{11}}(\mathcal{O}_{\omega_{1}^-,\leq T_{11}}(B)))) \]

\[ \mathcal{O}_{\mathcal{I}_b^+}(x) = \{ y \in M \mid \text{there exist } T \geq 0, u \in \mathcal{U}_{b}^+ \text{ with } \varphi(T, x, u) = y \} \]

\[ R_i \]

the \( i \)-th feedback region

\[ R_{K}(x) \]

regulation map to \( K \) at \( x \)

\[ T_{K}(x) \]

contingent cone to \( K \) at \( x \)

\[ \hat{U} \]

\( \{ \omega_1, \ldots, \omega_k \} \)

\[ \mathcal{U} \]

\( \{ u : \mathbb{R} \rightarrow U \mid \text{locally integrable} \} \)

\[ \mathcal{U}_{pc} \]

\( \{ u : \mathbb{R} \rightarrow U \mid u \text{ piecewise constant} \} \)

\[ \mathcal{U}_{pc}^\rho \]

\( \{ u : \mathbb{R} \rightarrow \rho U \mid u \text{ piecewise constant} \} \)

\[ \hat{U}_{pc} \]

\( \{ u : \mathbb{R} \rightarrow \hat{U} \mid u \text{ piecewise constant} \} \)

\[ \mathcal{U}^b \]

\( \{ u : M \rightarrow U \mid u \text{ measurable} \} \)

\[ \Pi(x) \]

all feasible controls at the state \( x \)

\[ \omega(x) \]

\( \omega \)-limit set of \( x \)
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I wish to thank the expert instruction of Dr. Wolfgang Kliemann during my graduate studies at Iowa State University. He introduced this research to me. Under his thoughtful guidance, I learned a great deal and enjoyed the learning. Thanks are gratefully extended to my dear friend Dr. Gerhard Häckl for his support and encouragement, especially for his help with the computer work, without his assistance the numerical results of this dissertation would not have been possible.
CHAPTER 1. INTRODUCTION

The science of controlling nonlinear dynamical systems has a strong standing in applied mathematics since the theory of control and differential equations/dynamical systems are closely linked and it is an area that has seen some major theoretical developments in last twenty years. Intuitively speaking, the concept of control is concerned with modifying the behavior of dynamical systems so as to achieve desired goals.

In this area, there are two research directions that are both practically important and mathematically interesting. The first central theme is concerned with the idea of feedback, a control scheme in which the inputs to the system are determined on the basis of the current states. From the technological point of view, feedback control leads to safer aircraft, more efficient cars, more accurate missiles, and so on. On the other hand, from the viewpoint of mathematical disciplines, topological structure associated with geometrical properties show up in the study of the stability and the dynamic behavior of the new “closed-loop” control systems.

The other central theme has been pointed to stabilization. It has been a problem of long standing to find implementable algorithms for stabilization of nonlinear control systems. A method to calculate stabilizing feedbacks would be of major importance. In engineering practice, nonlinear control systems are omnipresent; however,
most of them have been designed by using traditional linear regulation techniques. The situation is changing now because of the increasing availability of inexpensive computing power. More advanced mathematical techniques are beginning to find their way into applications.

A fundamental keystone in understanding the structure of control systems and the above two objectives is based on the study of controllability. In this context, much attention has been devoted to criteria for reachability of a point \( y \) starting from \( x \); that is, there exist an open loop control function \( u(t) \) and a time \( t \geq 0 \) such that \( \varphi(t, x, u) = y \). One of the key objects for this purpose has been introduced earlier as reachable sets, the set of all states where a given class of admissible controls can drive the system to (see e.g. Gayek and Vincent [27], Häckl [30, 31], Roxin [51]), which are certainly very important and interesting within control theory and many related questions can be investigated. For example, they can be used to describe regions of controllability and their domains of attraction. Of course, of great practical interest are numerical methods and related algorithms (see Häckl [30, 31]).

Before we discuss all aspects of this dissertation, we briefly review some important features about stabilization.

In fact, for the problem of stabilization (e.g. with respect to fixed points or periodic orbits of the uncontrolled systems), which requires controllability around these orbits, and convergence towards them, the literature reflects two viewpoints: local and global. The local results are commonly obtained by an "approximation" principle, that is, the attention has been devoted to the linearization around these orbits with normal form expansions and center manifold techniques. In many circumstances linear models of nonlinear systems will suffice for (local) controller design.
However, the realization that there are many other situations, such as some modern nonlinear mechanical or electrical systems, which cannot be analyzed by linear techniques has led to a renewed effort to understand the stabilization problem, and in particular focus on the cases for which there is a variety of topological and analytical tools available, such as center manifold reduction (see e.g. Aeyels [1, 2], Bacciotti [10], Brockett [13]), modification of zero dynamics by redefining the output function (see e.g. Bacciotti [10]), and partial feedback linearization (see e.g. Bacciotti [10]). Analysis of the stabilization problem for these classes of systems requires innovative techniques which have no counterparts in the theory of linear systems.

Even though differential geometry methods and operator theory have been successfully applied to the studies of nonlinear systems, most efforts are still now on analysis or synthesis of nonlinear systems locally around a point. Until recently, little attention has been paid to global qualitative studies of nonlinear control systems (see e.g. Bacciotti, Kalouptsidis and Tsinias [11], Colonius and Kliemann [20, 21], Lin [42]). Moreover, global stabilization results for nonlinear systems are often heavily based on Liapunov machinery (see e.g. Byrnes, Isidori and Williams [14], Saberi, Kokotović and Sussmann [52], Tsinias [56]), which serves as a tool to transform a given complicated differential system into a relatively simpler system and as a result, it is enough to study the properties of solutions of the simpler system.

An important and challenging research subject in the field of nonlinear control systems is the impact of input constraints on control performance, which is more realistic in practical problems but much more difficult in mathematical analysis. It is well known that if the control range $U$ is bounded, one cannot expect the whole state space to be the reachable set of any initial value $x$ in general.
The purpose of this dissertation is to study, via feedback controls, global control and stabilization problems of nonlinear systems with constrained control range. It is based on so called practical feedback controllability and practical feedback stabilization on some well-chosen feedback controlled invariant sets associated with an a priori closed loop control.

Our method is a different approach to the analysis introduced above. In particular, there are two manifest advantages of this theory, one is that it does not require the knowledge of linearization techniques or Liapunov’s theory, and the other is that, from a practical point of view, not only the proof of the existence theorem for global feedback controllers but also the construction of suitable feedback controlled invariant sets for (general) nonlinear systems are programmable algorithms for numerical calculation and therefore it has great power in application to real world problems from many disciplines.

More precisely, consider the following affine control system

\[ \dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i x_i(x(t)) \]

on a connected, smooth manifold \( M \) with dimension \( d \), where \( X_0, \ldots, X_m \) are smooth vector fields and \( U \subset \mathbb{R}^m \) compact, convex and \( 0 \in \text{int} U \), where \( \text{int} U \) is the interior of \( U \).

We assume that for any initial value \( x_0 \in M \) and for any feasible control \( u(\cdot) \in U_{pc}^{lb} \), the solution of the above equation is an absolutely continuous function given by

\[ \varphi(t, x_0, u) = x_0 + \int_0^t X_0(x(s)) \, ds + \sum_{i=1}^{m} \int_0^t u_i(x(s)) x_i(x(s)) \, ds, \]
whenever it exists.

It is well-known that as opposed to the time-invariant linear case, controllability of a nonlinear system does not imply the possibility of stabilization by a smooth feedback. In general, as far as existence of a global feedback control is concerned, it is impossible to have a continuous feedback that can fit this task (see for example, Sontag and Sussmann [54], Sussmann [55]).

Our first goal is to design a global (piecewise constant) feedback control law for any connected subset $B$, with the property that $\text{int}(B) = B$, of state space $M$ in the sense that it can steer any point in any compact subset of the domain of attraction of $B$ into $B$ in finite time. In fact, our approach gives an explicit construction of such feedback controller.

Our second task is to establish some (practical) stabilization results, namely, to characterize the sets, called feedback controlled invariant sets, to be as small as possible to keep the trajectories inside forever, whenever the trajectories have been steered into them. Particularly, the most interesting case is that for any component $L$ of those $\omega$—limit sets of uncontrolled trajectories, under some reasonable assumption, we can find a decreasing sequence of feedback controlled invariant sets with the limit equals $L$. In general, open loop controls will not result in robust stabilization (see e.g., Colonius and Kliemann [21]).

Putting these two aspects together, it turns out that the desired state of a system may be mathematically unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable. Many aircraft and missiles behave in this manner.

Our development of calculational algorithms is based heavily on Dr. Gerhard
Häckl's work [31]. Roughly speaking, our software for computing controlled invariant sets and their feedback regions is just the subsequent development of Dr. Häckl's software CS4.0, which was developed to compute control sets and their domain of attraction.

Based on our algorithm, one can numerically compute a small and suitable feedback controlled invariant set enclosing a component of the limit sets of a nonlinear system and design a global feedback controller steering all controllable points into this feedback controlled invariant set. In other words, this software is suitable for the purpose of practical controllability and practical stabilization of nonlinear control systems.

This dissertation is organized as follows. In Chapter 2 we analyze the basic ideas for practical controllability and give an explicit construction for practical feedback controllers. Chapter 3 is devoted to the theory of practical feedback stabilization. The theoretical background includes viability theory in set-valued analysis, the concept of control sets and the theory about their limit behavior. Many of the concepts that we introduce in this chapter are known in control theory. However some connections have never been explicitly mentioned in the cited literature. In Chapter 4 we apply the theory developed in Chapter 2 and Chapter 3 to investigate some two-dimensional and one three-dimensional model problems numerically. To conclude, in the final chapter of concluding remarks, we discuss several directions of our further research.
CHAPTER 2. PRACTICAL FEEDBACK CONTROLLABILITY

In this chapter we discuss both open loop and closed loop controllability of control affine systems. One of the key objects for this purpose is called reachable sets, which is defined as the set of all points that are reachable from an initial point $x_0$ with some admissible controls, from an initial point $x_0$. In order to construct a global feedback controller, we deal with a certain initial set $B$ rather than an initial state $x_0$. Since it is closely related to practical purpose, we called this kind of reachability "practical controllability".

The chapter is organized as follows. In Section 2, we give a precise mathematical statement of the system under consideration and present some basic definitions and concepts. Section 2 is devoted to the investigation of the existence of practical feedback controllers. A constructive algorithm for numerical calculations is proved. In Section 2, the main theorem which indicates that control-affine systems are practically feedback controllable is proved. In the final Section 2 we show how this construction of practical feedback controllers would apply to the problem of stabilization in dimension two without further knowledge from the next chapter.
Control Systems, Controllability and Accessibility

In the following, we consider affine control systems

\[ \dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t)X_i(x(t)), \]

(2.1)
on a connected $C^\infty$ manifold $M$ with $\dim M = d < \infty$.

Here the $X_0, \ldots, X_m$ are $C^\infty$ vector fields and

\[ u(\cdot) = (u_1(\cdot), \ldots, u_m(\cdot)) \in \mathcal{U} = \{ u : \mathbb{R} \to U \mid \text{locally integrable} \}, \]

(2.2)
with $U = \sigma(\omega_1, \ldots, \omega_k) \subset \mathbb{R}^m$, $k \in \mathbb{N}$.

We assume that (2.1) has a unique solution $\varphi(t, x, u)$ for all $x \in M$, $u \in \mathcal{U}$ with $\varphi(0, x, u) = x$ and being defined for all $t \in \mathbb{R}$.

Before we move on, let us first analyze the set $\mathcal{U}$. Because of the compactness of $\mathcal{U} \subset \mathbb{R}^m$, all measurable functions $u : \mathbb{R} \to \mathcal{U}$ are actually in $\mathcal{U}$ and vice versa, all locally integrable functions with values in $\mathcal{U}$ are measurable.

Moreover, if $\mathcal{U} \subset L^\infty(\mathbb{R}, \mathbb{R}^m)$ is equipped with the weak$^*$—topology, it has earlier been proved in Colonius and Kliemann [20, Lemma 2.1] that the following lemma holds.

**Lemma 2.1 (The Structure of $\mathcal{U}$)**

The set $\mathcal{U}$ is a compact and metrizable in the weak$^*$—topology of $L^\infty(\mathbb{R}, \mathbb{R}^m) = (L^1(\mathbb{R}, \mathbb{R}^m))^*$ and a metric is given by

\[ d(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{1 + | \int_{\mathbb{R}} (u(t) - v(t), x_n(t)) dt |}, \]

where $\{x_n, n \in \mathbb{N}\}$ is a countable, dense subset of $L^1(\mathbb{R}, \mathbb{R}^m)$. With this metric, $\mathcal{U}$ is a compact, complete, separable metric space. (Here $\langle \cdot, \cdot \rangle$ denotes an inner product in $\mathbb{R}^m$.)
It is convenient to know that \( \mathcal{U} \) is a compact metric space. Hence we do not only allow piecewise constant controls, but measurable ones.

Since an important question in this chapter is that of reachability, we define the set of points reachable from \( x \) and controllable to \( x \) with measurable controls in the following way.

**Definition 2.2 (Positive Orbits and Negative Orbits)**

Given the control system (2.1) and any subset \( B \subset M \), we define the sets of points reachable from \( B \) as

\[
\mathcal{O}^+_{t}(B) := \{ y \in M \mid \text{there are } x \in B, \text{ and } u \in \mathcal{U} \text{ with } y = \varphi(t, x, u) \},
\]

\[
\mathcal{O}^+_T(B) := \bigcup_{0 \leq t \leq T} \mathcal{O}^+_{t}(B),
\]

\[
\mathcal{O}^+(B) := \bigcup_{0 \leq t} \mathcal{O}^+_{t}(B).
\]

And analogously, the sets of points controllable to \( B \) are

\[
\mathcal{O}^-_{t}(B) := \{ y \in M \mid \text{there are } x \in B, \text{ and } u \in \mathcal{U} \text{ with } x = \varphi(t, y, u) \},
\]

\[
\mathcal{O}^-_{T}(B) := \bigcup_{0 \leq t \leq T} \mathcal{O}^-_{t}(B),
\]

\[
\mathcal{O}^-(B) := \bigcup_{0 \leq t} \mathcal{O}^-_{t}(B).
\]

Note that the closure of these orbits does not change if instead of measurable controls piecewise constant controls or piecewise continuous ones are employed (see Colonius and Kliemann [22]).

We say that the system is controllable from \( x \in M \) if \( \mathcal{O}^-(x) = M \), and it is completely controllable if it is controllable from every \( x \in M \).

Unlike the finite time controllability defined above, we study the set of points which can be steered approximately to the set \( B \).
Definition 2.3 (Domain of Attraction)

Given the control system (2.1) and any subset $B \subset M$, the domain of attraction of $B$ is given by

$$A(B) := \{ x \in M \mid O^+(x) \cap B \neq \emptyset \}.$$  

With this definition, one can expect that the domain of attraction of any set can be described as the negative orbit.

Lemma 2.4 Given the control system (2.1) and any subset $B \subset M$, then

$$A(B) = O^-(B).$$

Proof.

For any $x \in O^-(B)$, there exist $t > 0$, $y \in B$ and $u \in U$ such that $x = \varphi(t, y, u)$, this implies $y \in O^+(x) \cap B$ and hence $x \in A(B)$. On the other hand, for any $x \in A(B)$, by definition there exists $y \in O^+(x) \cap B$, therefore, there exist $t > 0$, and $u \in U$ such that $x = \varphi(t, y, u)$ and hence $x \in O^-(B)$. 

An important notion in nonlinear control theory has been introduced as (local) accessibility. We say that the system is locally accessible from $x \in M$ if for all neighborhoods $V \subset M$ of $x$ and all $T > 0$ it holds that $O^+_{\leq T}(x) \cap V$ and $O^-_{\leq T}(x) \cap V$ have nonvoid interior (see e.g. Isidori [33], Nijmeijer [45]). The system (2.1) is called locally accessible if every point in its state space has this property.

It is well known that for linear systems of the form

$$\dot{x} = Ax + Bu,$$  

where $x \in \mathbb{R}^d$ and $u \in U = \mathbb{R}^m$, a satisfactory criterion for controllability can be verified by checking the Kalman controllability rank condition. That is, (2.3) is
completely controllable (for all \( x, y \in \mathbb{R}^d \)) if and only if \( \text{rank}(B, AB, \ldots, A^{d-1}B) = d \). If the rank of the reachability matrix \((B, AB, \ldots, A^{d-1}B)\) is less than \( d \), one can steer \( x \) into \( y \) if both points are in the linear space generated by the columns of the reachability matrix.

For linear systems with \( U = \mathbb{R}^m \), complete controllability is equivalent to local accessibility. This equivalence is proved via the observation that accessibility holds if and only if the Lie algebra

\[
\mathcal{L}A \{ Ax + Bu \mid u \in \mathbb{R}^m \}
\]

has rank \( d \) for all \( x \in \mathbb{R}^d \). For \( x = 0 \) this is exactly the Kalman criterion.

In the case of nonlinear systems (2.1), there is a gap between these two concepts. In general, accessibility does not imply controllability. In other words, even if the orbits of any point in the state space are topologically "thick", they may be strictly contained in \( M \), that is, the system is not completely controllable.

However, local accessibility is guaranteed by the Lie algebra rank condition

\[
(H) \quad \dim \mathcal{L}A \{ X_0 + \sum u_i X_i \mid (u_i) = u \in U \}(x_0) = d \text{ for all } x_0 \in M,
\]

which can in principle be checked for a given system.

Throughout, we assume, unless otherwise specified, that all systems under consideration satisfy the Lie algebra condition \((H)\) and hence the local accessibility property.

Intuitively, the Lie algebra condition \((H)\) means that at any point \( x \in M \), the control system (2.1) can move in all directions of \( T_x M \); that is, \( M \) is the maximal integral manifold for the family of vector fields \( \{ X_0 + \sum u_i X_i \mid (u_i) \in U \} \).
Existence and Construction of Practical Feedback Controllers

What Is Practical Feedback Controllability?

In the previous section, we considered controllability of nonlinear systems by open loop controls, namely \( u \in \mathcal{U} \). In this section, we will construct an a priori (global) feedback controller for any connected subset \( B \subset M \) with the property that \( \mathbf{int}(\overline{B}) = \overline{B} \) in the sense that it can steer any point in any compact subset of the domain of attraction of \( B \) into \( B \) in finite time. Such a concept is called practical feedback controllability.

