Coloring count cones of planar graphs

Zdenek Dvorak
Charles University, Prague

Bernard Lidicky
Iowa State University, lidicky@iastate.edu

Follow this and additional works at: https://lib.dr.iastate.edu/math_pubs
Part of the Discrete Mathematics and Combinatorics Commons

The complete bibliographic information for this item can be found at https://lib.dr.iastate.edu/math_pubs/213. For information on how to cite this item, please visit http://lib.dr.iastate.edu/howtocite.html.
Coloring count cones of planar graphs

Abstract
For a plane near-triangulation G with the outer face bounded by a cycle C, let \( n^G \) denote the function that to each 4-coloring \( \psi \) of C assigns the number of ways \( \psi \) extends to a 4-coloring of G. The block-count reducibility argument (which has been developed in connection with attempted proofs of the Four Color Theorem) is equivalent to the statement that the function \( n^G \) belongs to a certain cone in the space of all functions from 4-colorings of C to real numbers. We investigate the properties of this cone for \(|C|=5\), formulate a conjecture strengthening the Four Color Theorem, and present evidence supporting this conjecture.

Disciplines
Discrete Mathematics and Combinatorics

Comments
This is a pre-print made available through arxiv: https://arxiv.org/abs/1907.04066.
Coloring count cones of planar graphs

Zdeněk Dvořák*  Bernard Lidický†

July 10, 2019

Abstract

For a plane near-triangulation $G$ with the outer face bounded by a cycle $C$, let $n^{\star}_G$ denote the function that to each 4-coloring $\psi$ of $C$ assigns the number of ways $\psi$ extends to a 4-coloring of $G$. The block-count reducibility argument (which has been developed in connection with attempted proofs of the Four Color Theorem) is equivalent to the statement that the function $n^{\star}_G$ belongs to a certain cone in the space of all functions from 4-colorings of $C$ to real numbers. We investigate the properties of this cone for $|C|=5$, formulate a conjecture strengthening the Four Color Theorem, and present evidence supporting this conjecture.

By the Four Color Theorem [1, 2, 5], every planar graph is 4-colorable. Nevertheless, many natural followup questions regarding 4-colorability of planar graphs are wide open. Even very basic precoloring extension questions, such as the one given in the following problem, are unresolved (a near-triangulation is a connected plane graph in which all faces except for the outer one have length three).

Problem 1. Does there exist a polynomial-time algorithm which, given a near-triangulation $G$ with the outer face bounded by a 4-cycle $C$ and a 4-coloring $\psi$ of $C$, correctly decides whether $\psi$ extends to a 4-coloring of $G$?

Note that there exist infinitely many near-triangulations $G$ with the outer face bounded by a 4-cycle $C$ such that not every precoloring of $C$ extends to a 4-coloring of $G$; and we do not have any good guess at how the near-triangulations with this property could be described.

Nevertheless, we do have some information about the precoloring extension properties of plane near-triangulations. For a plane near-triangulation $G$ with the outer face bounded by a cycle $C$, let $n^{\star}_C$ denote the function that to each 4-coloring $\psi$ of $C$ assigns the number of ways $\psi$ extends to a 4-coloring of $G$; hence, $\psi$ extends to a 4-coloring of $G$ if and only if $n^{\star}_C(\psi) \neq 0$. Suppose

*Computer Science Institute (CSI) of Charles University, Malostranské náměstí 25, 118 00 Prague, Czech Republic. E-mail: raktver@iuuk.mff.cuni.cz. Supported by the Neuron Foundation for Support of Science under Neuron Impuls programme.

†Department of Mathematics, Iowa State University, Ames, IA, USA. E-mail: bidicky@iastate.edu. Supported in part by NSF grants DMS-1600390 and DMS-1855653.
Figure 1: Precolorings \( \psi_1, \psi_2, \) and \( \psi_3 \) of a 4-cycle.

\( C = v_1v_2v_3v_4 \) is a 4-cycle and \( \psi_1, \psi_2 \) and \( \psi_3 \) are its 4-colorings such that 
\( \psi_i(v_j) = j \) for \( i \in \{1, 2, 3\} \) and \( j \in \{1, 2\} \), 
\( \psi_1(v_3) = \psi_3(v_3) = 1, \psi_2(v_3) = 3, \)
\( \psi_2(v_4) = \psi_3(v_4) = 2, \) and \( \psi_3(v_4) = 4; \) see Figure 1.

A standard Kempe chain argument shows that if \( n^*_G(\psi_1) \neq 0 \), then 
\( n^*_G(\psi_2) \neq 0 \) or \( n^*_G(\psi_3) \neq 0 \).

Actually, much more information can be obtained along these lines, using the idea of \textit{Block-count reducibility} \cite{3, 4} developed in connection with the attempts to prove the Four Color Theorem: Certain inequalities between linear combinations of 
\( n^*_G(\psi_1), n^*_G(\psi_2), \) and \( n^*_G(\psi_3) \) are satisfied for all near-triangulations \( G \), or equivalently, the vector \( (n^*_G(\psi_1), n^*_G(\psi_2), n^*_G(\psi_3)) \) is contained in a certain cone in \( \mathbb{R}^3 \). The main goal of this note is to present and motivate a conjecture regarding this cone in the case of near-triangulations with the outer face bounded by a 5-cycle; this conjecture strengthens the Four Color Theorem. We also provide evidence supporting this conjecture.

1 Definitions

In order to describe the cone we alluded to in the introduction, we need a number of definitions, which we introduce in this section. It is easier to state the idea in the dual setting of 3-edge-colorings of cubic plane graphs, which is well-known to be equivalent to 4-coloring of plane triangulations \cite{6}.

