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Minimum Rank, Maximum Nullity, and Zero Forcing Number of Graphs

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Minimum Rank, Maximum Nullity, and Zero Forcing Number of Graphs

Abstract
This chapter represents an overview of research related to a notion of the "*rank of a graph*" and the dual concept known as the "*nullity of a graph*," from the point of view of associating a fixed collection of symmetric or Hermitian matrices to a given graph. This topic and related questions have enjoyed a fairly large history within discrete mathematics, and have become very popular among linear algebraists recently, partly based on its connection to certain inverse eigenvalue problems, but also because of the many interesting applications (e.g., to communication complexity in computer science and to control of quantum systems in mathematical physics) and implications to several facets of graph theory.

This chapter is divided into eight parts, beginning in Section 1 with what we feel is the standard minimum rank problem concerning symmetric matrices over the real numbers associated with a simple graph. We continue with important variants of the standard minimum rank problem and related parameters, including the concept of minimum rank over other fields (Section 2), the positive semidefinite minimum rank of a graph (Section 3), graph coloring parameters known as *zero forcing numbers* (Section 4) and the more classical and celebrated parameters due to Y. Colin de Verdière (Section 5). Section 6 contains more advanced topics relevant to the previous five sections, and Section 7 discusses two well-known conjectures related to minimum rank. Whereas the first seven sections concern primarily symmetric matrices and the diagonal of the matrix is free, in Section 8 we discuss minimum rank problems with no symmetry assumption but the diagonal constrained.

NB: The topics discussed in this chapter are in an active research area and the facts presented here represent the state of knowledge as of the writing of this chapter in 2012.

Disciplines
Algebra | Applied Mathematics | Discrete Mathematics and Combinatorics

Comments
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Minimum Rank, Maximum Nullity, and Zero Forcing Number of Graphs

Shaun M. Fallat* Leslie Hogben†

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This chapter represents an overview of research related to a notion of the "*rank of a graph*" and the dual concept known as the "*nullity of a graph*," from the point of view of associating a fixed collection of symmetric or Hermitian matrices to a given graph. This topic and related questions have enjoyed a fairly large history within discrete mathematics, and have become very popular among linear algebraists recently, partly based on its connection to certain inverse eigenvalue problems, but also because of the many interesting applications (e.g., to communication complexity in computer science and to control of quantum systems in mathematical physics) and implications to several facets of graph theory.

This chapter is divided into eight parts, beginning in Section 1 with what we feel is the standard minimum rank problem concerning symmetric matrices over the real numbers associated with a simple graph. We continue with important variants of the standard minimum rank problem and related parameters, including the concept of minimum rank over other fields (Section 2), the positive semidefinite minimum rank of a graph (Section 3), graph coloring parameters known as *zero forcing numbers* (Section 4) and the more classical and celebrated parameters due to Y. Colin de Verdière (Section 5). Section 6 contains more advanced topics relevant to the previous five sections, and Section 7 discusses two well-known conjectures related to minimum rank. Whereas the first seven sections concern primarily symmetric matrices and the diagonal of the matrix is free, in Section 8 we discuss minimum rank problems with no symmetry assumption but the diagonal constrained.

NB: The topics discussed in this chapter are in an active research area and the facts presented here represent the state of knowledge as of the writing of this chapter in 2012.

1 Standard Minimum Rank of Graphs

For a given simple graph G, we associate, in a natural way, a collection of matrices whose entries reflect adjacency in G. We are interested in quantities such as the smallest possible rank and the largest possible nullity over all matrices in this collection. Of particular

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interest are the resulting connections between these extremal quantities and certain combinatorial properties of graphs, which partly motivates the fascination of this topic among many researchers in combinatorial matrix theory. All graphs in this section are simple.

Definitions 1.1. Definitions of common graph theory terms not given here can be found in Chapter 39 of *Handbook of Linear Algebra*, 2nd ed.
Notation: Let G be a simple graph with vertices $\{1, \dots, n\}$ and let $\mathcal{S}_n$ be the set of symmetric $n \times n$ real matrices.

For $B \in \mathcal{S}_n$, the **graph** of B, denoted $G(B)$, is the simple graph with vertices $\{1, \dots, n\}$ and edges $\{\{i, j\} : b_{ij} \neq 0 \text{ and } i \neq j\}$.

Let $\mathcal{S}(G) = \{B \in \mathcal{S}_n : G(B) = G\}$ be the **set of symmetric matrices associated with** G.

The **minimum rank** of G is $\operatorname{mr}(G) = \min\{\operatorname{rank} B : B \in \mathcal{S}(G)\}$.

The **maximum nullity** of G is $\operatorname{M}(G) = \max\{\operatorname{null} B : B \in \mathcal{S}(G)\}$.

The **maximum multiplicity** of G is $\max\{\alpha_B(\lambda) \mid B \in \mathcal{S}(G), \lambda \in \sigma(B))\}$.

The **path cover number** of G, $\operatorname{P}(G)$, is the minimum number of vertex disjoint paths occurring as induced subgraphs of G that cover all the vertices of G.

The **rank-spread** of G at vertex v is $r_v(G) = \operatorname{mr}(G) - \operatorname{mr}(G - v)$.

The **order** of G, denoted by $|G|$, is the cardinality of the vertex set of G.

For $W \subseteq \{1, \dots, n\}$, the subgraph of G induced by W is denoted $G[W]$.

A vertex v of a connected simple graph G is a **cut-vertex** if $G - v$ is disconnected.

The **girth** of G, denoted $\operatorname{g}(G)$, is the length of the shortest cycle in G (if G has a cycle); the girth of an acyclic graph is infinite.

An induced subgraph $G[W]$ of a graph G is a **clique** if $G[W]$ has an edge between every pair of vertices in W (i.e., $G[W]$ is isomorphic to $K_{|W|}$).

A set of subgraphs of G, each of which is a clique and such that every edge of G is contained in at least one of these cliques, is called a **clique covering** of G.

The **clique covering number** of G, denoted by $\operatorname{cc}(G)$, is the smallest number of cliques in a clique covering of G.

A graph is **planar** if it can be drawn in the plane without crossing edges.

A graph G is **outerplanar** it there exists a planar embedding of G in which all of the vertices of G lie in the boundary of a single face.

The **minimum degree** $\delta(G)$ of G is the minimum of the degrees of the vertices of G.

The **vertex connectivity** $\kappa(G)$ of G is the smallest number k such that there is a set of vertices S, with $|S| = k$, for which $G - S$ is disconnected, provided G is not complete. By convention, $\kappa(K_n) = n - 1$.

If $\kappa(G) \geq k$, then G is k-**connected**.

A k-**tree** is constructed inductively by starting with a complete graph on $k + 1$ vertices and connecting each new vertex to the vertices of an existing clique on k vertices.

A **partial** k-**tree** is a subgraph of a k-tree.

The **tree-width** $\operatorname{tw}(G)$ of G is the minimum k such that G is a partial k-tree.

The **Cartesian product** of graphs $G = (V, E)$ and $G' = (V', E')$, denoted by $G \times G'$ is the graph with vertex set $V \times V'$, where two distinct vertices (v, v') and (u, u') are adjacent if and only if (1) $v = u$ and $v'u' \in E(G')$, or (2) $v' = u'$ and $vu \in E(G)$.

The kth **hypercube** Q_k is defined recursively by $Q_1 = K_2, Q_{k+1} = Q_k \times K_2$.

The **dual** G^d of a plane embedding of a planar graph G is obtained as follows: Place a new vertex in each face of the embedding; these are the vertices of the dual. Two dual vertices are adjacent if and only if the two faces of G share an edge of G.

A **polygonal path** is a 2-connected graph G that can be embedded in the plane such that the graph obtained from the dual of G after deleting the vertex corresponding to the infinite face is a path. A polygonal path has been called an **LSEAC** graph, a **2-connected partial linear 2-tree**, or a linear 2-tree by some authors (the last of these terms is unfortunate, since a polygonal path need not be a 2-tree).

The graph H_n obtained from C_n by appending a leaf to each vertex on C_n is called an n**-sun**.

The **trimmed form** of G is the induced subgraph obtained by a sequence of trimming operations, namely deleting peripheral leaves, isolated paths and/or appropriate vertices until no further operations are possible (see [BFH05a] for details).

Let $e = \{u, v\}$ be an edge of G. Then G_e is the graph obtained from G by inserting a new vertex w into G, inserting the edges $\{u, w\}$ and $\{w, v\}$ and deleting e from G. We say that the edge e has been **subdivided** and call G_e an **edge subdivision of** G.

The **complete edge subdivision graph** of G, denoted $\overset{\circ}{\overline{G}}$ is obtained from G by subdividing each edge of G once.

The **union** of (simple) graphs $G_i = (V_i, E_i)$ is $\cup_{i=1}^h G_i = (\cup_{i=1}^h V_i, \cup_{i=1}^h E_i)$ (the vertex sets V_i are not required to be disjoint). In most of this chapter graphs are simple, so these are set unions, not multiset unions; multiple copies of the same edge are replaced by a single copy of the edge.

If G_1 and G_2 are disjoint graphs, the **join** of G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph defined by $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup E$, where E consists of all the edges $\{u, v\}$ with $u \in V(G_1)$, $v \in V(G_2)$.

Graph G is **decomposable** or a **co-graph** if it can be expressed as a sequence of joins or unions of isolated vertices.

A **Hamiltonian path** of G is a subgraph of G that is a path and includes every vertex of G.

Facts 1.1. Facts requiring proof for which no specific reference is given can be found in [FH07]. Additional relevant facts appear in Sections 4, 5, 6, and 7. Notation: G is a simple graph with vertices $\{1, \ldots, n\}$.

1. For any graph G, $\mathrm{M}(G) + \mathrm{mr}(G) = |G|$.

2. For any graph G, $\mathrm{M}(G)$ is equal to maximum multiplicity of G.

3. For any graph G, $0 \leq \mathrm{mr}(G) \leq |G| - 1$, and if G contains at least one edge, then $1 \leq \mathrm{mr}(G) \leq |G| - 1$.

4. If the connected components of G are $G_1, \ldots, G_t$, then

$$\mathrm{mr}(G) = \sum_{i=1}^{t} \mathrm{mr}(G_i) \qquad \text{and} \qquad \mathrm{M}(G) = \sum_{i=1}^{t} \mathrm{M}(G_i).$$

5. [AIM08], [HCY10a], [FP07], [ADH10] Minimum rank has been computed for many families of graphs (including many Cartesian products and complements) that are listed in an online catalog [AIM].

6. [DGH10] Minimum rank has been computed for all graphs of order at most 7.

7. [BDG] Free open-source *Sage* software is available online that implements many of the techniques listed here and in other sections for computing minimum rank.

8. If $G[W]$ is an induced subgraph of G, then $\mathrm{mr}(G[W]) \leq \mathrm{mr}(G)$.

9. For a connected graph G, $\mathrm{diam}(G) \leq \mathrm{mr}(G)$.

10. [BFS08] The trees and unicyclic graphs for which $\mathrm{diam}(G) = \mathrm{mr}(G)$ have been characterized.

11. For a graph G that contains a cycle, $\mathrm{g}(G) - 2 \leq \mathrm{mr}(G)$.

12. If $G = \cup_{i=1}^{h} G_i$ then $\mathrm{mr}(G) \leq \sum_{i=1}^{h} \mathrm{mr}(G_i)$.

13. If G is a graph, $\mathrm{mr}(G) \leq \mathrm{cc}(G)$.

14. [JL99] For any tree T, $M(T) = \mathrm{P}(T) = |T| - \mathrm{mr}(T)$. There are algorithms for computing these parameters for trees [FH07].

15. [Nyl96] For any vertex v of G, $0 \leq \mathrm{mr}(G) - \mathrm{mr}(G - v) \leq 2$. In other words, $r_v(G) \leq 2$.

16. [Nyl96] Adding or removing an edge from a graph G can change the minimum rank by at most 1.

17. [Hsi01], [BFH04] If G has a cut-vertex, the problem of computing the minimum rank of G can be reduced to computing minimum ranks of certain subgraphs. Specifically, let v be a cut-vertex of G. For $i = 1, \ldots, h$, let $W_i \subseteq V(G)$ be the vertices of the ith component of $G - v$ and let G_i be the subgraph induced by $\{v\} \cup W_i$. Then

$$r_v(G) = \min\left\{\sum_{i=1}^{h} r_v(G_i),\ 2\right\}$$

and thus

$$\mathrm{mr}(G) = \min\left\{\sum_{i=1}^{h} \mathrm{mr}(G_i),\ \sum_{i=1}^{h} \mathrm{mr}(G_i - v) + 2\right\}.$$

Equivalently,

$$\mathrm{M}(G) = \max\left\{\sum_{i=1}^{h} \mathrm{M}(G_i) - h + 1,\ \sum_{i=1}^{h} \mathrm{M}(G_i - v) - 1\right\}.$$

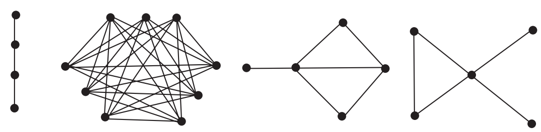
18. If $r_v(G_i) = 0$ for all but at most one of the G_i, then $\mathrm{mr}(G) = \sum_{i=1}^{h} \mathrm{mr}(G_i)$.

19. [Hol08a] A cut-set reduction formula, analogous to the one from Fact 17, has been established for cut-sets of size two. However, in this case the formula involves a minimum over six related graphs and multigraphs.

20. For $n \geq 2$, $\mathrm{mr}(K_n) = 1$, and if G is connected, $\mathrm{mr}(G) = 1$ implies $G = K_{|G|}$.

21. For $n \geq 1$, $\mathrm{mr}(P_n) = n - 1$ and [Fie69] if $\mathrm{mr}(G) = |G| - 1$, then $G = P_{|G|}$.

22. [BHL04] A connected graph G has $\mathrm{mr}(G) \leq 2$ if and only if G does not contain as an induced subgraph any of P_4, $K_{3,3,3}$ (the complete tripartite graph), Dart, or $\ltimes$ (all shown in Figure 1).



P_4 $K_{3,3,3}$ Dart $\ltimes$

Figure 1: Forbidden induced subgraphs for $\mathrm{mr}(G) \leq 2$.

23. [BHL04] A graph G has $\mathrm{mr}(G) \leq 2$ if and only if the complement of G is of the form

$$(K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$$

for appropriate nonnegative integers $k, s_1, s_2, p_1, q_1, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$ for all $i = 1, \ldots, k$.

24. [BF07] A connected graph G has minimum rank 2 if and only if $G = \bigvee_{i=1}^{r} G_i$, $r > 1$, where either

    (a) $G_i = K_{m_i} \cup K_{n_i}$, for suitable $m_i \geqslant 1$, $n_i \geqslant 0$, or

    (b) $G_i = \overline{K}_{m_i}$, for a suitable $m_i \geqslant 3$;

    and option (b) occurs at most twice.

25. [HH06] Let G be a 2-connected graph. Then $\mathrm{mr}(G) = |G| - 2$ if and only if G is a polygonal path.