Some Preparatory Technical Results

Consider here the control system (2.1) and any subset \( B \subset M \). In order to study in a simple and systematic manner the reachable sets of \( B \), we introduce the piecewise constant controls with only the extremal control values, that is,

\[
u(\cdot) \in \hat{\mathcal{U}}_{pc} = \{ u : \mathbb{R} \to \hat{U} \mid u \text{ piecewise constant} \},
\]

where \( \hat{U} = \{ \omega_1, \ldots, \omega_k \} \).

Analogously, given the control system (2.1) with respect to admissible controls replaced by \( \hat{\mathcal{U}}_{pc} \) or just \( \omega_i \), \( i = 1, \ldots, k \), we define the following negative orbits:

\[
\mathcal{O}_{\omega_i,t}(B) := \{ y \in M \mid \text{there is } x \in B \text{ with } x = \varphi(t, y, \omega_i) \},
\]

\[
\mathcal{O}_{\omega_i \leq t}(B) := \bigcup_{0 \leq t \leq T} \mathcal{O}_{\omega_i,t}(B),
\]

\[
\mathcal{O}_{\omega_i}(B) := \bigcup_{0 \leq t} \mathcal{O}_{\omega_i,t}(B),
\]

\[
\mathcal{O}_{\hat{U},t}(B) := \{ y \in M \mid \text{there are } x \in B, \text{ and } u \in \hat{\mathcal{U}}_{pc} \text{ with } x = \varphi(t, y, u) \},
\]
Given the control system (2.1) and any open connected subset \( B \subset M \), for \( i = 1, \ldots, k \), we have

1. for \( t \in [0, \infty) \), \( \mathcal{O}_{\omega_i,t}(B) \), \( \mathcal{O}_{\overline{U}_{i,t}}(B) \) and \( \mathcal{O}_{\overline{T}}(B) \) are all open and connected,

2. for \( T \in [0, \infty) \), \( \mathcal{O}_{\omega_i,T}(B) \), \( \mathcal{O}_{\overline{U}_{i,T}}(B) \) and \( \mathcal{O}_{\overline{T}}(B) \) are all open and connected,

3. \( \mathcal{O}_{\omega_i}(B) \), \( \mathcal{O}_{\overline{U}_{i}}(B) \) and \( \mathcal{O}_{\overline{T}}(B) \) are all backward invariant open connected sets,

4. for \( t \in [0, \infty) \), \( \mathcal{O}_{\omega_i,t}(B) = \overline{\mathcal{O}_{\omega_i,t}(B)} \), moreover, we have

\[ \partial \mathcal{O}_{\omega_i,t}(B) = \mathcal{O}_{\omega_i,t}(\partial B) \] and \[ \text{int} \mathcal{O}_{\omega_i,t}(B) = \mathcal{O}_{\omega_i,t}(B), \]

5. for \( T \in [0, \infty) \), \( \mathcal{O}_{\omega_i,T}(B) = \overline{\mathcal{O}_{\omega_i,T}(B)} \).

**Proof.**

(1) \( \mathcal{O}_{\omega_i,t}(B) \) is open and connected is due to the fact that \( \varphi(t, \cdot, \omega_i) \) is a diffeomorphism.

We show the openness of \( \mathcal{O}_{\overline{U}_{i,t}}(B) \): for \( x \in \mathcal{O}_{\overline{U}_{i,t}}(B) \), there exist \( y \in B \), \( u \in \hat{U} \), \( t \geq 0 \) such that \( y = \varphi(t, x, u) \). Let \( W \) be an open set with \( y \in W \subset B \). By continuity of \( \varphi(t, \cdot, u) \) there is a neighborhood \( V \) of \( x \) such that \( \varphi(t, z, u) \in W \) for all \( z \in V \), this implies \( V \subset \mathcal{O}_{\overline{U}_{i,t}}(B) \). Similarly \( \mathcal{O}_{\overline{T}}(B) \) is open.
(2), (3) These hold since arbitrary unions of open sets are again open. Backward invariance of these sets follows from the definition.

(4) For fixed $t \in [0, \infty)$, $\varphi(t, \cdot, \omega_i)$ is a diffeomorphism and hence takes the interior of $\bar{B}$ to the interior of $\overline{O_{\omega_i, t}(B)}$, and the boundary of $\bar{B}$ to the boundary of $\overline{O_{\omega_i, t}(B)}$.

(5) From the definition and part (4), $\overline{O_{\omega_i, \leq T}(B)} = \bigcup_{0 \leq t \leq T} \overline{O_{\omega_i, t}(B)} = \bigcup_{0 \leq t \leq T} \overline{O_{\omega_i, t}(B)}$. We only need to show $\overline{O_{\omega_i, \leq T}(B)} = \bigcup_{0 \leq t \leq T} \overline{O_{\omega_i, t}(B)}$. Since the "\( \supset \)" part is easy, we will show "\( \subseteq \)" part. For any $x \in \overline{O_{\omega_i, \leq T}(B)}$, there exists a sequence $x_n$ in $\bigcup_{0 \leq t \leq T} \overline{O_{\omega_i, t}(B)}$ with $x_n \to x$. For each $x_n$, there exist $t_n \in [0, T]$ and $y_n \in B$ such that $x_n = \varphi(t_n, y_n, \omega_i)$. The compactness of $[0, T] \times \bar{B}$ implies that there exists a subsequence $(t_{n_k}, y_{n_k})$ and $(t_0, y_0) \in [0, T] \times \bar{B}$ with $(t_{n_k}, y_{n_k}) \to (t_0, y_0)$. Now the continuity of $\varphi$ implies that $x = \varphi(t_0, y_0, \omega_i) \in \overline{O_{\omega_i, t_0}(B)}$ and hence the conclusion. \( \square \)

Consider the linear system

$$\dot{x} = A(t)x(t) + B(t)u(t);$$

$x(0)$ given.

It is well known that under very weak assumptions on the matrices $A(\cdot)$ and $B(\cdot)$ the reachable set up to any time $T \geq 0$ with controls constrained by $u \in \mathcal{U}$ is the same as the reachable set up to time $T$ with controls constrained by $u \in \mathcal{U}_{pc}$. This is called a "bang-bang theorem" because the controls need only take on their extreme values and not intermediate ones. For the case of nonlinear systems, the following lemma has previously been proven (see e.g. Krener [39], H"{a}ckl [30]).
Lemma 2.6 (Bang-Bang Theorem)

Given the control system (2.1) with the assumption (H) and \( x_0 \in M \), then
\[
\mathcal{O}_0^-(x_0) \subset \mathcal{O}^-(x_0) \subset \overline{\mathcal{O}_0^-(x_0)},
\]
that is,
\[
\mathcal{O}_0^-(x_0) \text{ is dense in } \mathcal{O}^-(x_0).
\]

For practical purposes, we would like to replace a single point \( x_0 \) by an open connected set.

Lemma 2.7 (Practical Bang-Bang Theorem)

Given the control system (2.1) with the assumption (H) and \( B \) be any open connected subset in \( M \), then
\[
\mathcal{O}_0^-(B) = \mathcal{O}^-(B).
\]

**PROOF.** Clearly \( \mathcal{O}_0^-(B) \subset \mathcal{O}^-(B) \). To show the opposite inclusion, we let \( x \in \mathcal{O}^-(B) \). Choose an open connected neighborhood \( V \) of \( x \) such that \( V \subset \mathcal{O}^-(B) \). The Lie algebra \( \mathcal{L} \{ X_0 + \sum u_i X_i \mid u \in U \} \) has the same dimension as \( \mathcal{L} \{ X_0 + \sum u_i X_i \mid u \in U \} \) since if one writes \( X_0 + \sum u_i X_i \) as \( f(x, u) \) then for each \( u \in U \), \( u = \sum \lambda_i \omega_i \), where \( \sum \lambda_i = 1 \) and \( f(x, \sum \lambda_i \omega_i) = \sum \lambda_i f(x, \omega_i) \). Hence \( \mathcal{O}_0^+(x) \cap V \) has a nonempty interior. Let \( y \in \text{int} \left( \mathcal{O}_0^+(x) \cap V \right) \), and since \( \text{int} \left( \mathcal{O}_0^+(x) \cap V \right) \subset V \subset \mathcal{O}^-(B) \), one can find a \( x_0 \in B \) such that \( y \in \mathcal{O}^-(x_0) \). Now from Lemma 2.6 there is a sequence \( y_m \in \mathcal{O}_0^-(x_0) \) such that \( y_m \) converges to \( y \). For \( m \) sufficiently large, \( y_m \in \text{int} \left( \mathcal{O}_0^+(x) \cap V \right) \). Hence \( y_m \in \mathcal{O}_0^+(x_0) \cap \mathcal{O}_0^-(x) \) and this implies \( x \in \mathcal{O}_0^-(x_0) \) and hence \( x \in \mathcal{O}_0^-(B) \).

We recall that the Hausdorff distance between two compact subsets \( K_1, K_2 \subset M \) is defined by
\[
d_H(K_1, K_2) := \max \left\{ \sup_{x \in K_1} d(x, K_2), \sup_{x \in K_2} d(x, K_1) \right\},
\]
where \( d(x, K_2) \) and \( d(x, K_1) \) are the point-to-set distance.

Occasionally, for nonclosed sets \( A, B \in 2^M \), we will use \( d_\mathcal{H}(A, B) \) to denote the Hausdorff distance between the closed sets \( \overline{A} \) and \( \overline{B} \).

For our next result, we will assume that \( M \) is compact.

Let \( \mathcal{K}(M) \) be the family of all nonempty compact subsets of \( M \). It is not difficult to see that \( d_\mathcal{H} \) is a metric on \( \mathcal{K}(M) \) and moreover, \( (\mathcal{K}(M), d_\mathcal{H}) \), the system of nonempty compact subsets of \( M \) endowed with the Hausdorff distance \( d_\mathcal{H} \), is a compact metric space if \((M, d)\) is a compact metric space (see e.g., Beer [12]).

The following lemma is a keystone for our construction of the global feedback controllers.

**Lemma 2.8**

*Given the control system (2.1) with the assumption (H) and that the state space \( M \) is compact. Fix \( i \in \{1, \ldots, k\} \) and consider the set-valued map

\[
\mathcal{G} : [0, \infty) \to (\mathcal{K}(M), d_\mathcal{H})
\]

\[
t \mapsto \overline{O_{\omega_i, \leq t}(B)}.
\]

Then

1. \( \mathcal{G} \) is monotone increasing (with respect to the set inclusion) and continuous,
2. the domain of \( \mathcal{G} \) can be extended to \([0, \infty]\) with the definition

\[
\mathcal{G}(\infty) := \lim_{t \to \infty} \mathcal{G}(t) := \lim_{t \to \infty} \overline{O_{\omega_i, \leq t}(B)},
\]
3. \( \mathcal{G} \) is uniform continuous, in particular, it is true that

\[
\mathcal{G}(\infty) = \overline{O_{\omega_i}(B)}.
\]
PROOF.

1. From the definition we notice that for $0 \leq t_1 \leq t_2$, $\overline{O_{\omega_i, \leq t_1}(B)} \subseteq \overline{O_{\omega_i, \leq t_2}(B)}$, that is, $G(t_1) \subseteq G(t_2)$. From Lemma 2.5 part (4), we have $\overline{O_{\omega_i, \leq t}(B)} = \overline{O_{\omega_i, \leq t}(B)}$ for $t \geq 0$, so we need to show that for all $t_0 \geq 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for any $t$ in the $\delta$-neighborhood of $t_0$,

$$d_H(\overline{O_{\omega_i, \leq t}(B)}, \overline{O_{\omega_i, \leq t_0}(B)}) < \varepsilon$$

For any $y \in \overline{B}$, by the continuity of $\varphi(t, y, \omega_i)$, there is $\delta_y > 0$ such that for any $t$ in the $\delta_y$-neighborhood of $t_0$, $d(\varphi(t, y, \omega_i), \varphi(t_0, y, \omega_i)) < \varepsilon$. By the compactness of $\overline{B}$, there exists $y_1 \in \overline{B}$ such that $\|y_1\| = \|y\|$. Let $\delta := \delta_{y_1} > 0$ and we have

$$d(\varphi(t, y, \omega_i), \varphi(t_0, y, \omega_i)) < \varepsilon \quad \text{for all } y \in \overline{B}, \ t \text{ in the } \delta \text{-neighborhood of } t_0.$$

On the other hand,

$$d(\varphi(t, y, \omega_i), \overline{O_{\omega_i, \leq t_0}(B)}) \leq d(\varphi(t, y, \omega_i), \varphi(t_0, y, \omega_i)),$$

since $\varphi(t_0, y, \omega_i) \in \overline{O_{\omega_i, \leq t_0}(B)}$.

This implies that

$$\sup_{y \in \overline{B}} d(\varphi(t, y, \omega_i), \overline{O_{\omega_i, \leq t_0}(B)}) < \varepsilon,$$

or

$$\sup_{z \in \overline{O_{\omega_i, \leq t_0}(B)}} d(z, \overline{O_{\omega_i, \leq t_0}(B)}) < \varepsilon.$$

Similarly, one can show that

$$\sup_{z \in \overline{O_{\omega_i, \leq t_0}(B)}} d(z, \overline{O_{\omega_i, \leq t}(B)}) < \varepsilon.$$
and hence
\[ d_H(\mathcal{O}_{\omega_i, \leq t}(\bar{B}), \mathcal{O}_{\omega_i, \leq t_0}(\bar{B})) = \max \left\{ \sup_{z \in \mathcal{O}_{\omega_i, \leq t}(\bar{B})} d(z, \mathcal{O}_{\omega_i, \leq t_0}(\bar{B})), \sup_{z \in \mathcal{O}_{\omega_i, \leq t}(\bar{B})} d(z, \mathcal{O}_{\omega_i, \leq t_0}(\bar{B})) \right\} < \varepsilon. \]

2. By the compactness of \((\mathcal{K}(M), d_H)\), and that \(\mathcal{G}(t) := \Omega_{\omega_i, \leq t}(B)\) is increasing, \(\lim_{t \to \infty} \Omega_{\omega_i, \leq t}(B)\) exists. Let us define
\[ \mathcal{G}(\infty) := \lim_{t \to \infty} \mathcal{G}(t) := \lim_{t \to \infty} \Omega_{\omega_i, \leq t}(B). \]

3. From part (2), \(\mathcal{G}\) is uniformly continuous. Now from Lemma 2.5 part (5) \(\Omega_{\omega_i, \leq t}(B) = \Omega_{\omega_i, \leq t}(\bar{B})\) for each \(t\) and since \(\Omega_{\omega_i, \leq t}(\bar{B})\) is an increasing sequence with \(\lim_{t \to \infty} \Omega_{\omega_i, \leq t}(\bar{B}) = \Omega_{\omega_i}(\bar{B})\), the equality holds.

\[ \square \]

**Construction of Global Practical Feedback Controllers**

**Definition 2.9 (Fundamental Sequence)**

A positive sequence \(\{\varepsilon_{ij}\}\) with \(i = 1, 2, \ldots; j = 1, \ldots, k\), where \(k\) is the number of the vertices of the convex hull \(\text{co}\{\omega_1, \ldots, \omega_k\}\), which decreases to 0 in the following dictionary order:

\[ \varepsilon_{ij} > \varepsilon_{ik} \quad \text{if} \quad j < k, \]
\[ \varepsilon_{ij} > \varepsilon_{lj} \quad \text{if} \quad i < l. \]

is called a fundamental sequence.
Consider the control system (2.1) with the assumption (H) and assume that the state space $M$ is compact. Let $B$ be any open connected subset in $M$. Moreover, associated with this system we preset a fundamental sequence $\{\varepsilon_{ij}\}$. From Lemma 2.8 and the continuity of $G$ we have:

For $\varepsilon_{11} > 0$, there is a time $T_{11} > 0$, such that for all $T \geq T_{11}$,

$$d_N(\overline{O_{\omega_1, \leq T}(B)}, \overline{O_{\omega_1}(B)}) < \varepsilon_{11}.$$ 

Furthermore, from Lemma 2.5 part (2), $O_{\omega_1, \leq T_{11}}(B)$ is again an open connected set in $M$. Applying Lemma 2.8 again with set $B$ replaced by $O_{\omega_1, \leq T_{11}}(B)$, we get

$$\lim_{T \to \infty} d_N(\overline{O_{\omega_2, \leq T}(O_{\omega_1, \leq T_{11}}(B))}, \overline{O_{\omega_2}(O_{\omega_1, \leq T_{11}}(B))}) = 0,$$

or for $\varepsilon_{12} > 0$, there is a time $T_{12} > 0$, such that for all $T \geq T_{12}$,

$$d_N(\overline{O_{\omega_2, \leq T}(O_{\omega_1, \leq T_{11}}(B))}, \overline{O_{\omega_2}(O_{\omega_1, \leq T_{11}}(B))}) < \varepsilon_{12}.$$ 

Repeat this procedure according to this fundamental sequence $\{\varepsilon_{ij}\}$ and we define

$$O_{\omega_1}^{-} (B) := O_{\omega_1, \leq T_{11}} (B),$$

$$O_{\omega_1, \omega_2}^{-} (B) := O_{\omega_2, \leq T_{12}} (O_{\omega_1, \leq T_{11}} (B)),$$

$$O_{\omega_1, \omega_2, \omega_3}^{-} (B) := O_{\omega_3, \leq T_{13}} (O_{\omega_2, \leq T_{12}} (O_{\omega_1, \leq T_{11}} (B))),$$

$$\ldots$$

$$O_{\omega_1, \ldots, \omega_k}^{-} (B) := O_{\omega_k, \leq T_{1k}} (\ldots (O_{\omega_{k-1}, \leq T_{2k}} (\ldots (O_{\omega_2, \leq T_{21}} (O_{\omega_1, \leq T_{11}} (B)))))))$$

$$\ldots$$
Moreover, we define the set $O^{-\omega_{1}^{11}, \ldots, \omega_{k}^{1k}, \omega_{1}^{21}, \ldots, \omega_{k}^{2k}, \ldots}(B)$ by

$$\{y \in M \mid \text{there exists an index } ij \text{ with } i \in \mathbb{N}; j \in \{1, \ldots, k\},$$

such that $y \in O^{-\omega_{1}^{11}, \ldots, \omega_{k}^{1k}, \omega_{1}^{21}, \ldots, \omega_{k}^{2k}, \ldots}(B)\}$. 

