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Leslie Hogben

Iowa State University, hogben@iastate.edu

Naomi Shaked-Monderer

Max Stern Yezreel Valley College

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A simple graph G is an SPN graph if every copositive matrix having graph G is the sum of a positive semidefinite and nonnegative matrix. SPN graphs were introduced in [Shaked-Monderer, SPN graphs: When copositive = SPN, *Linear Algebra Appl.*, 509(15):82-113, 2016], where it was conjectured that the complete subdivision graph of K_4 is an SPN graph. We disprove this conjecture, which in conjunction with results in the Shaked-Monderer paper show that a subdivision of K_4 is a SPN graph if and only if at most one edge is subdivided. We conjecture that a graph is an SPN graph if and only if it does not have an F_5 minor, where F_5 is the fan on five vertices. To establish that the complete subdivision graph of K_4 is not an SPN graph, we introduce rank-1 CP completions and characterize graphs that are rank-1 CP completable.

Keywords

Copositive, SPN, rank-1 CP completion, F_5 , fan, matrix, graph

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SPN GRAPHS AND RANK-1 CP-COMPLETABLE GRAPHS*

LESLIE HOGBEN[†] AND NAOMI SHAKED-MONDERER[‡]

Abstract. A simple graph G is an SPN graph if every copositive matrix having graph G is the sum of a positive semidefinite and nonnegative matrix. SPN graphs were introduced in [Shaked-Monderer, SPN graphs: When copositive = SPN, *Linear Algebra Appl.*, 509(15):82–113, 2016], where it was conjectured that the complete subdivision graph of K_4 is an SPN graph. We disprove this conjecture, which in conjunction with results in the Shaked-Monderer paper show that a subdivision of K_4 is a SPN graph if and only if at most one edge is subdivided. We conjecture that a graph is an SPN graph if and only if it does not have an F_5 minor, where F_5 is the fan on five vertices. To establish that the complete subdivision graph of K_4 is not an SPN graph, we introduce rank-1 CP completions and characterize graphs that are rank-1 CP completable.

Keywords: Copositive, SPN, rank-1 CP completion, F_5 , fan, matrix, graph

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1. Introduction. A real symmetric matrix A is *copositive* if $\mathbf{x}^T A \mathbf{x} \geq 0$ for every nonnegative vector \mathbf{x} . A matrix A is *SPN* if it is a sum of a positive semidefinite matrix and a symmetric nonnegative one. Every SPN matrix is copositive but not conversely. Determining whether a real symmetric matrix is copositive is hard (in complexity terms this problem is co-NP complete), while determining whether a matrix is SPN is easier (it can be done by a semidefinite program). Patterns of nonzero entries can play a role. For a symmetric matrix $A \in \mathbb{R}^{n \times n}$, the *graph of A* is $G(A) = ([n], E)$ where $ij \in E$ if and only if $i \neq j$ and a_{ij} is non-zero ($[n]$ denotes the set $\{1, \dots, n\}$). It was suggested by Shaked-Monderer [4] to characterize all graphs G with the property that a symmetric matrix A with $G(A) = G$ is copositive if and only if it is SPN. Such a graph is called an *SPN graph*; a graph that does not have this property is called *non-SPN*. In [4] the study of SPN graphs was initiated and numerous results that specific graphs were or were not SPN were established (see also the corrigendum [5]). In this paper we conjecture a characterization of SPN graphs, identify what needs to be done to establish this characterization, and make progress towards the proof of the following conjecture.

CONJECTURE 1.1. *A graph G is an SPN graph if and only if G does not have the fan graph F_5 shown in Figure 1.1 as a minor.*

Discussion and additional related conjectures are in Section 5; definitions of graph theory terms are given in Section 2. A main result established is the following.

THEOREM 1.2. *The complete subdivision \bar{K}_4 is a non-SPN graph.*

Theorem 1.2 shows that [4, Conjecture 9.13] is false. The next result follows from [4, Lemma 7.2] for SPN graphs, and [4, Lemma 7.2] and Theorem 1.2 for non-SPN graphs.

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[†]Department of Mathematics, Iowa State University, Ames, IA 50011, USA and American Institute of Mathematics, 600 E. Brokaw Road, San Jose, CA 95112, USA (hogben@aimath.org).

[‡]The Max Stern Yezreel Valley College, Yezreel Valley 1930600, Israel (nomi@technion.ac.il). The work of this author was supported by grant no. 2219/15 by ISF-NSFC joint scientific research program.

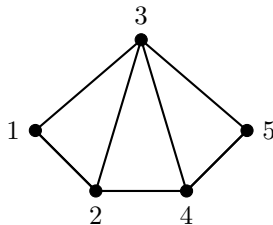


FIGURE 1.1. The fan F_5

32 COROLLARY 1.3. A subdivision of K_4 is an SPN graph if and only if at most one of the original edges
 33 was subdivided.

34 Theorem 1.2 is established in Section 4 by introducing the theory of rank-1 CP completions in Section 3,
 35 where we characterize graphs G that have the property that every partial rank-1 CP matrix A with specified
 36 entries corresponding to the edges of G can be completed to a rank-1 CP matrix.