26. [JLS09] All graphs G that satisfy $\mathrm{mr}(G) = |G| - 2$ (or, equivalently $\mathrm{M}(G) = 2$) have been characterized, including those from Fact 25 along with a list of additional 1-connected graphs.

27. [BF07] (Complete multipartite graph) Let $n_1 \geq n_2 \geq \cdots \geq n_k \geq 0$, $n_1 > 1$, and $K_{n_1,n_2,\ldots,n_k} := \overline{K}_{n_1} \vee \overline{K}_{n_2} \vee \cdots \vee \overline{K}_{n_k}$. Then

$$\mathrm{mr}(K_{n_1,n_2,\ldots,n_k}) = \begin{cases} 0 & \text{if } k = 1; \\ 2 & \text{if } k > 1, n_3 < 3; \\ 3 & \text{if } n_3 \geq 3. \end{cases}$$

28. [Sin10] For any outerplanar graph G, $\mathrm{M}(G) \leq \mathrm{P}(G)$.

29. [BFH05a] Let G be a unicyclic graph. Then

$$\mathrm{mr}(G) = \begin{cases} |G| - \mathrm{P}(G) + 1 & \text{if the trimmed form of } G \text{ is an } n\text{-sun, } n > 3, \text{ odd} \\ |G| - \mathrm{P}(G) & \text{otherwise.} \end{cases}$$

30. [BF07] The minimum rank of a decomposable graph can be computed recursively from its pieces as identified by the sequence of joins or unions of isolated vertices. Such a recursive formula exists for computing the minimum rank for the join of more general graphs in certain circumstances. See also Section 6.

31. [BBC09], [JLS09] If e is an edge of G, then $\mathrm{mr}(G) \leq \mathrm{mr}(G_e) \leq \mathrm{mr}(G) + 1$, or, equivalently, $\mathrm{M}(G) \leq \mathrm{M}(G_e) \leq \mathrm{M}(G) + 1$. If e is incident with a vertex of degree at most 2, then $\mathrm{mr}(G_e) = \mathrm{mr}(G) + 1$.

32. [BBC] For any graph G, $\mathrm{M}(\widehat{G}) = \mathrm{Z}(\widehat{G})$ and the value is determined. The first study of $\mathrm{mr}(\widehat{G})$ was in [BBC09] where the equality of maximum nullity and zero forcing number and $\mathrm{mr}(\widehat{G}) = 2|G| - 2$ were established for any graph with a Hamiltonian path; these results were extended to any graph with no cut-edge in [CCH12].

Examples 1.1. Additional relevant examples appear in Sections 4, 5, 6, and 7.

1. For the matrix

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 3.1 & -1.5 & 2 \\ 0 & -1.5 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix},$$

   $G(B)$ is shown in Figure 2

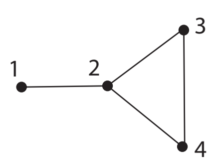


Figure 2: The graph $G = G(B)$ for matrix B in Example 1.

2. Let G be the graph in Figure 2 and let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

   Since $G \neq K_4$, $\mathrm{mr}(G) \geq 2$. Since $G(A) = G$ and the matrix A above has rank 2, $\mathrm{mr}(G) = 2$, and hence $\mathrm{M}(G) = 2$.

3. For the cycle C_n on n vertices, $\mathrm{mr}(C_n) = n-2$. Since C_n is not P_n, we have $\mathrm{mr}(C_n) \leq n-2$, and since C_n contains P_{n-1} as an induced subgraph it follows that $\mathrm{mr}(C_n) = n-2$. Observe that C_n is an example of a polygonal path.

4. For $p, q \geq 1$ and $p+q \geq 3$, $\mathrm{mr}(K_{p,q}) = 2$.

5. For P_n with $n \geq 2$, the rank spreads of the two pendant vertices are one, whereas the rank spreads of all remaining vertices are two.

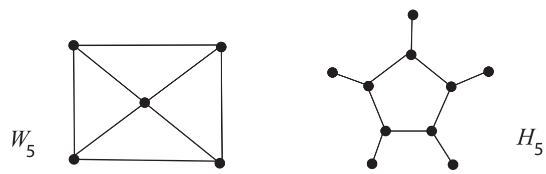


Figure 3: (a) The wheel W_5 and (b) the penta-sun H_5.

6. (Non-comparability of $\mathrm{M}(G)$ and $\mathrm{P}(G)$ for graphs in general) The wheel on 5 vertices W_5, shown in Figure 3(a), has $\mathrm{P}(W_5) = 2$ by inspection, and $\mathrm{mr}(W_5) = 2$ (by Fact 22), so $\mathrm{M}(W_5) = 3 > \mathrm{P}(W_5)$. The penta-sun, H_5, which is shown in Figure 3(b), has $\mathrm{P}(H_5) = 3$ by inspection. By repeated application of Fact 17, or Fact 29, we have $\mathrm{mr}(H_5) = 8$ and thus that $\mathrm{P}(H_5) > \mathrm{M}(H_5) = 2$.

7. This example illustrates the use of various facts for computing $\mathrm{mr}(G)$ when applicable. The graph G shown in Figure 4(a) has cut-vertex 4 and the induced subgraphs $G_1 = G[\{1,2,3,4\}]$ and $G_2 = G[\{4,5,6,7,8,9,10,11,12\}]$ associated with the two components (following the notation in Fact 17). By Facts 22 (or 23), and 21, $\mathrm{mr}(G_1) = 2 = \mathrm{mr}(G_1 - 4)$, so $r_4(G_1) = 0$. Thus by Fact 18, we know that $\mathrm{mr}(G) = \mathrm{mr}(G_1) + \mathrm{mr}(G_2)$. For G_2, using Fact 12 and observing that G_2 is the union of two copies of K_3 and one C_5, we have $\mathrm{mr}(G_2) \leq 1 + 1 + 3 = 5$. Further, since G_2 is outerplanar, we know that $\mathrm{mr}(G_2) \geq |G_2| - \mathrm{P}(G_2) = 5$. Thus, $\mathrm{mr}(G_2) = 5$, and $\mathrm{mr}(G) = 7$.

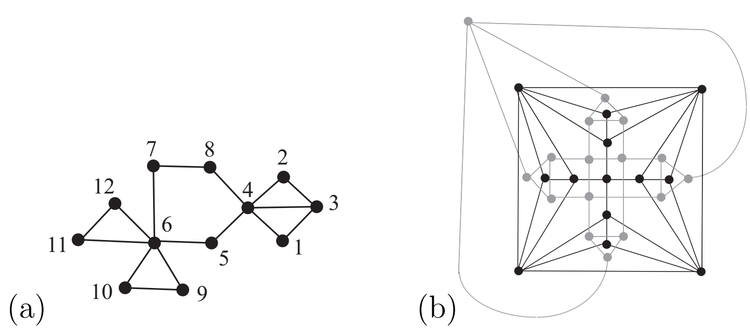


Figure 4: (a) A graph to which various facts are applied to compute minimum rank, and (b) a planar graph and its dual that have different maximum nullity.

8. [ADH10] A graph G is shown in Figure 4(b), with its dual G^d shown in gray. This graph G is a 3-connected planar graph and the maximum nullity of its dual graph G^d is different from the maximum nullity of G (such examples seem to be rare among small graphs). Specifically, $\mathrm{M}(G) = 7$ and $\mathrm{M}(G^d) = 5$.

2 Minimum Rank over Other Fields

In this section we survey the minimum rank of symmetric matrices described by a given simple graph G over an arbitrary field F. The two distinctions that become immediately apparent as sources of variation in minimum rank over arbitrary fields are the field characteristic (characteristic 2 versus everything else) and field cardinality (especially finite versus infinite). All graphs in this section are simple.

Definitions 2.1. Notation: Let G be a simple graph with vertices $\{1, \ldots, n\}$ and let F be a field. Let $\mathcal{S}_n(F)$ be the set of symmetric $n \times n$ matrices over F. In this section, $\mathcal{S}_n$ will be denoted by $\mathcal{S}_n(\mathbb{R})$.

For $B \in \mathcal{S}_n(F)$, the **graph** of B, denoted $G(B)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} |\ b_{ij} \neq 0 \text{ and } i \neq j\}$.

Let $\mathcal{S}^F(G) = \{B \in \mathcal{S}_n(F) : G(B) = G\}$ be the **set of symmetric matrices over** F **associated with** G.

The **minimum rank of** G **over** F is $\mathrm{mr}^F(G) = \min\{\mathrm{rank}\, B \,|\, B \in \mathcal{S}^F(G)\}$. In this section, $\mathrm{mr}(G)$ will be denoted by $\mathrm{mr}^{\mathbb{R}}(G)$.

The **maximum nullity of** G **over** F is $\mathrm{M}^F(G) = \max\{\mathrm{null}\, B : B \in \mathcal{S}^F(G)\}$. In this section, $\mathrm{M}(G)$ will be denoted by $\mathrm{M}^{\mathbb{R}}(G)$.

A graph G is **field independent** if $\mathrm{mr}^F(G) = \mathrm{mr}(G)$ for all fields F.

The **rank-spread of** G **over field** F at vertex v is defined as $r_v^F(G) = \mathrm{mr}^F(G) - \mathrm{mr}^F(G - v)$.

A **universally optimal matrix** is a symmetric integer matrix A such that every off-diagonal entry of A is 0, 1, or -1 and for all fields F, $\mathrm{rank}^F(A) = \mathrm{mr}^F(G(A))$.

Facts 2.1. Facts requiring proof (beyond that given over the real numbers) for which no specific reference is given can be found in [FH07].
Notation: G is a simple graph with vertices $\{1, \ldots, n\}$ and F is a field.

1. $\mathrm{M}^F(G) + \mathrm{mr}^F(G) = |G|$.

2. If F is a subfield of the field K, then $\mathrm{mr}^F(G) \geq \mathrm{mr}^K(G)$ and $\mathrm{M}^F(G) \leq \mathrm{M}^K(G)$. Fact 23 below, which is illustrated by the graph in Example 2 below, shows these inequalities can be strict. (See also Fact 3 next, where strict inqualities may occur for the case of infinite fields.)

3. [BFH08] Examples are known of graphs G_1 and G_2 such that $\mathrm{mr}^{\mathbb{R}}(G_1) > \mathrm{mr}^{\mathbb{C}}(G_1)$ and $\mathrm{mr}^{\mathbb{Q}}(G_2) > \mathrm{mr}^{\mathbb{R}}(G_2)$. The graph G_2 provides a counterexample to a conjecture in [AHK05].

4. For every G, $0 \leq \mathrm{mr}^F(G) \leq |G| - 1$, and if G contains at least one edge, then $1 \leq \mathrm{mr}^F(G) \leq |G| - 1$.

5. If the connected components of G are $G_1, \ldots, G_t$, then

$$\mathrm{mr}^F(G) = \sum_{i=1}^{t} \mathrm{mr}^F(G_i) \qquad \text{and} \qquad \mathrm{M}^F(G) = \sum_{i=1}^{t} \mathrm{M}^F(G_i).$$

6. [AIM08], [DGH09], [HCY10b] Minimum rank over fields has been computed for many families of graphs (see also Fact 20 below).

7. $\mathrm{mr}^F(G[W]) \leq \mathrm{mr}^F(G)$.

8. For a connected graph G, $\mathrm{diam}(G) \leq \mathrm{mr}^F(G)$.

9. For a graph G that contains a cycle, $\mathrm{g}(G) - 2 \leq \mathrm{mr}^F(G)$.

10. For any vertex v of G, $0 \leq \mathrm{mr}^F(G) - \mathrm{mr}^F(G - v) \leq 2$. In other words, $r_v^F(G) \leq 2$.

11. Adding or removing an edge from a graph G can change minimum rank over F by at most 1.

12. If F is infinite and $G = \cup_{i=1}^h G_i$, then $\mathrm{mr}^F(G) \leq \sum_{i=1}^h \mathrm{mr}^F(G_i)$.

13. [FH07] If F is infinite, then $\mathrm{mr}^F(G) \leq \mathrm{cc}(G)$. This need not be true for finite fields (see Example 2 below).

14. [CDH07] For any tree T, $\mathrm{M}^F(T) = \mathrm{P}(T) = |T| - \mathrm{mr}^F(T)$.

15. [FH07] Cut-vertex reduction remains valid for every field. With the notation of Fact 1.17,

$$r_v^F(G) = \min\left\{\sum_{i=1}^{h} r_v^F(G_i),\ 2\right\}$$

and thus

$$\mathrm{mr}^F(G) = \min\left\{\sum_{i=1}^{h} \mathrm{mr}^F(G_i),\ \sum_{i=1}^{h} \mathrm{mr}^F(G_i - v) + 2\right\}.$$

16. If $r_v^F(G_i) = 0$ for all but at most one of the G_i, then $\mathrm{mr}^F(G) = \sum_{i=1}^h \mathrm{mr}^F(G_i)$.

17. [Hol08a] A cut-set reduction formula, analogous to the one from Fact 15, has been established for cut-sets of size two. However, in this case the formula involves a minimum over six related graphs and multigraphs.

18. For $n \geq 2$, $\mathrm{mr}^F(K_n) = 1$, and if G is connected, $\mathrm{mr}^F(G) = 1$ implies $G = K_{|G|}$.

19. For $n \geq 1$, $\mathrm{mr}^F(P_n) = n - 1$, and [Fie69, RS84, BL05, CDH07] if $\mathrm{mr}^F(G) = |G| - 1$, then $G = P_{|G|}$.

20. [DGH09] The following graphs are field independent and have universally optimal matrices:

   (a) The path, $\mathrm{mr}^F(P_n) = n - 1$.

   (b) The complete graph, $\mathrm{mr}^F(K_n) = 1$ $(n \geq 2), \mathrm{mr}^F(K_1) = 0$.

   (c) The cycle, $\mathrm{mr}^F(C_n) = 2$.

   (d) The complete bipartite graph with at least one of $p, q > 1$, $\mathrm{mr}^F(K_{p,q}) = 2$.

   (e) Every tree T, $\mathrm{mr}^F(T) = P(T)$.

   (f) Every polygonal path G, $\mathrm{mr}^F(G) = |G| - 2$.

   (g) [Pet12], [AIM08] The kth hypercube, Q_k, satisfies $\mathrm{mr}^F(Q_k) = 2^{k-1}$.

   (h) [BHL04] The graphs Dart and $\ltimes$ shown in Figure 1 both have minimum rank 3 over any field.

   (i) [BBC] The complete subdivision graph $\overset{\circ}{G}$ for any graph G.

21. [BHL04] Let F be an infinite field of characteristic $\neq 2$. The following statements are equivalent:

   (a) $\mathrm{mr}^F(G) \leq 2$.

   (b) $\overline{G}$ is of the form $(K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$ for appropriate nonnegative integers $k, s_1, s_2, p_1, q_1, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$ for all $i = 1, \ldots, k$.

   (c) G does not contain as an induced subgraph any of $P_4, \mathrm{Dart}, \ltimes, P_3 \cup K_2, 3K_2,$ or $K_{3,3,3}$ ($P_4, \mathrm{Dart}, \ltimes, K_{3,3,3}$ are shown in Figure 1).