1.1 Near-cubic graphs and their edge-colorings

Let \( G \) be a connected graph and let \( v \) be a vertex of \( G \). We consider each edge of \( G \) as consisting of two half-edges. Let \( \nu \) be a bijection between the half-edges incident with \( v \) and \( \{0, \ldots, \deg(v) - 1\} \) (so, if \( v \) is incident with a loop, each half of the loop is assigned a different number by \( \nu \)). If all vertices of \( G \) other than \( v \) have degree three, we say that \( \tilde{G} = (G, v, \nu) \) is a near-cubic graph. We say that \( G \) is a \textit{plane near-cubic graph} if \( G \) is a plane graph and the half-edges incident with \( v \) are drawn around it in the clockwise cyclic order \( \nu^{-1}(0), \ldots, \nu^{-1}((\deg(v) - 1)) \). We define \( d(\tilde{G}) = \deg(v) \).

A \textit{3-edge-coloring} of \( \tilde{G} \) is an assignment of colors 1, 2, and 3 to edges of \( G \) such that any two edges incident with a common vertex other than \( v \) have different colors. For an integer \( d \geq 2 \), a function \( \psi : \{0, \ldots, d-1\} \rightarrow \{1, 2, 3\} \) is
a d-precoloring if $|\psi^{-1}(1)| \equiv |\psi^{-1}(2)| \equiv |\psi^{-1}(3)| \equiv d \pmod{2}$. We say that a 3-edge-coloring $\varphi$ of $\tilde{G}$ extends a $d(\tilde{G})$-precoloring $\psi$ if for any edge $e$ incident with $v$ and a half-edge $h$ of $e$ incident with $v$, we have $\varphi(e) = \psi(v(h))$. Let $n_{\tilde{G}}(\psi)$ denote the number of 3-edge-colorings of $\tilde{G}$ which extend $\psi$. Via the theory of nowhere-zero flows [7], it is easy to establish the following correspondence between 4-colorings of near-triangulations and 3-edge-colorings in their duals.

**Observation 2.** Let $\tilde{G} = (G,v,\nu)$ be a plane near-cubic graph, and let $G^*$ be the dual of $G$ drawn so that the outer face of $G^*$ corresponds to $v$. Suppose the outer face of $G^*$ is bounded by a cycle $C$. Then there exists a mapping $f$ from 4-colorings of $C$ to $d(\tilde{G})$-precolorings such that

- $f$ maps exactly four 4-colorings of $C$ to each $d(\tilde{G})$-precoloring, and
- every 4-coloring $\psi$ of $C$ satisfies $n_{\tilde{G}}^*(\psi) = n_G(f(\psi))$.

Given two near-cubic graphs $\tilde{G}_1 = (G_1, v_1, \nu_1)$ and $\tilde{G}_2 = (G_2, v_2, \nu_2)$ with $\deg(v_1) = \deg(v_2)$, let $\tilde{G}_1 \oplus \tilde{G}_2$ denote the graph obtained from $\tilde{G}_1$ and $\tilde{G}_2$ by, for $0 \leq i \leq \deg(v_1) - 1$, removing the half-edges $\nu_1^{-1}(i)$ and $\nu_2^{-1}(i)$ and connecting the other halves of the edges. Note that $\tilde{G}_1 \oplus \tilde{G}_2$ is a cubic graph, and if $\tilde{G}_1$ and $\tilde{G}_2$ are plane near-cubic graphs, then $\tilde{G}_1 \oplus \tilde{G}_2$ is a planar graph. Observe that the number of 3-edge-colorings of $\tilde{G}_1 \oplus \tilde{G}_2$ is

$$\sum_{\psi} n_{\tilde{G}_1}(\psi) n_{\tilde{G}_2}(\psi),$$

where the sum goes over all $\deg(v_1)$-precolorings $\psi$. For any integer $n \geq 3$, let $\tilde{C}_n$ denote the plane near-cubic graph $W_n, v, \nu$, where $W_n$ is the wheel with the central vertex $v$ adjacent to all vertices of an $n$-cycle.

### 1.2 Signatures and Kempe chains

For an integer $d \geq 2$, a $d$-signature is a set $S$ of pairs $(m,s)$, where $m$ is an unordered pair of integers in $\{0, \ldots, d - 1\}$ and $s \in \{-1, 1\}$, satisfying the following conditions:

(i) for any distinct $(m_1, s_1), (m_2, s_2) \in S$ we have $m_1 \cap m_2 = \emptyset$, and

(ii) $S$ does not contain elements $\{(a,b), s_1\}$ and $\{(c,d), s_2\}$ such that $a < c < b < d$.

A $d$-precoloring $\psi$ is compatible in (distinct) colors $i,j \in \{1,2,3\}$ with a $d$-signature $S$ if

- $\psi^{-1}(\{i,j\}) = \bigcup_{(m,s) \in S} m$, and
- for each $\{(a_1,a_2), s\} \in S$, $\psi(a_1) = \psi(a_2)$ holds if and only if $s = -1$. 


Now, consider a 3-edge-coloring \( \varphi \) of a near-cubic graph \( \tilde{G} = (G, v, \nu) \). Each vertex other than \( v \) is incident with edges of all three colors. Hence, for any distinct \( i, j \in \{1, 2, 3\} \), the subgraph \( G_{ij} \) of \( G \) consisting of edges of colors \( i \) or \( j \) is a union of pairwise edge-disjoint cycles, vertex-disjoint except for possible intersections in \( v \). An \( ij \)-Kempe chain of \( \varphi \) is a cycle \( C \) in \( G_{ij} \) containing \( v \); the sign \( \sigma(C) \) of the \( ij \)-Kempe chain \( C \) is 1 if the length of \( C \) is even and \(-1\) if the length of \( C \) is odd. If \( h_1 \) and \( h_2 \) are the half-edges in \( C \) incident with \( v \), we let \( \mu(C) = \{ \nu(h_1), \nu(h_2) \} \). The \( ij \)-Kempe chain signature \( \sigma_{ij}(\varphi) \) of \( \varphi \) is defined as \( \{ (\mu(C), \sigma(C)) : C \text{ is an } ij \text{-Kempe chain of } \varphi \} \).