In the next proposition we show that the negative reachable set of an open connected set $B$ can be reconstructed by the limit of the sets we just defined above.

**Proposition 2.10**

Consider the control system (2.1) with the assumption (H) and assume that the state space $M$ is compact. Letting $B$ be any open connected subset in $M$, then

$$O^{-}(B) = O^{-\omega_{1}^{11}, \ldots, \omega_{k}^{1k}, \omega_{1}^{21}, \ldots, \omega_{k}^{2k}, \ldots}(B).$$

**Proof.** According to Lemma 2.7 we only need to show that $O^{-\omega_{1}^{11}, \ldots, \omega_{k}^{1k}, \omega_{1}^{21}, \ldots, \omega_{k}^{2k}, \ldots}(B)$.

Clearly " $\subseteq$ " holds. To show the opposite inclusion we let $y \in O^{-\omega_{1}^{11}, \ldots, \omega_{k}^{1k}, \omega_{1}^{21}, \ldots, \omega_{k}^{2k}, \ldots}(B)$. then there exist $x \in B$, a piecewise constant control $\omega \in \hat{U}_{pc}$ and $t \geq 0$ such that $x = \varphi(t, x, \omega)$, say

$$x \leftarrow x_{1} \leftarrow x_{2} \cdots x_{n-1} \leftarrow x_{n} = y.$$  \hspace{1cm} (2.4)$$

It can be directly noted that $x_{1} \in O_{\omega_{1}^{11}}^{-}(B)$. Let $\delta_{1} = \min_{z \in \partial O_{\omega_{1}^{11}}^{-}(B)} d(x_{1}, z)$, since the sequence $\epsilon_{ij}$ decreases to 0 there exists a nature number $i_1$ such that $\epsilon_{i_1 i_1} < \delta_{1}$. We claim that $x_{1} \in O_{\omega_{1}^{11}, \ldots, \omega_{i_1}^{11}, \omega_{i_1}^{11}, \ldots}^{-}(B)$. This is because that $O_{\omega_{1}^{11}, \ldots, \omega_{i_1}^{11}, \omega_{i_1}^{11}, \ldots}^{-}(B)$ contains any point in $O_{\omega_{1}^{11}}^{-}(B)$ whose distance to the boundary of $O_{\omega_{1}^{11}}^{-}(B)$ is more than $\epsilon_{i_1 i_1}$. Next, let $\delta_{2} = \min_{z \in \partial O_{\omega_{1}^{11}, \ldots, \omega_{i_1}^{11}, \omega_{i_1}^{11}, \ldots}} d(x_{2}, z)$, there exists a natural number $i_2$ such
that $\varepsilon_{i_2l_2} < \min(\varepsilon_{i_1l_1}, \delta_2)$. We claim that $z_2 \in \mathcal{O}_{\omega_{i_1l_1} \ldots \omega_{i_2l_2}}^{-}(B)$. This is because that $\mathcal{O}_{\omega_{i_1l_1} \ldots \omega_{i_2l_2}}^{-}(B)$ contains any point in $\mathcal{O}_{\omega_{i_2l_2}}^{-}(\mathcal{O}_{\omega_{i_1l_1} \ldots \omega_{i_2l_2}}^{-}(B))$ whose distance to the boundary of $\mathcal{O}_{\omega_{i_1l_1} \ldots \omega_{i_2l_2}}^{-}(\mathcal{O}_{\omega_{i_1l_1} \ldots \omega_{i_2l_2}}^{-}(B))$ is more than $\varepsilon_{i_2l_2}$. Repeat this process, we show that $y$ in $\mathcal{O}_{\omega_{i_1l_1} \ldots \omega_{i_2l_2}}^{-}(B)$ and hence in $\mathcal{O}_{\omega_{i_1l_1} \ldots \omega_{i_2l_2} \ldots \omega_{i_kl_k}}^{-}(B)$.

The next proposition shows that with finitely many steps, the set $\mathcal{O}_{B}^{-}(B)$ can be approximated arbitrarily close.

**Proposition 2.11**

Given the control system (2.1) with the same assumption as in Proposition 2.10, then for any $\varepsilon > 0$, there exist natural numbers $i$ and $j$ with $1 \leq j \leq k$ such that

$$d_{\mathcal{H}}(\mathcal{O}_{\omega_{i_1l_1} \ldots \omega_{i_jl_j}}^{-}(B), \mathcal{O}_{B}^{-}(B)) < \varepsilon.$$

**Proof.** For any compact subset $Q \subset \mathcal{O}_{B}^{-}(B)$ with $d_{\mathcal{H}}(Q, \mathcal{O}_{B}^{-}(B)) < \varepsilon$, there is a countable open cover $\{\mathcal{O}_{\omega_{i_1l_1}}^{-}(B), \mathcal{O}_{\omega_{i_1l_1} \ldots \omega_{i_jl_j}}^{-}(B), \ldots\}$. By compactness of $Q$ there exists a finite subcover. Moreover, since $\mathcal{O}_{\omega_{i_1l_1}}^{-}(B) \subseteq \mathcal{O}_{\omega_{i_1l_1} \ldots \omega_{i_jl_j}}^{-}(B) \subseteq \cdots$, there are natural numbers $i$ and $j$ with $1 \leq j \leq k$ such that $\mathcal{O}_{\omega_{i_1l_1} \ldots \omega_{i_jl_j}}^{-}(B) \supset Q$ and hence the assertion holds.

With all these preparations, we are now ready to answer the question about the existence of a practical feedback controller.

Note it is not possible to steer a point $z$ outside the domain of attraction of $B$ into $B$ so the existence of a feedback controller on the domain of attraction of $B$ is the best one we can hope to get.

Furthermore, from a practical point of view, Lemma 2.7 and Proposition 2.11 show that it is good enough to construct a feedback controller $F(x)$ on $\mathcal{O}_{\omega_{i_1l_1} \ldots \omega_{i_jl_j}}^{-}(B)$.
for some natural numbers \( i \) and \( j \) with \( 1 \leq j \leq k \).

For convenience we define a sequence of sets \((R_i)_{i=0}^{\infty}\) as follows:

\[
R_0 := \overline{B}, \\
R_1 := \overline{\mathcal{O}_{\omega_1^{e_{11}}}(B)} \setminus \overline{B}, \\
R_2 := \overline{\mathcal{O}_{\omega_1^{e_{11}}, \omega_2^{e_{12}}}(B)} \setminus \overline{\mathcal{O}_{\omega_1^{e_{11}}}(B)}, \\
\ldots \\
R_{k+1} := \overline{\mathcal{O}_{\omega_1^{e_{11}}, \omega_k^{e_{1k}}, \omega_j^{e_{1j}}}(B)} \setminus \overline{\mathcal{O}_{\omega_1^{e_{11}}}(B)}, \\
\ldots \\
\]

and associated with \((R_i)_{i=0}^{\infty}\) we define a sequence of controls \((\tilde{\omega}_i)_{i=0}^{\infty}\) in the following order

\[
\tilde{\omega}_1 = \omega_p \text{ if } \bigcup_{s=0}^{l-1} R_s = \overline{\mathcal{O}_{\omega_1^{e_{11}}, \ldots, \omega_m^{e_{1m}}}(B)} \text{ for some } r \in \mathbb{N} \text{ and some } p = 1, \ldots, k,
\]

where \( k \) is the number of the vertices of the convex hull \( \text{co}\{\omega_1, \ldots, \omega_k\} \). That is, if \( R_l \) is generated via control \( \omega_p \) from \( \bigcup_{s=0}^{l-1} R_s \), we define \( \tilde{\omega}_l = \omega_p \).

We notice that \( R_r \cap R_s = \emptyset \) if \( r \neq s \), and so if \( \overline{\mathcal{O}_{\omega_1^{e_{11}}, \ldots, \omega_j^{e_{1j}}}(B)} = \bigcup_{l=0}^{n} R_l \), we get a mutually disjoint decomposition \( \{R_l\}_{l=0}^{n} \) of \( \overline{\mathcal{O}_{\omega_1^{e_{11}}, \ldots, \omega_j^{e_{1j}}}(B)} \).

Define a feedback controller \( K(x) \) on \( \bigcup_{l=1}^{n} R_l \) by

\[
K(x) := \tilde{\omega}_l \text{ if } x \in R_l \text{ for } l = 1, \ldots, n.
\]

resulting in the system in \( \overline{\mathcal{O}_{\omega_1^{e_{11}}, \ldots, \omega_j^{e_{1j}}}(B)} \setminus \overline{B} \)

\[
\dot{x} = X_0(x) + \sum_{i=1}^{m} K_i(x) \cdot X_i(x). \quad (2.5)
\]

The (discontinuous) feedback function \( K(x) \) is well-defined since \( R_r \cap R_s = \emptyset \) if \( r \neq s \). Furthermore, we have
Figure 2.1: The feedback regions in the proof of Proposition 2.12.

Proposition 2.12 (Existence of Global Feedback Controllers)

Given the control system (2.1) with the assumption (H) and that the state space $M$ is compact. For every initial point $x \in \overline{C_{\omega_{11},\omega_{11}}^{-\infty}}(B) \setminus \overline{B}$ there exists a unique trajectory $\varphi(t, x) \in \overline{C_{\omega_{11},\omega_{11}}^{-\infty}}(B) \setminus \overline{B}$ of the feedback system (2.5) with $\varphi(0, x) = x$. Moreover, there exist finite time $T$ and a point $y \in \partial \overline{B}$ such that $\varphi(T, x) = y$.

Proof. Without loss of generality, say $\overline{C_{\omega_{11},\omega_{11}}^{-\infty}}(B) \setminus \overline{B} = \bigcup_{l=1}^{n} R_l$ (see Figure 2.1). And for any $x \in \overline{C_{\omega_{11},\omega_{11}}^{-\infty}}(B) \setminus \overline{B}$, there is a $m_1 \in \{1, 2, \cdots, n\}$ such that $x \in R_{m_1}$. From Lemma 2.5 part (5) and the construction of $R_{m_1}$, we know that

$$\min\{t; \varphi(t, x) \in \bigcup_{l=1}^{m_1-1} R_l\} := \eta_1 < \infty$$

and $\varphi(\eta_1, x) \in \partial R_{m_2}$ where $m_2 \in \{0, 1, \cdots, m_1 - 1\}$. If $m_2 = 0$ we stop, otherwise again

$$\min\{t; \varphi(t, \varphi(\eta_1, x)) \in \bigcup_{l=1}^{m_2-1} R_l\} := \eta_2 < \infty$$
and \(\varphi(\eta_2, \varphi(\eta_1, x)) \in \partial R_{m_3}\) where \(m_3 \in \{0, 1, \ldots, m_2 - 1\}\). If \(m_3 = 0\) we stop, otherwise repeat this procedure and there is \(r \in \{1, 2, \ldots, m\}\) such that

\[
\varphi(\eta_r, (\varphi(\eta_{r-1}, \cdots, (\varphi(\eta_1, x))) \cdots)) := y \in \partial R_0 = \partial B.
\]

and \(\eta_1 + \eta_2 + \cdots + \eta_r := T < \infty\).

### Practical Feedback Controllability

First we give a precise definition of practical feedback controllability.

**Definition 2.13 (Practical Feedback Controllability)**

The control system (2.1) is said to be **practically feedback controllable** in \(M\) if for any compact subset \(Q \subset M\) and for any connected subset \(B \subset Q\) with the property \(\text{int}(B) = \overline{B}\), there exists a piecewise constant feedback control law \(u = K(x)\) defined on \(A(B) \cap Q\), such that for any initial state \(x_0\) in \(A(B) \cap Q\), the trajectory can be steered into \(B\) in finite time with at most finitely many switches on the control values.

The following theorem is our main result in this chapter.

**Theorem 2.14 (Affine Systems Are Practically Feedback Controllable)**

Given the assumption \((H)\), then the control system (2.1) is practically feedback controllable.

**Proof.** First, if the state space \(M\) is not compact, one can restrict the state space to any compact subset \(Q \subset M\) and construct all the negative orbits \(O^{-}_{\omega_{11}}(B)\), \(O^{-}_{\omega_{11}} \omega_{22}(B)\), \(\cdots\) inside this compact set \(Q\). From Lemma 2.5 (4), (5), Proposition 2.12 can be extended to the case that \(B\) is any connected subset of \(M\) with the
property $\overline{\text{int}(B)} = \overline{B}$. By Proposition 2.11 and Proposition 2.12 the system (2.1) is practically feedback controllable. \hfill \Box

Feedback Stabilization of Nonlinear Systems in Dimension Two

In this section, we will consider a simple application of our earlier construction of global feedback controllers, namely, a stabilization result at a regular fixed point, in the sense that the Lie algebra rank condition ($H$) holds, in a two dimensional manifold.

Consider a simplified affine control system with single input

$$\dot{x} = X_0(x) + u X_1(x), \quad x \in M,$$

(2.6)

where $X_0, X_1$ are $C^\infty$ vector fields on a two dimensional smooth manifold $M$, and $u(\cdot) \in U = [-\rho, \rho]$.

Moreover, we assume that

1. $X_0(0) = 0$, i.e., the origin is a fixed point of the uncontrolled equation $\dot{x} = X_0(x)$,

2. there is an one dimensional stable manifold $W^s(0)$ and the origin 0 is in the interior of some $W^s(0)$-neighborhood $N$ of 0 with respect to the relative topology of $W^s(0)$,

3. $X_1(0) \neq 0$, so that with no loss of generality we suppose that $X_1$ does not vanish on a neighborhood $B$ of the origin 0 and

4. $X_1(0)$ is not a tangent vector of $W^s(0)$. 
The problem considered is the following:

Does there exist a (global) piecewise constant stabilizing feedback function defined in any compact subset of the domain of attraction of the origin such that the origin is asymptotically stable?

The following theorem gives a confirmative answer.

**Theorem 2.15 (The Existence of a Global Feedback Stabilizer)**

*Under the assumptions (1) – (4), there exists a piecewise constant feedback controller \( K(x) \) defined in any compact subset \( Q \) of \( \mathcal{A}(0) \) such that for any initial point \( x \in Q \), there is a unique trajectory \( \varphi(t, x) \in Q, t \geq 0 \), of the feedback system

\[
\dot{x} = X_0(x) + K(x) \cdot X_1(x) \tag{2.7}
\]

with \( \varphi(0, x) = x \). The origin is a stable fixed point of (2.7), i.e. \( \varphi(t, x) \to 0 \) as \( t \to \infty \).

**Proof.** By Lemma 2.5 part (2) and assumption (3), \( \mathcal{O}_{\rho \leq T}(B_\varepsilon) \) is an open subset of \( M \), for any \( T > 0 \) and any open neighborhood \( B_\varepsilon \) with diameter \( \varepsilon \) of the origin. Moreover, let \( N_\varepsilon := B \cap W^s(0) \). By assumption (4) and the diffeomorphism property of \( \varphi(t, \cdot, \rho) \), if we shrink the diameter \( \varepsilon \) of \( B_\varepsilon \) and reduce time \( T \), there exist \( \varepsilon_1 > 0 \) and \( T_1 > 0 \) sufficiently small, such that \( \text{int} \mathcal{O}_{\rho \leq T_1}(N_{\varepsilon_1}) \) is an open subset of \( M \), where \( N_{\varepsilon_1} := B_{\varepsilon_1} \cap W^s(0) \) (see Figure 2.2).

It is easy to see that \((X_0 - \rho X_1)(0)\) is at the opposite direction of \((X_0 + \rho X_1)(0)\) and similarly, there exist \( \varepsilon_2 > 0 \) and \( T_2 > 0 \) small enough, such that \( \text{int} \mathcal{O}_{\rho \leq T_2}(N_{\varepsilon_2}) \) is an open subset of \( M \), where \( N_{\varepsilon_2} := B_{\varepsilon_2} \cap W^s(0) \).

Letting \( \tau := \min(T_1, T_2) \) and \( \delta := \min(\varepsilon_1, \varepsilon_2) \), we have

\[
\mathcal{O}_{\rho \leq \tau}(N_\delta) \cup \mathcal{O}_{\rho \leq \tau}(N_\delta) := B(0) \text{ is an open neighborhood of the origin.}
\]
Furthermore, for any initial point \( y \in \partial B(0) \setminus W^s(0) \), either the constant control \( \rho \) or \( -\rho \) can drive \( y \) to the stable manifold \( W^s(0) \). That is, if we define a feedback \( K_1(x) \) in \( \overline{B(0)} \) by

\[
K_1(x) := \begin{cases} 
\rho & \text{for } x \in \overline{O_{\rho, \leq \tau}(N_\delta)} \setminus N_\delta, \\
0 & \text{for } x \in N_\delta, \\
-\rho & \text{for } x \in \overline{O_{-\rho, \leq \tau}(N_\delta)} \setminus N_\delta,
\end{cases}
\] (2.8)

then for any initial point \( y \in \overline{B(0)} \), the solution of the feedback equation

\[
\dot{x} = X_0(x) + K_1(x) \cdot X_1(x)
\]

is well defined in \( \overline{B(0)} \), and \( \varphi(t, y) \to 0 \) as \( t \to \infty \).

From open set \( B(0) \), we can apply those techniques we developed in Section 2, especially Proposition 2.10, Proposition 2.11 and Theorem 2.14, and define a feedback controller \( K_2 \) in \( A(0) \setminus \overline{B(0)} \) to drive all initial point \( z \in A(0) \setminus \overline{B(0)} \) to \( \partial B(0) \).

With the combination of \( K_1 \) and \( K_2 \), we proved the assertion. \( \square \)
CHAPTER 3. PRACTICAL FEEDBACK STABILIZATION

This chapter is devoted to the investigation of practical feedback stability of nonlinear control systems and set-valued differential equations.