37 **2. Preliminaries.** A (simple) graph G is a pair (V, E) , where $V = V(G)$ is the set of vertices and
 38 $E = E(G)$ is the set of edges, i.e., two element subsets of V . The edge $\{i, j\}$ is often denoted by ij . A
 39 graph $H = (V(H), E(H))$ is a subgraph of $G = (V(G), E(G))$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a
 40 graph G and $e = uv \in E(G)$, subdividing edge e creates a new graph G_e from G by adding a new vertex w
 41 adjacent to u and v and deleting e , so $G_e = (V(G) \cup \{w\}, (E(G) \setminus \{e\}) \cup \{uw, vw\})$. Any graph obtained
 42 by successively subdividing edges of G is a subdivision of G . The graph obtained from G by subdividing
 43 each edge once is denoted by \tilde{G} . A vertex v of a graph G is a cut vertex of G if deleting v disconnects G .
 44 A block of G is a subgraph that has no cut vertex and is maximal with respect to this property. A graph
 45 is 2-connected if it has at least three vertices and does not have a cut vertex. The contraction of an edge
 46 $e = uv$ of G is obtained by identifying the vertices u and v , deleting any loops that arise in this process, and
 47 replacing any multiple edges by a single edge. A graph H is a minor of a graph G if H can be obtained from
 48 G by a series of the following operations (in any order): delete an isolated vertex; delete an edge; contract
 49 an edge.

50 A signed graph is a graph of the form $\mathcal{G} = (V, E, \Sigma)$ where V is the set of vertices, E is the set of
 51 edges, and $\Sigma : E \rightarrow \{+, -\}$ signs the edges. The signed graph of a symmetric matrix A , denoted by $\mathcal{G}(A)$,
 52 is obtained from $G(A)$ by assigning the sign of a_{ij} to the edge ij . A signed subgraph \mathcal{H} of a signed graph \mathcal{G}
 53 is a subgraph \mathcal{H} of \mathcal{G} with $\Sigma(\mathcal{H})$ being the restriction of $\Sigma(\mathcal{G})$ to $E(\mathcal{H})$. In signed graph drawings, a dashed
 54 line denotes a negative edge and a solid line a positive edge. For a graph $G = (V, E)$ (or a signed graph
 55 $\mathcal{G} = (V, E, \Sigma)$) and $S \subseteq V$, $G[S] = (S, E[S])$ (or $\mathcal{G}[S] = (S, E[S], \Sigma[S])$) is the induced subgraph of G (or \mathcal{G})
 56 on the vertices S , where $E[S] = \{uv \in E : u, v \in S\}$ (and $\Sigma[S]$ is the restriction of Σ to $E[S]$). For a signed
 57 graph \mathcal{G} and such S , $G_-[S]$ denotes the graph on vertices S whose edges are the negative edges of $\mathcal{G}[S]$.

58 The space of $n \times n$ real symmetric matrices is denoted by \mathcal{S}_n . The set of all copositive matrices in \mathcal{S}_n
 59 is denoted by \mathcal{COP}_n . This is a closed convex cone in \mathcal{S}_n , and it contains the closed convex cone \mathcal{SPN}_n
 60 consisting of the SPN matrices of order n . The inclusion $\mathcal{SPN}_n \subseteq \mathcal{COP}_n$ is an equality if and only if $n \leq 4$.

61 The cone \mathcal{COP}_5 , as the first to differ from its subcone \mathcal{SPN}_5 , has been studied in [1, 3]. As in [1], we define

$$62 \quad S(\boldsymbol{\theta}) = \begin{pmatrix} 1 & -\cos(\theta_1) & \cos(\theta_1 + \theta_2) & \cos(\theta_4 + \theta_5) & -\cos(\theta_5) \\ -\cos(\theta_1) & 1 & -\cos(\theta_2) & \cos(\theta_2 + \theta_3) & \cos(\theta_1 + \theta_5) \\ \cos(\theta_1 + \theta_2) & -\cos(\theta_2) & 1 & -\cos(\theta_3) & \cos(\theta_3 + \theta_4) \\ \cos(\theta_4 + \theta_5) & \cos(\theta_2 + \theta_3) & -\cos(\theta_3) & 1 & -\cos(\theta_4) \\ -\cos(\theta_5) & \cos(\theta_1 + \theta_5) & \cos(\theta_3 + \theta_4) & -\cos(\theta_4) & 1 \end{pmatrix}$$

63 for $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$. For $\boldsymbol{\theta} > \mathbf{0}$ such that $\sum_{i=1}^5 \theta_i < \pi$ the matrices $S(\boldsymbol{\theta})$ are (extremal) *Hildebrand*
64 *matrices* [3], $S(\mathbf{0})$ is the (extremal) *Horn matrix*, and for other $\boldsymbol{\theta} \geq \mathbf{0}$ with $\sum_{i=1}^5 \theta_i < \pi$ the matrix $S(\boldsymbol{\theta})$ is
65 a copositive matrix that is not SPN [1].

66 The submatrix of the $n \times n$ matrix A in rows indexed by $R \subseteq [n]$ and columns indexed by $C \subseteq [n]$ is
67 denoted by $A[R|C]$. When $R = C$, we write $A[R]$ for $A[R|R]$, and $A[R]$ is called a *principal submatrix* of
68 A . We sometimes omit the set brackets of R and C in these notations, writing $A[2]$ instead of $A[\{2\}]$, for
69 example. If $M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ and A is nonsingular, the *Schur complement* of A in M is $M/A = C - B^T A^{-1} B$;
70 a permutation similarity is applied to define the Schur complement $A/A[S]$ for $S \subset [n]$.

71 We will use the following results from the literature.

72 LEMMA 2.1. [4, Lemma 3.3(b)] Let $A = \begin{pmatrix} c & \mathbf{a}^T \\ \mathbf{a} & B \end{pmatrix}$ with $c > 0$ and $\mathbf{a} \leq \mathbf{0}$. Then $A \in \mathcal{SPN}_n$ (respectively,
73 $A \in \mathcal{COP}_n$) if and only if $A/A[1] \in \mathcal{SPN}_{n-1}$ (respectively, $A/A[1] \in \mathcal{COP}_{n-1}$).