Note that if \( \tilde{G} \) is plane, then the \( ij \)-Kempe chains do not cross and the \( ij \)-Kempe chain signature of \( \varphi \) satisfies the condition (ii); and thus \( \sigma_{ij}(\varphi) \) is a \( d(\tilde{G}) \)-signature.

### 2 Coloring count cones

Let \( \tilde{G} = (G, v, \nu) \) be a plane near-cubic graph and let \( \psi \) be a \( d(\tilde{G}) \)-precoloring. Suppose that \( \psi \) is compatible (in colors \( i, j \in \{1, 2, 3\} \)) with a \( d(\tilde{G}) \)-signature \( S \). We define \( n_{\tilde{G},S}(\psi) \) as the number of 3-edge-colorings \( \varphi \) of \( \tilde{G} \) extending \( \psi \) such that \( \sigma_{ij}(\varphi) = S \). Note that swapping the colors \( i \) and \( j \) on any set of \( ij \)-Kempe chains of \( \varphi \) results in another 3-edge-coloring with the same \( ij \)-Kempe chain signature. Furthermore, clearly for any permutation \( \pi \) of colors, we have \( n_{\tilde{G},S}(\psi \circ \pi) = n_{\tilde{G},S}(\psi) \). This establishes bijections implying the following.

**Observation 3.** Let \( \tilde{G} \) be a plane near-cubic graph and let \( S \) be a \( d(\tilde{G}) \)-signature. Any \( d(\tilde{G}) \)-precolorings \( \psi_1 \) and \( \psi_2 \) compatible with \( S \) satisfy

\[
n_{\tilde{G},S}(\psi_1) = n_{\tilde{G},S}(\psi_2).
\]

Hence, we can define an integer \( n_{\tilde{G},S} \) to be equal to \( n_{\tilde{G},S}(\psi) \) for an arbitrarily chosen \( d(\tilde{G}) \)-precoloring \( \psi \) compatible with \( S \).

Let \( d \geq 2 \) be an integer and let \( i, j \in \{1, 2, 3\} \) be distinct colors. For a \( d \)-precoloring \( \psi \), let us define \( S_{\psi,ij} \) as the set of \( d \)-signatures compatible with \( \psi \) in colors \( ij \). Since every 3-edge-coloring of \( \tilde{G} \) has an \( ij \)-Kempe chain signature, we have

\[
n_{\tilde{G}}(\psi) = \sum_{S \in S_{\psi,ij}} n_{\tilde{G},S}(\psi) = \sum_{S \in S_{\psi,ij}} n_{\tilde{G},S}.
\]

Let \( P_d \) denote the set of all \( d \)-precolorings and \( S_d \) the set of all \( d \)-signatures. We will work in the vector spaces \( \mathbb{R}^{P_d} \) and \( \mathbb{R}^{S_d} \) with coordinates corresponding to the \( d \)-precolorings and to the \( d \)-signatures, respectively. For each integer \( d \geq 2 \), the coloring count cone \( B_d \) is the set of all \( x \in \mathbb{R}^{P_d} \) such that

- \( x(\psi) \geq 0 \) for every \( d \)-precoloring \( \psi \), and
- there exists \( y \in \mathbb{R}^{S_d} \) such that

\[
(1)
\]
- \( y(S) \geq 0 \) for every \( d \)-signature \( S \), and
- \( x(\psi) = \sum_{S \in S_v} y(S) \) for every \( d \)-precoloring \( \psi \) and distinct colors \( i, j \in \{1, 2, 3\} \).

Note that \( B_d \) is indeed a cone, i.e., an unbounded polytope closed under linear combinations with non-negative coefficients. By (1), the vector of precoloring extension counts for any plane near-cubic graph belongs to the corresponding coloring count cone.

**Theorem 4.** For each plane near-cubic graph \( \tilde{G} \), we have

\[ n_{\tilde{G}} \in B_{d(\tilde{G})}. \]

Each cone is uniquely determined as the set of non-negative linear combinations of its rays. For \( d \in \{2, 3, 4, 5\} \), the rays of \( B_d \) are easy to enumerate by hand or using polytope-manipulation software such as Sage Math or the Parma Polyhedra Library (a program doing so for \( d = 5 \) can be found at [http://lidicky.name/pub/4cone/] link). For a near-cubic graph \( \tilde{G} \) such that \( n_{\tilde{G}} \) is not the zero function, let \( \text{ray}(\tilde{G}) \) denote the set of all non-negative multiples of \( n_{\tilde{G}} \).

**Lemma 5.** Refering to graphs in Figure 3:
- the cone \( B_2 \) has exactly one ray equal to \( \text{ray}(\tilde{R}_{2,1}) \);
- the cone \( B_3 \) has exactly one ray equal to \( \text{ray}(\tilde{R}_{3,1}) \);
- the cone \( B_4 \) has exactly four rays equal to \( \text{ray}(\tilde{R}_{4,1}), \ldots, \text{ray}(\tilde{R}_{4,4}) \); and
- the cone \( B_5 \) has exactly 12 rays equal to \( \text{ray}(\tilde{R}_{5,1}), \ldots, \text{ray}(\tilde{R}_{5,12}) \).

Let us remark that \( B_6 \) has 208 rays; the direct method we employ is too slow to enumerate all rays for \( d \geq 7 \) on current workstations.

### 3 The cone \( B_5 \) and the conjecture

Note that while \( \tilde{R}_{5,1}, \ldots, \tilde{R}_{5,11} \) are plane, \( \tilde{R}_{5,12} \) is not. Indeed, the following holds.

**Lemma 6.** The following claims are equivalent.

(a) Every planar cubic 2-edge-connected graph is 3-edge-colorable.

(b) For every plane near-cubic graph \( \tilde{G} \) with \( d(\tilde{G}) = 5 \), if \( n_{\tilde{G}} \in \text{ray}(\tilde{R}_{5,12}) \), then \( n_{\tilde{G}} \) is the zero function.