22. [BHL04] Let F be an infinite field of characteristic 2. The following statements are equivalent:

   (a) $\mathrm{mr}^F(G) \leq 2$.

   (b) $\overline{G}$ is of one of the forms $(K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_k}) \vee K_r$ or $(K_{s_1} \cup K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$ for appropriate nonnegative integers $k, s_1, s_2, p_1, q_1, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$ for all $i = 1, \ldots, k$.

   (c) G does not contain as an induced subgraph any of $P_4, \mathrm{Dart}, \ltimes, P_3 \cup K_2, 3K_2,$ or $\overline{P_3 \cup 2K_3}$.

23. [BHL05] Let F be a finite field of prime characteristic $p \neq 2$ having p^t elements. The following statements are equivalent:

   (a) $\mathrm{mr}^F(G) \leq 2$.

   (b) $\overline{G}$ is of one of the forms:
   $(K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$ for appropriate nonnegative integers $k, p_1, q_1, \ldots, p_k, q_k$, r, with $p_i + q_i > 0$ for all $i = 1, \ldots, k$ and $k \leq \frac{p^t+1}{2}$;
   or $(K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$ for appropriate nonnegative integers $k, s_1, s_2, p_1, q_1, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$, for all $i = 1, \ldots, k$ and $k \leq \frac{p^t-1}{2}$.

(c) G does not contain as an induced subgraph any of P_4, Dart, $\ltimes$, $P_3 \cup K_2$, $3K_2$, $K_{3,3,3}$, $\overline{(m+2)K_2 \cup K_1}$, $\overline{K_2 \cup 2K_1 \cup mP_3}$, or $\overline{K_1 \cup (m+1)P_3}$ where $m = \frac{p^t-1}{2}$.

24. [BHL05] Let F be a finite field of characteristic 2 having 2^t elements. The following statements are equivalent:

    (a) $\mathrm{mr}^F(G) \leq 2$.

    (b) $\overline{G}$ is of one of the forms:
    $(K_{s_1} \cup \cdots \cup K_{s_k}) \vee K_r$ for appropriate nonnegative integers $k, s_1, \ldots, s_k, r$, with $k \leq 2^t + 1$;
    or $(K_{s_1} \cup K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$ for appropriate nonnegative integers $k, s_1, p_1, q_1, \ldots, p_k, q_k, r$ with $k \leq 2^{t-1}$.

    (c) G does not contain as an induced subgraph any of P_4, Dart, $\ltimes$, $P_3 \cup K_2$, $3K_2$, $\overline{P_3 \cup 2K_3}$, $\overline{(2^{t-1}+1)K_2 \cup (2^{t-1}+1)K_1}$, $\overline{P_3 \cup 2^{t-1}K_2 \cup K_1}$, $\overline{2K_3 \cup 2^t K_1}$, or $\overline{2^{t-1}P_3 \cup 2K_1}$.

25. [JLS09] Over an infinite field F, all simple graphs G that satisfy $\mathrm{mr}^F(G) = |G| - 2$ (or, equivalently $\mathrm{M}^F(G) = 2$) have been characterized.

26. [Sin10][1] For any outerplanar graph G, $\mathrm{M}^F(G) \leq \mathrm{P}(G)$.

27. [BBC09] If e is an edge of G, then $\mathrm{mr}^F(G) \leq \mathrm{mr}^F(G_e) \leq \mathrm{mr}^F(G) + 1$. If e is incident with a vertex of degree at most 2, then $\mathrm{mr}^F(G_e) = \mathrm{mr}^F(G)$.

28. [BBC] For any graph G, $\mathrm{M}^F(\hat{\overline{G}}) = \mathrm{Z}(\hat{\overline{G}})$, the value is determined, and it is shown that $\hat{\overline{G}}$ has a universally optimal matrix. The first study of $\mathrm{mr}^F(\hat{\overline{G}})$ was in [BBC09] where the equality of maximum nullity and zero forcing number and $\mathrm{mr}^F(\hat{\overline{G}}) = 2|G| - 2$ were established for any graph with a Hamiltonian path; these results were extended to any graph with no cut-edge in [CCH12].

29. A symmetric integer matrix A can be interpreted as a matrix in $\mathbb{Q}$, or as a matrix in $\mathbb{Z}_p$ with p prime by considering entries mod p, and hence as in any field. Some nonzero integers (multiples of p) are interpreted as zero, but 1 and -1 are always nonzero. Thus if every off-diagonal entry of an integer matrix A is $0, 1$, or -1, then $G(A)$ does not depend on the field in which A is interpreted.

30. If $A \in \mathbb{Z}^{n\times n}$, then for every prime p, $\mathrm{rank}^{\mathbb{Z}_p} A \leq \mathrm{rank}^{\mathbb{Q}} A = \mathrm{rank}^{\mathbb{R}} A$.

Examples 2.1.

1. The simple graph G shown in Figure 2 is field independent and $\mathrm{mr}^F(G) = 2$, as can be seen by considering the (universally optimal) matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

[1]The case $F \neq \mathbb{Z}_2$ is Theorem 4.4 in [Sin10]. The case $F = \mathbb{Z}_2$ then follows from Fact 2.

2. For the full house graph H shown in Figure 5(a) $\mathrm{mr}^{\mathbb{R}}(H) = 2$, as can be seen by considering the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$, whereas $\mathrm{mr}^{\mathbb{Z}_2}(H) = 3$ by computation with all possible diagonal elements or by Fact 24.

3. $\mathrm{mr}^{\mathbb{Z}_3}(\overline{3K_2 \cup K_1}) = 3$ by Fact 23 and $\mathrm{mr}^{F}(\overline{3K_2 \cup K_1}) = 2$ where F is an infinite field of characteristic 3 by Fact 21. $\overline{3K_2 \cup K_1}$ is shown in Figure 5(b).

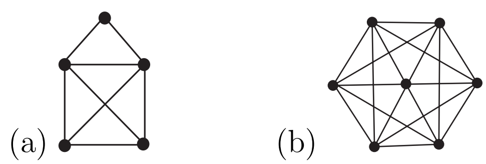


Figure 5: (a) The full house graph H for Example 2 and (b) $\overline{3K_2 \cup K_1}$ for Example 3.

3 Minimum Positive Semidefinite Rank

Orthogonal labelings of the vertices of a simple graph has long been a topic of interest among researchers in graph theory. Such labelings lead to an association between graphs and positive semidefinite matrices via the Gram matrix corresponding to the labeling vectors. In this section we explore this concept further including a list of many recent advances that have sparked additional current interest in this topic. All graphs in this section are simple except where noted otherwise.

Definitions 3.1. Notation: Let G be a simple graph with vertices $\{1, \ldots, n\}$.

$\mathcal{S}_+(G)$ denotes the subset of $\mathcal{S}(G)$ consisting of all real positive semidefinite matrices.

$\mathcal{H}_+(G)$ denotes the subset of positive semidefinite matrices among all complex Hermitian matrices A such that $G(A) = G$.

For any graph G, $\mathrm{mr}_+(G) = \min\{\operatorname{rank} A : A \in \mathcal{S}_+(G)\}$ is called the **(real) minimum positive semidefinite rank of** G.

$\mathrm{mr}_+^{\mathbb{C}}(G) = \min\{\operatorname{rank} A : A \in \mathcal{H}_+(G)\}$ is called the **complex (or Hermitian) minimum positive semidefinite rank of** G; $\mathrm{mr}_+^{\mathbb{C}}(G)$ is also denoted by $\mathrm{msr}(G)$ or $\mathrm{hmr}_+(G)$.

$\mathrm{M}_+(G) = \max\{\operatorname{null} A : A \in \mathcal{S}_+(G)\}$ is called the **maximum positive semidefinite nullity of** G.

$\mathrm{M}_+^{\mathbb{C}}(G) = \max\{\operatorname{null} A : A \in \mathcal{H}_+(G)\}$ is called the **complex (or Hermitian) maximum positive semidefinite nullity of** G.

If G is a graph and each vertex $i \in V$ is assigned the vector $\mathbf{v}_i \in \mathbb{R}^d$ such that $\{i, j\} \notin E$ implies $\mathbf{v}_i^T \mathbf{v}_j = 0$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an **orthogonal representation of** G. If in addition $\mathbf{v}_i^T \mathbf{v}_j = 0$ implies $\{i, j\} \notin E$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a **faithful orthogonal representation of** G.

The **tree cover number** of G, $\mathrm{T}(G)$, is the minimum number of vertex disjoint trees occurring as induced subgraphs of G that cover all the vertices of G.

A set of vertices is called **independent** if the subgraph induced on these vertices has no edges.

The **independence number of** G, $\iota(G)$, is the size of the largest independent set of vertices in G. (The standard notation for independence number is $\alpha(G)$ but $\iota(G)$ is used here due to conflict with the use of $\alpha(G)$ to denote the algebraic connectivity of G in Chapter 48.)

A vertex v in a graph G is called **simplicial** if the subgraph induced by the neighbors of v forms a clique.

For a given graph G and a vertex v of G, the multigraph corresponding to the **orthogonal removal of** v, denoted by $G \ominus v$, is obtained from the induced subgraph $G - v$, by adding $e - 1$ edges between any two neighbors of v, say u and w, where e is the sum of the number of edges between u and v and the number of edges between w and v. (Note: The graph $G \ominus v$ may contain multiple edges. See, for example, [Hol03] or [BFM11] for definitions of $\mathcal{S}_+(G)$ and $\mathrm{mr}_+(G)$ for multigraphs.)

Facts 3.1. Facts requiring proof for which no specific reference is given can be found in [FH07]. Additional relevant facts appear in Sections 4, 5, 6, and 7.
Notation: Let G be a simple graph with vertices $\{1, \ldots, n\}$.

1. For any graph G, $\mathrm{mr}(G) \leq \mathrm{mr}_+(G)$.
2. For any graph G, $\mathrm{mr}_+(G) + \mathrm{M}_+(G) = |G|$.
3. If the connected components of G are $G_1, \ldots, G_t$, then
$$\mathrm{mr}_+(G) = \sum_{i=1}^{t} \mathrm{mr}_+(G_i) \qquad \text{and} \qquad \mathrm{M}_+(G) = \sum_{i=1}^{t} \mathrm{M}_+(G_i).$$
4. $0 \leq \mathrm{mr}_+(G) \leq |G| - 1$, and if G contains an edge, then $1 \leq \mathrm{mr}_+(G) \leq |G| - 1$.
5. For $n \geq 2$, $\mathrm{mr}_+(K_n) = 1$, and if G is connected, $\mathrm{mr}_+(G) = 1$ implies $G = K_{|G|}$.
6. [Hol03] $\mathrm{mr}_+(G) = |G| - 1$ if and only if G is a tree.
7. If $G[W]$ is an induced subgraph of G, then $\mathrm{mr}_+(G[W]) \leq \mathrm{mr}_+(G)$.
8. For a connected graph G, $\mathrm{diam}(G) \leq \mathrm{mr}_+(G)$
9. The smallest d such that the graph G admits a faithful orthogonal representation with vectors lying in $\mathbb{R}^d$ is equal to $\mathrm{mr}_+(G)$.
10. If $G = \cup_{i=1}^{h} G_i$, then $\mathrm{mr}_+(G) \leq \sum_{i=1}^{h} \mathrm{mr}_+(G_i)$.
11. For any vertex v of G, $0 \leq \mathrm{mr}_+(G) - \mathrm{mr}_+(G - v) \leq \delta(v)$ (where $\delta(v)$ denotes the degree of v).
12. If G is a graph, $\mathrm{mr}_+(G) \leq \mathrm{cc}(G)$.
13. [BHH08] If G is a chordal graph, then $\mathrm{mr}_+(G) = \mathrm{cc}(G)$.

14. [LSS89, LSS00] $\mathrm{mr}_+(G) \leq |G| - \kappa(G)$.

15. [Hol09], [BHH11] Let v be a cut-vertex of G. For $i = 1, \ldots, h$, let $W_i \subseteq V(G)$ be the vertices of the ith component of $G - v$ and let G_i be the subgraph induced by $\{v\} \cup W_i$. Then
$$\mathrm{mr}_+(G) = \sum_{i=1}^{h} \mathrm{mr}_+(G_i).$$

16. [Dea11] For triangle free graphs, $\mathrm{mr}_+(G) \geq \lceil n/2 \rceil$.

17. [BHL04] A graph G has $\mathrm{mr}_+(G) \leq 2$ if and only if the complement of G is of the form $(K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$ for appropriate nonnegative integers $k, p_1, q_1, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$ for all $i = 1, \ldots, k$.

18. [BBF12], [HHL09] If G and H are two graphs, then
$$\mathrm{mr}_+(G \vee H) = \max\{\mathrm{mr}_+(G \vee K_1), \mathrm{mr}_+(H \vee K_1)\},$$
where K_1 is the complete graph on a single vertex.

19. If G and H do not contain any isolated vertices, then
$$\mathrm{mr}_+(G \vee H) = \max\{\mathrm{mr}_+(G), \mathrm{mr}_+(H)\}.$$

20. For any graph G, $\mathrm{mr}_+(G \vee K_1) = \mathrm{mr}_+(G)$ plus the number of isolated vertices of G.

21. Let G be a decomposable graph with s isolated vertices. Then $\mathrm{mr}_+(G) = \iota(G) - s$.

22. [Hol09] If G_e is obtained from G by an edge subdivision, then $\mathrm{mr}_+(G_e) = \mathrm{mr}_+(G) + 1$. (This fact also follows easily from Fact 27 in this section.)

23. [BFM11][2] If G is outerplanar or a partial 2-tree, then $\mathrm{M}_+(G) = \mathrm{T}(G)$.

24. [SH11] If G is a partial k-tree, then $\mathrm{mr}_+(\overline{G}) \leq k + 2$.

25. [BHH08] For any vertex v of G, $M_+(G) \leq M_+(G \ominus v)$. However, $M_+(G \ominus v) - M_+(G)$ may be arbitrarily large.

26. [BHH08] For any simplicial vertex v of a connected graph G, $M_+(G) = M_+(G \ominus v)$.

27. [JMN08] If v is a vertex of degree two in a connected graph G, then $M_+(G) = M_+(G \ominus v)$.

28. [BBF10] For any graph G, $\mathrm{mr}_+^{\mathbb{C}}(G) \leq \mathrm{mr}_+(G)$ and there exist graphs for which $\mathrm{mr}_+^{\mathbb{C}}(G) < \mathrm{mr}_+(G)$.

[2] The proof given in [BFM11] for outerplaner graphs remains valid for partial 2-trees [EEH12].

29. The results in Facts 1 through 27 also hold for $\operatorname{mr}_+^{\mathbb{C}}(G)$ (adapted in the obvious way). For Fact 18, the first reference deals with the case of $\operatorname{mr}_+(G)$ and the second reference deals with the case of $\operatorname{mr}_+^{\mathbb{C}}(G)$. In other cases, the reference gives the result for $\operatorname{mr}_+^{\mathbb{C}}(G)$ and that implies the result for $\operatorname{mr}_+(G)$, or vice versa. Finally, for some facts the proof is clearly valid for both $\mathbb{R}$ and $\mathbb{C}$ (even if it is stated only for one).