A roughly equivalent formulation of the control problem is by a "differential inclusion", where the control function \( u(\cdot) \) is absent but the equation defining the evolution of the system contains a set-valued function. We follow this general differential inclusion setup to prove that there is a perfect candidate, called the control set, for the purpose of practical feedback stabilization. In Section 3 we introduce the concept of feedback controlled invariant sets, control sets and feedback control sets. We will show that in general control sets are feedback control sets. Section 3 is devoted to the analysis of the controlled invariance property via the theory of differential inclusion, in which we prove that the closure of control sets are actually feedback controlled invariant sets. A keystone to practically stabilize the complicated behavior of nonlinear dynamical systems is the study of limit behavior of control sets, which will be explored in Section 3. In Section 3 a more general situation about the continuity property of control sets depending on a parameter will be discussed. Our main results on practical feedback stabilization will be shown in the final Section 3.
Feedback Controlled Invariant Sets, Control Sets and Feedback Control Sets

In this section, we again concentrate on the following class of affine control systems
\[ \dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i X_i(x(t)), \] (3.1)
on a connected \( C^\infty \) manifold with \( \dim M = d < \infty \) and \( C^\infty \) vector fields \( X_0, \ldots, X_m \).

We will consider either
\[ u(\cdot) = (u_1(\cdot), \ldots, u_m(\cdot)) \in \mathcal{U} = \{ u : \mathbb{R} \rightarrow U \mid u \text{ locally integrable} \}, \] (3.2)
or
\[ u(\cdot) = (u_1(\cdot), \ldots, u_m(\cdot)) \in \mathcal{U}^b = \{ u : M \rightarrow U \mid u \text{ measurable} \}, \] (3.3)
with \( U = \text{co}\{\omega_1, \ldots, \omega_k\} \subset \mathbb{R}^m \), \( k \in \mathbb{N} \).

As same as in Chapter 2, if the feasible controls \( u(\cdot) \in \mathcal{U} \), we assume that (3.1) has a unique solution \( \varphi(t, x_0, u) \) for all \( x_0 \in M \), \( u \in \mathcal{U} \) with \( \varphi(0, x_0, u) = x_0 \) and being defined for all \( t \in \mathbb{R} \). The solution is an absolutely continuous function which is given by
\[ \varphi(t, x_0, u) = x_0 + \int_0^t X_0(x(s)) \, ds + \sum_{i=1}^{m} \int_0^t u_i(s) X_i(x(s)) \, ds. \]

On the other hand, for any initial value \( x_0 \in M \), if the feasible controls \( u(\cdot) \in \mathcal{U}^b \), any solution of (3.1) is an absolutely continuous function given by
\[ \varphi(t, x_0, u) = x_0 + \int_0^t X_0(x(s)) \, ds + \sum_{i=1}^{m} \int_0^t u_i(x(s)) X_i(x(s)) \, ds, \]
whenever it exists.
What is a Feedback Controlled Invariant Set?

Definition 3.1 (Feedback Controlled Invariant Set)

Given system (3.1) with the feasible controls \( u(\cdot) \in U^b \), a subset \( K \subseteq M \) is called a feedback controlled invariant set if there exists a control \( u(\cdot) \in U^b \) such that for any initial state \( x_0 \in K \), the solution \( \varphi(t, x_0, u) \in K \) for all \( t \geq 0 \).

Remark 3.2

In Section 3 we will see that under a reasonable assumption on the vector fields of system (3.1), a stronger result about the existence of a continuous feedback \( u(\cdot) \) in \( K \subseteq M \) is possible.

A similar concept of feedback controlled invariant sets has been introduced under the name positively weakly invariant sets over two decades ago, (see e.g., Yorke [58], Feuer and Heymann [26]). In fact, our feedback controlled invariant sets are the feedback control theoretic analogue of positively weakly invariant sets as studied in Roxin [50], which in turn generalize the concept of positively invariant sets of ordinary stability theory of dynamical systems (see e.g. Amann [3, Chapter 4], Pavel [47]).

More recently, a number of concepts of varying invariance properties have been studied in the viability theory literature, (see e.g., Aubin [5], Aubin and Cellina [6], Aubin and Da Prato [7]), which will be introduced in next section.

Control Sets

A possible candidate to fit the criterion of a feedback controlled invariant set of system (3.1) with \( u \in U \) is called control set (it will be proved in next section that the closure of a control set is a feedback controlled invariant set), these have
been introduced by Arnold and Kliemann in [4] and [37] to describe the support of invariant measures for degenerate stochastic diffusions. Roughly speaking, control sets are maximal subsets of the state space where, via open loop controls, complete controllability holds. More precisely, control sets are maximal subsets of the state space in which the system is approximately controllable, i.e., for any two points \( x \) and \( y \) of a control set, we can find a control function \( u(\cdot) \in \mathcal{U} \), such that the corresponding solution, starting in \( x \) will reach \( y \) approximately.

**Definition 3.3 (Control Sets)**

A set \( D \subset M \) is called a control set of system (3.1) (with the admissible controls in \( \mathcal{U} \)) if

1. \( D \subset \bar{O}^+(x) \) for all \( x \in D \),
2. for all \( x \in D \) there exists a \( u \in \mathcal{U} \) such that \( \varphi(t, x, u) \in D \) for all \( t \geq 0 \),
3. \( D \) is maximal (w.r.t. set inclusion) with these properties.

A particular important class of control sets are invariant control sets:

**Definition 3.4 (Invariant Control Sets)**

A control set \( C \subset M \) is called an invariant control set of system (3.1) (with admissible controls in \( \mathcal{U} \)) if

\[ C = \bar{O}^+(x) \text{ for all } x \in C. \text{ All other control sets are called variant.} \]

We are often interested in determining how far from a control set \( D \) can the trajectory still be steered into \( D \). This gives rise to the definition of domain of attraction.
Definition 3.5 (Domain of Attraction of Control Sets)

The domain of attraction of a control set $D$ is given by

$$\mathcal{A}(D) = \{ x \in M \mid \mathcal{O}^+(x) \cap D \neq \emptyset \}. $$

Actually, from Definition 2.3 and the next proposition, one can define the domain of attraction of a control set $D$ by

$$\mathcal{A}(D) = \{ x \in M \mid \overline{\mathcal{O}^+(x)} \cap D \neq \emptyset \}. $$

The following proposition describes that for a control set $D$ with nonvoid interior the domain of attraction $\mathcal{A}(D)$ can be described as a negative orbit (see Häckl [31]).

**Proposition 3.6** Let $D$ be a control set with nonvoid interior and $x \in \text{int}(D)$ then

$$\mathcal{A}(D) = \mathcal{O}^-(x).$$

The next definition provides an order relation between control sets.

**Definition 3.7**

For two control sets $D_1$ and $D_2$, we define

$$D_1 \prec D_2 \iff D_1 \cap \mathcal{A}(D_2) \neq \emptyset.$$

The control sets and the order relation between them give a clue about the global picture of the controllability of the system, we refer to Colonius and Kliemann [22] for more details.

Here we state some important properties of control sets and invariant control sets. One can find most of the proofs in Colonius and Kliemann [22].
Proposition 3.8

Consider the nonlinear system (3.1) with admissible controls in $U$ and assume that hypothesis (H) holds. Then

1. every invariant control set has nonvoid interior,

2. a control set with nonvoid interior is invariant if and only if it is closed,

3. if the state space $M$ is compact, there are at least one closed control set with nonvoid interior and one open control set,

4. there are at most countably many control sets with nonvoid interior,

5. for any control set $D$ with nonvoid interior we have $D$ is connected, $\overline{\text{int}(D)} = D$ and $\text{int}(D) \subseteq \overline{O^+(x)}$ for any $x \in D$,

6. for any control set $D$ with nonvoid interior, if $p \in \partial D \cap D$, then for any $x \in \text{int}(D)$, $x$ cannot reach $p$ in finite time.

Notice that from the definition of control sets, once a solution has been steered into a control set $D$, it can be held inside with an open loop control and that a solution cannot leave the invariant control set after entering it. Moreover, the maximality indicates that control sets are either disjoint or identical. Furthermore, a solution cannot leave a variant control set and enter it again. More precisely, we quote a proposition from H"{a}ckl [31].

Proposition 3.9

Given a variant control set $D$ and a point $x \in D$, then we have

$$\overline{O^+(y)} \cap D = \emptyset \quad \text{for all } y \in \overline{O^+(x)} \setminus D.$$
Our main interest is focused on control sets with nonvoid interior. They can be described as the intersection of a negative orbit and the closure of a positive orbit as shown in Colonius and Kliemann [22]. One can also refer to Geyek and Vincent [27] and Roxin [51].

**Proposition 3.10**

*For any control set $D$ with nonvoid interior we have*

$$D = \overline{O^+(x)} \cap O^-(x) \quad \text{for all } x \in \text{int}(D).$$

On the other hand, a nonvoid interior of an intersection of a positive and a negative orbits indicates the existence of a control set (see Hӓckl [31] or Colonius and Kliemann [22]).

**Proposition 3.11**

*For any $x \in M$, if $\text{int}(O^+(x) \cap O^-(x)) \neq \emptyset$ then there exists a control set $D$ with*

$$\text{int}(D) = \text{int}(O^+(x) \cap O^-(x)).$$

**Feedback Control Sets**

Consider system (3.1) with the admissible controls replaced by

$$u(\cdot) = (u_1(\cdot), \ldots, u_m(\cdot)) \in \mathcal{U}^b = \{u : M \to U \mid u \text{ measurable}\}.$$

We introduce the concept of positive feedback orbit of $x \in M$ as follows:

$$\mathcal{O}^{fb,+}(x) := \{y \in M \mid \text{there exist } T \geq 0, u \in \mathcal{U}^b \text{ with } \phi(T, x, u) = y\}.$$ 

In analogy to control sets, feedback control sets are defined as maximal regions of feedback controllability.
Definition 3.12 (Feedback Control Sets)

A set $D^f \subset M$ is called a feedback control set of system (3.1) with the admissible controls in $\mathcal{U}^f$ if

1. $D^f \subset \overline{\mathcal{O}^{f,+}(x)}$ for all $x \in D^f$,

2. for all $x \in D^f$ there exists a $u \in \mathcal{U}^f$ such that $\varphi(t, x, u) \subset D^f$ for all $t \geq 0$,

3. $D^f$ is maximal (w.r.t. set inclusion) with these properties.

Moreover, a set $C^f \subset M$ is called an invariant feedback control set if $C^f = \overline{\mathcal{O}^{f,+}(x)}$ for all $x \in C^f$.

The following lemma is important for our next result.

Lemma 3.13

Given system (3.1) then for all $x \in M$ we have

$$ \mathcal{O}^+(x) = \mathcal{O}^{f,+}(x). $$

PROOF. "\(\supset\)" part is clear since if $y = \varphi(T, x, u), u \in \mathcal{U}^f$, for $t \in [0, T]$. Then one can define $u_1(t) := u(z)$, where $z = \varphi(t, x, u)$, which implies $u_1(\cdot) \in \mathcal{U}$ and $y \in \varphi(T, x, u_1)$. On the other hand, for "\(\subset\)" part: suppose $y \neq x$ and $y \in \mathcal{O}(x)$. One can find $u \in \mathcal{U}$ with $y = \varphi(T, x, u)$ and $\{\varphi(t, x, u) \mid t \in [0, T]\}$ is a simple curve (i.e. no self-intersection) on the state space $M$ where $\varphi(\cdot, x, u)$ is the solution of equation (3.1) with the initial condition $\varphi(0, x, u) = x$. For $z \in \{\varphi(t, x, u) \mid t \in [0, T]\}$ we define $u_1(z) := u(t)$, where $z = \varphi(t, x, u)$. Since the trajectory is a measurable set, its complement is measurable too, and hence we can have a measurable extension of the domain of $u_1(\cdot)$ to the whole of $M$. Then $u_1(\cdot) \in \mathcal{U}^f$ and $y \in \varphi(T, x, u_1)$. □
The following theorem specifies the relation between control sets and feedback control sets.

**Theorem 3.14 (Control Sets are Feedback Control Sets)**

*Given system (3.1), then any (invariant) control set \(D\) is a (invariant) feedback control set and vice versa, any (invariant) feedback control set \(D^{fb}\) is a (invariant) control set.*

**Proof.** Let \(D \subseteq M\) be a control set of system (3.1). Then for any \(x \in D\), by Lemma 3.13 we have \(D \subseteq \overline{\mathcal{O}^+(x)} = \overline{\mathcal{O}^{fb,+}(x)}\). Next, from the definition of the control set \(D\), for any \(x \in D\), there exists an \(u \in \mathcal{U}\) such that \(\varphi(t,x,u) \in D\) for all \(t \geq 0\). For \(z \in \{\varphi(t,x,u) \mid t \in [0,\infty)\}\) we define \(u_1(z) := u(t)\), where \(z = \varphi(t,x,u)\), and with a similar way to the proof in Lemma 3.13, one can have a measurable extension of the domain of \(u_1(\cdot)\) to whole \(M\). Then \(u_1(\cdot) \in \mathcal{U}^{fb}\) and \(\varphi(t,x,u_1) \in D\) for all \(t \geq 0\). Furthermore, the maximality of \(D\) to be a feedback control set is easy to see due to the maximality of \(D\) being a control set. Hence we prove that every control set is a feedback control set. Similarly, every feedback control set is a control set. \(\Box\)

We will show in next section that actually control sets (or feedback control sets) are feedback controlled invariant sets.
More About Feedback Controlled Invariant Sets

Differential Inclusion Set-Up

In this subsection, we consider the following general nonlinear control system:

\[ \dot{x} = f(x, u), \quad (3.4) \]

on a connected smooth manifold \( M \) with \( \dim M = d < \infty \).

Here the controls are functions \( u \in U = \{ u : \mathbb{R} \rightarrow U \mid u \text{ locally integrable} \} \) with \( U \subset \mathbb{R}^m \) is compact and convex. Furthermore, the vector field \( f \) is Lipschitz continuous on \( M \).

An equivalent formulation of the control problem is by a "differential inclusion", which was developed by Ważewski [57] in the early 1960s. Indeed if we introduce a set-valued map

\[ F(x) := \{ f(x, u) \mid u \in U \}, \quad (3.5) \]

and consider the associated differential inclusion

\[ \dot{x} \in F(x(t)), \quad (3.6) \]

then the Filippov Theory (see e.g. Aubin and Cellina [6, Chapter 2]) states that the solutions of (3.6) and (3.4) do coincide.

The differential inclusion (3.6) provides a convenient way to treat not only usual control systems of the form

\[ \dot{x} = f(x, u), \ u \in \mathcal{U}, \]

but also control systems with feedbacks,

\[ \dot{x} = f(x, u), \ u \in \mathcal{U}^b. \]
In this section we will investigate the invariance properties of system (3.4), with respect to feedback controls in stead of open loop controls, in the framework of differential inclusions.

Since the basic theory of set-valued maps and viability analysis can be found in several excellent books (e.g. Aubin [5], Aubin and Cellina [6], Aubin and Frankowska [8]) we do not want to give a self-contained introduction to this subject here. It may however be useful to fix our notation, and to collect briefly some important definitions and results.

We introduce a measurable feedback set-valued map $\|I : M \rightarrow U$ associating with any state $x$, the subset $I(x)$ of feasible controls when the state of the system is $x$. In other words, we assume that the available controls of the system are required to obey constraints which may depend upon the state.

The dynamics of the system are described by a map $f : \text{Graph}(I) \rightarrow \mathbb{R}^d$ which assigns to each state-control pair $(x, u) \in \text{Graph}(I)$ the velocity $f(x, u)$ of the state.

Hence the set
$$F(x) := \{f(x, u)\}_{u \in I(x)}$$
is the set of available velocities to the system when its state is $x$.

Therefore, the nonlinear control system (3.4) with the set of admissible controls replaced by $U^b := \{u : M \rightarrow U \mid u \text{ measurable}\}$ can be denoted by $(I, f)$ including

- a measurable feedback set-valued map $I : M \rightarrow U$, and

- a map $f : \text{Graph}(I) \rightarrow \mathbb{R}^d$ describing the dynamics of the system.
The evolution of the system $F := (\Pi, f)$ is governed by the differential inclusion

$$
F := \begin{cases}
\dot{x}(t) = f(x(t), u(t)), & \text{where} \\
u(t) \in \Pi(x(t)).
\end{cases}
$$

(3.7)

First we recall the definition of the contingent cone introduced by Bouligand in the early 1930s.

**Definition 3.15 (Contingent Cone $T_K(x)$)**

For a nonempty subset $K \subset M$ and $x \in \bar{K}$, the (Bouligand) contingent cone to $K$ at $x$ is given by

$$
T_K(x) := \left\{ v \in \mathbb{R}^d \mid \liminf_{h \to 0^+} \frac{d(x + hv, K)}{h} = 0 \right\}.
$$

In other words (see Aubin and Cellina [6, Proposition 4.1.2]), $v$ belongs to $T_K(x)$ if and only if there exist a positive sequence $h_n > 0$ converging to 0 and a sequence $v_n \in M$ converging to $v$ such that

for all $n \geq 0$, $x + h_nv_n \in K$.

We see that

for all $x \in \text{int}(K)$, $T_K(x) = \mathbb{R}^d$,

and that the contingent cone $T_K(x)$ is the upper limit of the differential quotients

$$
\frac{K - x}{h} := \left\{ \frac{y - x}{h} \mid y \in K \right\},
$$

so that $T_K(x)$ is a closed cone (see Aubin and Frankowska [9]).

Moreover, when $K$ is a differential manifold, the contingent cone $T_K(x)$ coincides with the tangent space to $K$ at $x$ and when $K$ is convex, it coincides with the tangent cone of convex analysis (see e.g., Aubin [5], Aubin and Frankowska [9]).
Note that the condition \( \liminf_{h \to 0^+} \frac{d(x + hv, K)}{h} = 0 \) has also been introduced by several authors over the past decades under the name "subtangentiality" (see for example, Yorke [58], Feuer and Heymann [26]).

Our first goal in this section is to characterize the subsets \( K \subset M \) which are viable under \( F \) in the following sense:

**Definition 3.16**

*Given system (3.7), a subset \( K \subset M \) is said to be viable under \( F \) if for any initial state \( x_0 \in K \), there exists a solution \( x(t) \) which is viable in the sense that*

\[
\text{for all } t \geq 0, \ x(t) \in K.
\]

Intuitively speaking, a subset \( K \subset M \) is viable under \( F \) if at each state \( x \in K \) there is a velocity \( f(x, u) \in F(x) \) which is "tangent" to \( K \) at \( x \), i.e. brings back a solution inside \( K \). This motivates the next definition.