**Proof.** Let us first prove that (a) implies (b). Consider a plane near-cubic graph \( \tilde{G} = (\tilde{G}, v, \nu) \) such that \( n_{\tilde{G}} \in \text{ray}(\tilde{R}_{5,12}) \), and thus for some constant \( c \geq 0 \), we have \( n_{\tilde{G}}(\psi) = c \cdot n_{\tilde{R}_{5,12}}(\psi) \) for every 5-precoloring \( \psi \). Observe that
Figure 2: Graphs $\tilde{R}_{2,1}, \ldots , \tilde{R}_{5,12}$. The dashed circle intersects the half-edges incident with the vertex $v$, which is not depicted; the values of $\nu$ are written at the respective half-edges.
$n_{\tilde{R}_{5,12}}(\psi)n_{\tilde{C}_5}(\psi) = 0$ for every 5-precoloring $\psi$ (since $\tilde{R}_{5,12} \oplus \tilde{C}_5$ is the Petersen graph, which is not 3-edge-colorable), and thus the number of 3-edge-colorings of $G \oplus \tilde{C}_5$ is

$$\sum_{\psi} n_G n_{\tilde{C}_5}(\psi) = c \sum_{\psi} n_{\tilde{R}_{5,12}} n_{\tilde{C}_5}(\psi) = 0.$$ 

Hence, the planar cubic graph $\tilde{G} \oplus \tilde{C}_5$ is not 3-edge-colorable. By (a), $\tilde{G} \oplus \tilde{C}_5$ has a bridge, and thus $G$ has a bridge. But then a standard parity argument implies that $\tilde{G}$ has no 3-edge-coloring, and thus $n_{\tilde{G}}$ is the zero function.

Next, let us prove that (b) implies (a). Suppose for a contradiction that (b) holds, but there exists a plane cubic 2-edge-connected graph that is not 3-edge-colorable, and let $H$ be one with the smallest number of vertices. By Euler’s formula, $H$ has a face $f$ of length $d \leq 5$; hence, we can write $H = \tilde{G} \oplus \tilde{C}_d$ for a plane near-cubic graph $\tilde{G}$. By Theorem 4, we have $n_{\tilde{G}} \in B_d$, and by Lemma 5, there exist non-negative real numbers $c_i$ such that $n_{\tilde{G}} = \sum_i c_i n_{\tilde{R}_{d,i}}$.

Observe there exists a plane near-cubic graph $\tilde{P}$ with $d - 1$ vertices such that $\tilde{G} \oplus \tilde{P}$ is 2-edge-connected. By the minimality of $H$, $\tilde{G} \oplus \tilde{P}$ is 3-edge-colorable, and in particular $n_{\tilde{G}}$ is not the zero function. By (b), $n_{\tilde{G}}$ is not a positive multiple of $n_{\tilde{R}_{5,12}}$, and thus there exists an index $k \leq 11$ such that $c_k > 0$.

Observe that $\tilde{R}_{d,k} \oplus \tilde{C}_d$ is 3-edge-colorable, and thus there exists a $d$-precoloring $\psi_0$ such that $n_{\tilde{R}_{d,k}}(\psi_0)n_{\tilde{C}_d}(\psi_0) > 0$. However, then the number of 3-edge-colorings of $H$ is

$$\sum_{\psi} n_{\tilde{G}}(\psi)n_{\tilde{C}_d}(\psi) \geq c_k \sum_{\psi} n_{\tilde{R}_{d,k}}(\psi)n_{\tilde{C}_d}(\psi) \geq c_k n_{\tilde{R}_{d,k}}(\psi_0)n_{\tilde{C}_d}(\psi_0) > 0.$$ 

This contradicts the assumption that $H$ is not 3-edge-colorable.

Note that (a) is well-known to be equivalent to the Four Color Theorem [6], and thus indeed there is no plane near-cubic graph $\tilde{G}$ with $d(\tilde{G}) = 5$ such that $n_{\tilde{G}}$ is not the zero function and $n_{\tilde{G}} \in \text{ray}(\tilde{R}_{5,12})$; and furthermore, a direct proof of this fact would imply the Four Color Theorem. Motivated by this observation (and experimental evidence), we propose the following conjecture, a strengthening of the Four Color Theorem. Let $B'_5$ denote the cone in $\mathbb{R}P^d$ with rays $\text{ray}(R_{5,1})$, $\ldots$, $\text{ray}(R_{5,11})$.

**Conjecture 7.** Every plane near-cubic graph $\tilde{G}$ with $d(\tilde{G}) = 5$ satisfies $n_{\tilde{G}} \in B'_5$.

For $i \in \{0, \ldots, 4\}$, let $\psi^{5,a}_i$ and $\psi^{5,b}_i$ denote the 5-precolorings whose values at $j \in \{0, \ldots, 4\}$ are defined by the following table; see also Figure 3.
Figure 3: Precolorings $\psi_0^{5,a}$ and $\psi_0^{5,b}$.

<table>
<thead>
<tr>
<th>$(j - i) \mod 5$</th>
<th>$\psi_i^{5,a}(j)$</th>
<th>$\psi_i^{5,b}(j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Note that each 5-precoloring is obtained from one of these ten by a permutation of colors. The cone $B_5'$ has exactly one facet which is not also a facet of $B_5$, giving an equivalent formulation of Conjecture 7.

Conjecture 8. Every plane near-cubic graph $\tilde{G}$ with $d(\tilde{G}) = 5$ satisfies

$$3 \sum_{i=0}^{4} n_{\tilde{G}}(\psi_i^{5,a}) \geq \sum_{i=0}^{4} n_{\tilde{G}}(\psi_i^{5,b}).$$

In the rest of the note, we provide some evidence supporting Conjecture 7; in particular, we show there are no counterexamples to the conjecture with less than 30 vertices.