74 LEMMA 2.2. [4, Lemma 8.1] Let \mathcal{G} is a signed graph, and let $\widehat{\mathcal{G}}$ be a signed graph obtained by subdividing
75 a negative edge of \mathcal{G} , replacing it by a negative path of length 2. Then \mathcal{G} is SPN if and only if $\widehat{\mathcal{G}}$ is SPN.

76 **3. Rank-1 CP completion.** In Section 4 we will use rank-1 CP matrix completions (Theorem 3.9) in
77 the proof that the complete subdivision of K_4 is not SPN. In this section we define rank-1 CP-completable
78 graphs, establish some of their properties, and characterize rank-1 CP-completable graphs, in addition to
79 proving Theorem 3.9.

80 Recall that a real symmetric matrix A is *completely positive* (CP) if $A = BB^T$, where $B \geq 0$. A rank-1
81 completely positive matrix is of the form $\mathbf{b}\mathbf{b}^T$, where $\mathbf{0} \neq \mathbf{b} \geq \mathbf{0}$.

82 A *partial matrix* is a matrix in which some entries specified and others are not (no entries specified or all
83 entries specified are permitted). We use $A^?$ to denote a partial matrix, and a_{ij} to denote a specified entry of
84 A ; ? can be used to denote an unspecified entry. An $n \times n$ matrix $B = [b_{ij}]$ is a *completion* of a partial matrix
85 $A^?$ if $b_{ij} = a_{ij}$ for every specified entry a_{ij} of $A^?$. A partial matrix is symmetric if whenever a_{ij} is specified,
86 so is a_{ji} and $a_{ji} = a_{ij}$. For a graph G , a *partial matrix described by G* is a symmetric partial matrix that
87 has the i, j -entry specified if and only if ij is an edge of G . We note that this is a different association of a
88 graph and (partial) matrices than that used throughout the rest of the paper, where matrices are complete
89 and an nonedge of G is associated with a *zero* off-diagonal entry of the matrix.

90 A symmetric partial matrix is a *partial rank-1 CP matrix* if every specified entry is positive and all
91 diagonal entries are unspecified. A graph G is *rank-1 CP-completable* if every partial rank-1 CP matrix
92 described by G can be completed to a rank-1 CP matrix. A graph G is *uniquely rank-1 CP-completable* if
93 every partial rank-1 CP matrix described by G has a unique rank-1 CP completion.

94 Note that our definition of rank-1 CP-completable is designed for the purpose of showing certain graphs
 95 are non-SPN and is different from the definition of CP-completable in [2], where the diagonal is fully specified
 96 and any fully specified principal submatrix is required to be CP in order for the partial matrix to be considered
 97 a partial CP matrix (in addition to being symmetric and having all specified off-diagonal entries nonnegative).
 98 The differences in the definitions arise naturally from the differences in problems being considered. Without
 99 the rank-1 restriction, every partial CP matrix with unspecified diagonal can be completed to a CP matrix,
 100 whereas we need the freedom to set the diagonal for our application in Theorem 3.9.

101 A graph G is the *vertex sum (at v)* of G_1 and G_2 if $G = G_1 \cup G_2$, $G_1 \cap G_2 = \{v\}$ is a single vertex, and
 102 $G \neq G_i, i = 1, 2$. If G is the vertex sum at v of G_1 and G_2 , then v is a *cut vertex* of G . A cycle is *odd* or
 103 *even* according as its order is odd or even. We start with a few simple observations.

104 REMARK 3.1. Clearly a graph of order one is rank-1 CP-completable (and not uniquely so).
 105 Any partial rank-1 CP matrix described by K_2 has the form $\begin{pmatrix} ? & a \\ a & ? \end{pmatrix}$, $a > 0$, and $A_x = \begin{pmatrix} x & a \\ a & \frac{a^2}{x} \end{pmatrix}$ is a rank-1
 106 CP completion of $A^?$ for any $x > 0$. This form describes all the rank-1 CP completions of K_2 , which is
 107 therefore rank-1 CP-completable but not uniquely so.

108 A disjoint union of rank-1 CP-completable graphs is also rank-1 CP-completable: If $\mathbf{a}\mathbf{a}^T$, $\mathbf{a} \geq \mathbf{0}$, is a
 109 rank-1 CP completion of $A^?$ and $\mathbf{b}\mathbf{b}^T$, $\mathbf{b} \geq \mathbf{0}$, is a rank-1 CP completion of $B^?$, then $\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}^T$ is a
 110 rank-1 CP completion of $A^? \oplus B^?$.

111 Finally, note that if $\mathbf{b}\mathbf{b}^T$ is a rank-1 CP completion of a partial rank-1 matrix described by a graph G ,
 112 and $b_i = 0$, then the vertex i is necessarily an isolated vertex of G , and setting $b_i = x$ for any $x > 0$ will also
 113 yield a rank-1 CP completion of the same matrix. Thus if a graph G is rank-1 CP-completable, any partial
 114 rank-1 CP matrix described by G has a completion $\mathbf{b}\mathbf{b}^T$ with \mathbf{b} a *positive* vector.