30. $\operatorname{mr}_+(G) = \operatorname{mr}_+^{\mathbb{C}}(G) = \operatorname{mr}(G)$ has been computed for the following graphs (when the result is not obvious, the first reference is for minimum positive semidefinite rank and the second for minimum rank, or the single reference does both).

    (a) For $n \geq 1$, $\operatorname{mr}_+(P_n) = n - 1$.

    (b) For $n \geq 3$, $\operatorname{mr}_+(C_n) = n - 2$.

    (c) For $p, q \leq 1$, $\operatorname{mr}_+(K_{p,q}) = \max\{p, q\}$.

    (d) For $n \geq 4$, $\operatorname{mr}_+(W_n) = n - 3$, where $W_n = C_{n-1} \vee K_1$ is the wheel on n vertices.

    (e) [Pet12] $\operatorname{mr}_+(Q_k) = 2^{k-1}$, where Q_k is the kth hypercube.

    (f) [Pet12], [HH06] If G is a polygonal path, then $\operatorname{mr}_+(G) = |G| - 2$.

    (g) [Pet12], [AIM08] $\operatorname{mr}_+(K_s \times P_t) = s(t - 1)$.

    (h) [AIM08] $\operatorname{mr}_+(T_k) = \frac{1}{2}k(k - 1)$, where T_k is the kth supertriangle (see [AIM08] for the definition of supertriangle).

    (i) [Pet12], [DGH09] $\operatorname{mr}_+(\mathrm{Half}_k) = k$, where Half_k is the kth half-graph (see [DGH09] for the definition of half-graph).

    (j) [Pet12], [DGH09] $\operatorname{mr}_+(N_k) = 3k - 2$, where N_k is the kth necklace (see [DGH09] for the definition of necklace).

Examples 3.1. Additional relevant examples appear in Sections 4, 5, 6, and 7.

1. If G is the graph from Figure 2, then assigning the standard basis vector $\mathbf{e}_1$ from $\mathbb{R}^2$ to vertex 1, $\mathbf{e}_2 \in \mathbb{R}^2$ to vertices 3 and 4, and $\mathbf{e}_1 + \mathbf{e}_2$ to vertex 2, is a faithful orthogonal representation of G. Further, if

$$B = \begin{bmatrix} \mathbf{e}_1, & \mathbf{e}_1 + \mathbf{e}_2, & \mathbf{e}_2, & \mathbf{e}_2 \end{bmatrix},$$

   then B is a 2×4 real matrix such that $B^T B$ is a positive semidefinite matrix in $\mathcal{S}(G)$ (in particular, $B^T B = A$ where A is the matrix in Example 2 in Section 1).

2. For the graph in Figure 4(a), we may compute $\operatorname{mr}_+(G)$ by using Facts 15 and/or 23. As noted in Example 1.7, G has cut-vertex 4 and the induced subgraphs $G_1 = G[\{1, 2, 3, 4\}]$ and $G_2 = G[\{4, 5, 6, 7, 8, 9, 10, 11, 12\}]$ associated with the two components. By Facts 12 and 5, $\operatorname{mr}_+(G[\{1, 2, 3, 4\}]) = 2$. Since G_2 is outerplanar, we know that $\mathrm{M}_+(G_2) = \mathrm{T}(G_2)$. Since every cycle contains a tree that is disjoint from the other cycles, it is easy to see that $\mathrm{T}(G_2) = 4$. Thus, $\operatorname{mr}_+(G_2) = 9 - 4 = 5$, and $\operatorname{mr}_+(G) = 2 + 5 = 7$.

4 Zero Forcing Parameters

Many of the parameters defined and utilized thus far are algebraic in nature and are therefore potentially dependent on the underlying field over which they are defined. In fact, this is easy to see for the parameter $\operatorname{mr}(\cdot)$ relative to finite fields and fields of characteristic two, but it is also true that minimum rank can differ over the fields $\mathbb{Q}$ or $\mathbb{C}$. Given that minimum rank relies heavily on graphs and their combinatorial properties, it is also appropriate to approach this issue from the point of view of purely combinatorial parameters.

One such avenue is the concept of *zero forcing parameters*. The underlying technique had been used previously, but the introduction of coloring notation at the AIM workshop "Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns" [AIM06] led to much wider use of this technique. Zero forcing is the same as graph infection used by physicists to study control of quantum systems [BG07], [Sev08] (see Application 1 for more information).

To begin with we consider a basic result for bounding the nullity of a given matrix. Suppose the dimension of the null space of a given matrix A is greater than k. Then, we may conclude that for any set of k indices I from $\{1, \ldots, n\}$, there exists a nonzero null vector of A for which all of its coordinates from I are zero. Indeed, if for a given index set I of size k, the only null vector for which the coordinates that correspond to the positions from I are zero is the zero vector, then it follows that the dimension of the null space is at most k. The operation of the color change rule reflects the forcing of zeros in a null vector, starting with zeros in the positions labelled by a zero forcing set, and arguing sequentially that the entries in all other positions must be zero. There is also an analogous zero forcing parameter for minimum positive semidefinite rank (discussed in this section) and other variants (discussed in Section 6). All graphs in this section are simple.

Definitions 4.1. Let $G = (V, E)$ be a simple graph.

A subset $Z \subseteq V$ defines an initial set of black[3] vertices (with all vertices not in Z white); this is called a **coloring of** G.

The **color change rule** is to change the color of a white vertex w to black if w is the unique white neighbor of a black vertex u; in this case, we say u **forces** w and write $u \to w$.

Given a coloring of G, the **derived set** or **final coloring** is the set of black vertices obtained by applying the color change rule until no more changes are possible.

A **zero forcing set** for G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived set is V.

The **zero forcing number** $\operatorname{Z}(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V$.

For a given zero forcing set, we construct the derived set, listing the forces in the order in which they were performed. This list is a **chronological list of forces**.

A **forcing chain** (for a particular chronological list of forces $\mathcal{F}$) is a sequence of vertices $(v_1, v_2, \ldots, v_s)$ such that for $i = 1, \ldots, k-1$, $v_i \to v_{i+1}$ in $\mathcal{F}$.

A **maximal forcing chain** is a forcing chain that is not a proper subsequence of another forcing chain.

[3] In the literature, "black" is the term commonly used to indicate a zero entry in the null vector, but some authors are changing to "blue" to avoid negative social implications. Here we follow the literature as of 2012.

The **forcing chain cover** (for a chronological list of forces $\mathcal{F}$) is the set of maximal forcing chains.

The **positive semidefinite color change rule** is: Let B be the set consisting of all the black vertices. Let $W_1, \dots, W_k$ be the sets of vertices of the $k \geq 1$ components of $G - B$. Let $w \in W_i$. If $u \in B$ and w is the only white neighbor of u in $G[W_i \cup B]$, then change the color of w to black; in this case, we say u **forces** w and write $u \to w$.

A **positive semidefinite zero forcing set** for G is a subset of vertices B such that if initially the vertices in B are colored black and the remaining vertices are colored white, the set of black vertices obtained by applying the color change rule until no more changes are possible is V.

The **positive semidefinite zero forcing number** $\mathrm{Z}_+(G)$ is the minimum of $|X|$ over all positive semidefinite zero forcing sets $X \subseteq V$ (using the positive semidefinite color change rule).

For a given positive semidefinite zero forcing set, we perform forces to color all vertices black, listing the forces in the order in which they were performed. This list is a **chronological list of forces**.

For a vertex x in a positive semidefinite zero forcing set X, and chronological list of forces $\mathcal{F}$ of X, define V_x to be the set of vertices w such that there is a sequence of forces $x = v_1 \to v_2 \to \cdots \to v_k = w$ in $\mathcal{F}$ (the empty sequence of forces is permitted, i.e., $x \in V_x$). The **forcing tree** T_x is the induced subgraph $T_x = G[V_x]$.

The **forcing tree cover** (for a chronological list of forces $\mathcal{F}$) is $\{T_b \,|\, b \in B\}$.

Suppose $S = (v_1, v_2, \dots, v_m)$ is an ordered subset of vertices from a given graph G. For each k with $1 \leq k \leq m$, let G_k be the subgraph of G induced by $\{v_1, v_2, \dots, v_k\}$, and let H_k be the connected component of G_k that contains v_k. If for each k, there exists a vertex w_k that satisfies: $w_k \neq v_l$ for $l \leq k$, $\{w_k, v_k\} \in E$, and $\{w_k, v_s\} \notin E$, for all v_s in H_k with $s \neq k$, then S is called an **ordered set of vertices** in G, or an **OS-set**.

The **OS number** of a graph G, denoted by $\mathrm{OS}(G)$, is the maximum of $|S|$ over all OS-sets S of G.

If G is a graph, then H is called a **supergraph** of G, if H is obtained from G by adding some edges.

A graph G is called a **graph on two parallel paths** if V can be partitioned into disjoint nonempty subsets U_1 and U_2 so that the induced subgraphs $P_i = G[U_i], i = 1, 2$ are paths, G can be drawn in the plane with the paths P_1 and P_2 as parallel line segments, and edges between the two paths (drawn as line segments, not curves) do not cross. A single path is not considered a graph on two parallel paths, but a graph consisting of two disjoint paths is a graph on two parallel paths.

A graph G is a **unit interval graph** if there is an ordering on the vertices of G such that for each vertex v, the closed neighborhood of v (i.e., the union of v and its neighbors) is a set of consecutive vertices in that order.

Facts 4.1. Additional relevant facts appear in Sections 6 and 7.
Notation: Let G be a simple graph.

Facts about $\mathrm{Z}(G)$

1. For any graph G, $1 \leq \mathrm{Z}(G) \leq |G|$, and if G contains at least one edge, $1 \leq \mathrm{Z}(G) \leq |G| - 1$.

2. If the connected components of G are $G_1, \dots, G_t$, then
$$\mathrm{Z}(G) = \sum_{i=1}^{t} \mathrm{Z}(G_i).$$

3. [AIM08] For any graph G, $\mathrm{M}(G) \le \mathrm{Z}(G)$. (In fact, $\mathrm{M}^F(G) \le \mathrm{Z}(G)$.)

4. [BBF10] For any graph G, $\delta(G) \le \mathrm{Z}(G)$.

5. [BBF10] For any graph G, $\mathrm{P}(G) \le \mathrm{Z}(G)$. For any chronological list of forces of a zero forcing set, the forcing chain cover is a path cover.

6. [AIM08] For any tree T, $\mathrm{Z}(T) = \mathrm{M}(T)$ $(= \mathrm{P}(T))$.

7. [BDG] Free open-source *Sage* software is available online that computes $\mathrm{Z}(G)$.

8. [BBF10] If G is connected of order greater than one, then G does not have a unique minimum zero forcing set, and no single vertex is a member of every minimum zero forcing set for G.

9. [HCY10a] Let v be a cut-vertex of G. Suppose that $G - v$ is the disjoint union of two graphs induced on the vertices W_1 and W_2. For $i = 1, 2$, let G_i be the subgraph induced by $\{v\} \cup W_i$. Then
$$\mathrm{Z}(G) \le \min\left\{\mathrm{Z}(G_1) + \mathrm{Z}(G_2 - v), \mathrm{Z}(G_2) + \mathrm{Z}(G_1 - v)\right\}.$$

10. [EHH12] For any vertex v and edge e of G, we have $-1 \le \mathrm{Z}(G) - \mathrm{Z}(G - v) \le 1$ and $-1 \le \mathrm{Z}(G) - \mathrm{Z}(G - e) \le 1$.

11. [AIM08] For all simple graphs G and H, $\mathrm{Z}(G \times H) \le \min\{\mathrm{Z}(G)|H|,\ \mathrm{Z}(H)|G|\}$.

12. [AIM08] $\mathrm{Z}(G) \ge |G| - 2$ if and only if G does not contain P_4, $P_3 \cup K_2$, Dart, $\ltimes$, or $3K_2$ as an induced subgraph.

13. $\mathrm{Z}(G) = 1$ if and only if G is a path.

14. [Row12] $\mathrm{Z}(G) = 2$ if and only if G is a graph on two parallel paths.

15. [HCY10a] If G is a connected unit interval graph, then $\mathrm{cc}(G) = \mathrm{mr}(G)$ and $\mathrm{Z}(G) = \mathrm{M}(G)$.

16. For $p, q \ge 1$, $\mathrm{Z}(K_{p,q}) = p + q - 2 = \mathrm{M}(K_{p,q})$.

17. For every graph listed in Fact 3.30, $\mathrm{Z}(G) = \mathrm{M}(G)$.

Facts about $\mathrm{Z}_+(G)$ **and** $\mathrm{OS}(G)$

18. Since any zero forcing set is a positive semidefinite zero forcing set, $\mathrm{Z}_+(G) \le \mathrm{Z}(G)$.

19. For any graph G, $1 \leq \mathrm{Z}_+(G) \leq |G|$, and if G contains at least one edge, $1 \leq \mathrm{Z}_+(G) \leq |G| - 1$.

20. For any graph G that is the disjoint union of connected components $G_i, i = 1, 2, \ldots, k$, $\mathrm{Z}_+(G) = \sum_{i=1}^{k} \mathrm{Z}_+(G_i)$.

21. [BBF10] For any graph G, $\mathrm{M}_+(G) \leq \mathrm{M}_+^{\mathbb{C}}(G) \leq \mathrm{Z}_+(G)$.

22. [MNZ10], [BBF10] For every graph G, $\delta(G) \leq \mathrm{Z}_+(G)$.

23. [EEH13] For any graph G, $\mathrm{T}(G) \leq \mathrm{Z}_+(G)$. For any chronological list of forces of a positive semidefinite zero forcing set, the forcing tree cover is a tree cover.

24. [EEH12] If G is a partial 2-tree (this includes every outerplanar graph), then $\mathrm{Z}_+(G) = \mathrm{M}_+(G)$ $(= \mathrm{T}(G))$.

25. [BDG] Free open-source *Sage* software is available online that computes $\mathrm{Z}_+(G)$.

26. [HHL09] For any graph G, $\mathrm{OS}(G) \leq \mathrm{mr}_+^{\mathbb{C}}(G) \leq \mathrm{mr}_+(G)$.

27. [Mit11] For any graph G, there exists a chordal supergraph H of G such that $\mathrm{OS}(G) = \mathrm{mr}_+(H) = \mathrm{cc}(H)$.

28. [BBF10] $\mathrm{Z}_+(G) + OS(G) = |G|$. Furthermore, for any ordered set S, $V \setminus S$ is a positive semidefinite forcing set for G, and for any positive semidefinite forcing set X for G, there is an order that makes $V \setminus X$ an ordered set for G.

29. [MNZ10], [EEH13] No connected graph G of order greater than one has a unique minimum positive definite zero forcing set. No single vertex is a member of every minimum positive definite zero forcing set for G. For any vertex v of G, there is a minimum positive definite zero forcing set containing v.