4 Evidence

Before we present the experimental evidence for the validity of Conjecture 7, we need a few more definitions. A vector $x \in P_d$ is *invariant with respect to permutation of colors* if all $d$-precolorings $\psi$ and $\psi'$ that only differ by a permutation of colors satisfy $x(\psi) = x(\psi')$.

See Figure 4 for an illustration of the following definitions. The *rotation by $t$* of a $d$-precoloring $\psi$ is the $d$-precoloring $r_t(\psi)$ such that $r_t(\psi)((i+t) \mod d) = \psi(i)$ for $i \in \{0,\ldots,d-1\}$. The *flip* of a $d$-precoloring $\psi$ is the $d$-precoloring $f(\psi)$ such that $f(\psi)(i) = \psi(d-1-i)$ for $i \in \{0,\ldots,d-1\}$. For $x \in \mathbb{R}^P_d$, let $r_t(x)$ be defined as $y \in \mathbb{R}^P_d$ such that $y(r_t(\psi)) = x(\psi)$ for every $d$-precoloring $\psi$, and let $f(x)$ be defined as $z \in \mathbb{R}^P_d$ such that $z(f(\psi)) = x(\psi)$ for every $d$-precoloring $\psi$.

A set $K \subseteq \mathbb{R}^P_d$ is *closed under rotations and flips* if we have $x \in K$ if and only
Let $\nu$ be a near-cubic graph $(G,v,\nu)$ with $\deg(v) = d$, let $r_1(G)$ denote the near-cubic graph $(G,v,\nu_1)$, where $\nu^{-1}_1((i+t) \mod d) = \nu^{-1}_1(i)$ for $i \in \{0,\ldots,d-1\}$, and let $f(G)$ denote the near-cubic graph $(G,v,\nu_2)$, where $\nu^{-1}_2(i) = \nu^{-1}_2(d-1-i)$ for $i \in \{0,\ldots,d-1\}$.

Observation 9. Let $\tilde{G}$ be a near-cubic graph, $d = d(\tilde{G})$ and $t \in \{0,\ldots,d-1\}$. Then $n_{r_t(\tilde{G})} = r_t(n_{\tilde{G}})$ and $n_{f(\tilde{G})} = f(n_{\tilde{G}})$.

Let $\psi_1$ be a $d_1$-precoloring and $\psi_2$ a $d_2$-precoloring. For an integer $k \leq \min(d_1,d_2)$, we say that $\psi_1$ $k$-matches $\psi_2$ if $\psi_1(d_1 - k + i) = \psi_2(d_2 - 1 - i)$ for $i \in \{0,\ldots,k-1\}$. By $\gamma_k(\psi_1,\psi_2)$, we denote the $(d_1 + d_2 - 2k)$-precoloring $\gamma$ such that $\gamma(i) = \psi_1(i)$ for $i \in \{0,\ldots,d_1 - k - 1\}$ and $\gamma(i) = \psi_2(i - (d_1 - k))$ for $i \in \{d_1 - k,\ldots,d_1 + d_2 - 2k - 1\}$. For $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$, we define $\gamma_k(x_1,x_2)$ as the vector $y \in \mathbb{R}^{d_1 + d_2 - 2k}$ such that

$$y = \sum_{\psi_1,\psi_2 : \gamma_k(\psi_1,\psi_2) = \psi} x_1(\psi_1)x_2(\psi_2),$$

where the sum is over all $k$-matching $d_1$-precolorings $\psi_1$ and $d_2$-precolorings $\psi_2$.

For near-cubic graphs $G_1 = (G_1,v_1,\nu_1)$ with $\deg(v_1) = d_1$ and $G_2 = (G_2,v_2,\nu_2)$ with $\deg(v_2) = d_2$, let $\gamma_k(G_1,G_2)$ denote the near-cubic graph $(G,v,\nu)$, where $G$ is obtained from $G_1$ and $G_2$ by identifying $v_1$ with $v_2$ to a single vertex $v$ and for $i \in \{0,\ldots,k-1\}$ removing the half-edges $\nu^{-1}_1(d_1 - k + i)$ and $\nu^{-1}_2(d_2 - 1 - i)$ and connecting the other halves of the edges; and $\nu^{-1}(i) = \nu^{-1}_1(i)$ for $i \in \{0,\ldots,d_1 - k - 1\}$ and $\nu^{-1}(i) = \nu^{-1}_2(i - (d_1 - k))$ for $i \in \{d_1 - k,\ldots,d_1 + d_2 - 2k - 1\}$. See Figure 5 for an illustration.

Observation 10. Let $\tilde{G}_1$ and $\tilde{G}_2$ be near-cubic graphs. For every integer $k \in \{0,\ldots,\min(d(\tilde{G}_1),d(\tilde{G}_2))\}$, we have $n_{\gamma_k(\tilde{G}_1,\tilde{G}_2)} = \gamma_k(n_{\tilde{G}_1},n_{\tilde{G}_2})$.

By a computer-assisted enumeration, we verified the following claim.

Lemma 11. There exists cones $K_d \subseteq \mathbb{R}^d$ for $d = 2,\ldots,8$ such that the following claims hold.
Figure 5: $\gamma_k(\tilde{G}_1, \tilde{G}_2)$

(a) $K_d = B_d$ when $d \leq 4$ and $K_5 = B'_5$.

(b) For all $d \in \{2, \ldots, 8\}$, the elements of $K_d$ are invariant with respect to permutation of colors.

(c) For $d \in \{2, \ldots, 7\}$, the cone $K_d$ is closed under rotations and flips.

(d) If $2 \leq d_1 \leq d_2$ and $d_1 + d_2 \leq 7$, then for all $x_1 \in K_{d_1}$ and $x_2 \in K_{d_2}$ we have $\gamma_0(x_1, x_2) \in K_{d_1 + d_2}$.

(e) If $2 \leq d \leq 5$, then for all $x \in K_d$ we have $\gamma_1(n_{\tilde{R}_3}, x) \in K_{d+1}$.