**Definition 3.17 (Viability Domain)**

*A subset \( K \subset M \) is called a viability domain of \( F \) if and only if*

\[
\text{for all } x \in K, \ F(x) \cap T_K(x) \neq \emptyset.
\]

Among all candidates which are viable under \( F \), for example, viability domain is the one to require that for any state \( x \), there exists at least one velocity \( v \in F(x) \) which is contingent to \( K \) at \( x \). The other one, called invariance domain (for the definition, see Aubin [5]), demands that all velocities \( v \in F(x) \) are contingent to \( K \) at \( x \). The set defined above will fit our requirement as we will see later.

We shall associate with each viability domain \( K \) the regulation map \( R_K \subset \Pi \) as follows.
Definition 3.18 (Regulation Map)

Consider the system (3.7). We associate with any subset $K \subset M$ the regulation map $R_K : K \rightarrow U$ defined by

$$
\text{for all } x \in K, \quad R_K(x) := \{ u \in \Pi(x) \mid f(x,u) \in T_K(x) \}.
$$

Controls $u$ belonging to $R_K(x)$ are called viable.

We observe that $K$ is a viability domain if and only if the regulation map $R_K$ is strict (i.e., has nonempty values).

Are Control Sets Feedback Controlled Invariant Sets?

We translate the control systems in the language of differential inclusions in the previous subsection and in this subsection, we will continue our investigations in this framework.

Let we again focus on the affine control system (3.1) with $u(\cdot) = (u_1(\cdot), \ldots, u_m(\cdot)) \in \mathcal{U}^b$ and its equivalent differential inclusion form:

$$
\dot{x} \in F_1(x),
$$

where

$$
F_1 := \begin{cases} 
\dot{x}(t) &= X_0(x(t)) + \sum_{i=1}^m u_i X_i(x(t)), \\
u(t) &= (u_1(t), \ldots, u_m(t)) \in \Pi(x) = U.
\end{cases}
$$

Naturally, one can see that $F_1$ is a continuous set-valued map (see Aubin and Cellina [6, Proposition 1.2.1]) with compact convex images since $U$ is the convex hull generated by $\{\omega_1, \ldots, \omega_k\}$ and the control-affine feature of this system.

The fundamental relation between the viability domain, which is based on the geometric property of the contingent cone or the so called subtangentiality condition,
and the dynamic property of controlled invariance is given in the theorem as followed, see Aubin & Cellina [6, Theorem 4.2.1] and Aubin [5, Theorem 6.1.4].

**Theorem 3.19 (Viability Theorem)**

Consider the affine control system (3.8). For any subset $K \subset M$ with $F_1(K)$ bounded, $K$ is viable under $F_1$ if and only if it is a viability domain.

**Corollary 3.20**

Given the system (3.1), then any bounded control set $D$ is a viability domain of (3.8).

**Proof.**

By Theorem 3.14, any control set $D$ is a feedback control set. And from the definition of feedback control set, we know that any feedback control set is viable under $F_1$, and hence by the above Viability Theorem the assertion holds.

The next theorem indicates that not only the control sets themselves are viability domain but also the closure of any control set is a viability domain. Moreover, if a control set $D$ has nonvoid interior, then $\text{int}(D)$ is also a viability domain.

**Theorem 3.21**

If $D \subset M$ is a bounded control set with nonvoid interior of system (3.1), then

1. Its closure $\overline{D}$ is a viability domain.

2. $\text{int}(D)$ is a viability domain.
PROOF.

1. Let \( \dot{x} \in \overline{D} \), we will show that there exists \( \hat{u} \in \mathcal{U} \) such that \( \varphi(t, \dot{x}, \hat{u}) \in \overline{D} \) for all \( t \geq 0 \). Then similarly to the proof in Lemma 3.13, one can construct a measurable feedback \( \hat{u}_1 \in \mathcal{U}^b \) such that \( \varphi(t, \dot{x}, \hat{u}_1) = \varphi(t, \dot{x}, \hat{u}) \) and hence the trajectory \( \varphi(t, \dot{x}, \hat{u}_1) \) will stay in \( \overline{D} \) for all \( t \geq 0 \).

First, there is a sequence \( x_n \in D \) with \( x_n \to \dot{x} \). By the definition of a control set, for each \( x_n \) there is a control \( u_n \in \mathcal{U} \) such that \( \varphi(t, x_n, u_n) \in D \) for all \( t \geq 0 \). Now by Lemma 2.1, \( \mathcal{U} \) is compact, and hence by the compactness of \( \overline{D} \times \mathcal{U} \), there exists \( \hat{u} \in \mathcal{U} \) such that \( (x_n, u_n) \to (\dot{x}, \hat{u}) \).

We claim that

\[ \varphi(t, \dot{x}, \hat{u}) \in \overline{D} \quad \text{for all} \quad t \geq 0. \]

If this is not the case, there exists \( T > 0 \) such that \( \varphi(T, \dot{x}, \hat{u}) = y \notin \overline{D} \), say the distance \( d(y, \overline{D}) = \delta > 0 \).

Now by the continuity of \( \varphi(T, \cdot, \cdot) \) and that \( (x_n, u_n) \to (\dot{x}, \hat{u}) \), one has \( \varphi(T, x_n, u_n) \to \varphi(T, \dot{x}, \hat{u}) \). Moreover, by the continuity of \( d(\cdot, \overline{D}) \), one gets

\[ 0 = d(\varphi(T, x_n, u_n), \overline{D}) \to d(\varphi(T, \dot{x}, \hat{u}), \overline{D}) = \delta > 0, \]

which is a contradiction. This proves our claim and part (1).

2. This is true because for all \( x \in \text{int}(D) \), the collection of all feasible velocities

\[ F_1(x) := \{(X_0 + \sum_{i=1}^{m} u_i X_i)(x) \mid u \in U\} \]

at \( x \) has a nonempty intersection with the contingent cone \( T_{\text{int}(D)}(x) \) at \( x \) and hence by the Viability Theorem (Theorem 3.19) \( \text{int}(D) \) is a viability domain.
For our particular interest, the question now is as follows:

Given a bounded control set \( D \) with nonvoid interior of system (3.1), does there exist a measurable feedback (or even a continuous feedback), which is a single-valued function \( r_D(\cdot) \) from \( \bar{D} \) to \( U \), such that for any initial state \( x_0 \in \bar{D} \), the trajectories \( \varphi(t, x_0, r_D) \in \bar{D} \) for all \( t \geq 0 \)?

The question boils down to finding some appropriate selections. More precisely, given a set-valued map \( \Gamma \) from \( X \) to \( Y \). A map \( \gamma : X \to Y \) is said to be a selector for \( \Gamma \) if \( \gamma(x) \in \Gamma(x) \) for all \( x \in X \).

Our first task is to find a measurable selection procedure of the map \((F_1 \cap T_D)(\cdot)\) on the closure \( \bar{D} \) of a control set \( D \). We quote a standard measurable selection theorem (see e.g. Aubin and Frankowska [9, Theorem 8.1.3] or Kisielewicz [36, pp. 46-48]) as followed:

**Theorem 3.22 (Measurable Selection)**

Consider a measure space \((X, \mathcal{F}, \mu)\) and a separable complete metric space \((Y, \rho)\). If \( \Gamma \) is a set-valued map from \( X \) to closed nonempty subsets of \( Y \), then \( \Gamma \) has a measurable selector.

Using the theorem above, we obtain

**Theorem 3.23**

Given a bounded control set \( D \) with nonvoid interior of system (3.1). There exists a measurable selection \( \tilde{r}_D(x) \in F_1(x) \cap T_D(x) \) in \( \bar{D} \).
PROOF.

First we notice that \( F_1(x) \cap T_D(x) \neq \emptyset \) for all \( x \in \overline{D} \). Furthermore, \( F_1(\cdot) \) has compact and convex images and \( T_D(\cdot) \) has closed and convex images implies that \( (F_1 \cap T_D)(\cdot) \) has compact and convex images for all \( x \in \overline{D} \). Hence \( F_1 \cap T_D \) has a measurable selector \( f_D(x) \in F_1(x) \cap T_D(x) \) by Theorem 3.22. In particular, since the set of \( x \) with \( f_D(x) \notin F_1(x) \cap T_D(x) \) has only measure 0, one can reassign the values of \( f_D(x) \) at those \( x \) and make \( f_D(x) \in F_1(x) \cap T_D(x) \) for all \( x \in \overline{D} \) without changing the measurability of \( f_D(\cdot) \).

In order to show that there is a measurable feedback to guarantee \( \overline{D} \) is a feedback controlled invariant set, we quote a very useful measurable selection theorem from Aubin and Frankowska [9, Theorem 8.2.10].

**Theorem 3.24 (Filippov Measurable Selection Theorem)**

Consider a complete \( \sigma \)-finite measurable space \((K, \mathcal{A}, \mu)\), complete separable metric spaces \( U, X \) and a measurable set-valued map \( \Pi : K \to U \) with closed nonempty images. Let \( f : K \times U \to X \) be a Carathéodory map. Then for every measurable map \( g : K \to X \) satisfying

\[
g(x) \in f(x, \Pi(x)) \text{ for almost all } x \in K,
\]

there exists a measurable selection \( r(x) \in \Pi(x) \) such that

\[
g(x) = f(x, r(x)) \text{ for almost all } x \in K.
\]
Our second preliminary task is the following corollary.

**Corollary 3.25**

Consider the measurable function $\hat{f}_D(x)$ defined in Theorem 3.23. There exists a measurable selection $r_D: D \rightarrow U$ such that

$$\hat{f}_D(x) = X_0(x) + \sum_{i=1}^{m} r_{D_i}(x)X_i(x),$$

where $r_D(x) = (r_{D_1}(x), \cdots, r_{D_m}(x))$.

**PROOF.**

In Filippov’s Theorem, one chooses $X = \mathbb{R}^d$, $g(x) = \hat{f}_D(x)$, $\Pi(x) = U$ and $K = \overline{D}$ to obtain the conclusion. 

Finally, we can state one of our main results in this section in the following theorem, which indicates that the closure of any control set is a feedback controlled invariant set.

**Theorem 3.26**

Given a bounded control set $D$ with nonvoid interior of system (3.1), there exists a measurable feedback $r_D(\cdot)$ from $D$ to $U$, such that for any initial state $x_0 \in \overline{D}$, the trajectories $\varphi(t, x_0, r_D) \in \overline{D}$ for all $t \geq 0$.

**PROOF.**

From Theorem 3.23 and Corollary 3.25 we have a measurable function $\hat{f}_D: \overline{D} \rightarrow \mathbb{R}^d$ and a measurable function $r_D: \overline{D} \rightarrow U$ satisfying

$$\dot{x} = \hat{f}_D(x) = X_0(x) + \sum_{i=1}^{m} r_{D_i}(x)X_i(x).$$
Since $\dot{f}_D(x) \in (F_1 \cap T_D)(x)$ for all $x \in \overline{D}$, the feasible velocity $\dot{f}_D(x_0)$ for any point $x_0 \in \overline{D}$ lies inside the contingent cone $T_D(x_0)$ and hence by the Viability Theorem, any trajectory starting from $x_0$ would not leave $\overline{D}$. In other words, for any $x_0 \in \overline{D}$, $\varphi(t, x_0, r_{\overline{D}}) \in \overline{D}$ for all $t \geq 0$. □

Actually, with some reasonable assumption, we can have a much stronger result, namely, the existence of a continuous feedback in $\overline{D}$ which still enjoys the properties of above theorem.

We start with the definition of lower semi-continuity and the celebrated Michael's theorem (see Michael [44], Aubin and Frankowska [9]).

Definition 3.27 (Lower Semi-continuity)

A set-valued map $G : X \to Y$ is called lower semi-continuous at $x \in \text{Dom}(G)$ if and only if for any $y \in G(x)$ and for any sequence of elements $x_n \in \text{Dom}(G)$ converging to $x$, there exists a sequence of elements $y_n \in G(x_n)$ converging to $y$. It is said to be lower semi-continuous if it is lower semi-continuous at every point $x \in \text{Dom}(G)$.

Theorem 3.28 (Michael's Theorem)

Let $G$ be a lower semi-continuous set-valued map with closed convex values from a compact metric space $X$ to a Banach space $Y$. It does have a continuous selection. In particular, for every $y_1 \in G(x_1)$ there exists a continuous section $g$ of $G$ such that $g(x_1) = y_1$.

Our goal here is to find a continuous selection procedure of the map $(F_1 \cap T_D)(\cdot)$ in the closure $\overline{D}$ of a control set $D$. With some reasonable assumption, we can prove a stronger result than Theorem 3.23.
Theorem 3.29

Consider a bounded control set $D$ with nonvoid interior of system (3.1) and assume that the set-valued map $T_{D}(\cdot)$ is lower semi-continuous in $\overline{D}$, then there exists a continuous selection $\hat{f}_{D}(x) \in F_{1}(x) \cap T_{D}(x)$ in $\overline{D}$.

Proof.

First, $F_{1}(\cdot)$ has compact and convex images and $T_{D}(\cdot)$ has closed and convex images implies that $(F_{1} \cap T_{D})(\cdot)$ has compact and convex images for all $x \in \overline{D}$. Next, the lower semi-continuity of $F_{1} \cap T_{D}(\cdot)$ is guaranteed by the continuity of $F_{1}(\cdot)$ and the lower semi-continuity of $T_{D}(\cdot)$. Hence $(F_{1} \cap T_{D})(\cdot)$ satisfies the conditions of Michael’s Theorem in $\overline{D}$. There exists a continuous selection $f_{D}(x) \in (F_{1} \cap T_{D})(x)$ in $\overline{D}$.

From Theorem 3.29 we notice that the continuous selection $\hat{f}_{D}(\cdot)$ in $\overline{D}$ satisfying

$$\dot{x} = \hat{f}_{D}(x) = X_{0}(x) + \sum_{i=1}^{m} u_{i}(x) X_{i}(x).$$

Our question now can be converted to an inverse problem as follows:

Given a bounded control set $D$ with nonvoid interior of system (3.1), a continuous function

$$f(x) := [f_{1}(x), \cdots, f_{d}(x)]^{T} := \hat{f}_{D}(x) - X_{0}(x) \text{ in } \overline{D},$$

and a $C^{\infty}$ function

$$G(x) := \begin{pmatrix}
X_{11}(x) & \cdots & X_{1m}(x) \\
\vdots & \ddots & \vdots \\
X_{d1}(x) & \cdots & X_{dm}(x)
\end{pmatrix} \text{ in } \overline{D}.\quad (3.11)$$
Under what conditions does the following equation

\[ f(x) = G(x)u(x) \tag{3.12} \]

have a continuous solution \( u(x) := [u_1(x), \ldots, u_m(x)]^T \) in \( \overline{D} \)?

It turns out that with a reasonable assumption which can be verified by an algebraic criterion, there exists a continuous function \( \hat{u}(x) \) in \( \overline{D} \) such that equation (3.12) holds. The most surprising result is the following theorem.

**Theorem 3.30**

*Given a bounded control set \( D \) with nonvoid interior of system (3.1). If there exists a continuous function

\[
\hat{G}(x) := \begin{pmatrix}
\dot{X}_{11}(x) & \cdots & \dot{X}_{1d}(x) \\
\cdots & \cdots & \cdots \\
\dot{X}_{m1}(x) & \cdots & \dot{X}_{md}(x)
\end{pmatrix}
\text{in } \overline{D}
\]

such that \( \hat{G}(\cdot)G(\cdot) = I_{mxm} \), the \( m \times m \) identity matrix, then there exists a continuous feedback \( \hat{u}(\cdot) \) from \( \overline{D} \) to \( U \), defined by \( \hat{u}(x) = \hat{G}(x)f(x) \), such that for any initial state \( x_0 \in \overline{D} \), the trajectories \( \varphi(t, x_0, \hat{u}) \in \overline{D} \) for all \( t \geq 0 \).

**Proof.**

Starting from equation (3.12), we multiply \( \hat{G}(x) \) on both sides to obtain

\[ \hat{G}(x)f(x) = u(x). \]

So if one defines \( \hat{u}(x) = \hat{G}(x)f(x) \) in \( \overline{D} \), the following equation holds:

\[ \dot{x} = f_{\overline{D}}(x) = X_0(x) + \sum_{i=1}^{m} \hat{u}_i(x)X_i(x) \text{ in } \overline{D}, \]
where \( \hat{f}_T(\cdot) \) is a continuous selection from \((F_1 \cap T_D)(\cdot)\) and \( \hat{u}(\cdot) = [\hat{u}_1(\cdot), \cdots, \hat{u}_m(\cdot)]^T \).

Similarly to Theorem 3.26, the feasible velocity \( \hat{f}_T(x_0) := (X_0 + \sum_{i=1}^{m} \hat{u}_i X_i)(x_0) \) for any point \( x_0 \in \overline{D} \) lies inside the contingent cone \( T_{\overline{D}}(x_0) \) and hence by the Viability Theorem, any trajectory starting from \( x_0 \) would not leave \( \overline{D} \). In other words, for any \( x_0 \in \overline{D} \), \( \varphi(t, x_0, \hat{u}) \in \overline{D} \) for all \( t \geq 0 \).

\( \square \)

**Limit Sets and the Limit Behavior of Control Sets**

In this section we consider the differential equation

\[
\dot{x}(t) = X_0(x(t)),
\]

(3.13)
on a compact manifold \( M \), together with the following family of control-affine non-linear systems depending on a parameter \( \rho > 0 \) which indicates the size of the control range:

\[
\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t) X_i(x(t)),
\]

(3.14)

\( u = (u_i) \in U^\rho := \{ u : \mathbb{R} \rightarrow \mathbb{R}^m \mid u(t) \in U^\rho = \rho \cdot U, \text{ locally integrable} \} \) with \( \rho > 0 \) and \( U \subset \mathbb{R}^m \) compact, convex and \( 0 \in \text{int} U \). In particular, \( U = \{ u \in \mathbb{R}^m ; \ | u | \leq \rho \} \) for any norm \( | \cdot | \) on \( \mathbb{R}^m \) is a possible choice.

For \( \rho = 0 \), we obtain an interpretation of (3.13) as control system, with the one-point control range \( U^0 = \{0\} \). Recall that a point \( x \) in the state space of a dynamical system is called recurrent if it is an element of its \( \omega \)-limit set. Roughly speaking, it turns out that the sets of chain recurrent points of ordinary differential equation (i.e. \( \rho = 0 \)) expand to control sets for the control system (i.e. \( \rho > 0 \)).