115 PROPOSITION 3.2. *If H is a subgraph of a rank-1 CP-completable graph G , then H is rank-1 CP-*
 116 *completable.*

117 *Proof.* Let $A^?$ be a partial rank-1 CP matrix described by H . For every $ij \in E(G) \setminus E(H)$ choose
 118 any positive number for a_{ij} . This creates a partial rank-1 CP matrix $B^?$ described by G . A rank-1 CP
 119 completion B of $B^?$ yields a rank-1 CP-completion of $A^?$ (the principal submatrix of B on H 's vertices). \square

120 LEMMA 3.3. *Suppose that G is the vertex sum of K_2 and H . If H is rank-1 CP-completable (respectively,*
 121 *uniquely rank-1 CP-completable), then G is rank-1 CP-completable (uniquely rank-1 CP-completable).*

122 *Proof.* Without loss of generality assume $V(K_2) = \{1, 2\}$ and $V(H) = \{2, \dots, n\}$. Then a partial rank-1
 123 CP matrix described by G has the form:

$$124 \quad A^? = \begin{pmatrix} ? & a & ? & \cdots & ? \\ a & & & & \\ ? & & & & \\ \vdots & & & B^? & \\ ? & & & & \end{pmatrix},$$

125 where $a > 0$ and $B^?$ is an partial rank-1 CP matrix described by H . Let $B = \mathbf{b}\mathbf{b}^T$ be a completion of $B^?$,

126 where $\mathbf{b}^T = (b_2, \dots, b_n)$ is a positive vector. Then appending $b_1 = \frac{a}{b_2}$ to \mathbf{b} we get that

$$127 \begin{pmatrix} b_1 \\ \mathbf{b} \end{pmatrix} \begin{pmatrix} b_1 \\ \mathbf{b} \end{pmatrix}^T$$

128 is a rank-1 CP completion of $A^?$. If the completion B of $B^?$ is unique, then so is this completion. \square

129 For any partial rank-1 CP matrix A such that A is described by a connected graph G , Algorithm 3.4
 130 below provides a method to determine whether or not A has a rank-1 CP completion and finds all such
 131 completions. A *rooted tree* is a tree with a distinguished vertex, called the *root*. The *parent* of a vertex v
 132 (that is not the root) in a rooted tree is the neighbor of v in the path between v and the root.

133 **ALGORITHM 3.4.** *Given a partial rank-1 CP matrix $A^?$ such that $A^?$ is described by a connected graph*
 134 *G , find all possible rank-1 CP completions of $A^?$ or determine there are none.*

- 135 1. *Initialize:*
 - 136 (a) *Choose a spanning tree T of G and a root, k .*
 - 137 (b) *Denote every unspecified i, j -entry of A by x_{ij} .*
 - 138 (c) *Set the true/false variable $COMPLETION := \text{true}$.*
- 139 2. *Assign a (positive) real variable x to the diagonal entry x_{kk} corresponding to the root: $x_{kk} := x$.*
- 140 3. *Proceeding outwards from the root, successively assign each diagonal entry x_{ii} from its parent x_{jj}*
 141 *and a_{ij} : $x_{ii} := \frac{a_{ij}^2}{x_{jj}}$.*
- 142 4. *Assign every unspecified off diagonal entry x_{ij} from the diagonal entries of the associated 2×2*
 143 *submatrix: $x_{ij} := \sqrt{x_{ii}x_{jj}}$.*
- 144 5. *For every edge ij of G such that ij is not an edge of T , compare a_{ij} to x_{ij} :*
 - 145 (a) *Case: x is still a variable*
 - 146 i. *Case: $a_{ij} = x_{ij}$ can be solved for x*
 - 147 A. *Solve $a_{ij} = x_{ij}$ for x and assign the resulting value to x .*
 - 148 B. *Update all entries x_{st} accordingly.*
 - 149 ii. *Case: The equation $a_{ij} = x_{ij}$ is true for all values of x*
 150 *No action taken.*
 - 151 iii. *Case: The equation $a_{ij} = x_{ij}$ is false for all values of x*
 - 152 A. *Set $COMPLETION := \text{false}$.*
 - 153 B. *Exit Step 5.*
 - 154 (b) *Case: x has already been assigned a numerical value*
 - 155 i. *Case: $a_{ij} = x_{ij}$*
 156 *No action taken.*
 - 157 ii. *Case: $a_{ij} \neq x_{ij}$*
 - 158 A. *Set $COMPLETION := \text{false}$.*
 - 159 B. *Exit Step 5.*

160 **PROPOSITION 3.5.** *When Algorithm 3.4 is applied to a partial rank-1 CP matrix $A^?$ such that $A^?$ is*
 161 *described by a connected graph G , the outcome produced by Algorithm 3.4 is correct:*

- 162 • *If $COMPLETION = \text{true}$, then all possible completions have been found (either a one parameter*
 163 *family with positive real variable x or a unique rank-1 CP completion).*
- 164 • *If $COMPLETION = \text{false}$, then $A^?$ does not have a rank-1 CP completion.*

165 Furthermore, Step 5(a)i occurs at most once. When Step 5(a)i occurs, there is no longer any variable and
 166 the completion, if it exists, is unique.

167 *Proof.* It is clear that $\text{COMPLETION} = \text{false}$ implies $A^?$ does not have a rank-1 CP completion. So
 168 assume $\text{COMPLETION} = \text{true}$. By Lemma 3.3, the completion produced in Steps 2 and 3 is a rank-1 CP
 169 completion of the entries of $A^?$ described by T , and A is a completion of $A^?$ by Step 5. \square

170 LEMMA 3.6. A cycle C_n is rank-1 CP-completable if and only if n is odd; in this case it is uniquely
 171 rank-1 CP-completable.

172 *Proof.* Label the vertices of C_n in cycle order as $1, 2, \dots, n$ and let $A^?$ be a partial rank-1 CP matrix
 173 described by the cycle; the specified entries of $A^?$ are $a_{i,i+1}$ for $i = 1, \dots, n-1$ and $a_{1,n}$. Apply Algorithm
 174 3.4, choosing the path $1, 2, \dots, n$ as the spanning tree and 1 as its root in Step 1.