(f) If $3 \leq d \leq 7$, then for all $x \in K_d$ we have $\gamma_2(n_{\tilde{R}_3}, x) \in K_{d-1}$.

(g) If $2 \leq d_1 \leq 6$ and $1 \leq c \leq d_1/2$, then for all $x_1 \in K_{d_1}$ and $x_2 \in K_{7+2c-d_1}$, we have $\gamma_c(x_1, x_2) \in K_{7}$.

(h) For every $x_1 \in K_8$ and $x_2 \in K_7$, we have $\gamma_4(x_1, x_2) \in K_7$.

(i) For every $x_1, x_2 \in K_6$, we have $r_2(\gamma_2(x_1, x_2)) \in K_8$.

Proof. The proof and the program to verify the proof can be found at [http://lidicky.name/pub/4cone/](http://lidicky.name/pub/4cone/). The cones are described by their rays, enumerated in the file. Cone $K_6$ has 102 rays, $K_7$ has 22605 rays, and $K_8$ has 4330 rays. It suffices to verify all the claims for $x, x_1, x_2$ being the rays of the cones specified in the claims; the inclusion of the resulting vectors in the appropriate cone is certified by expressing them as a linear non-negative combination of the rays of the cone.

Parts (e) and (f) of Lemma 11 have the following corollary.

Lemma 12. Let $\tilde{G} = (G, v, \nu)$ be a plane near-cubic graph and let $d = d(\tilde{G})$. If $d \in \{2, \ldots, 7\}$ and $n_\tilde{G} \not\in K_d$, then there exists a plane near-cubic graph $G_0 = (G_0, v_0, \nu_0)$ such that $d(G_0) = 7$, $n_{\tilde{G}_0} \not\in K_7$, $G_0 - v_0$ is an induced subgraph of $G - v$, and $|V(G_0)| \leq |V(G)| - (7 - d)$. 

10
Proof. We prove the claim by induction on the number of vertices of $G$. When $d \leq 4$, the claim is vacuously true by Theorem 4 since $K_d = B_d$. When $d = 7$, we can set $G_0 = G$. Hence, suppose that $d \in \{5, 6\}$. Since $n_{\tilde{G}} \notin K_d$, the function $n_{\tilde{G}}$ is not identically zero.

If $G - v$ is disconnected, we can by symmetry assume that $\tilde{G} = \gamma_0(\tilde{G}_1, \tilde{G}_2)$ for plane near-cubic graphs $\tilde{G}_1$ and $\tilde{G}_2$ such that $d = d(\tilde{G}_1) + d(\tilde{G}_2)$ and $d(\tilde{G}_1) \leq d(\tilde{G}_2)$. Since $n_{\tilde{G}}$ is not the zero function, $n_{\tilde{G}_1}$ is not the zero function either, and thus $d(\tilde{G}_1) \neq 1$. Hence $d(\tilde{G}_1) \geq 2$, and thus $d(\tilde{G}_2) \leq 4$. Hence, $n_{\tilde{G}_1} \in K_{d(\tilde{G}_1)}$ and $n_{\tilde{G}_2} \in K_{d(\tilde{G}_2)}$, and $n_{\tilde{G}} \in K_d$ by Lemma 11(d), which is a contradiction.

Hence, $G - v$ is connected (and the same argument shows that no loop is incident with $v$). Consequently, $v$ is not incident with a triple edge. If $v$ is incident with a double edge, then we can by symmetry assume that $\tilde{G} = \gamma_1(\tilde{G}_1, \tilde{G}_2)$ for a plane near-cubic graph $\tilde{G}_1 = (G_1, v_1, \nu_1)$ with $d(\tilde{G}_1) = d - 1 \leq 5$. By Lemma 11(e), since $n_{\tilde{G}_1} \notin K_d$, we have $n_{\tilde{G}_1} \notin K_{d-1}$. By the induction hypothesis, there exists a plane near-cubic graph $\tilde{G}_0 = (G_0, v_0, \nu_0)$ with $d(\tilde{G}_0) = 7$, such that $n_{\tilde{G}_0} \notin K_7$, $G_0 - v_0$ is an induced subgraph of $G_1 - v_1$, and thus also of $G - v$, and $|V(G_0)| \leq |V(G_1)| - (7 - (d - 1)) < |V(G)| - (7 - d)$, as required.

Hence, we can assume $v$ is not incident with a double edge. Consequently, we can by symmetry assume that $\tilde{G} = \gamma_1(\tilde{G}_1, \tilde{G}_2)$ for a plane near-cubic graph $\tilde{G}_1 = (G_1, v_1, \nu_1)$ with $d(\tilde{G}_1) = d + 1$. By Lemma 11(f), since $n_{\tilde{G}_1} \notin K_d$, we have $n_{\tilde{G}_1} \notin K_{d+1}$. By the induction hypothesis, there exists a plane near-cubic graph $\tilde{G}_0 = (G_0, v_0, \nu_0)$ with $d(\tilde{G}_0) = 7$, such that $n_{\tilde{G}_0} \notin K_7$, $G_0 - v_0$ is an induced subgraph of $G_1 - v_1$, and thus also of $G - v$, and $|V(G_0)| \leq |V(G_1)| - (7 - (d + 1)) = |V(G)| - (7 - d)$. Hence, the claim of the lemma follows.

We will say that a plane near-cubic graph $\tilde{G} = (G, v, \nu)$ is extremal if $d(\tilde{G}) = 7$, $n_{\tilde{G}} \notin K_7$, and there does not exist any plane near-cubic graph $\tilde{G}_0 = (G_0, v_0, \nu_0)$ with $d(\tilde{G}_0) = 7$ such that $n_{\tilde{G}_0} \notin K_7$ and $G_0 - v_0$ is a proper minor of $G - v$.