We recall some notions from the theory of dynamical systems (see e.g. Conley [23] or Mañé [43]).
Definition 3.31

Consider the flow \((M, \varphi)\) of the system (3.13) and positive constants \(\varepsilon > 0\) and \(T > 0\)

1. **(Limit Set)** The limit set \(\omega(x)\) of \(x \in M\) is given by

\[
\omega(x) = \{y \in M \mid \text{there is } t_k \to \infty \text{ with } \varphi(t_k, x) \to y\},
\]

2. **((\varepsilon, T)-chain)** An \((\varepsilon, T)\)-chain from \(x\) to \(y\) is a finite sequence of points \(x_0 = x, x_1, \ldots, x_{n-1}, x_n = y\) and a sequence of times \(t_0 > T, \ldots, t_{n-1} > T\) with the property

\[
d(\varphi(t_i, x_i), x_{i+1}) < \varepsilon \quad \text{for all } i = 0, \ldots, n - 1,
\]

here \(d(\cdot, \cdot)\) is the metric on \(M\),

3. **(Chain Limit Set)** The chain limit set of \(x \in M\) is defined as

\[
\Omega(x) = \{y \in M \mid \text{for all } \varepsilon, T > 0 \text{ there is an } (\varepsilon, T)\text{-chain from } x \text{ to } y\},
\]

4. **(Chain Recurrent Point)** A point \(x \in M\) is called chain recurrent if for any \(\varepsilon > 0\) and any \(T > 0\) there is an \((\varepsilon, T)\)-chain from \(x\) to \(x\),

5. **(Chain Recurrent Set)** The chain recurrent set is defined as

\[
CR = \{x \in M \mid x \in \Omega(x)\},
\]

that is, the set of all chain recurrent points.
Moreover, a closed connected maximal subset of $\mathcal{CR}$ is called a component of $\mathcal{CR}$. And we call the flow $(M, \varphi)$ is chain recurrent, if $M = \mathcal{CR}$, and chain transitive, if $y \in \Omega(x)$ for all $x, y \in M$.

We notice that all the limit points of bounded trajectories e.g. fixed points, periodic orbits and homoclinic orbits, etc, are contained in the set $\mathcal{CR}$.

In the sequel the correspondence between the components of the chain recurrent set $\mathcal{CR}$ and the control sets $D^\rho$ play an important role. In order to clarify when limit sets are contained in control sets, the following notion turns out to be crucial.

**Definition 3.32 (Inner-Pair-Condition)**

A pair $(x, u) \in M \times U$ is called an inner pair to the control system (3.14) if there exists $T > 0$ such that

\[ \varphi(T, x, u(\cdot)) \in \text{int}O^+(x). \]  

(3.15)

To get nice results, the pair $(x, 0)$ need to be an inner pair for $\rho > 0$ and for every $x \in \mathcal{CR}$. It is well known that controllability of the linearized system is a sufficient condition for local controllability (in fixed time) around trajectories, hence in this case the inner pair condition holds. However, in general this property is difficult to verify. A reasonable sufficient condition for this property can be formulated using Lie-brackets which is particularly simple to verify:

Let $Y(x) = X_0(x) + \sum_{i=1}^{m} u_i^0 X_i(x)$ and denote by $ad_X^k X_i$ the $k$-th Lie derivative of the vectorfield $X_i$ along $Y$, that is,

\[ ad_X^0 X_i(x) := X_i(x) \]
\[ ad_X^k X_i(y) := [Y, ad_X^{k-1} X_i](y) = \frac{\partial}{\partial x} \frac{\partial}{\partial y} (Y(y) - \frac{\partial}{\partial x} Y(y)) ad_Y^{k-1} X_i(y) \]

According to Corollary 4.6 in Colonius and Kliemann [21] we have
Lemma 3.33

Let $u^0 \in \text{int}(U^p)$ be a constant control and $x \in M$ and consider $Y = X_0 + \sum_{i=1}^{m} u_i^0 X_i$. Instead of (3.15) we assume the following stronger condition:

for all $y \in \omega(x, u^0)$ the equality

$$\text{linear span } \{(ad_{y}^k X_i)(y); \ i = 1, \ldots, m, \ k = 0, 1, \ldots\} = T_y M$$

holds. Then every pair $(y, u^0)$ with $y \in \omega(x, u^0)$ is an inner pair.

Now we are ready to formulate the theorems about the relation between the chain recurrent components of (3.13) and the control sets of (3.14)$^p$ which are immediate consequences of Corollary 5.3 in Colonius and Kliemann [21].

Theorem 3.34

If for every $x \in CR$ of the ordinary differential equation (3.13) $(x, 0)$ is an inner pair, then for every isolated invariant component $L$ of $CR$ there is a $\rho^0 > 0$ and a decreasing sequence of control sets $D^p$ with respect to decreasing $\rho$ such that $L \subset \text{int}(D^p)$ for each $\rho > 0$ and

$$L = \bigcap_{0<\rho<\rho^0} D^\rho.$$

Conversely, we obtain the following result.

Theorem 3.35

If there exists a sequence of control sets $D^{p_k}$ of the control-affine systems (3.14)$^{p_k}$ such that

1. $\rho_k \to 0$ as $k \to \infty$, 

2. $L \subset \bigcap_{0<\rho<\rho^0} D^\rho$. 

3. $\lim_{k \to \infty} \rho_k = 0$. 

then every isolated invariant component $L$ of $CR$ there is a $\rho^0 > 0$ and a decreasing sequence of control sets $D^p$ with respect to decreasing $\rho$ such that $L \subset \text{int}(D^p)$ for each $\rho > 0$ and

$$L = \bigcap_{0<\rho<\rho^0} D^\rho.$$
2. The set $L := \{ y \in M \mid \text{there is a sequence } x_k \in D^{\rho_k} \text{ with } x_k \to y \text{ as } k \to \infty \}$ is nonempty.

Then $L$ is a component of the chain recurrent set of the ordinary differential equation (3.13).

The Behavior of Control Sets Under Varying Control Range

In this section we first state a continuity property of control sets in parameter dependent control systems. Recall the definition of lower semi-continuity of a set-valued map (Definition 3.27) and consider the following family of control systems depending on a parameter $\rho \in A \subset \mathbb{R}^k$:

$$\dot{x}(t) = X(\rho, x(t), u(t)), \quad t \in \mathbb{R},$$

$$u \in \mathcal{U}_{pc}$$

where $\mathcal{U}_{pc} = \{ u : \mathbb{R} \to \mathbb{R}^m \mid u(t) \in U \text{ for all } t \in \mathbb{R}, \text{ piecewise constant} \}$.

Since the control sets do not change if instead of piecewise constant controls, piecewise continuous or measurable ones are employed (cp. see Colonius and Kliemann [22]), we cite the following theorem from Colonius and Kliemann [22, Theorem 41] which states that control sets depend lower semi-continuously on $\rho$:

Theorem 3.36

Let $D^{\rho}$ be a control set of (3.16)$^{\rho}$ with $\rho^0 \in \text{int}A$.

1. If $L \subset \text{int}D^{\rho}$ is a compact set such that for all points $x_0 \in L$ the Lie algebra condition

$$(H^*) \quad \dim \mathcal{L}A\{X(\rho, \cdot, u) \mid (u_i) = u \in U\}(x_0) = d$$
is satisfied then there exists $\delta > 0$, such that for all $\rho \in A$ with $d(\rho, \rho^0) < \delta$ there is a control set $D^\rho$ of $(3.16)^\rho$ with $L \subseteq \text{int} D^\rho$.

2. If, additionally, the control systems $(3.16)^\rho$ are locally accessible for all $\rho$ in a neighborhood of $\rho^0$, then the map $\rho \mapsto \overline{D^\rho}$ from $A$ into the set of compact subsets of $M$ (endowed with the Hausdorff metric) is lower semi-continuous at $\rho = \rho^0$, where $D^\rho$ is given above.

Now we consider the following family of control systems depending on a parameter $\rho \geq 0$ which indicates the size of the control range:

$$\dot{x}(t) = X(x(t), u(t)), \quad t \in \mathbb{R}, \quad u \in \mathcal{U}_{pc}^\rho,$$

where $\mathcal{U}_{pc}^\rho = \{ u : \mathbb{R} \to \mathbb{R}^m | u(t) \in \rho U \text{ for all } t \in \mathbb{R}, \text{ piecewise constant} \}$.

This can be reformulated as a special case of the perturbed family $(3.16)$ with $A = [0, \infty)$ in the following way:

$$\dot{x}(t) = X(x(t), \rho u(t)), \quad t \in \mathbb{R}, \quad u \in \mathcal{U}_{pc}.$$

Hence Theorem 3.36 is valid for equation (3.17).

For the purpose of deriving a nice stabilization result, we consider the ordinary differential equation

$$\dot{x}(t) = X_0(x(t)),$$

on a compact manifold $M$. 

Associated with equation (3.19) we are, in particular, interested in the following control affine systems:

$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t)X_i(x(t)),$$

$$u \in U^\rho_{pc},$$

where $U^\rho_{pc} = \{ u : \mathbb{R} \to \mathbb{R}^m \mid u(t) \in \rho U \text{ for all } t \in \mathbb{R}, \text{ piecewise constant} \}$ with $\rho > 0$ and $U \subset \mathbb{R}^m$ compact, convex and $0 \in \text{int} U$.

The next lemma is from elementary point-set topology.

**Lemma 3.37**

*Given an open set $A \subset M$ and a compact subset $K$ of $A$. Then there exists $\varepsilon > 0$ such that

$$d_H(K, \partial A) > \varepsilon > 0.$$*

**Proof.**

Suppose this is not true. Then there exist sequences $x_n \in K$ and $y_n \in \partial A$ such that $d(x_n, y_n) \to 0$, as $n \to \infty$. By the compactness of $K \times \partial A$, there is $(x, y) \in K \times \partial A$, such that $(x_n, y_n) \to (x, y)$, as $n \to \infty$. This implies that $d(x, y) = 0$ and hence $y = x \in K$, where $y \in \partial A$, and this gives a contradiction. $\Box$

The following main result of this section shows that under the local accessibility condition and the inner pair condition on a limit set of equation (3.20), this limit set is contained in the interior of a feedback controlled invariant set which can be chosen as small as possible.
Theorem 3.38

Consider a component \( L \) of the chain recurrent set of the ordinary differential equation (3.19), assume that

1. for every \( x \in L \), \((x,0)\) is an inner pair of equation (3.20) and in addition,

2. the control system (3.20) satisfies the Lie algebra condition \((H)\) for all \( \rho \in (0,\rho^0) \), where \( \rho^0 \) is some positive number.

Then there is a decreasing sequence of control sets \( D^{\rho^k}, k = 0,1,2, \ldots \), with nonvoid interior such that

1. \( L \subseteq \text{int}(D^{\rho_k}) \) for each \( k \) and \( L = \lim_{k \to \infty} D^{\rho_k}, \)

2. \( \overline{D^{\rho_k+1}} \subseteq \text{int}(D^{\rho_k}) \) for \( k = 0,1,2, \ldots, \)

as \( \rho^k \searrow 0. \)

Proof.

First, we notice that if \( x \in \text{int}D^\rho \cap \text{int}D^\rho' \), where \( \rho > \rho' > 0 \), one has \( O^\pm(x) \subseteq O^\pm(x) \) and then Proposition 3.10 indicates that \( D^{\rho'} \subseteq D^\rho \). Now from Theorem 3.34 there exists a control set \( D^{\rho^0} \) with nonvoid interior such that \( L \) is the only chain recurrent set of the ordinary differential equation (3.19) inside \( \text{int}D^{\rho^0} \). By Lemma 3.37 there exists \( \varepsilon > 0 \) such that

\[ d_H(L, \partial(\text{int}D^{\rho^0})) = d_H(L, \partial D^{\rho^0}) > \varepsilon > 0, \]

say \( d_H(L, \partial D^{\rho^0}) = \delta_1. \)
Now apply Theorem 3.34 again with Theorem 3.36 part (1): There exists $\rho^1$ with $\rho^0 > \rho^1 > 0$ such that $L \subset \text{int}D^\rho$ and $d_H(L, \partial D^\rho) < \frac{\delta_1}{2}$ or in other words, \[ \min_{x \in \partial D^\rho} d(x, \partial D^\rho) > \frac{\delta_1}{2}, \] which implies $D^\rho \subset \text{int}D^\rho$.

Again, by Lemma 3.37 we assume $d_H(L, \partial D^\rho) = \delta_2 > 0$. Here we notice that $\delta_2 < \delta_1$. Similarly, one may find $\rho^2$ with $\rho^1 > \rho^2$ and a control set $D^\rho$ such that $L \subset \text{int}D^\rho$ and $d_H(L, \partial D^\rho) < \frac{\delta_2}{2}$, which implies \[ \min_{x \in \partial D^\rho} d(x, \partial D^\rho) > \frac{\delta_2}{2} \] and hence $D^\rho \subset \text{int}D^\rho$.

Continuing with this procedure of construction of $D^\rho$ with respect to a decreasing sequence $\delta_n$ with $\delta_{n+1} < \frac{\delta_n}{2}$ for each $n$, the assertion is proved. \[ \square \]

**Remark 3.39**

Consider a component $L$ of the chain recurrent set of the ordinary differential equation (3.19), if we use a control set with nonvoid interior, which contains $L$, as a “practical” region and apply the techniques we developed in Chapter 2 to design an a-priori global feedback controller (see Section 3). Then the feedback controller can guarantee that it could steer all trajectories with initial points in any compact subset of $A(L)$ to $\overline{D}$ and remain within it.

**Practical Feedback Stabilization**

The class of nonlinear control systems which do not admit a continuous stabilizing feedback is rich (see e.g. Brockett [13], Sontag and Sussmann [54], Sontag [53]), so it is often the case that discontinuous control laws must be considered. In this section we will study one of our two techniques of discontinuous stabilization, namely, “practical stabilization” which deals with bringing states close to a certain
set with nonvoid interior rather than to the particular limit sets (e.g. an equilibrium or a periodic orbit) and keeping the trajectory inside this certain set forever.

From a practical point of view, the desired state of a system may oscillate sufficiently near this state and its performance is acceptable. Actually, many problems fall into this category, for example, keeping the temperature within certain bounds in a chemical process and the behavior of a traveling vehicle between two points, the behavior of an aircraft or a missile may all be in this manner.

To deal with such situations, the notion of practical stability is more useful, which we define in a general set up below.

Assume the state space $M$ has been restricted to be a compact forward invariant set and consider a component $L$ of the chain recurrent set of the autonomous system

$$\dot{x}(t) = X_0(x(t)).$$

(3.21)

$N_\varepsilon(L)$ is called an open $\varepsilon$-cover of $L$ if $N_\varepsilon(L)$ is open, containing $L$ and

$$d_H(L, N_\varepsilon(L)) < \varepsilon,$$

where $d_H$ is the Hausdorff distance between two sets.

For any subset $A \subseteq M$ with nonvoid interior, we define a compact $\varepsilon$-subset of $A$ by $A_\varepsilon$ (if it exists) whenever

$$d_H(A, A_\varepsilon) < \varepsilon,$$

and $A_\varepsilon$ is a compact subset of $A$.

Let us introduce the concept of practical feedback stabilization as followed. Consider a component $L$ of the chain recurrent set of the nonlinear system (3.21) and the associated affine control system

$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t)X_i(x(t)),$$

(3.22)
For a measurable feedback control \( u = K(x) \in \mathcal{U}^b \), the resulting closed loop system is given by

\[
\dot{x} = X_0(x) + \sum_{i=1}^{m} K_i(x) \cdot X_i(x),
\]

and the solution (if it exists) of (3.23) starting from an initial point \( x_0 \in M \) is written as \( \varphi(t, x_0, K) \).

**Definition 3.40 (Practical Feedback Stabilization)**

Given a component \( L \) of the chain recurrent set of the nonlinear system (3.21) and let \( \mathcal{A}(L) \subset M \) be the domain of attraction of \( L \) in the associated affine control system (3.23). The system (3.23) is said to be practically feedback stabilizable (at the component \( L \) in the region \( \mathcal{A}(L) \)) if there are positive decreasing sequences \( \varepsilon_n \searrow 0 \) and \( \delta_n \searrow 0 \). Such that associated with each \( n \) there are an \( \varepsilon_n \)-cover \( N_{\varepsilon_n}(L) \) of \( L \) and a \( \delta_n \)-subset \( \mathcal{A}_{\delta_n}(L) \) of \( \mathcal{A}(L) \) such that for each \( n \)

1. there exists a piecewise constant feedback control law \( u^1_n = K^1_n(x) \) defined on \( \mathcal{A}_{\delta_n}(L) \setminus N_{\varepsilon_n}(L) \) such that the closed loop solution \( \varphi(\cdot, x_0, K^1_n) \) starting from any point \( x_0 \in \mathcal{A}_{\delta_n}(L) \setminus N_{\varepsilon_n}(L) \) will reach \( \partial N_{\varepsilon_n}(L) \) in finite time with at most finitely many switches on the control values, and

2. there exists a measurable feedback control law \( u^2_n = K^2_n(x) \) defined on \( \overline{N_{\varepsilon_n}(L)} \) such that the closed loop solution \( \varphi(\cdot, x_0, K^2_n) \) starting from any point \( x_0 \in \overline{N_{\varepsilon_n}(L)} \) remains in \( \overline{N_{\varepsilon_n}(L)} \) forever.
The main result of this section is as follows.

**Theorem 3.41 (Affine Systems Are Practically Feedback Stabilizable)**

Consider a component $L$ of the chain recurrent set of the ordinary differential equation (3.21) and assume that

1. for every $x \in L$, $(x, 0)$ is an inner pair of equation (3.23) and in addition,

2. the control system (3.23) satisfies the Lie algebra condition (H).

Then the control system (3.23) is practically feedback stabilizable.

**Proof.**

From the proof of Theorem 3.38, one can construct a decreasing sequence of control sets with nonvoid interior $\{D_n\}_{n=0}^\infty$ and $L \subset \text{int}(D_n)$ for each $n$. Without
loss of generality, we can start from $D_0$ with $\inf_{x \in D_0} d(x, \partial A(L)) > \alpha > 0$, where $\alpha$ is a positive constant.

Associated with $\{D_n\}_{n=0}^{\infty}$ is a decreasing sequence $\varepsilon_n \searrow 0$ such that $d_H(D_n, L) < \varepsilon_n$. We assign each $\text{int}(D_n)$ as an open $\varepsilon$-cover of $L$.