175 In Step 2 we get that

$$176 \quad x_{ii} = \begin{cases} \frac{a_{1,2}^2 a_{3,4}^2 \dots a_{2k-1,2k}^2}{x a_{2,3}^2 a_{4,5}^2 \dots a_{2k-2,2k-1}^2} & \text{if } i = 2k, \\ \frac{x a_{2,3}^2 a_{4,5}^2 \dots a_{2k,2k+1}^2}{a_{1,2}^2 a_{3,4}^2 \dots a_{2k-1,2k}^2} & \text{if } i = 2k + 1. \end{cases}$$

177 for $i = 2, \dots, n$.

178 Then in Step 3 we get, in particular, that

$$179 \quad x_{1,n} = \begin{cases} \frac{x a_{2,3} a_{4,5} \dots a_{n-1,n}}{a_{1,2} a_{3,4} \dots a_{n-2,n-1}} & \text{if } n \text{ is odd,} \\ \frac{a_{1,2} a_{3,4} \dots a_{n-1,n}}{a_{2,3} a_{4,5} \dots a_{n-2,n-1}} & \text{if } n \text{ is even.} \end{cases}$$

180 In Step 5 there is only one extra edge to check, namely $\{1, n\}$. For n odd, we solve the resulting equation
 181 and obtain

$$182 \quad x = \frac{a_{1,2} a_{3,4} \dots a_{n,1}}{a_{2,3} a_{4,5} \dots a_{n-1,n}},$$

183 yielding a unique rank-1 CP completion for any $A^?$. For n even, $A^?$ has a rank-1 CP completion if and only
 184 if $a_{1,n} = x_{1,n} = \frac{a_{1,2} a_{3,4} \dots a_{n-1,n}}{a_{2,3} a_{4,5} \dots a_{n-2,n-1}}$, or equivalently,

$$185 \quad \prod_{i=1}^{n/2} a_{2i-1,2i} = \prod_{i=1}^{n/2} a_{2i,2i+1}$$

186 where \dagger denotes addition modulo n . In this case, a one-parameter family of rank-1 CP completions is
 187 obtained. \square

188 COROLLARY 3.7. Every graph of order at most three is rank-1 CP-completable.

189 THEOREM 3.8. A connected graph G is rank-1 CP-completable if and only if G is a tree or G is unicyclic
 190 with an odd cycle. A unicyclic graph with an odd cycle is uniquely rank-1 CP-completable.

191 *Proof.* Observe that any tree (of order at least two) can be constructed by starting with a K_2 and
 192 performing a (possibly empty) sequence of vertex sums with additional copies of K_2 . Any unicyclic graph
 193 can be constructed from its cycle by performing a (possibly empty) sequence of vertex sums with copies of
 194 K_2 . So a tree or a unicyclic graph with odd cycle is rank-1 CP-completable by Lemmas 3.3 and 3.6, and in
 195 the unicyclic case uniquely so.

196 By Proposition 3.2 and Lemma 3.6, any graph that contains an even cycle is not rank-1 CP-completable.
 197 Suppose a connected graph G has no even cycles and has at least two odd cycles $C^{(1)}$ and $C^{(2)}$. Then $C^{(1)}$
 198 and $C^{(2)}$ are either disjoint or intersect in a single vertex. Let P be the shortest path from any vertex in $C^{(1)}$
 199 to any vertex in $C^{(2)}$ (if $C^{(1)}$ and $C^{(2)}$ intersect in a single vertex, P is the common vertex). Without loss of
 200 generality, assume the vertices of $C^{(1)}$ are $1, \dots, t$, the vertices of P are t, \dots, s , and the vertices of $C^{(2)}$ are
 201 s, \dots, r . Choose any positive values for $a_{i,i+1}$ for $1 \leq i \leq s-1$ and for $a_{t,1}$ to obtain $A^?[\{1, \dots, s\}]$. Compute
 202 the unique rank-1 CP completion B of $A^?[\{1, \dots, s\}]$. Choose positive values for $a_{i,i+1}$ for $s \leq i \leq r-1$
 203 and for $a_{r,s}$ so that the unique rank-1 CP completion of $A^?[\{s, \dots, r\}]$ has a value for the diagonal entry
 204 associated with vertex s that is different from b_{ss} . Then $A^?[\{1, \dots, r\}]$ does not have a rank-1 CP completion.
 205 Thus, $G[\{1, \dots, r\}]$ is not rank-1 CP-completable. By Proposition 3.2, G is not rank-1 CP-completable. \square

206 The next theorem will be used in Section 4. It shows a connection between the rank-1 CP-completable
 207 graphs discussed in this section and the study of SPN (signed) graphs. Theorem 3.9 generalizes [4, Lemma
 208 8.4]. See Figure 3.1 for illustrations of the graph transformation described in this theorem, generalizing the
 209 Λ -paw transformation described in [4]. In this figure, \mathcal{H} is a non-SPN signed F_5 and $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are non-SPN
 210 signed graphs that may be obtained by the generalized graph transformation in Theorem 3.9.