30. [MNZ10], [EEH13] Suppose $G_i, i = 1, \ldots, h$ are simple graphs of order at least two, $h \geq 2$, there is a vertex v such that for all $i \neq j$, $G_i \cap G_j = \{v\}$, and $G = \bigcup_{i=1}^{h} G_i$ (v is a cut-vertex of G). Then

$$\begin{aligned} \mathrm{OS}(G) &= \sum_{i=1}^{h} \mathrm{OS}(G_i), \\ \mathrm{Z}_+(G) &= \left(\sum_{i=1}^{h} \mathrm{Z}_+(G_i) \right) - h + 1. \end{aligned}$$

31. [EEH13] For any vertex v and edge e of G, $\mathrm{Z}_+(G) - \mathrm{Z}_+(G - v) \leq 1$ and $-1 \leq \mathrm{Z}_+(G) - \mathrm{Z}_+(G - e) \leq 1$.

32. [BBF10] For all simple graphs G and H,

$$\mathrm{Z}_+(G \times H) \leq \min\{\mathrm{Z}_+(G)|H|,\ \mathrm{Z}_+(H)|G|\},$$

(where $G \times H$ is the Cartesian product of G and H).

33. [MNZ12], [BBF10] Let G be a graph. Then $\mathrm{mr}_+^{\mathbb{C}}(G) = 1, 2, |G| - 2, |G| - 1$ if and only if $\mathrm{Z}_+(G) = |G| - 1, |G| - 2, 2, 1$, respectively.

34. [MNZ12] Let G be a graph. If $\mathrm{OS}(G) = |G| - 3$, then $\mathrm{mr}_+^{\mathbb{C}}(G) = |G| - 3$, and if $\mathrm{mr}_+^{\mathbb{C}}(G) = 3$, then $\mathrm{OS}(G) = 3$.

35. For $1 \le p \le q$, $\mathrm{Z}_+(K_{p,q}) = p = \mathrm{M}_+(K_{p,q})$.

36. For every graph listed in Fact 3.30, $\mathrm{Z}_+(G) = \mathrm{M}_+(G) = \mathrm{Z}(G) = \mathrm{M}(G)$ and $\mathrm{OS}(G) = \mathrm{mr}_+(G) = \mathrm{mr}(G)$.

Examples 4.1.

1. Either pendant vertex of P_n is a zero forcing set for P_n, and so $\mathrm{Z}(P_n) = 1$. Hence $\mathrm{M}(P_n) = 1$. Moreover, the only graph G with $Z(G) = 1$ is $G = P_n$.

2. For $n \ge 2$, $\mathrm{Z}(K_n) = \mathrm{Z}_+(K_n) = n - 1$. The only connected graph G with $\mathrm{Z}(G) = |G| - 1$ is the complete graph.

3. Let T be a tree. Then $\mathrm{Z}_+(T) = 1$ because any one vertex v is a positive semidefinite zero forcing set. Observe that $\mathrm{M}_+(T) = \mathrm{M}_+^{\mathbb{C}}(T) = 1$ is an immediate consequence of $\mathrm{Z}_+(T) = 1$.

4. For $n \ge 3$, any two consecutive vertices of C_n form a zero forcing set, and any two vertices form a positive definite zero forcing set. Thus, $\mathrm{Z}(C_n) = \mathrm{Z}_+(C_n) = 2$.

5. (Petersen graph) Consider the initial coloring given for the Petersen graph in Figure 6(a). Observe that the 5 black vertices form a zero forcing set for the Petersen graph, and thus the zero forcing number for the Petersen graph is at most 5. On the other hand, the adjacency matrix $\mathcal{A}$ has an eigenvalue with multiplicity five, and so there exists a translate of $\mathcal{A}$ with nullity 5. Thus, the zero forcing number of the Petersen graph (and the maximum nullity) is 5.

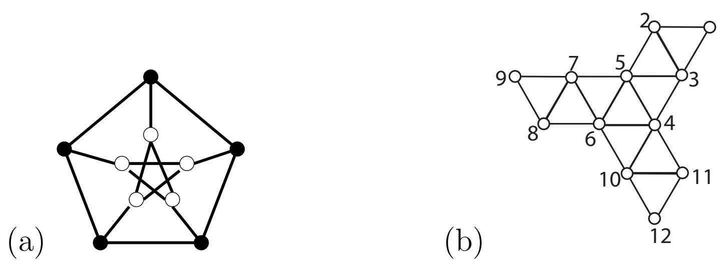


Figure 6: (a) Initial coloring for the Petersen graph, and (b) the pinwheel on 12 vertices, G_{12}.

6. Let G_{12} be the graph shown in Figure 6(b), called the pinwheel on 12 vertices. Note that G_{12} is an outerplanar 2-tree. The set $\{1, 2, 6, 10\}$ is a zero forcing set for G_{12}, so $\mathrm{Z}(G_{12}) \le 4$. In fact, $\mathrm{Z}(G_{12}) = 4$ [BBF10]. Since $\mathrm{cc}(G_{12}) = 9$, we have $\mathrm{mr}_+(G_{12}) \le \mathrm{mr}(G_{12}) \le \mathrm{cc}(G_{12}) = 9$. But since G_{12} is also chordal, equality holds, and hence

$M_+(G_{12}) = M(G_{12}) = 3$. For an alternate way to determine maximum positive semidefinite nullity, observe that since G_{12} is outerplanar and $T(G_{12}) = 3$, we deduce that $M_+(G_{12}) = 3$. Furthermore, G_{12} has $Z_+(G_{12}) = 3 = M_+(G_{12})$ because $X = \{4, 5, 6\}$ is a positive semidefinite zero forcing set ($G_{12} - X$ is disconnected, and X is a zero forcing set for $G[\{1, 2, 3, 4, 5, 6\}]$, etc.). Thus, for the pinwheel G_{12} we have $Z(G_{12}) = 4 > Z_+(G_{12}) = 3 = M_+(G_{12}) = M(G_{12}) = 3$.

7. There are examples of graphs for which Z_+ is not equal to $M_+^{\mathbb{C}}$. For example, in [MNZ10] it was established that the Möbius ladder on 8 vertices, shown in Figure 7 and sometimes denoted by ML_8 or V_8, satisfies $OS(ML_8) = 4$ and $\mathrm{mr}_+^{\mathbb{C}}(ML_8) = 5$. In this case, it follows that $Z_+(ML_8) = 4$, and hence $Z_+(ML_8) > 3 = M_+^{\mathbb{C}}(ML_8) = M_+(ML_8)$.

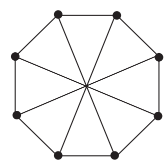


Figure 7: The graph $ML_8 = V_8$.

Applications 4.1.

1. Whether a quantum system is controllable is determined by a Lie algebra associated with the system (see Chapter 87 for definitions and background information on Lie algebras). The Lie algebra rank condition says that a necessary and sufficient condition for complete controllability of a system is that the Lie algebra generated by matrices associated with the set of admissible values for the control is the Lie algebra $u(n)$ of $n \times n$ skew-Hermitian matrices (or the subalgebra of $u(n)$ of matrices having trace 0). In [BDH13] it is shown that if G is a connected graph, $A \in \mathcal{S}_n$, and $Z \subseteq V$ is a zero forcing set of G, then the Lie algebra generated by iA and the matrices $i\mathbf{e}_j\mathbf{e}_j{}^T$ with $j \in Z$ is equal to $u(n)$ and thus the corresponding quantum system is controllable. However, the converse is false, i.e., having the Lie algebra generated by iA and $\{i\mathbf{e}_j\mathbf{e}_j{}^T : j \in Z\}$ being equal to $u(n)$ does not imply that Z is a zero forcing set.

5 Colin de Verdière Parameters

Recall that minimum rank is monotone on induced subgraphs (i.e., if $G[W]$ is an induced subgraph of a graph G, then $\mathrm{mr}(G[W]) \leq \mathrm{mr}(G)$), and this property can be useful in bounding $\mathrm{mr}(G)$ from below. Unfortunately, maximum nullity is not monotone for induced subgraphs (see Example 1 below). In 1990, Colin de Verdière ([CdV93] in English) introduced the graph parameter μ equal to the maximum nullity among certain generalized Laplacian matrices having a given graph and satisfying the Strong Arnold Hypothesis. The Colin de Verdière number μ was the first of several related parameters that are both minor monotone and bound the maximum nullity from below. In this section we discuss several Colin de Verdière-type parameters and their use for computing the maximum nullity (or equivalently, the minimum rank) of a graph. These parameters are most useful when the graph has a large number of edges (since a matrix with many nonzero entries is more likely to satisfy the Strong Arnold Hypothesis), and least useful for trees, where a convenient method already exists for evaluation of maximum nullity and minimum rank. For bounding maximum nullity, these parameters should be used only for connected graphs (each component should be analyzed separately). Reference [HLS96] provides an excellent introduction to the parameter μ and the Strong Arnold Hypothesis from a linear algebra perspective. All graphs in this section are simple.

Definitions 5.1. Notation: Let G be a simple graph with vertices $\{1, \dots, n\}$.

A graph parameter ζ is **monotone on induced subgraphs** if for any induced subgraph $G[W]$ of G, $\zeta(G[W]) \le \zeta(G)$.

A **contraction** of an edge in a (simple) graph G is obtained by identifying the two endpoints of the edge (which are adjacent vertices of G) and suppressing any loops or multiple edges that arise in this process.

A **minor** of G arises by performing a series of deletions of edges, deletions of isolated vertices, and/or contraction of edges.

A graph parameter ζ is **minor monotone** if for any minor G' of G, $\zeta(G') \le \zeta(G)$.

For any graph G, the **Hadwiger number** $\mathrm{h}(G)$ is the maximum size of a clique minor in G.

A real symmetric matrix M satisfies the **Strong Arnold Hypothesis** provided there does not exist a nonzero real symmetric matrix X satisfying:

(i) $MX = \mathbf{0}$,

(ii) $M \circ X = \mathbf{0}$,

(iii) $I \circ X = \mathbf{0}$.

where $\circ$ denotes the Hadamard (entrywise) product and I is the identity matrix.

A matrix $L \in \mathcal{S}(G)$ is a **generalized Laplacian matrix** of G if L is a Z-matrix (i.e., all off-diagonal entries of L are non-positive).

The **Colin de Verdière number** $\mu(G)$ is the maximum nullity among matrices L that satisfy the following:

- L is a generalized Laplacian matrix of G,
- L has exactly one negative eigenvalue (of multiplicity 1),
- L satisfies the Strong Arnold Hypothesis.

The **μ-minimum rank** of G, $\mathrm{mr}_\mu(G)$, is the minimum of the ranks of this set of matrices.

The parameter $\nu(G)$ (also denoted $\nu^{\mathbb{R}}(G)$) is the maximum nullity among matrices A that satisfy the following:

- $A \in \mathcal{S}_+(G)$,
- A satisfies the Strong Arnold Hypothesis.

The **ν-minimum rank** of G, $\mathrm{mr}_\nu(G)$, is the minimum of the ranks of this set of matrices.

The parameter $\xi(G)$ is the maximum nullity among matrices A that satisfy the following:

- $A \in \mathcal{S}(G)$,
- A satisfies the Strong Arnold Hypothesis.

The ξ-**minimum rank** of G, $\mathrm{mr}_\xi(G)$, is the minimum of the ranks of this set of matrices.

Facts 5.1. Additional relevant facts appear in Sections 6 and 7.
Notation: Let G be a simple graph.

1. $\mathrm{mr}_\mu(G) = |G| - \mu(G)$, $\mathrm{mr}_\nu(G) = |G| - \nu(G)$, and $\mathrm{mr}_\xi(G) = |G| - \xi(G)$.
2. $\mu(G) \leq \xi(G) \leq \mathrm{M}(G)$, $\nu(G) \leq \xi(G)$, and $\nu(G) \leq \mathrm{M}_+(G)$ and these inequalities can be strict (see Example 3 and Facts 10, 11, and 17 below for examples).
3. $\mathrm{mr}_\mu(G) \geq \mathrm{mr}_\xi(G) \geq \mathrm{mr}(G)$, $\mathrm{mr}_\nu(G) \geq \mathrm{mr}_\xi(G)$, and $\mathrm{mr}_\nu(G) \geq \mathrm{mr}_+(G)$ and these inequalities can be strict.
4. A subgraph is a minor. A minor monotone graph parameter is monotone on subgraphs and thus on induced subgraphs.
5. [CdV93], [HLS96] The Strong Arnold Hypothesis is equivalent to the requirement that certain manifolds intersect transversally.
6. [CdV93], [HLS96] The parameter μ is minor monotone.
7. [CdV98] The parameter ν is minor monotone.
8. [BFH05b] The parameter ξ is minor monotone.
9. [LSS89],[LSS00], [Hol08b] $\kappa(G) \leq \nu(G)$.
10. [BFH05b] If G is the disjoint union of components $G_i, i = 1, \ldots, k$ then $\xi(G) = \max_{i=1}^k \xi(G_i)$ (whereas $\mathrm{M}(G) = \sum_{i=1}^k \mathrm{M}(G_i)$).
11. [Hol02b] If G is the disjoint union of components $G_i, i = 1, \ldots, k$ then $\nu(G) = \max_{i=1}^k \nu(G_i)$ (whereas $\mathrm{M}_+(G) = \sum_{i=1}^k \mathrm{M}_+(G_i)$).
12. [HLS96] If G is the disjoint union of components $G_i, i = 1, \ldots, k$ and G has at least one edge, then $\mu(G) = \max_{i=1}^k \mu(G_i)$.
13. For a path P_n, $\mu(P_n) = \nu(P_n) = \xi(P_n) = 1$.
14. For $n \geq 2$, $\mu(K_n) = \nu(K_n) = \xi(K_n) = n - 1$
15. [BFH05b], [CdV93], [HLS96], [Pet12] For $p \leq q$ and $q \geq 3$, $\mu(K_{p,q}) = \xi(K_{p,q}) = p + 1$ and $\nu(K_{p,q}) = p$.
16. $\mathrm{h}(G) - 1 \leq \mu(G)$ and $\mathrm{h}(G) - 1 \leq \nu(G)$. These inequalities can be strict (see Examples 4 and 5 below).

17. [BFH05b], [CdV93], [HLS96], [CdV98] If T is a tree that is not a path, then

$$\xi(T) = \mu(T) = 2 \text{ and } \nu(T) = 1.$$

18. [CdV93], [HLS96] μ is used to characterize planarity:
    (a) $\mu(G) \le 1$ if and only if G is a disjoint union of paths,
    (b) $\mu(G) \le 2$ if and only if G is outerplanar,
    (c) $\mu(G) \le 3$ if and only if G is planar.

19. [KLV97][4]
    (a) If $\overline{G}$ is a disjoint union of paths, then $\mu(G) \ge |G| - 3$.
    (b) If $\overline{G}$ is outerplanar, then $\mu(G) \ge |G| - 4$.
    (c) If $\overline{G}$ is planar, then $\mu(G) \ge |G| - 5$.

20. [GB11] $\mu(G) \le \mathrm{tw}(G) + 1$.

21. [BBF13] $\nu(G) \le \mathrm{tw}(G) + 1$.

22. [GB11] If G is chordal, then $\mu(G) \le \omega(G)$ (where ω is the maximum size of a clique subgraph of G).

23. [Gol09] Assume G_1 and G_2 are connected. Then $\mu(G_1 \times G_2) \ge \mu(G_1) + \mathrm{h}(G_2) - 1$, and it is conjectured there that $\mu(G_1 \times G_2) \ge \mu(G_1) + \mu(G_2)$.