For each $\varepsilon_n$, we can find $\delta_n > 0$ such that there is a $\delta_n$-subset $A_{\delta_n}(L)$ of $A(L)$ and

$$L \subset N_{\varepsilon_n}(L) \subset A_{\delta_n}(L) \subset A(L),$$

see Figure 3.1. (3.24)

Moreover, the sequence $\delta_n$ can be chosen as a decreasing one and the corresponding $A_{\delta_n}(L)$ still satisfies the set inclusion relation (3.24).

For each $n$, Proposition 2.11 and Theorem 2.14 indicate the existence of a piece-wise feedback controller $K^1_n(x)$ defined on $A_{\delta_n}(L) \setminus N_{\varepsilon_n}(L)$ such that part (1) holds. Furthermore, since from our definition, $N_{\varepsilon_n}(L)$ is the closure of a control set and hence is a feedback controlled invariant set by Theorem 3.21. Now Theorem 3.26 implies the existence of a measurable feedback controller $K^2_n(x)$ defined on $N_{\varepsilon_n}(L)$ such that part (2) holds.

Remark 3.42

1. A stronger result about the existence of a continuous feedback $u^2_n = K^2_n(x)$ in $N_{\varepsilon_n}(L)$ is possible, see Theorem 3.30.

2. For initial values $x_0$ outside the domain of attraction $A(L)$ of $L$, the system (3.22) cannot be steered into any control set $D$ containing $L$ with open loop controls $u \in U$ nor with feedbacks $u \in U^f$.

3. The theory developed here takes two important aspects into account: The facts
that systems may not be completely controllable, and that feedback gains may be bounded.

4. For practical purpose, we can choose the parameters $\varepsilon$ and $\delta$ as small as we wish, so that the $\varepsilon$-cover $N_\varepsilon(L)$ is close to $L$ and the $\delta$-subset $A_\delta(L)$ of $A(L)$ is close to $A(L)$.

5. The construction of the feedback controller that led to the closed loop system (3.23) has the following robustness property: (For convenience), assume that the possible feedback range is $U := [-\rho, \rho]^m$, but that a computed feedback can be followed by the system only up to an accuracy of $\pm \varepsilon \rho$, where $\varepsilon$ is a constant with $0 < \varepsilon < 1$. In this situation, our construction of feedback law still steers the system (3.23) from any point in some $\delta$-subset $A_\delta(L)$ of $A(L)$ into the closure of the control set $D^p$ which contains $L$ in finite time and remains there forever. In fact, our construction is optimally robust in the sense that it steers the system (3.23) from any point in some $\delta$-subset $A_\delta(L)$ of $A(L)$ into the control set $D^p$ containing $L$, which deviates from $L$ as little as possible under the given disturbance range for feedback, and that it does so for the largest possible range of disturbances, such that system (3.23) can still be kept in $D^p$. The largest range of this kind is $[-\rho, \rho]^m$ for our discussion here.
CHAPTER 4. NUMERICAL EXAMPLES AND APPLICATIONS

In this chapter we present several case studies of real-world model problems that involve different types of unstable limits sets.

Our development of calculational algorithms is based heavily on Dr. Gerhard Häckl's work [31]. Roughly speaking, our software for computing feedback controlled invariant sets and their feedback regions is just the subsequent development of Dr. Häckl's software CS4.0, which was developed to compute control sets and their domain of attraction. The analysis of the computational algorithms is generally a very mathematically involved process and hence we will ignore the mathematics of our calculational algorithms. For those who are interested in this topic, we refer to Häckl [31].

Again, roughly speaking, the algorithm we compute the particularly small feedback controlled invariant set in each example is basically the one to compute a control set with small control range. This is because from Chapter 3, we know the closure of each control set is a feedback controlled invariant set. Moreover, it is possible to shrink the feedback controlled invariant set as small as we want if we shrink the control range carefully.

The main idea of computing those feedback regions of this feedback controlled invariant set are the following.
1. We assume the state space $Q$ is a rectangle $[a_1, b_1] \times \cdots \times [a_d, b_d]$, which is big enough under our consideration,

2. Instead of choosing a fundamental sequence $\epsilon_{ij} \searrow 0$ (see Section 2), we assign all $\epsilon_{ij}$ equal to a small positive constant $\epsilon$,

3. Discretize the state space $Q$ and use piecewise constant control functions with the extremal control values $\{\omega_1, \ldots, \omega_k\}$ to compute the $\epsilon$-invariant approximation (for the definition, see Häckl [31]) of the corresponding negative reachable set of each feedback region.

We start with a simple linear system and then we analyze some more complicated two-dimensional control systems. In the last section, we will briefly discuss a simple three-dimensional system in which we will see that due to some technical gaps, there is still much future work.

All the computation has been done on a DEC 3000 Model 300 AXP workstation with a DECchip 21064 RISC-style microprocessor. The graphical outputs for all our examples have been produced with MATLAB. At the border of some feedback controlled invariant sets we indicate the discrete convex hull (see Häckl [31]) and we use dots in the interior.
Two-Dimensional Linear System with Control Constraints

In the present section, we consider a simple two-dimensional linear dynamical system

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\end{pmatrix} = \begin{pmatrix}
-2 & 1 \\
0 & 1 \\
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
\end{pmatrix},
\]

where \( x = 0 \) is a saddle, see Figure 4.1, which is therefore described as an unstable equilibrium point.

Associated with (4.1), let us consider the control system

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\end{pmatrix} = \begin{pmatrix}
-2 & 1 \\
0 & 1 \\
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
\end{pmatrix} + \begin{pmatrix}
0 \\
1 \\
\end{pmatrix} u,
\]

where \( u(t) \in U := [-1, 1] \).

First, we notice that without the input constraint \( u(t) \in [-1, 1] \), system (4.2) is (completely) controllable (see e.g. Chen [16] for details) and hence the domain of attraction of the origin is the whole \( \mathbb{R}^2 \) plane and one can design a (high-gain) feedback controller \( u = k(x) \) such that the origin is globally stabilizable, see e.g., Chen [16].

The assumption that controls can take arbitrary large values, from a practical point of view, is quite restrictive. More natural considerations imply that controls have to be bounded by some known function. In our example (4.2), \( u(t) \in [-1, 1] \). With this restriction, the domain of attraction of the origin is shown in Figure 4.2.

Next, we compute the control sets containing the origin with respect to different control ranges, see Figure 4.3.

From the previous chapter, we know that not only these control sets are the subsets in which the system is approximately controllable but also their closure are
Figure 4.1: Trajectories near the saddle point \( x = 0 \) for (4.1).

Figure 4.2: Domain of attraction of the origin for (4.2).
Figure 4.3: Control sets w.r.t. different control ranges for (4.2).

Figure 4.4: Feedback region $R_1$ for (4.2).
feedback controlled invariant sets.

Again, from practical point of view and based on the theory developed in last chapters, we would like to shrink these control sets as small as possible and pick up a particularly small one $D$. Then design a (global) piecewise constant feedback controller such that for any initial point $x_0 \in A(D) \setminus D$, the trajectory can be steered to the closure of this control set $D$ and stay inside $\overline{D}$ forever.

In our case, we choose $D = D_3$, where $D_3$ is the control set with respect to the control range $U = [-0.2, 0.2]$. Then first we apply control $u = -1$ to compute the feedback region $R_1$, which is the difference between the $\epsilon$-invariant approximation (for the definition, see Häckl [31]) of the negative orbit $\mathcal{O}_-^1(D)$ of $D$ and $D$, see Figure 4.4.

Next, we apply control $u = 1$ to compute the feedback region $R_2$, which is the difference between the $\epsilon$-invariant approximation of the negative orbit $\mathcal{O}_-^1(D \cup R_1)$ and $D \cup R_1$, see Figure 4.5.

Again, we apply control $u = -1$ to compute the feedback region $R_3$, which is the difference between the $\epsilon$-invariant approximation of the negative orbit $\mathcal{O}_-^1(D \cup R_1 \cup R_2)$ and $D \cup R_1 \cup R_2$, see Figure 4.6.

We can see from the pictures of these three feedback regions, that it is a good approximation of the domain of attraction of the origin and one can design a global piecewise constant feedback controller by assigning the feedback equals the control we applied to get the corresponding feedback region, respectively.
Figure 4.5: Feedback regions $R_1$ and $R_2$ for (4.2).

Figure 4.6: Feedback regions $R_1$, $R_2$ and $R_3$ for (4.2).
Consider the tunnel diode circuit shown in Figure 4.7 (a), where the tunnel diode is characterized by $i_R = h(v_R)$, as shown in Figure 4.7 (b) (see Chua, Desoer and Kuh [17], Khalil [38]).

The energy-storing elements in this circuit are the capacitor $C$ and the inductor $L$. Assuming they are linear and time-invariant, we can model them by the equations

\[
\frac{di}{dt} = C \frac{dv}{dt},
\]

\[
\frac{dv}{dt} = \frac{1}{L} \frac{di}{dt},
\]

where $i$ and $v$ are the current through and the voltage across an element, with the subscript specifying the element. To write a state-space model for the system, let us take $x_1 = v_R$ and $x_2 = i_R$ as the state variables. Measuring time in nanoseconds, the currents $x_2, h(x_1)$ in mA, voltages $x_1, u$ in V, and applying Kirchhoff’s current law

Figure 4.7: (a) Tunnel diode circuit. (b) Tunnel diode $v_R - i_R$ characteristic and equilibrium points.

Tunnel Diode Circuit
and Kirchhoff’s voltage law, we can write the state-space model for the circuit as

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
\frac{1}{C}[h(x_1) + x_2] \\
\frac{1}{L}[-x_1 - Rx_2]
\end{pmatrix} + u(t) \begin{pmatrix}
0 \\
\frac{1}{L}
\end{pmatrix},
\]

(4.3)

Here \( h(\cdot) \) is given by

\[
h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5,
\]

and the circuit parameters are \( R = 1.5 \times 10^3 \Omega, C = 2 \times 10^{-12} F, L = 5 \times 10^{-6} H \) and \( u(t) \in U = [1.0 V, 1.4 V] \).

For the parameters above, the fixed points of the systems with constant control \( u(t) \equiv u \in U \) can be determined by setting \( \dot{x}_1 = \dot{x}_2 = 0 \) and solving for \( x_1 \) and \( x_2 \). We get that the fixed points correspond to the roots of the equation

\[
h(x_1) = \frac{1}{1.5}(u - x_1).
\]

(4.4)

Figure 4.7 (b) shows graphically that for certain value of \( u \) this equation has three isolated roots which correspond to three isolated fixed points of the system. The three fixed points are denoted by \( Q_1, Q_2 \) and \( Q_3 \), respectively.

Notice that from (4.4) and Figure 4.7 (b), we can see that the number of fixed points might change if values of \( u \not\in U = [1.0 V, 1.4 V] \). Furthermore, there is no bifurcation occurs when \( u \in U = [1.0 V, 1.4 V] \).

The phase portrait of the system (4.3) for \( u = 1.2 V \), is shown in Figure 4.8.

The fixed points \( Q_1 \) and \( Q_3 \) in Figure 4.8 are said to be asymptotically stable because all trajectories originating from points in a small neighborhood of \( Q_1 \) or \( Q_3 \) tend to \( Q_1 \) or \( Q_3 \) as \( t \to \infty \). In contrast, the fixed point \( Q_2 \) is said to be unstable.
Figure 4.8: Phase portrait of the tunnel diode circuit for $u = 1.2V$.

Figure 4.9: Domain of attraction of $Q_2$ for (4.3).
because there exist points arbitrarily close to $Q_2$ whose trajectories diverge from $Q_2$ as $t \to \infty$.

Incidently, this tunnel diode circuit is referred to as a bistable circuit, because it has two steady-state operating points. It has been used as a computer memory, where the fixed point $Q_1$ is associated with the binary state "0" and the fixed point $Q_3$ is associated with the binary state "1".

In terms of the control system, a point is multistable if the trajectories starting from this point exhibit different limit behavior with respect to different controls. We notice that the domain of attraction of $Q_2$ (see Figure 4.9) is the bistable region of system (4.3).

To continue our analysis, we first compute the corresponding Lie derivatives of $X_0$ and $X_1$, with these vector fields defined as in (4.3):

$$ad_{X_0}^{0}X_1 = X_1 = [0, \frac{1}{L}]^T,$$

$$ad_{X_0}^{1}X_1 = [X_0, ad_{X_0}^{0}X_1] = \left[\frac{-1}{LC}, \frac{R}{L}\right]^T.$$

Particularly, for the choice of the parameters above, one sees easily that the vectorfields $X_1(x)$ and $ad_{X_0}^{1}X_1(x)$ span the tangent space $T_x\mathbb{R}^2$ for all $x \in \mathbb{R}^2$. Hence we can apply the theory we developed in the previous chapters.

Here it is of interest to analyze feedback stabilizing the system around $Q_2$. In other words, we would like to choose a relatively small control set $D$ containing $Q_2$ and design a global piecewise constant feedback controller such that for any initial point $x_0$ in any compact subset of $\mathcal{A}(D) \setminus \overline{D}$, the trajectory can be steered into this feedback controlled invariant set $\overline{D}$ in finite time and stay inside $\overline{D}$ forever. Actually, from Theorem 3.30, one can verify that $\mathcal{G} = [0, \frac{1}{L}]^T$ and it is easy to find the function $\hat{\mathcal{G}} = [0, L]$ which can guarantee the existence of a continuous feedback in $\overline{D}$ and make
$\bar{D}$ be a feedback controlled invariant set.

Here we choose $D = D_2$, which is the control set with respect to the control range $U = [1.18V, 1.22V]$, see Figure 4.10.

With a similar procedure to Section 4, we first apply control $u = 1.0V$ to compute the feedback region $R_1$, which is the difference between the $\epsilon$-invariant approximation of the negative orbit $O_{-\epsilon}(D)$ of $D$ and $D$, see Figure 4.11.

Next, we apply control $u = 1.4V$ to compute the feedback region $R_2$, which is the difference between the $\epsilon$-invariant approximation of the negative orbit $O_{-\epsilon}(D \cup R_1)$ and $D \cup R_1$, see Figure 4.12.

Again, we apply control $u = 1.0V$ to compute the feedback region $R_3$, which is the difference between the $\epsilon$-invariant approximation of the negative orbit $O_{-\epsilon}(D \cup R_1 \cup R_2)$ and $D \cup R_1 \cup R_2$, see Figure 4.13.

We can see from the pictures of these three feedback regions, it is a good approximation of the domain of attraction of fixed point $Q_2$ and one can design a global piecewise constant feedback controller by assigning the feedback equals the control we applied to get the corresponding feedback region, respectively.
Control Sets Around Q2 w.r.t. Different Control Ranges

\[ U_1 = [1.17, 1.23] \]
\[ U_2 = [1.18, 1.22] \]

Figure 4.10: The control sets w.r.t. different control ranges for (4.3).

Feedback Region \( R_1 \) Under \( U = [1.0, 1.4] \)

Figure 4.11: Feedback region \( R_1 \) for (4.3).
Figure 4.12: Feedback regions $R_1$ and $R_2$ for (4.3).

Figure 4.13: Feedback regions $R_1$, $R_2$, and $R_3$ for (4.3).
A Chemical Reactor Model

The model of a well-stirred chemical reactor can be described by the equations

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
-x_1 + B\alpha(1 - x_2)e^{x_1} \\
-x_2 + \alpha(1 - x_2)e^{x_1}
\end{pmatrix} + u(t) \begin{pmatrix}
x_e - x_1 \\
o
\end{pmatrix}, \quad (4.5)
\]

This is a simplified model, where \(x_1\) is the temperature, \(x_2\) is the product concentration, and \(\alpha, B, x_e\) are positive technical constants. The control input \(u\) is the heat transfer coefficient, and the state space is \(M = (0, \infty) \times (0, 1)\).

Here, our analysis followed from Colonius and Kliemann [18, 19]. For the numerical results we have chosen \(x_e = 1.0, \alpha = 0.05, B = 10.0\), and taken \(u(t) \in U = [0.9, 1.0] \subset \mathbb{R}\).

For the parameters above, the Lie algebra rank condition is satisfied (see [18]) and the equation with constant control \(u(t) \equiv u \in U\) has three fixed points in \(M\), namely

\[
Q_1(u) = \begin{pmatrix}
\alpha \\
\frac{0.05e^\alpha}{1 + 0.05e^\alpha}
\end{pmatrix}: \text{ stable},
\]

\[
Q_2(u) = \begin{pmatrix}
\beta \\
\frac{0.05e^\beta}{1 + 0.05e^\beta}
\end{pmatrix}: \text{ hyperbolic, i.e. the linearization about } Q_2 \text{ has a positive and a negative eigenvalue},
\]

\[
Q_3(u) = \begin{pmatrix}
\gamma \\
\frac{0.05e^\gamma}{1 + 0.05e^\gamma}
\end{pmatrix}: \text{ stable}.
\]

Here \(\alpha < \beta < \gamma\) are the zeros of the transcendental equation

\[-x - u(x - 1) + \frac{0.5e^x}{1 + 0.05e^x} = 0.\]
Figure 4.14 shows the phase portrait of (4.5) for the parameter values chosen above. There are two stable equilibrium points $Q_1$ and $Q_3$ and one saddle point $Q_2$.

The interesting feature of this system is that the fixed point $Q_2$ is unstable and hence it cannot be used for a technical realization of the system without modification. However, if one can embed $Q_2$ into the interior of a control set $D$, then applying the theory from the previous chapters, we can steer the system to $D$ from the entire domain of attraction, and stabilize the system there.

Figure 4.15 shows the domain of attraction of $Q_2$ if we allow the control function $u(t)$ to have values in $U := [0.9, 1.0]$.

Figure 4.16 shows some controlled invariant sets which are the control sets containing $Q_2$ with respect to different control ranges.

From practical point of view, we choose a relative smaller control set $D = D_3$ as shown in Figure 4.16 and would like to design a feedback law $u = F(x)$ on $A(Q_2) \setminus D$ such that for any initial point $x_0 \in A(Q_2) \setminus D$, the trajectory can be steered to this controlled invariant set $D$ and stay inside $D$ forever.

With a similar procedure to the previous two sections, we first apply control $u = 0.9$ to compute the feedback region $R_1$, see Figure 4.17. Then we apply control $u = 1.0$ to compute the feedback region $R_2$, as depicted in Figure 4.18. And again, we apply control $u = 0.9$ to compute the feedback region $R_3$, see Figure 4.19.