211 **THEOREM 3.9.** *Let $\mathcal{H} = (V, E, \Sigma)$ be a non-SPN signed graph. Let $S \subseteq V$. Let \mathcal{G} be a signed graph
 212 obtained by adding a vertex v adjacent to each of the vertices in S by a negative edge, (possibly) deleting
 213 some of the negative edges of $\mathcal{H}[S]$, (possibly) changing some negative edges between vertices in S to positive
 214 edges, and adding a positive edge between each pair of vertices in S that are not adjacent in \mathcal{H} . If $H_-[S]$ is
 215 rank-1 CP-completable, then \mathcal{G} is non-SPN.*

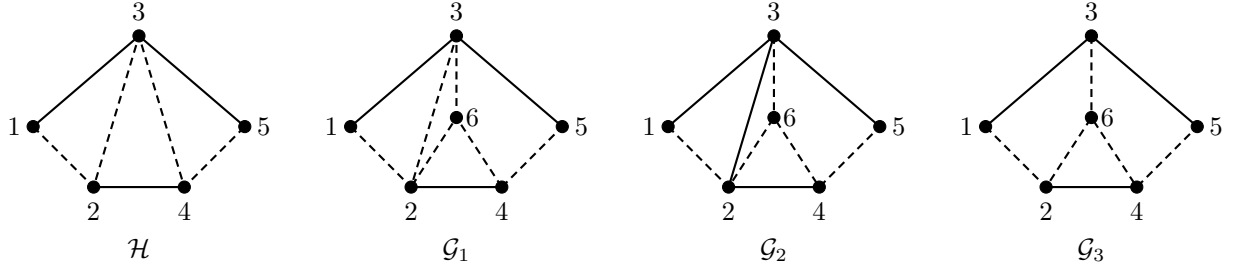


FIGURE 3.1. Examples for Theorem 3.9 with $S = \{2, 3, 4\}$

216 *Proof.* Observe that $G_-[S] \subseteq H_-[S]$ and S is a clique in $H \cup G$ (where H and G are the underlying
 217 graphs of \mathcal{H} and \mathcal{G}). Without loss of generality assume $S = \{1, \dots, k\}$, and $V(\mathcal{H}) = [n]$. Let A be a
 218 copositive non-SPN matrix with $\mathcal{G}(A) = \mathcal{H}$, and let $B^?$ be a partial rank-1 CP $k \times k$ matrix defined as
 219 follows on $H_-[S]$: For an edge ij of $H_-[S]$ that is not an edge of \mathcal{G} , set $b_{ij} = -a_{ij}$. For an edge ij of $H_-[S]$
 220 that is a negative edge of \mathcal{G} , choose b_{ij} such that $0 < b_{ij} < -a_{ij}$. For an edge ij of $H_-[S]$ that is a positive
 221 edge of \mathcal{G} , choose $b_{ij} > -a_{ij}$. Then $B^?$ has a rank-1 CP completion $\mathbf{b}\mathbf{b}^T$ where $\mathbf{b} \in \mathbb{R}_+^k$ is a positive vector.
 222 Let $\mathbf{a} \in \mathbb{R}^n$ be the non-positive vector obtained by appending $n - k$ zeros to $-\mathbf{b}$. Consider the matrix

$$223 \quad C = \begin{pmatrix} A + \mathbf{a}\mathbf{a}^T & \mathbf{a} \\ \mathbf{a}^T & 1 \end{pmatrix}.$$

224 Then, $C/C[n+1] = A$ is a non-SPN copositive matrix. Therefore, C is a non-SPN copositive matrix by
 225 Lemma 2.1, and $\mathcal{G}(C) = \mathcal{G}$. \square

226 **4. Subdivisions of K_4 .** In this section we show that the complete subdivision of K_4 is a non-SPN
 227 graph. Together with the results in [4], this shows that a subdivision of K_4 is an SPN graph if and only if
 228 at most one of the original edges was subdivided (that one edge may be subdivided as much as desired). We
 229 begin with some tools.

230 LEMMA 4.1. For $0 < \alpha, \beta, \gamma < \pi/2$, the partial positive semidefinite matrix

$$231 \quad A^? = \begin{pmatrix} 1 & -\cos(\alpha) & \cos(\alpha + \beta) & ? \\ -\cos(\alpha) & 1 & -\cos(\beta) & \cos(\beta + \gamma) \\ \cos(\alpha + \beta) & -\cos(\beta) & 1 & -\cos(\gamma) \\ ? & \cos(\beta + \gamma) & -\cos(\gamma) & 1 \end{pmatrix}$$

232 has a unique positive semidefinite completion obtained by setting $? = -\cos(\alpha + \beta + \gamma)$.

233 *Proof.* Since

$$234 \quad \begin{pmatrix} 1 & -\cos(\alpha) & \cos(\alpha + \beta) & -\cos(\alpha + \beta + \gamma) \\ -\cos(\alpha) & 1 & -\cos(\beta) & \cos(\beta + \gamma) \\ \cos(\alpha + \beta) & -\cos(\beta) & 1 & -\cos(\gamma) \\ -\cos(\alpha + \beta + \gamma) & \cos(\beta + \gamma) & -\cos(\gamma) & 1 \end{pmatrix} =$$

$$235 \quad \begin{pmatrix} 1 \\ -\cos(\alpha) \\ \cos(\alpha + \beta) \\ -\cos(\alpha + \beta + \gamma) \end{pmatrix} \begin{pmatrix} 1 \\ -\cos(\alpha) \\ \cos(\alpha + \beta) \\ -\cos(\alpha + \beta + \gamma) \end{pmatrix}^T + \begin{pmatrix} 0 \\ -\sin(\alpha) \\ \sin(\alpha + \beta) \\ -\sin(\alpha + \beta + \gamma) \end{pmatrix} \begin{pmatrix} 0 \\ -\sin(\alpha) \\ \sin(\alpha + \beta) \\ -\sin(\alpha + \beta + \gamma) \end{pmatrix}^T,$$

237 $A^?$ has a (rank-2) positive semidefinite completion. This is the unique positive semidefinite completion of
 238 $A^?$ because $A[\{1, 2, 3\}|\{4\}]$ must be orthogonal to $\ker A[\{1, 2, 3\}] = \text{span}((\sin \beta, \sin(\alpha + \beta), \sin \alpha)^T)$ by the
 239 Column Inclusion Property of positive semidefinite matrices. \square