Examples 5.1.

1. The maximum nullity $\mathrm{M}(\cdot)$ is not, in general, monotone on induced subgraphs. Consider the graph G shown in Figure 8. By Fact 1.17 we have that $\mathrm{mr}(G) = 4$, so $\mathrm{M}(G) = 2$. Deleting the vertex of degree three produces $K_{1,4}$ and $\mathrm{M}(K_{1,4}) = 3$. In

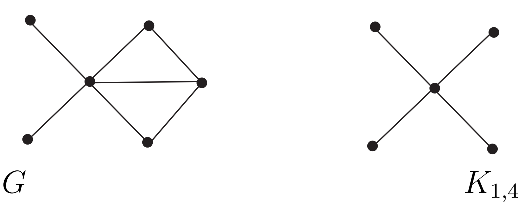


Figure 8: A graph G for which maximum nullity is not monotone on induced subgraphs

general, the Strong Arnold Hypothesis is relevant for minor-monotonicity. Any matrix

[4]The reader is warned that as used in [KLV97], the notation $\nu(G)$ is defined using certain vector representations and is equal to $|G| - \mu(\overline{G}) + 1$ (for $|G| > 2$), *not* the Colin de Verdière parameter $\nu(G)$ used in this chapter.

realizing $\mathrm{M}(K_{1,4}) = 3$ does not satisfy the Strong Arnold Hypothesis: If $A \in \mathcal{S}(K_{1,4})$ and $\operatorname{rank} A = 2$, then

$$A = \begin{bmatrix} x & a & b & c & d \\ a & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 0 \end{bmatrix},$$

where a, b, c and d are all nonzero. Then setting

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{cd}{ab} & \frac{bd}{ac} & \frac{-b-c}{a} \\ 0 & \frac{cd}{ab} & 0 & \frac{-bd-cd}{bc} & 1 \\ 0 & \frac{bd}{ac} & \frac{-bd-cd}{bc} & 0 & 1 \\ 0 & \frac{-b-c}{a} & 1 & 1 & 0 \end{bmatrix},$$

shows that A does not satisfy the Strong Arnold Hypothesis.

2. This example demonstrates the use of minor monotonicity to compute the minimum rank of the graph G shown in Figure 9(a). Since $G[\{3,4,5,6\}] = P_4$ is an induced subgraph of G, $3 = \mathrm{mr}(P_4) \le \mathrm{mr}(G)$. Observe that the graphs shown in Figure 9(b),(c) are both minors of G (delete vertex 6 to obtain (b) and then contract on edge $\{1,5\}$ in (b) to obtain (c)). Hence, $3 = \mathrm{h}(K_4) - 1 = \xi(K_4) \le \xi(G) \le M(G)$. Thus, $\mathrm{mr}(G) \le 6 - 3 = 3$, which implies $\mathrm{mr}(G) = 3$.

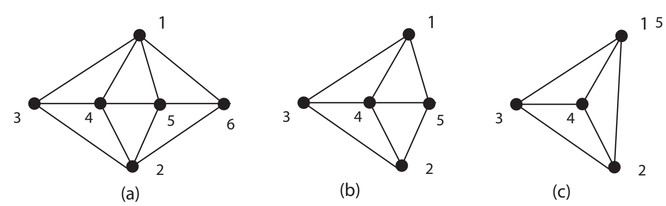


Figure 9: Using ξ to compute minimum rank.

3. Let G be the graph shown in Figure 10(a). Then $\xi(G) = 3$ because the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has nullity 3, $A \in \mathcal{S}(G)$, and A satisfies the Strong Arnold Hypothesis. Since G is outerplanar, $\mu(G) = 2$. In addition, we see that $\nu(G) \le \mathrm{M}_+(G) \le \mathrm{Z}_+(G) \le 2$ because $X = \{1, 4\}$ is a positive semidefinite zero forcing set. Since K_3 is a minor of G, $\nu(G) = \mathrm{M}_+(G) = \mathrm{Z}_+(G) = 2$.

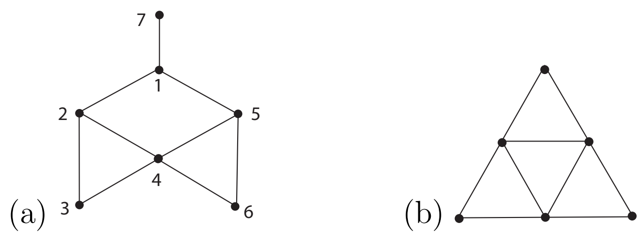


Figure 10: (a) The graph G for Example 3 and (b) the supertriangle T_3.

4. The supertriangle T_3 is shown in Figure 10(b). Observe that $\mathrm{h}(T_3) = 3$, so $\mathrm{h}(T_3) - 1 = 2$. The clique covering of T_3 by three triangles gives a matrix that satisfies the Strong Arnold Hypothesis, so $\nu(T_3) \geq 3$. Since P_4 is an induced subgraph of T_3, $\mathrm{mr}_+(T_3) \geq 3$, so $\nu(T_3) = 3$.

5. Observe that $\mathrm{h}(K_{2,3}) = 3$, so $\mathrm{h}(K_{2,3}) - 1 = 2$. Since $K_{2,3}$ is planar but not outerplanar, $\mu(K_{2,3}) = 3$.

6 Advanced Topics

All graphs in this section are simple.

Definitions 6.1. Notation: Let G be a simple graph with vertices $\{1, \ldots, n\}$.

For any $n \times n$ real symmetric matrix A, we define the **inertia** of A as the triple $(\pi(A), \nu(A), \delta(A))$, consisting of the number of positive, negative, and zero eigenvalues (counting multiplicity) of A, respectively.

Suppose A is an $n \times n$ real symmetric matrix. A **nonzero (h, k)-representation of** A is a $(h + k) \times n$ matrix

$$\begin{bmatrix} P_A \\ N_A \end{bmatrix}$$

with no zero columns such that P_A has h rows, N_A has k rows, and $A = P_A^T P_A - N_A^T N_A$.

A symmetric matrix $A \in \mathbb{R}^{n \times n}$ has **balanced inertia** if $\nu(A) \leq \pi(A) \leq \nu(A) + 1$.

A graph G is **inertia-balanced** if there exists $A \in \mathcal{S}(G)$ that satisfies $\mathrm{rank}(A) = \mathrm{mr}(G)$, and A has balanced inertia.

The **join minimum rank** of G is $\mathrm{jmr}(G) = \mathrm{mr}(G \vee K_1)$.

Let $G = \bigvee_{i=1}^{r} G_i$ be a join of r graphs. Then G is said to be **anomalous** if for each i, $\mathrm{jmr}(G_i) \leqslant 2$ and $K_{3,3,3} = \overline{K}_3 \vee \overline{K}_3 \vee \overline{K}_3$ is a subgraph of G.

A matrix $Q \in \mathbb{R}^{n \times n}$ of order $h + k$ is said to be **(h, k)-unitary** if $Q^T \tilde{I} Q = \tilde{I}$, where

$$\tilde{I} = \begin{bmatrix} I_h & 0 \\ 0 & -I_k \end{bmatrix}.$$

A set S of edges of a graph G (with $|G| > 1$) is a **disconnecting set** if $G - S$ has more than one component.

The **edge connectivity** of G, denoted $\kappa_e(G)$ is the minimum size of a disconnecting set of G.

A real matrix X is **generic** if every square submatrix of X is nonsingular.
The **generic nullity of a nonzero $n \times n$ matrix** A is defined by

$$GN(A) = \max\{k : AX = 0, X \text{ is a generic } n \times k \text{ matrix}\}.$$

The **maximum generic nullity of a graph** G is

$$GM(G) = \max\{GN(A) : A \in \mathcal{S}(G)\}.$$

The **average minimum rank** of all (labeled) graphs of order n is defined to be

$$\operatorname{amr}(n) = \frac{\sum_{|G|=n} \operatorname{mr}(G)}{2^{\binom{n}{2}}}.$$

The graph $G(n,p)$ denotes the Erdös-Rényi random graph of order n with edge probability p.

Facts 6.1. Notation: G is a simple graph.

Facts about inertia balanced graphs and joins of graphs

1. [BF07] Trees, K_n, and C_n are all inertia balanced.
2. [BHL09] Not all graphs are inertia balanced.
3. [BF07] $\operatorname{mr}(G) \leq \operatorname{jmr}(G) \leq \operatorname{mr}(G) + 2$.
4. [BF07] Let $G = \bigvee_{i=1}^{r} G_i$, $r > 1$, where each G_i is inertia-balanced. Then G is inertia-balanced, and
$$\operatorname{mr}(G) = \begin{cases} \max_i\{\operatorname{jmr}(G_i)\} & \text{if } G \text{ is not anomalous;} \\ 3 & \text{if } G \text{ is anomalous.} \end{cases}$$
5. [BF07] Let $G = \bigvee_{i=1}^{r} G_i$, $r > 1$, be a connected decomposable graph. Then G is inertia-balanced, and
$$\operatorname{mr}(G) = \begin{cases} \max_i\{\operatorname{jmr}(G_i)\} & \text{if } G \text{ is not anomalous;} \\ 3 & \text{if } G \text{ is anomalous.} \end{cases}$$
6. [BF07] [BBF12] Let G and H be two graphs and let $A \in \mathcal{S}(G)$ and $B \in \mathcal{S}(H)$. There exists an (h,k)-unitary matrix Q such that
$$\begin{bmatrix} P_A & P'_B \\ N_A & N'_B \end{bmatrix}$$
is a nonzero (h,k)-representation of a matrix in $\mathcal{S}(G \vee H)$ with
$$\begin{bmatrix} P'_B \\ N'_B \end{bmatrix} = Q \begin{bmatrix} P_B \\ N_B \end{bmatrix},$$
whenever $h \geq \max\{\pi(A), \pi(B)\}$ and $k \geq \max\{\nu(A), \nu(B)\}$.

7. [BBF12] Let G and H be graphs. If
   (a) G and H each have an edge, or
   (b) either G has an edge and $H = \overline{K_r}$, and $\text{mr}_\nu(G) \geq r$;
   or the same is true with the roles of G and H reversed,

   then
   $$\text{mr}_\nu(G \vee H) = \max\{\text{mr}_\nu(G), \text{mr}_\nu(H)\}.$$
   Otherwise,
   $$\text{mr}_\nu(G \vee H) = \max\{\text{mr}_\nu(G), \text{mr}_\nu(H)\} + 1.$$

Facts about maximum generic nullity

8. [HS10] If G is connected, then $GM(G) \leq \kappa_e(G)$.
9. [HS10] For any graph G, $\kappa(G) \leq GM(G)$.
10. [HS10] Examples of graphs are known to show that both of the inequalities $GM(G) \leq \kappa_e(G)$ and $\kappa(G) \leq GM(G)$ can be strict.
11. [HS10] As n goes to infinity almost all graphs satisfy
$$\kappa(G) = GM(G) = \kappa_e(G) = \delta(G).$$

Facts about average minimum rank

12. [HHM10] For any n, $\text{amr}(n)$ is equal to the expected value of the minimum rank of $G(n, 1/2)$.
13. [HHM10] For n sufficiently large,
    (a) $0.146907n < \text{amr}(n) < 0.5n + \sqrt{7n \ln n}$, and
    (b) $|\,\text{mr}(G(n, 1/2)) - \text{amr}(n)| < \sqrt{n \ln \ln n}$ with probability approaching 1 as $n \longrightarrow \infty$.

Facts about relationships between parameters

14. For any graph G, $\delta(G) \leq \text{tw}(G)$.
15. [BBF13] Figure 11 describes the relationships between the zero forcing parameters, maximum nullity parameters, and other graph parameters (for graphs that have at least one edge). In Figure 11, a line between two parameters q and p where q is below p in the diagram means that for all graphs G, $q(G) \leq p(G)$. Furthermore, it is known in all cases that inequalities represented in Figure 11 can be strict (see [BBF13]). The strongest form of the δ conjecture ($\delta(G) \leq \nu(G)$, see Section 7) appears as a dashed line of small triangles.

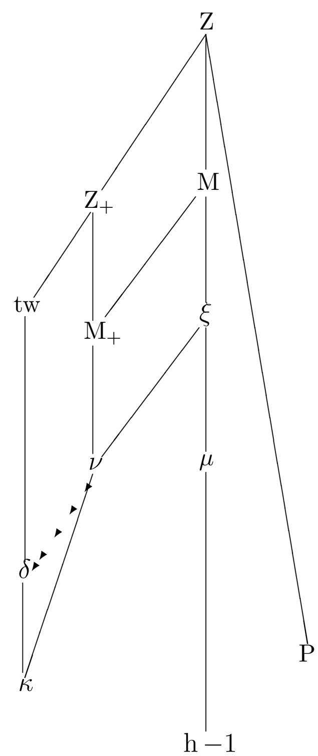


Figure 11: Relationships between zero forcing parameters, parameters related to maximum nullity, and other graph parameters.

Examples 6.1. 1. For the inertia-balanced graph $G = K_{3,3,3} = \overline{K}_3 \vee \overline{K}_3 \vee \overline{K}_3$, we have $3 = \operatorname{mr}(G) \neq \max\{\operatorname{jmr}(\overline{K}_3)\} = 2$.

2. Consider the decomposable G graph in Figure 12: Using the recursive formula from Fact 5 we can compute the minimum rank of G as follows:

$$\operatorname{mr}(G) = \max\{\operatorname{jmr}(G_1), \operatorname{jmr}(G_2)\},$$

where $G_1 = K_1 \cup K_1$ based on vertices 1,2 and $G_2 = P_3 \cup P_2$ based on vertices 3,4,5 and 6,7. It is not difficult to determine that $\operatorname{jmr}(G_1) = \operatorname{mr}(P_3) = 2$ and that $\operatorname{jmr}(G_2) = \operatorname{mr}(P_3) + \operatorname{mr}(P_2) = 3$. Hence by Fact 5, we have that $\operatorname{mr}(G) = 3$.

3. For $n \geq 2$, $GM(K_n) = n - 1$ and if G is obtained from K_n by deleting a single edge, then $GM(G) = n - 2$.

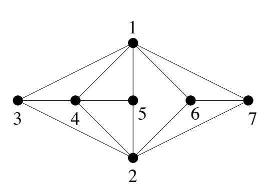


Figure 12: Decomposable graph G.

7 Conjectures and Open Problems

In this section we present definitions, facts, and examples related to two well-known conjectures, the δ Conjecture and the Graph Complement Conjecture, in that order. To avoid having two facts called Fact 1 in this section, the numbering of facts for the Graph Complement Conjecture begins with the next number after the last δ Conjecture fact, and similarly for examples. A comment about terminology: Throughout this volume, $\delta(v)$ denotes the degree of the vertex v in any graph and $\delta(G)$ denotes the minimum degree over all vertices in the graph G. In this section, the abbreviation δ will mean the minimum degree over all vertices in the graph G. This choice was made here, as δ typically denotes the minimum degree, and the use of δ in this section coincides with current terminology used in works that describe this conjectured inequality involving the minimum degree. All graphs in this section are simple.