Lemma 13. If $\tilde{G} = (G, v, \nu)$ is an extremal plane near-cubic graph and $\tilde{G}' = (G', v', \nu')$ is a plane near-cubic graph with $d(\tilde{G}') \leq 7$ such that $G' - v'$ is a proper minor of $G - v$, then $n_{\tilde{G}'} \in K_{d(\tilde{G}')}$.

Proof. If $n_{\tilde{G}} \notin K_{d(\tilde{G}')}$, then by Lemma 11 there would exist a plane near-cubic graph $\tilde{G}_0 = (G_0, v_0, \nu_0)$ such that $d(\tilde{G}_0) = 7$, $n_{\tilde{G}_0} \notin K_7$ and $G_0 - v_0$ is an induced subgraph of $G' - v'$. However, then $G_0 - v_0$ would be a proper minor of $G - v$, contradicting the assumption that $\tilde{G}$ is extremal.

Next, let us explore consequences of part (g) of Lemma 11.

Lemma 14. If $\tilde{G} = (G, v, \nu)$ is an extremal plane near-cubic graph, then $v$ is not incident with loops or parallel edges and $G - v$ is 2-edge-connected.

Proof. Analogously to the proof of Lemma 12 if $v$ were incident with a loop or a parallel edge or if $G - v$ were not 2-edge-connected, we would have $\tilde{G} = \gamma_0(\tilde{G}_1, \tilde{G}_2)$ for plane near-cubic graphs $\tilde{G}_1$ and $\tilde{G}_2$ such that $d = d(\tilde{G}_1) + d(\tilde{G}_2)$ and $d(\tilde{G}_1) \leq d(\tilde{G}_2)$. Since $n_{\tilde{G}}$ is not the zero function, $n_{\tilde{G}_1}$ is not the zero function either, and thus $d(\tilde{G}_1) \neq 1$. Hence $d(\tilde{G}_1) \geq 2$, and thus $d(\tilde{G}_2) \leq 4$. Hence, $n_{\tilde{G}_1} \in K_{d(\tilde{G}_1)}$ and $n_{\tilde{G}_2} \in K_{d(\tilde{G}_2)}$, and $n_{\tilde{G}} \in K_d$ by Lemma 11(d), which is a contradiction.
γc(˜G1, ˜G2) for plane near-cubic graphs ˜G1 and ˜G2 such that $2 \leq d(˜G1) \leq d(˜G2)$, $d(˜G1) + d(˜G2) = 7 + 2c$, and $c \leq 1$; in particular, $d(˜G2) \leq 7$ and $d(˜G1) \leq \left\lfloor (7 + 2c)/2 \right\rfloor \leq 4$. By Lemma 13, we have $n_{˜G_i} \in K_{d(˜G_i)}$ for $i \in \{1, 2\}$. By Lemma 11(g), we conclude $n_{˜G} \in K_7$, which is a contradiction.

Suppose $A$ and $B$ form a partition of the vertex set of a graph $H$, and let $S$ be the set of edges of $H$ with one end in $A$ and the other end in $B$. In this situation, we say $S$ is an edge cut of $H$ with sides $A$ and $B$.

**Lemma 15.** If $G = (G,v,ν)$ is an extremal plane near-cubic graph, then $G - v$ does not contain an edge cut $S$ such that $v$ has at least $|S|$ neighbors in each side of the cut.

**Proof.** Suppose for a contradiction $G - v$ contains such an edge cut $S$ of size $c$, and thus $G = γc(˜G1, ˜G2)$ for plane near-cubic graphs ˜G1 and ˜G2 such that $2c \leq d(˜G1) \leq d(˜G2)$ and $d(˜G1) + d(˜G2) = 7 + 2c$. Since $v$ has 7 neighbors and at least $c$ of them are contained in each of the sides of the cut, we have $c \leq 3$. Note that $d(˜G2) \leq 7$ and $d(˜G1) \leq \left\lfloor (7 + 2c)/2 \right\rfloor \leq 6$. By Lemma 13 we have $n_{˜G_i} \in K_{d(˜G_i)}$ for $i \in \{1, 2\}$. By Lemma 11(g), we conclude $n_{˜G} \in K_7$, which is a contradiction.

An edge cut $S$ of size at most five in a near-cubic graph $G = (G,v,ν)$ is essential if the side of $S$ containing $v$ contains at least one other vertex and the other side $B$ of $S$ induces neither a tree nor a 5-cycle.

**Lemma 16.** If $G = (G,v,ν)$ is an extremal plane near-cubic graph, then $G$ does not contain an essential edge cut $S$ of size at most five.

**Proof.** Suppose for a contradiction $G$ contains an essential edge-cut $S$ of size $k \leq 5$, and choose one with minimum $k$, and subject to that one for which the side $B$ not containing $v$ is minimal. We claim $G[B]$ is 2-edge-connected. Otherwise, $B$ is a disjoint union of non-empty sets $B_1$ and $B_2$, where $G$ contains $r \leq 1$ edges with one end in $B_1$ and the other end in $B_2$. For $i \in \{1, 2\}$, let $S_i$ denote the set of edges of $G$ with exactly one end in $B_i$. Since $G$ is extremal, $n_{˜G} \notin K_7$ is not identically zero, and thus $G$ is 2-edge-connected, implying $|S_i| \geq 2$. Hence, $|S_i| = k + 2r - |S_{3-i}| \leq k$. By the minimality of $B$, we conclude that $B_1$ induces a tree or a 5-cycle, and thus $|S_1| \geq 3$. Hence $5 \geq k = |S_1| + |S_2| - 2r \geq 6 - 2r$, and thus $r = 1$ and $|S_1|, |S_2| \leq 4$. This implies that neither $B_1$ nor $B_2$ induces a 5-cycle, and thus both of them induce trees; and $G$ contains an edge between them, implying that $B$ induces a tree, contrary to the assumption that $S$ is an essential edge cut.