240 PROPOSITION 4.2. The signed graph \mathcal{H}_5 shown in Figure 4.1 is non-SPN. Specifically, with $\theta_i > 0$ for
 241 $i = 1, 2, 3, 4$ and $\sum_{i=1}^4 \theta_i < \frac{\pi}{2}$, the matrix

$$242 \quad (1) \quad A = \begin{pmatrix} 1 & -\cos(\theta_1) & \cos(\theta_1 + \theta_2) & 0 & 0 \\ -\cos(\theta_1) & 1 & -\cos(\theta_2) & \cos(\theta_2 + \theta_3) & \cos(\pi/2 + \theta_1) \\ \cos(\theta_1 + \theta_2) & -\cos(\theta_2) & 1 & -\cos(\theta_3) & \cos(\theta_3 + \theta_4) \\ 0 & \cos(\theta_2 + \theta_3) & -\cos(\theta_3) & 1 & -\cos(\theta_4) \\ 0 & \cos(\pi/2 + \theta_1) & \cos(\theta_3 + \theta_4) & -\cos(\theta_4) & 1 \end{pmatrix},$$

243 is a non-SPN copositive matrix such that $\mathcal{G}(A) = \mathcal{H}_5$.

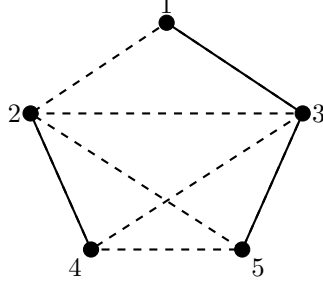


FIGURE 4.1. The non-SPN signed graph \mathcal{H}_5

244 *Proof.* Observe that $A \geq S(\boldsymbol{\theta})$, where $\boldsymbol{\theta}^T = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, with $\theta_5 = \frac{\pi}{2}$ (A differs from $S(\boldsymbol{\theta})$ only in
 245 the 1, 4 and 4, 1 entries). Since the Hildebrand matrix $S(\boldsymbol{\theta})$ is copositive, A is copositive.

246 Suppose that the matrix A is SPN. Then, $A \geq P$ for some positive semidefinite matrix P . Each of
 247 the submatrices $A[i, i+1, i+2]$ is positive semidefinite of rank 2 for $i = 1, 2, 3, 5$. The vector $\mathbf{v}^{(i)} =$
 248 $(\sin \theta_{i+1}, \sin(\theta_i + \theta_{i+1}), \sin \theta_i)^T$ spans the nullspace of $A[i, i+1, i+2]$ (with $\theta_5 = \frac{\pi}{2}$). Since $0 =$
 249 $\mathbf{v}^{(i)T} A[i, i+1, i+2] \mathbf{v}^{(i)} \geq \mathbf{v}^{(i)T} P[i, i+1, i+2] \mathbf{v}^{(i)} \geq 0$, and $\mathbf{v}^{(i)}$ is a positive vector, we get that $P[i, i+1, i+2] =$
 250 $A[i, i+1, i+2]$, $i = 1, 2, 3, 5$. Thus

$$251 \quad P = \begin{pmatrix} 1 & -\cos(\theta_1) & \cos(\theta_1 + \theta_2) & p & 0 \\ -\cos(\theta_1) & 1 & -\cos(\theta_2) & \cos(\theta_2 + \theta_3) & \cos(\pi/2 + \theta_1) \\ \cos(\theta_1 + \theta_2) & -\cos(\theta_2) & 1 & -\cos(\theta_3) & \cos(\theta_3 + \theta_4) \\ p & \cos(\theta_2 + \theta_3) & -\cos(\theta_3) & 1 & -\cos(\theta_4) \\ 0 & \cos(\pi/2 + \theta_1) & \cos(\theta_3 + \theta_4) & -\cos(\theta_4) & 1 \end{pmatrix},$$

252 for some $-1 \leq p \leq 1$. Since P is positive semidefinite, so is $P[\{2, 3, 4, 5\}]$. Therefore, $\cos(\pi/2 + \theta_1) =$
 253 $-\cos(\theta_2 + \theta_3 + \theta_4)$ by Lemma 4.1, contradicting $\sum_{i=1}^4 \theta_i < \pi/2$ and $\theta_i > 0$ for $i = 1, 2, 3, 4$. Thus, no such
 254 positive semidefinite P exists and A is not SPN. \square

255 We now prove Theorem 1.2.

256 *Proof.* Let $\mathcal{G}_i, i = 1, \dots, 4$ be the graphs in Figure 4.2. Observe that $\mathcal{G}_4 = \overline{K_4}$. The graph \mathcal{G}_1 is \mathcal{H}_5 ,
 257 which is non-SPN by Proposition 4.2. We next show that \mathcal{G}_2 is non-SPN. Let A be a non-SPN matrix with
 258 graph \mathcal{G}_1 , defined as in (1), where $\theta_i > 0$ for every $1 \leq i \leq 4$, $\sum_{i=1}^4 \theta_i < \frac{\pi}{2}$, and

$$259 \quad \sin(\theta_1) < \frac{\cos(\theta_2) \cos(\theta_4)}{\cos(\theta_3)}.$$

260 Let

$$261 \quad \mathbf{a}^T = (0, -\frac{\sin(\theta_1)}{\cos(\theta_4)}, -\cos(\theta_3), -1, -\cos(\theta_4)).$$

262 Then

$$263 \quad B = \begin{pmatrix} A + \mathbf{a}\mathbf{a}^T & \mathbf{a} \\ \mathbf{a}^T & 1 \end{pmatrix}$$

264 is a copositive matrix (as a sum of the copositive $A \oplus 0$ and a rank-1 positive semidefinite matrix). Since the
 265 off-diagonal entries in the 6th column of A are nonpositive, $A = B/B[6]$ is also copositive by Lemma 2.1.
 266 Since A is non-SPN, B is non-SPN. As $\mathcal{G}(B) = \mathcal{G}_2$, \mathcal{G}_2 is non-SPN.