We can see from the pictures of these three feedback regions, it is a good approximation of the domain of attraction of the fixed point $Q_2$ and one can design a global piecewise constant feedback controller by assigning the value of the feedback control equals the exact bang-bang control we applied to get the corresponding feedback region, respectively.
Figure 4.14: Phase portrait of (4.5) for $u = 0.95$.

Figure 4.15: Domain of attraction of $Q_2$ of (4.5) for $U = [0.9, 1.0]$. 
Control sets w.r.t. different control ranges

\[ U_1 = [0.9, 1.0] \]
\[ U_2 = [0.92, 0.98] \]
\[ U_3 = [0.94, 0.96] \]

Figure 4.16: Control sets of (4.5) w.r.t. different control ranges.

Feedback region \( R_1 \) for \( U = [0.9, 1.0] \)

Figure 4.17: Feedback region \( R_1 \) for (4.5).
Figure 4.18: Feedback regions $R_1$ and $R_2$ for (4.5).

Figure 4.19: Feedback regions $R_1$, $R_2$ and $R_3$ for (4.5).
Bacterial Respiration Model

In this section, we are going to consider the stabilization problem around periodic orbits. We consider the following two-dimensional system

\[
\begin{align*}
\dot{x} &= b(t) - x - \frac{xy}{1+qz^2}, \\
\dot{y} &= a - \frac{xy}{1+qz^2}.
\end{align*}
\] (4.6)

Here \(a\) and \(q\) are positive constants, and \(b\) is the critical parameter, depending on the concentration rates in the underlying chemical reaction scheme.

This system has been proposed by Degn and Harrison in [24] as a model for the existence of a maximal oxygen consumption rate at low oxygen concentration in Klebsiella Aerogen cultures. Fairén and Velarde [25] analyzed this model with respect to its limit cycle and bifurcation behavior and they showed that for \(b = 20\), \(q = 0.5\) and \(a = 11\) there are an unstable limit cycle \(L_1\), a stable fixed point \(P_0\) and a stable limit cycle \(L_2\).

The phase portrait of system (4.6) for \(b = 20\), \(q = 0.5\) and \(a = 11\) is shown in Figure 4.20.

In our analysis, we treat the case where the parameter \(b\) is a control function. Using the values \(a = 11.0\), \(q = 0.5\), \(b(t) \in U = [19.97, 20.03]\), we get a control affine system where local accessibility holds. With these values, Håckl [31] showed that there are a variant control set \(D\) containing the unstable limit cycle \(L_1\) in its interior, an invariant control set \(C_1\) containing the stable fixed point \(P_0\) in its interior, and another invariant control set \(C_2\) that contains the outer stable limit cycle \(L_2\). We notice that the order of the limit sets is given by \(L_1 \prec P_0\) and \(L_1 \prec L_2\) and as we know it will be preserved by the corresponding control sets containing the limit sets.
The phase portrait for $u = 20$

Figure 4.20: The phase portrait of system (4.6).

Control sets for the bacterial respiration model

Figure 4.21: The control sets of system (4.6).
in their interior, respectively. The control sets of the associated control system (4.6) is depicted in Figure 4.21.

In what follows our main interest is to design global feedback controllers and practically stabilize the system (4.6) around those limit sets.

**Around the Stable Fixed Point \( P_0 \)**

We choose the control set \( C_1 \), as a reasonably small feedback controlled invariant set containing the fixed point \( P_0 \). The domain of attraction \( \mathcal{A}(P_0) \), for \( U = [19.97,20.03] \), of \( P_0 \) and this particular feedback controlled invariant set \( C_1 \) are shown in Figure 4.22. We notice that since \( D \prec C_1 \), one has \( \mathcal{A}(P_0) \) contains the whole control set \( D \), which is a bistable region.

Applying the bang-bang controls of \( U = [19.97,20.03] \), one can design a global feedback controller on \( \mathcal{A}(C_1) \setminus C_1 \) and stabilize the system (4.6) around \( P_0 \). Figure 4.23 shows the feedback regions \( R_1 \) through \( R_5 \) of the feedback controlled invariant set \( C_1 \).

**Around the Stable Limit Cycle \( L_2 \)**

Similar to the previous subsection, We choose the control set \( C_2 \), as a reasonably small feedback controlled invariant set containing the stable limit cycle \( L_2 \). The domain of attraction \( \mathcal{A}(L_2) \), for \( U = [19.97,20.03] \), of \( L_2 \) and this particular feedback controlled invariant set \( C_2 \) are shown in Figure 4.24. And that since we have the order \( D \prec C_2 \), \( \mathcal{A}(L_2) \) contains the whole control set \( D \), which is a bistable region.

Applying the bang-bang controls of \( U = [19.97,20.03] \), one can design a global feedback controller on \( \mathcal{A}(C_2) \setminus C_2 \) and stabilize the system (4.6) around \( L_2 \). Figure
The Invariant control set $C_1$ and its domain of attraction $A(C_1)$ for (4.6).

Figure 4.22: $C_1$ and its domain of attraction $A(C_1)$ for (4.6).

Feedback regions $R_1$ through $R_5$ for the invariant control set $C_1$.

Figure 4.23: Feedback regions $R_1$ through $R_5$ of $C_1$ of system (4.6).
The invariant control set $C_2$ and its domain of attraction $A(C_2)$ for (4.6).

Figure 4.24: $C_2$ and its domain of attraction $A(C_2)$ for (4.6).

Figure 4.25: Feedback regions $R_1$, $R_2$ and $R_3$ of $C_2$ of system (4.6).
4.25 shows the feedback regions $R_1$ through $R_3$ of the feedback controlled invariant set $C_2$. From this we can see that the union of $R_1$ and $R_2$ is a good approximation of $\mathcal{A}(C_2) \setminus C_2$.

**Around the Unstable Limit Cycle $L_1$**

The most challenging problem is the stabilization around the unstable limit cycle $L_1$. We notice that the control set $D$ containing $L_1$ is both a variant control set and a bistable region in the sense that not only $D$ is contained in $\mathcal{A}(P_0)$ but also in $\mathcal{A}(L_2)$. Moreover, we know that the domain of attraction $\mathcal{A}(D)$ of the control set $D$ equals $D$ itself if we only apply controls $u \in U = [19.97, 20.03]$.

First, we would like to find a smaller feedback controlled invariant set rather than $\overline{D}$ and a good candidate for our purpose is the closure of control set $D_1$ with respect to the control range $U = [19.97, 20.03]$. Next, in order to enlarge the domain of attraction of the limit cycle $L_1$ we would apply controls $u \in U_1 = [19.95, 20.05]$ instead of $U = [19.97, 20.03]$. The domain of attraction $\mathcal{A}(L_1)$, for $U_1 = [19.95, 20.05]$, of $L_1$ and this particularly small feedback controlled invariant set $D_1$ are shown in Figure 4.26.

Applying the bang-bang controls of $U = [19.95, 20.05]$, one can design a global feedback controller on $\mathcal{A}(D_1) \setminus \overline{D_1}$ and stabilize the system (4.6) around $L_1$. Figure 4.27 shows the feedback regions $R_1$ through $R_5$ of the feedback controlled invariant set $D_1$. Unfortunately, there is significant chattering from this feedback design. Especially, we can see from Figure 4.27, the union of the feedback regions $R_1$ through $R_5$ is a good approximation of the outer part of $\mathcal{A}(D_1) \setminus \overline{D_1}$ but it is still not a good approximation of the inner part of $\mathcal{A}(D_1) \setminus \overline{D_1}$. 
A small variant control set and its domain of attraction

Figure 4.26: The control set $D_1$ of (4.6) and its domain of attraction.

Feedback regions $R_1 - R_5$ for the variant control set $D_1$

Figure 4.27: Feedback regions $R_1$ through $R_5$ of $D_1$ of system (4.6).
Consider a two-parameter differential system

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= \lambda_1 + \lambda_2 x + x^2 + xy.
\end{align*}
\] (4.7)

Mathematically, this is the simplest example for a codimension two bifurcation where \(\lambda_1\) and \(\lambda_2\) are the bifurcation parameters. A qualitative study of (4.7), see Guckenheimer and Holmes [29], Carr [15], Kopell and Howard [38], Perko [48], shows that the singularity of this system arises naturally as the common endpoint (or start point) of a Hopf-bifurcation curve and a homoclinic bifurcation curve (separatrix loop). This system occurs when modeling the motion of a thin panel in a flow, for shock waves, population dynamics, or solar gravity.

The asymptotic behavior of the perturbed Takens-Bogdanov oscillator (4.7) has been studied by Jankovsky and Plecháč in [34]. It is also possible to study the asymptotic behavior of the perturbed Takens-Bogdanov oscillator by looking at the control sets of the associated control system, see Häckl and Schneider [32].

In [32], the authors consider the perturbed Takens-Bogdanov system associated with the control system

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= \lambda_1 + \lambda_2 x + x^2 + xy + u(t).
\end{align*}
\] (4.8)

where \(u(\cdot) \in \mathcal{U}^\rho := \{u : \mathbb{R} \to [-\rho, \rho] \mid u \text{ measurable}\}\). There they analyze how the behavior of the limit sets for the uncontrolled system (4.7) affects the behavior of the control sets for the controlled system (4.8) as the parameters are varied. We notice that this system is regular, i.e., it satisfies the Lie algebra rank condition. For more discussion of system (4.8) we refer the reader to Häckl and Schneider [32].
To order to avoid the bifurcation phenomena, we particularly choose \((\lambda_1, \lambda_2) = (-0.3, -1)\) with two different control ranges \(\rho = 0.04, 0.09\). The phase portrait of the uncontrolled system (4.7) for \((\lambda_1, \lambda_2) = (-0.3, -1)\), is shown in Figure 4.28.

According to [32], for \((\lambda_1, \lambda_2) = (-0.3, -1)\) and \(\rho = 0.04, 0.09\), the system (4.8) has two control sets \(D^\rho\) and \(C^\rho\) with nonvoid interior where \(D^\rho\) is a variant control set containing the homoclinic orbit of the uncontrolled system (4.7), and \(C^\rho\) is an invariant control set containing the stable focus inside the homoclinic orbit of the uncontrolled system (4.7), (see Figure 4.29 and Figure 4.30). We notice that if the control range becomes larger, the influence of the control exceed the underlying dynamics and the control sets merge.

In what follows, our main interest is to design global feedback controllers and practically stabilize the system (4.8) around the limit sets of system (4.7).

**Around the Fixed Point**

We choose the control set \(C^{0.04}\) as a reasonably small feedback controlled invariant set \(C\) containing the fixed point of the system (4.7). The domain of attraction, for \(U = [-0.09, 0.09]\), of this fixed point and this particular feedback controlled invariant set \(C\) are shown in Figure 4.31.

First we apply \(u = -0.09\) to get the feedback region \(R_1\) then we apply \(u = 0.09\) to get the feedback region \(R_2\) and we can see that actually combining with these two feedback regions, it is a good approximation of the domain of attraction of this fixed point. See Figure 4.32 and Figure 4.33.
Figure 4.28: The phase portrait of the uncontrolled system (4.7).

Figure 4.29: The control sets of (4.7) for $U = [-0.04, 0.04]$. 
The control sets for $U = (-0.09, 0.09)$. 

Figure 4.30: The control sets of (4.7) for $U = [-0.09, 0.09]$. 

The invariant control set $C$ and its domain of attraction. 

Figure 4.31: The control set $C$ of (4.7) and its domain of attraction.
Figure 4.32: The feedback region $R_1$ of $C$ for (4.7).

Figure 4.33: The feedback regions $R_1$ and $R_2$ of $C$ for (4.7).
Around the Homoclinic Orbit

Similarly, We choose the control set \( D^{0.04} \) as a reasonably small control set \( D \) containing the homoclinic orbit of the system (4.7). The domain of attraction, for \( U = [-0.09, 0.09] \), of this homoclinic orbit and this particular control set \( D \) are shown in Figure 4.34.

First we apply \( u = -0.09 \) to get the feedback region \( R_1 \) then we apply \( u = 0.09 \) to get the feedback region \( R_2 \) and again we apply \( u = -0.09 \) to get the feedback region \( R_3 \). We can see that actually combining with these three feedback regions, it is a good approximation of the domain of attraction of this homoclinic orbit. See Figure 4.35, Figure 4.36 and Figure 4.37.
Figure 4.34: The control set $D$ of (4.7) and its domain of attraction.

Figure 4.35: The feedback region $R_1$ of $D$ for (4.7).
Figure 4.36: The feedback regions $R_1$ and $R_2$ of $D$ for (4.7).

Figure 4.37: The feedback regions $R_1$, $R_2$ and $R_3$ of $D$ for (4.7).
Three-Dimensional Linear System

In this section we briefly discuss the numerical simulation of a three-dimensional system. For the convenience of our discussion (although this is not necessary from the numerical aspect), we concentrate on a simple linear system with two-dimensional control:

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{pmatrix} =
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} +
\begin{pmatrix}
u_1(t) \\
u_2(t)
\end{pmatrix},
\]

(4.9)

where \( u(t) = (u_1(t), u_2(t)) \in U := [-2, 2]^2 \).

For this simple example, it is easy to see that the origin \( 0 = (0, 0, 0)^T \) is an unstable fixed point of the uncontrolled system

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{pmatrix} =
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}.
\]

(4.10)

The main goal here is similar to those of the two-dimensional examples; that is, first to find a relatively smaller control set \( D \) containing the origin and then to design a global feedback controller to stabilize the system (4.9) around this particular controlled invariant set \( \overline{D} \).

It is easy to write down the exact solution for any initial value \( x^0 = (x_1^0, x_2^0, x_3^0) \in \mathbb{R}^3 \) and any \( u(\cdot) \in U \) since there is no coupling involved:

\[
\begin{pmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{pmatrix} =
\begin{pmatrix}
x_1^0 e^{-t} + \int_0^t u_1(s) e^{s-t} \, ds \\
x_2^0 e^t + \int_0^t u_1(s) e^{t-s} \, ds \\
x_3^0 e^{-t} + \int_0^t u_2(s) e^{s-t} \, ds
\end{pmatrix}.
\]

(4.11)
Hence for the origin $0 = (0,0,0)^T$ and a small control range $U_1 := [-1,1]^2$, one can write down the positive orbit $\mathcal{O}^+(0)$ and the negative orbit $\mathcal{O}^-(0)$ explicitly:

$$\mathcal{O}^+(0) = (-1,1) \times [-2,2] \times (-1,1),$$
$$\mathcal{O}^-(0) = [-2,2] \times (-1,1) \times [-2,2].$$

Since the intersection is not empty, from Proposition 3.10, there is a variant control set $D = \mathcal{O}^+(0) \cap \mathcal{O}^-(0) = [-1,1] \times (-1,1) \times [-1,1]$ containing the origin $0$. The computational control set $D$ is shown in Figure 4.38.

In order to design a global feedback controller, we apply the bang-bang controls...
of $U = [-2, 2]^2$ to compute the feedback regions in the following order:

$\omega_{4n+1} = (-2, 0), \omega_{4n+2} = (2, 0), \omega_{4n+3} = (0, -2) \text{ and } \omega_{4n+4} = (0, 2), n = 0, 1, \ldots$

With a constrained state space $[-3, 3]^3$, we computed several feedback regions, but unfortunately, due to the following technical problems:

1. The numerical expense is too high. For three-dimensional systems, the data of some feedback regions may be around 1MB – 100MB. It is difficult to generate the postscript files and print them out, especially the color printer we used has only 1MB memory.

2. The picture of all our examples were produced with MATLAB, which is not an advanced graphic package for three-dimensional systems. Even if we put two regions together in one file, the output is not well presented. Obviously, we will need a better visualization software for our future research.

Figure 4.39 and Figure 4.40 show the result of the computation of the feedback region $R_1$ and the control set $D$. While Figure 4.41 and Figure 4.42 show the feedback region $R_2$ and the corresponding $R_1$ and $D$. Figure 4.43 and Figure 4.44 show the feedback region $R_3$ and $R_4$, respectively.
The feedback region $R_1$ and the control set $D$ for (4.9).

Figure 4.39: The feedback region $R_1$ of $D$ for (4.9).

The feedback region $R_1$ and the control set $D$

Figure 4.40: The feedback region $R_1$ and the control set $D$ for (4.9).
Figure 4.41: The feedback region $R_2$ of $D$ for (4.9).

Figure 4.42: The feedback regions $R_1, R_2$ and the control set $D$ for (4.9).
Figure 4.43: The feedback regions $R_3$ for (4.9).
The feedback region $R_4$

Figure 4.44: The feedback regions $R_4$ for (4.9).
CHAPTER 5. CONCLUSIONS AND FURTHER RESEARCH

This dissertation is primarily devoted to developing theoretical methods as well as the numerical methods of practical feedback control and stabilization.

For the theoretical development, we have obtained results for general multidimensional nonlinear systems with constrained multi-inputs, which are useful in designing global practical feedback controllers. We have successfully applied these results to some known nonlinear control systems and practically stabilize the unstable limit sets of their associated dynamical systems. These results may be useful for solving problems in various disciplines, such as industrial and technological applications.

The fundamental reason for using feedback is to accomplish performance objectives in the presence of uncertainty. Following with this direction, we consider the nonlinear system of the form $\dot{x} = f(x, w, u)$, where $w$ is a (time-varying) perturbation with values in $W \subset \mathbb{R}^p$ and $u$ is a (feedback or open loop) control with values in $U \subset \mathbb{R}^m$. There are important and challenging questions related with controllability, stabilization, and the existence and characterization of invariant sets in the state space of the system under all possible perturbations with values in $W$. For the one-dimensional case, we have obtained complete results, see Lai and Lin [40]. One direction of our further research is to investigate the higher dimensional case based
on the theory we developed in this dissertation.

On the other hand, as for the numerical aspects, we have developed a software which can compute the feedback controlled invariant sets and the feedback regions of two-dimensional and three-dimensional nonlinear systems. For the extension to higher dimensional case, the key algorithms are related to the computation of the convex hull of a given set of points \( \{x_1, \ldots, x_l\} \) and the distance of given point \( y \) to this convex hull, which is a quadratic programming problem, see H"ackl [31] for precise description. The present author and Dr. H"ackl would have a joint work on this problem in the near future.