δ Conjecture

As mentioned earlier, the 2006 AIM workshop [AIM06] produced many new and interesting directions for investigating the minimum rank of a graph. One such line of study is the so-called δ conjecture, which suggests that the maximum nullity of a graph is bounded below by the minimum degree of a graph. A stronger version of the δ conjecture was made by Maehara in 1987 [LSS89]. Since 2006 these conjectures and an even stronger related version have produced many interesting advances even though these conjectures still remain open.

Definitions 7.1. Notation: G is a simple graph.

The δ **conjecture** [BHS07] is the following inequality relating the maximum nullity of a graph G to the minimum degree $\delta(G)$:

$$\mathrm{M}(G) \geq \delta(G).$$

Equivalently, $\mathrm{mr}(G) \leq |G| - \delta(G)$ holds for all graphs G.

For the δ conjecture and each variant δ_x below, to say **the δ_x conjecture holds for G** means that the inequality in the δ_x conjecture is true for G. The actual δ_x conjecture is that the δ_x conjecture is true for every graph G.

The **positive semidefinite δ conjecture** (or δ_+ **conjecture**) [LSS89, attributed to Maehara] is the following inequality:

$$\mathrm{M}_+(G) \geq \delta(G).$$

Equivalently, $\mathrm{mr}_+(G) \leq |G| - \delta(G)$ holds for all graphs G.

The δ **conjecture for ν** (or δ_ν **conjecture**) [BBF13] is the following inequality:

$$\nu(G) \geq \delta(G).$$

Facts 7.1.

1. [LSS89], [Hol08b] For any graph G, $\mathrm{M}_+(G) \geq \nu(G) \geq \kappa(G)$.

2. [BFH08] The δ conjecture is known to hold for many classes of graphs including trees, graphs with $\delta(G) \leq 3$, $\delta(G) \geq |G| - 2$, and bipartite graphs.

3. [BFH08] The δ conjecture is known to hold for any graph G with a cut vertex v provided that it holds for the components of $G - v$.

4. [BFH08] If symmetry is replaced by combinatorial symmetry, then the δ conjecture holds for all graphs.

5. It is shown[5] in Example 1 below that with certain additional conditions, the complement of a bipartite graph satisfies the δ_+ conjecture.

6. At the time of this writing, the δ_+ conjecture is still unresolved for bipartite graphs.

7. [BFH08] The δ conjecture can fail for finite fields such as $\mathbb{Z}_2$; $K_3 \times K_2$ provides an example:
$$\mathrm{M}^{\mathbb{Z}_2}(K_3 \times K_2) = 2 < 3 = \delta(K_3 \times K_2).$$

Examples 7.1.

1. Suppose the complement $\overline{G}$ of G is a bipartite graph with bipartition $V(\overline{G}) = U \cup W$, and suppose that every vertex of U is adjacent in G to some vertex in W, and likewise every vertex of W is adjacent in G to some vertex in U. Let $m = |U|$ and $n = |W|$, and assume the vertices in U precede the vertices in W in the ordering used to associate matrices to G. Then a matrix in $\mathcal{S}_+(G)$ has the nonzero pattern $\begin{bmatrix} * & Y \\ Y^T & * \end{bmatrix}$ where $*$ denotes an all-nonzero pattern of appropriate size and Y is an $m \times n$ nonzero pattern with every row and column having a nonzero entry. In [BFH09] it is shown that $\mathrm{mr}_+(G)$ is equal to the minimum rank among matrices having nonzero pattern Y. In [BFH08] it is shown that if r is the minimum number of nonzero entries in a row of Y, then there is a matrix of rank at most $n-r+1$ having nonzero pattern Y, so $\mathrm{mr}_+(G) \leq n-r+1$. Observe that $\delta(G) \leq (m-1)+r$, and thus
$$\mathrm{M}_+(G) = |G| - \mathrm{mr}_+(G) \geq n + m - (n - r + 1) = m - 1 + r \geq \delta(G).$$

Graph Complement Conjecture (GCC)

An interesting conjecture that arose from the 2006 AIM workshop [AIM06] has become known as the graph complement conjecture or GCC for short (see [BHS07]). A stronger variant involving the Colin de Verdière number μ had been conjectured previously [KLV97], and several other stronger variants have been conjectured recently. Numerous partial results supporting the conjectures have been obtained. However, it remains unresolved at present.

GCC and its variants are what graph theorists call Nordhaus-Gaddum type problems, in that they involve bounding the sum of a graph parameter evaluated at a graph G and its complement $\overline{G}$. Nordhaus-Gaddum type problems have been studied for many different graph parameters, including chromatic number, independence number, domination number, Hadwiger number, etc. (see, for example, [AH13] or [CN71]).

[5] The brief proof is included there because a proof does not appear in the literature.

Definitions 7.2. Notation: G is a simple graph and $\overline{G}$ is the complement of G.

The **Graph Complement Conjecture (GCC)** [BHS07] is the following inequality on the minimum rank of G and its complement:

$$\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) \leq |G| + 2.$$

For GCC and each variant GCC_x below, to say **GCC_x is true for** G (or that G **satisfies GCC_x**) means that the GCC_x inequality is true for G. The actual conjecture GCC_x is that GCC_x is true for every graph G.

The **positive semidefinite Graph Complement Conjecture (GCC_+)** [BBF12] is the following inequality:

$$\mathrm{mr}_+(G) + \mathrm{mr}_+(\overline{G}) \leq |G| + 2.$$

The **Graph Complement Conjecture for ν (GCC_ν)** [BBF12] is the following inequality:

$$\nu(G) + \nu(\overline{G}) \geq |G| - 2.$$

The **Graph Complement Conjecture for ξ (GCC_ξ)** is the following inequality:

$$\xi(G) + \xi(\overline{G}) \geq |G| - 2.$$

The **Graph Complement Conjecture for μ (GCC_μ)** [KLV97] is the following inequality:

$$\mu(G) + \mu(\overline{G}) \geq |G| - 2.$$

GCC for tree-width ($\mathrm{GCC}_{\mathrm{tw}}$) [EEH13] is

$$\mathrm{tw}(G) + \mathrm{tw}(\overline{G}) \geq |G| - 2.$$

Facts 7.2. Notation: G is a simple graph.

8. [BBF12] GCC and GCC_+ have maximum nullity equivalents,

$$\mathrm{M}(G) + \mathrm{M}(\overline{G}) \geq |G| - 2 \qquad \text{and} \qquad \mathrm{M}_+(G) + \mathrm{M}_+(\overline{G}) \geq |G| - 2.$$

9. [BBF12] GCC_ν implies GCC_+, and GCC_+ implies GCC.

10. GCC_ν implies GCC_ξ, GCC_μ implies GCC_ξ, and GCC_ξ implies GCC.

11. [EEH13] $\mathrm{GGC}_{\mathrm{tw}}$ is true for all graphs, and thus $\mathrm{Z}_+(G) + \mathrm{Z}_+(\overline{G}) \geq |G| - 2$ ($\mathrm{GCC}_{\mathrm{Z}_+}$) and $\mathrm{Z}(G) + \mathrm{Z}(\overline{G}) \geq |G| - 2$ ($\mathrm{GCC}_{\mathrm{Z}}$) are true for all graphs.

12. [SH11] GCC_ν is true for graphs of tree-width at most 3.

13. [SH11] If G is a k-connected partial k-tree, then GCC_ν is true for G. In particular, if G is a k-tree, then G satisfies GCC_ν.

14. [BBF12] If G is a graph with $|G| \leq 8$, then G satisfies GCC_ν (and hence GCC_+ and GCC).

15. [BBF12] If G is a graph with $|G| \leq 10$, then G satisfies GCC_ξ (and hence GCC).

16. [Hog08] If $\mathrm{mr}(G) \leq 4$ or $\mathrm{mr}(\overline{G}) \leq 4$, then G satisfies GCC. If $\mathrm{mr}_+(G) \leq 4$ or $\mathrm{mr}_+(\overline{G}) \leq 4$, then G satisfies GCC_+.

17. The following are shown[6] in Example 5 next: If $\nu(G) \geq |G| - 4$ or $\nu(\overline{G}) \geq |G| - 4$, then G satisfies GCC_ν. If $\xi(G) \geq |G| - 4$ or $\xi(\overline{G}) \geq |G| - 4$, then G satisfies GCC_ξ.

18. [KLV97] If $\mu(G) \geq |G| - 6$ or $\mu(\overline{G}) \geq |G| - 6$, then G satisfies GCC_μ (see Example 6 below for explanation).

19. [BBF12] If $\mathrm{mr}(G) \geq |G| - 3$ or $\mathrm{mr}(\overline{G}) \geq |G| - 3$, then G satisfies GCC. If $\mathrm{mr}_+(G) \geq |G| - 3$ or $\mathrm{mr}_+(\overline{G}) \geq |G| - 3$, then G satisfies GCC_+. If $\nu(G) \leq 2$ or $\nu(\overline{G}) \leq 2$, then G satisfies GCC_ν.

20. [KLV97], [BBF12] GCC_μ is true for all planar graphs.

21. [BBF12] If G is decomposable, then G satisfies GCC_ν.

22. [Mit11] If G is chordal, then G satisfies GCC_ν.

23. [BBF12] If G and H are graphs that satisfy GCC_ν, then the join $G \vee H$ and disjoint union $G \cup H$ satisfy GCC_ν.

24. [BBF12] If G and H are graphs such that their inductive cores satisfy GCC, then the join $G \vee H$ and disjoint union $G \cup H$ satisfy GCC (see [BBF12] for the definition of inductive core).

25. If the δ-conjecture (respectively, δ_+-conjecture, δ_ν-conjecture) is true, then GCC (respectively, GCC_+, GCC_ν) is true for all regular graphs.

26. If GCC fails for a simple graph G, then at least one of $\mathrm{M}(G) < \mathrm{tw}(G)$ or $\mathrm{M}(\overline{G}) < \mathrm{tw}(\overline{G})$ must hold. See Example 7 for a graph having $\mathrm{M}(\overline{G}) < \mathrm{tw}(\overline{G})$ (however, GCC_ν does hold for this graph).

27. The natural extension of GCC to minimum degree, $\delta(G) + \delta(\overline{G}) \geq |G| - 2$, is false. See Example 9.

28. The natural extension of GCC to path cover number, $\mathrm{P}(G) + \mathrm{P}(\overline{G}) \geq |G| - 2$, is false. See Example 10.

29. [Kos84] The natural extension of GCC to Hadwiger number, $\mathrm{h}(G) + \mathrm{h}(\overline{G}) \geq |G|$, is false (this is the natural extension because $\mathrm{h}(G) - 1 \leq \nu(G)$). See Example 8.

Examples 7.2. 2. $\overline{P_4} = P_4$ and $\mathrm{mr}(P_4) = 3$, so P_4 satisfies GCC with equality (P_4 also satisfies GCC_+ and GCC_ν with equality).

3. For $n \geq 5$, $\overline{C_n} = n - 3$ [AIM08]. Since $\mathrm{M}(C_n) = 2$, $\mathrm{M}(C_n) + \mathrm{M}(\overline{C_n}) = n - 1$ and C_n satisfies GCC. (Since C_n is a 2-connected partial 2-tree, C_n satisfies GCC_ν by Fact 13.)

[6]The brief proof is included there because a proof does not appear in the literature.

4. Let G be a connected strongly regular graph with parameters (n, k, λ, μ) (see Section 39.3). Then the adjacency matrix $\mathcal{A}_G$ of G has exactly three eigenvalues, one of which is k with multiplicity one (see Facts 39.3.13 and 39.3.14). Thus, $\mathcal{A}_G$ has an eigenvalue of multiplicity at least $\lceil \frac{n-1}{2} \rceil$. Since $\overline{G}$ is also strongly regular, $\mathrm{M}(G) + \mathrm{M}(\overline{G}) \geq n - 1$ and GCC is true for G.

5. In this example we show why $\nu(G) \geq |G| - 4$ implies G satisfies GCC_ν, and similarly why $\xi(G) \geq |G| - 4$ implies G satisfies GCC_ξ.

   Clearly if $|G| \leq 3$, then G satisfies GCC_ν. Assume $\nu(G) \geq |G| - 4$ and GCC_ν fails, i.e., $\nu(G) + \nu(\overline{G}) \leq |G| - 3$. Thus, $|G| \geq 4$, $\nu(G) = |G| - 4$ and $\nu(\overline{G}) = 1$; the latter implies each connected component of $\overline{G}$ is a tree [CdV98], so $\overline{G}$ is a forest. (For GCC_ξ, the analogous argument shows that $\xi(\overline{G}) = 1$, and this implies each connected component of $\overline{G}$ is a path [BFH05b].) Add edges to $\overline{G}$ to obtain a tree T and observe that $\overline{T}$ is a subgraph of G. Thus, $\nu(G) \geq \nu(\overline{T})$ and it suffices to show $\nu(\overline{T}) \geq |T| - 3$ for every tree of order at least 4, contradicting $\nu(G) = |G| - 4$.

   Let $T = (V, E)$ be a tree of order $n \geq 3$. In the proof of Theorem 3.16 in [AIM08], a faithful orthogonal representation of T is constructed in $\mathbb{R}^3$ for $\overline{T}$ with the additional property that if $i \neq j \in V$, then $\mathbf{v}_i$ and $\mathbf{v}_j$ are linearly independent (call such a representation an *independent faithful orthogonal representation*). Let $B = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ be an independent faithful orthogonal representation of $\overline{T}$ in $\mathbb{R}^3$. We show by induction on $n = |T|$ that $B^T B$ satisfies the Strong Arnold Hypothesis, and thus $\nu(\overline{T}) \geq |T| - 3$.

   Since every nonsingular matrix trivially satisfies the Strong Arnold Hypothesis, the result is clear for $n = 2$. Assume true for all trees of order at most $n - 1$. Without loss of generality (by renumbering if necessary), assume n is a vertex of degree one in T and $n - 1$ is its neighbor in T. Define $B' = [\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}]$, $A = B^T B$, and $A' = B'^T B'$. We can partition $A = \begin{bmatrix} A' & \mathbf{c} \\ \mathbf{c}^T & \alpha \end{bmatrix}$ where $\mathbf{c} = [\gamma_1, \ldots, \gamma_{n-2}, 0] \in \mathbb{R}^{n-1}$ and $\alpha \in \mathbb{R}$. Let X be a symmetric matrix satisfying (i) $AX = \mathbf{0}$, (ii) $A \circ X = \mathbf{0}$, and (iii) $I \circ X = \mathbf{0}$. Partition X conformally with A as $X = \begin{bmatrix} X' & \mathbf{z} \\ \mathbf{z}^T & 0 \end{bmatrix}$ where $\mathbf{z} = [0, \ldots, 0, \zeta] \in \mathbb{R}^{n-1}$ by (ii) and (iii). By examining the 1,2-block of $AX = 0$, we see that $0 = A'\mathbf{z} = B'^T B' \mathbf{z}$. Thus, $0 = B'\mathbf{z} = \zeta \mathbf{v}_{n-1}$. Since $\mathbf{v}_{n-1} \neq 0$ (all vectors in an independent faithful orthogonal representation are nonzero), $\zeta = 0$, i.e., $\mathbf{z} = 0$. Then by examining the 1,1-block of $AX = 0$, we have $0 = A'X' + \mathbf{c}\mathbf{z}^T = A'X'$. Since $A' = B'^T B'$ and B' is an independent faithful orthogonal representation of $T \setminus n$, A' satisfies the Strong Arnold Hypothesis by the induction assumption. Thus, $X' = 0$, so $X = 0$, and A satisfies the Strong Arnold Hypothesis.