267 Since \mathcal{G}_2 is non-SPN, Theorem 3.9 (applied to $\mathcal{H} = \mathcal{G}_2$ and $S = \{2, 3, 6\}$) implies that the signed graph
 268 \mathcal{G}_3 is also non-SPN. The signed graph \mathcal{G}_4 is obtained from the non-SPN \mathcal{G}_3 by subdividing negative edges,
 269 and therefore is also non-SPN by Lemma 2.2. \square

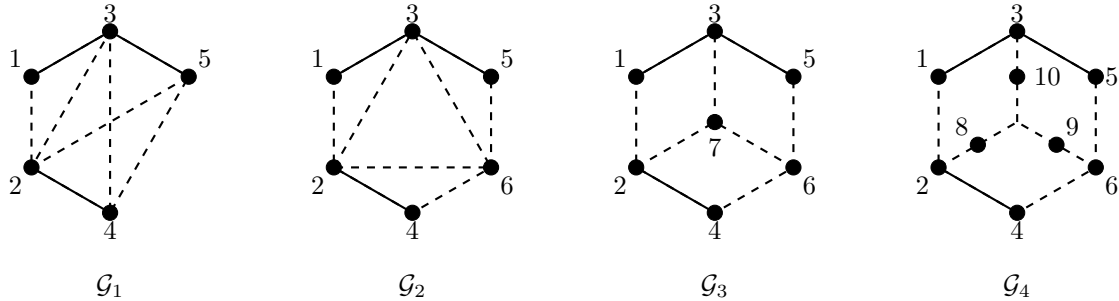


FIGURE 4.2. Signed graphs used in the proof of Theorem 1.2

270 In view of Corollary 1.3 and the discussion in [4, Remark 9.11], for a full description of SPN graphs (either
 271 by forbidden subgraphs or by their possible blocks) it remains to determine whether, or which, subdivisions
 272 of T_n are SPN, where T_n is the graph on n vertices consisting of $n - 2$ triangles sharing a common base. So
 273 far it is only known that T_n is SPN for $n = 3, 4, 5$ and subdivisions of T_n are SPN for $n = 3, 4$ (see [4, 5]).
 274 This is discussed further in Section 5.

275 **5. Conjectures.** Here we list some conjectures, in increasing order of strength, leading up to Conjecture
 276 1.1, which is the strongest. We also discuss possible methods to establish the conjectures.

277 CONJECTURE 5.1. *Any subdivision of a non-SPN graph is non-SPN.*

278 Conjecture 5.1 has been established in the case that the subdivided edge corresponds to a negative entry
 279 in a realizing non-SPN matrix in [4, Lemma 8.1] (see Lemma 2.2). So it remains to consider the case that
 280 the corresponding entry is positive.

281 If ζ is a graph property, and G has property ζ , then we say G is a ζ graph (e.g., SPN graph). A graph
 282 property ζ is *minor-closed* if G is a ζ graph implies that H is a ζ graph for every minor H of G . If ζ is
 283 minor-closed, then ζ -graphs are characterized by a finite set of forbidden graphs.

284 CONJECTURE 5.2. *Being an SPN graph is minor-closed, i.e., if H is a minor of an SPN graph G , then
 285 H is SPN.*

286 Since subgraphs of SPN graphs are SPN [4, Lemma 4.2], proving Conjecture 5.2 amounts to showing
 287 that the class of SPN graphs is closed under edge contractions. If Conjecture 5.2 is true, then the SPN
 288 graphs are characterized by a finite set of forbidden minors. Our Conjecture 1.1 is that the only forbidden
 289 minor is F_5 .

290 From [4, Theorem 9.10] and Corollary 1.3 we obtain the next result.

291 THEOREM 5.3. Let G be an SPN graph. Then G does not contain the following subgraphs:

- 292 (1) a subdivision of the fan F_5 ,
 293 (2) a subdivision of CD_6 , where CD_6 is the graph shown in figure 5.1.
 294 (3) a subdivision of K_4 in which at least two edges were subdivided at least once each.

295 Since CD_6 and K_4 with two edges subdivided have an F_5 minor, every graph that is known to be non-SPN
 296 has an F_5 minor.

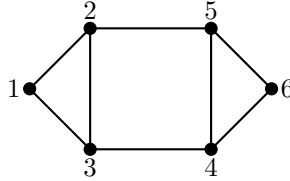


FIGURE 5.1. CD_6

297 Conjecture 1.1 is established for graphs on five vertices in [4, Theorem 7.3]; see also the corrigendum [5].
 298 As discussed in [4, Remark 9.11], if a graph G is a 2-connected graph that contains no subdivision of F_5 , no
 299 subdivision of CD_6 , and no subdivision of K_4 where at least two edges have been subdivided, then G is either
 300 a subdivision of T_k , $k \geq 3$, or a DR_k , $k \geq 4$ (DR_k is a K_4 with one edge subdivided $k - 4$ times). So if every
 301 subdivision of T_k is SPN, Conjecture 1.1 is true and SPN-graphs are determined. In this case, we would
 302 have not only the forbidden minor characterization in Conjecture 1.1, but also the block characterization
 303 suggested in [4], that a graph is SPN if and only if every block is one of an edge, a subdivision of T_k , $k \geq 3$,
 304 or a DR_k , $k \geq 4$.

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