6. In this example we explain why $\mu(G) \geq |G| - 6$ implies G satisfies GCC_μ. Assume $\mu(G) \geq |G| - 6$ and G does not satisfy GCC_μ. Then $\mu(\overline{G}) \leq 3$, so $\overline{G}$ must be planar. Then by results in [KLV97] (see Fact 20), G satisfies GCC_μ, contradicting the hypothesis.

7. [BBF13] The Heawood graph H is shown in Figure 13(a); we consider its complement $\overline{H}$. It is well known that H is the incidence graph of the Fano projective plane (the

numbering in Figure 13(a) follows [BFH09, Figure 4.1] with vertices $1-7$ interpreted as the lines and vertices $8-14$ interpreted as points). Any matrix in $\mathcal{S}(\overline{H})$ has the form $\begin{bmatrix} * & X_F \\ X_F^T & * \end{bmatrix}$ where * indicates every entry is nonzero (except possibly on the diagonal) and X_F is the zero-nonzero pattern of the complement of the incidence pattern in the Fano projective plane. So by [BFH09, Theorem 3.1], $\mathrm{mr}_+(\overline{H}) = \mathrm{mr}(\overline{H})$ is equal to the minimum of the ranks of the (not necessarily symmetric) matrices described by X_F, i.e., $\mathrm{mr}_+(\overline{H}) = \mathrm{mr}(\overline{H}) = 4$, so $\mathrm{M}_+(\overline{H}) = \mathrm{M}(\overline{H}) = 10$. Since $\mathrm{g}(H) = 6$, $\mathrm{tw}(\overline{H}) \geq 14-3 = 11$ [BBF13, Theorem A.19] (in fact $\mathrm{tw}(\overline{H}) = 11$). Thus, $\mathrm{M}(\overline{H}) < \mathrm{tw}(\overline{H})$. It is straightforward to verify that the matrix

$$A = \begin{bmatrix}
44 & 26 & -34 & 60 & 78 & 18 & 11 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
26 & 70 & -59 & 129 & 85 & -44 & 3 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
-34 & -59 & 54 & -113 & -88 & 25 & -6 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
60 & 129 & -113 & 242 & 173 & -69 & 9 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\
78 & 85 & -88 & 173 & 166 & -7 & 17 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\
18 & -44 & 25 & -69 & -7 & 62 & 8 & 1 & -1 & 0 & -1 & 1 & 0 & 0 \\
11 & 3 & -6 & 9 & 17 & 8 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 21 & 33 & 34 & 17 & 5 & 4 & 38 \\
0 & 1 & 0 & 1 & 0 & -1 & 1 & 33 & 54 & 59 & 45 & 24 & 19 & 78 \\
0 & 0 & 1 & -1 & -1 & 0 & 1 & 34 & 59 & 70 & 77 & 52 & 41 & 111 \\
0 & 1 & 1 & 0 & -1 & -1 & 0 & 17 & 45 & 77 & 178 & 150 & 118 & 195 \\
1 & 0 & 1 & -1 & 0 & 1 & 0 & 5 & 24 & 52 & 150 & 131 & 103 & 155 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 4 & 19 & 41 & 118 & 103 & 81 & 122 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 38 & 78 & 111 & 195 & 155 & 122 & 233
\end{bmatrix}$$

satisfies the Strong Arnold Hypothesis, $\operatorname{rank} A = 4$, and $A \in \mathcal{S}(\overline{H})$, so $\nu(\overline{H}) \geq 10 = |\overline{H}| - 4$. Thus, $\overline{H}$ satisfies GCC_ν by Fact 17.

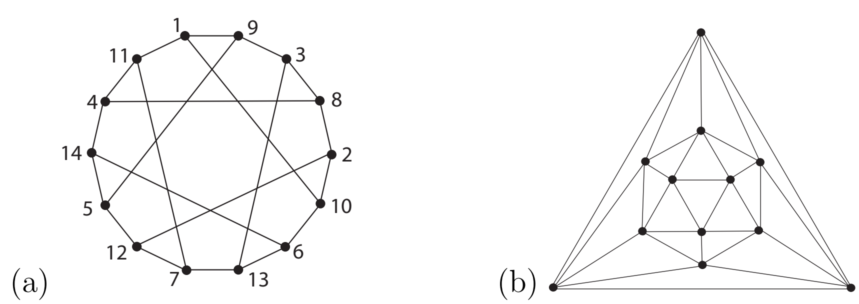


Figure 13: (a) The Heawood graph H and (b) the icosahedral graph G_{12}.

8. [BBF12] The icosahedral graph G_{12} shown in Figure 13(b) has order 12, is 5-regular, and is planar. Thus, G_{12} cannot have a K_5 minor. So in order for G_{12} to satisfy $\mathrm{h}(G_{12}) + \mathrm{h}(\overline{G_{12}}) \geq |G_{12}|$, $\overline{G_{12}}$ would need to have a K_8 minor. To obtain a minor, we delete a (possibly empty) subset of vertices, delete a (possibly empty) subset of edges, and (possibly) perform edge contractions. The contractions induce a partition

of the subset of vertices that are not deleted. Thus, it is impossible to have a K_8 minor of $\overline{G_{12}}$, since for any minor that has 8 vertices, we must partition a subset of the 12 vertices of $\overline{G_{12}}$ into 8 sets (associated with the 8 vertices of the minor), requiring that there be a set with only one vertex of $\overline{G_{12}}$, hence a vertex of degree at most 6 in the minor, because $\overline{G_{12}}$ is 6-regular. Note that $\kappa(G_{12}) = 5$ and $\kappa(\overline{G_{12}}) = 6$, so by Fact 5.9, $\nu(G_{12}) + \nu(\overline{G_{12}}) \geq 11 > |G_{12}| - 2$. So G_{12} satisfies GCC$_\nu$ and hence GCC and GCC$_+$.

9. A graph G and its complement $\overline{G}$ are shown in Figure 14. For this graph, $\delta(G) = 1 = \delta(\overline{G})$ and $|G| = 5$, so $\delta(G) + \delta(\overline{G}) < |G| - 2$.

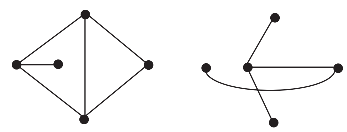


Figure 14: A graph and its complement.

10. For the path on six vertices, $\mathrm{P}(P_6) = 1$ and $\mathrm{P}(\overline{P_6}) = 2$, so $\mathrm{P}(P_6) + \mathrm{P}(\overline{P_6}) < |P_6| - 2$.

8 Minimum Rank without Symmetry

The previous sections present a relatively complete description of the state of knowledge of the topics covered at the time of this writing (2012). By contrast, this section is a brief overview that is not comprehensive.

The families of matrices discussed in the previous sections have had off-diagonal nonzero patterns described by edges of simple undirected graphs, and the matrices have been symmetric (or sometimes Hermitian in the case of minimum positive semidefinite rank). In this section we give a brief overview of the more limited work that has been done on the minimum rank problem for matrices without symmetry that are described by more general patterns of nonzero entries, sometimes eliminating the requirement of positional symmetry by using directed graphs, allowing the pattern to (more fully) constrain the diagonal, and including sign patterns in addition to nonzero patterns.

A **nonzero pattern** is a matrix (not necessarily square) whose entries are elements of $\{*, 0\}$, where $*$ denotes a nonzero entry. A square nonzero pattern can be naturally associated with a digraph (see Chapter 40 for definitions and properties of digraphs, which are directed graphs that allow loops but not multiple edges). A **sign pattern** is a matrix (not necessarily square) having entries in $\{+, -, 0\}$ (see Chapter 42 for definitions and properties of sign patterns). The definitions of minimum rank and maximum nullity are extended in the natural way to a nonzero pattern or sign pattern. Definitions of related parameters such as zero forcing number and path cover number have also been extended to patterns — see Example 1 next, [BFH09], and [Hog11]. A ***t*-triangle** of an $m \times n$ nonzero pattern Y is a $t \times t$ subpattern that is permutation similar to a pattern that is upper triangular with all diagonal entries nonzero. The **triangle number** of pattern Y, denoted $\mathrm{tri}(Y)$, is the maximum size of a triangle in Y. The triangle number and t-triangles have been used as a lower bound for minimum rank in both the symmetric and asymmetric minimum rank problems, see e.g., [BHL04], [CJ06], [JL08]; triangle number is generally more useful in the asymmetric case.

The minimum rank of full sign patterns has important applications to communication complexity in computer science (a sign pattern is **full** if all entries are nonzero), and some progress on minimum rank of full sign patterns has been obtained through work on communication complexity. See Application 1 for a discussion of the connection between minimum rank and communication complexity.

For a simple (undirected) graph G, the **Haemers minimum rank** $\eta(G)$ is the smallest rank of any (not necessarily symmetric) matrix (over any field) having all diagonal entries nonzero and having zero in every position that corresponds to a nonedge of G [Hae81]; facts about η can be found in Section 39.5, an more recent results can be found in [Tim12].

Facts 8.1.

1. It is straightforward to determine the minimum rank of any tree pattern by computing a related parameter such as triangle number or zero forcing number, but techniques vary with the type of pattern. For a survey of all types of trees and ditrees, see [Hog10]. For the original work on nonezero patterns, see [BFH09]. For the original work on trees with loops and positionally symmetric tree sign patterns, see [DHH06]. For tree sign patterns in general, see [Hog11].

2. [For02] Let X be a $m \times n$ full sign pattern and let M_X be the $m \times n$ matrix obtained from X by replacing $+$ by 1 and $-$ by -1. Then

$$\operatorname{mr}(X) \geq \frac{\sqrt{mn}}{\|M_X\|_2},$$

where $\|M_X\|_2$ is the spectral norm of M_X.

Examples 8.1.

1. Let Γ be a digraph. Vertex v is an *out-neighbor* of u if (u, v) is an arc of Γ, and we say that Γ *requires nonsingularity* if every matrix having the nonzero pattern of entries described by Γ is nonsingular. The zero forcing number $\mathrm{Z}(\Gamma)$ is the minimum number of black vertices needed to color all the vertices black using the following *color change rule*: If exactly one out-neighbor v of u is white, then change the color of v to black (the possibility that $u = v$ is permitted). The extension of the definition of path cover number requires a fundamental change in the definition: The *path cover number* $\mathrm{P}(\Gamma)$ of Γ is the minimum number of vertex disjoint paths whose deletion leaves a digraph that requires nonsingularity (or the empty set) [BFH09]. In [BFH09] it is shown that for a tree digraph T, $\operatorname{mr}(T) = \operatorname{tri}(T)$ and $\mathrm{M}(T) = \mathrm{Z}(T) = \mathrm{P}(T)$. Here we use the latter to compute $\mathrm{M}(T)$ and hence $\operatorname{mr}(T)$ for the tree digraph shown in Figure 15. Observe that the black vertices $\{1, 13\}$ are a zero forcing set for T, with the chronological list of forces $1 \to 2, 2 \to 3, 13 \to 12, 12 \to 4, 11 \to 11, 5 \to 6, 8 \to 7, 9 \to 9, 6 \to 5, 7 \to 8, 4 \to 10$. Similarly, the paths $(1, 2, 3), (13, 12, 4, 10)$ are a path cover because $T[\{5, 6, 7, 8, 9\}]$ and $T[\{11\}]$ both require nonsingulararity. In fact, both of these are minimal, so $\mathrm{M}(T) = 2$ and $\operatorname{mr}(T) = 11$.

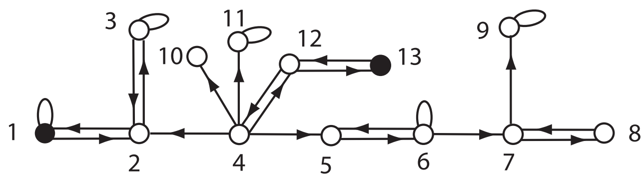


Figure 15: A tree digraph T.

2. An $n \times n$ *Hadamard matrix* is a ± 1-matrix H such that $H^T H = nI_n$ (see Chapter 44). For an $n \times n$ Hadamard matrix H, $\frac{\sqrt{n^2}}{\|H\|_2} = \sqrt{n}$, so $\mathrm{mr}(\mathrm{sgn}(H)) \geq \sqrt{n}$ [For02]. For the 4×4 Hadamard matrix
$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$
it is shown in [Hog11] using results from [BCS94] that $\mathrm{mr}(\mathrm{sgn}(H)) = 3 > 2 = \sqrt{4}$.

Applications 8.1.

1. In a simple model of communication, described in [CCM00], there are two processors A and B, each of which receives its own input (a string of bits that are 0 or 1), and the goal is to compute a value that is a function of both inputs. The computation function can be described by a $\{0,1\}$-matrix M with rows indexed by the possible inputs of A, columns indexed by the possible inputs for B, and the entry representing the value computed. A *(deterministic) protocol* tells the processors how to exchange information to enable this computation. The *(deterministic) communication complexity* $c(M)$ associated to the $\{0,1\}$ function matrix M is the minimum number of bits that must be transmitted in any protocol associated with M. Melhorn and Schmidt [MS82] showed that $\log_2 \operatorname{rank} M \leq c(M) \leq \operatorname{rank} M$ [CCM00].

   Communication complexity is also studied from a probabilistic point of view; this approach is described in [Lok09] and [DS12]. An *unbounded error probabilistic protocol* tells the processors how to exchange information to enable computation that will be accurate with probability $> \frac{1}{2}$. The *unbounded error probabilistic communication complexity* upp-cc(M) associated to the function matrix M is the minimum number of bits that must be transmitted in any unbounded error probabilistic protocol associated with M. When studying upp-cc, it is common to use a $\{+1,-1\}$-matrix. A $\{0,1\}$-matrix M can be converted to a $\{+1,-1\}$-matrix by replacing entry m_{ij} by $(-1)^{m_{ij}}$, or equivalently, using $J - 2M$, where J is the all ones matrix. If M is an $m \times n$ $\{+1,-1\}$-matrix, then $\mathrm{sgn}(M)$ is a full sign pattern. For a $\{+1,-1\}$-matrix M, the *sign rank* of M is sign-rank$(M) = \mathrm{mr}(\mathrm{sgn}(M))$. Paturi and Simon [PS84], [Lok09, p. 106] showed that
$$\log_2 \text{sign-rank}(M) \leq \text{upp-cc}(M) \leq \log_2 \text{sign-rank}(M) + 1.$$

Thus, the computation of sign-rank(M) = mr(sgn(M)) is of interest in the study of communication complexity. A more thorough introduction to communication complexity and its connections to sign-rank and minimum rank can be found in Deaett and Srinivasan's recent survey [DS12] or Lokam's book [Lok09].

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