Since $G[B]$ is 2-edge-connected and subcubic, each face of $G[B]$ is bounded by a cycle. Let $C_S$ denote the cycle bounding the face $f$ of $G[B]$ whose interior contains $v$. Observe that all edges of $S$ are drawn inside $f$. Otherwise, the set $S'$ of edges of $S$ drawn inside $C$ forms an edge cut of order smaller than $k$ and by the minimality of $k$, its side $B' \supseteq B$ induces a tree or a 5-cycle; this is not possible, since $G[B]$ is 2-edge connected and not a tree.
Let $\tilde{G}_c$ be the plane near-cubic graph obtained from $G$ by contracting the side of the cut containing $v$ to a single vertex. By Lemma 13 we have $n_{\tilde{G}_c} \in K_k$. Since $K_d = B_d$ for $d \leq 4$ and $K_5 = B'_5$, 

$$n_{\tilde{G}_c} = \sum_i c_i n_{R_{k,i}},$$

where $i \leq 11$ if $k = 5$ and the coefficients $c_i$ are non-negative. Let $G_i = (G_i, v_i, \nu_i)$ denote the plane near-cubic graph obtained from $G$ by replacing the side of the cut $S$ not containing $v$ by $R_{k,i}$. Note that $n_{\tilde{G}} = \sum_i c_i n_{\tilde{G}_i}$, and since $K_7$ is a cone and $n_{\tilde{G}} \notin K_7$, there exists $i$ such that $n_{\tilde{G}_i} \notin K_7$. Because $B$ contains the cycle $C_S$ and all edges of $S$ are incident with vertices of $C_S$, we see $G_i - v_i$ is a proper minor of $G - v$, contradicting the extremality of $G$. \hfill $\square$

In Lemma 14 we argued that if $\tilde{G} = (G, v, \nu)$ is an extremal plane near-cubic graph, then $G - v$ is 2-edge-connected, and thus its face containing $v$ is bounded by a cycle $C$. Let us now argue that the graph stays 2-edge-connected after removing $V(C)$ as well.

**Lemma 17.** Let $\tilde{G} = (G, v, \nu)$ be an extremal plane near-cubic graph and let $C$ be the cycle bounding the face of $G - v$ containing $v$. The cycle $C$ is induced, no two neighbors of $v$ in $C$ are adjacent, and the graph $G - (V(C) \cup \{v\})$ is 2-edge-connected and has more than one vertex.

**Proof.** Consider a simple closed curve $c$ in the plane intersecting $G$ in two edges of $C$, $b \leq 4$ edges incident with $v$, and $r \leq 1$ edges of $E(G - v) \setminus E(C)$, where each edge is intersected at most once. The curve $c$ separates the plane into two parts; let $A$ and $B$ be the corresponding partition of vertices of $G$, where $v \in A$, and let $S$ be the edge cut in $G$ consisting of the edges with one end in $A$ and the other end in $B$. By Lemma 15 applied to the edge cut in $G - v$ obtained from $S$ by removing the edges incident with $v$, it follows that $b \leq r + 1$, and thus $|S| \leq 3 + 2r \leq 5$. By Lemma 16 we conclude that the edge cut satisfies one of the following conditions.

1. $r = 0$, $b = 1$, $|S| = 3$, and $B$ consists of a single vertex of $C$, or
2. $r = 1$ and $G[B]$ is a subpath of $C$, or
3. $r = 1$, $b = 2$, and $G[B]$ is a 5-cycle containing exactly one vertex not in $V(C)$.

If $C$ had a chord $e$, this would give a contradiction by considering a curve $c$ (with $r = 0$) drawn next to the chord so that $e \in E(G[B])$ and $b \leq 3$; hence, $C$ is an induced cycle. If two neighbors of $v$ in $C$ were adjacent, we would obtain a contradiction by considering a curve $c$ (with $r = 0$ and $b = 2$) drawn around them. If the graph $G - (V(C) \cup \{v\})$ were not connected, we would obtain a contradiction by considering a curve $c$ (with $r = 0$ and $b \leq 3$) chosen so that both $A$ and $B$ contain a vertex of $G - (V(C) \cup \{v\})$. Finally, if the graph $G - (V(C) \cup \{v\})$ were not 2-edge-connected, then we could choose $c$ so
that \( r = 1 \), \( b \leq 3 \), and \( B \) contains a vertex of \( G - (V(C) \cup \{v\}) \). But then \( G[B] \) would be a 5-cycle containing exactly one vertex not in \( V(C) \) and consequently two adjacent vertices of \( C \) would be neighbors of \( v \), which is a contradiction.

Therefore, the graph \( G - (V(C) \cup \{v\}) \) is 2-edge-connected. Since no two neighbors of \( v \) in \( C \) are adjacent, \( G \) contains at least 7 edges between \( V(C) \) and \( V(G) \setminus (V(C) \cup \{v\}) \), and thus \( G - (V(C) \cup \{v\}) \) has more than one vertex. \( \square \)

Finally, let us apply the parts (h) and (i) of Lemma 11.

**Lemma 18.** If \( \tilde{G} = (G, v, \nu) \) is an extremal plane near-cubic graph, then \( G \) has at least 28 vertices.

**Proof.** By Lemma 14 the face of \( G - v \) containing \( v \) is bounded by a cycle \( C \). Let \( v_1, \ldots, v_7 \) be the neighbors of \( v \) in \( C \) in order. For \( i \in \{1, \ldots, 7\} \), let \( P_i \) denote the subpath of \( C \) from \( v_i \) to \( v_{i+1} \) (where \( v_8 = v_1 \)).

By Lemma 14, the cycle \( C \) is induced, no two neighbors of \( v \) in \( C \) are adjacent, and the graph \( G - (V(C) \cup \{v\}) \) is 2-edge-connected and has more than one vertex. Hence, the face of \( G - (V(C) \cup \{v\}) \) containing \( v \) is bounded by a cycle \( C' \). For a subgraph \( G' \subseteq G \) containing \( C \cup C' \), let \( X(G') \) denote the set of faces of \( G' \) separated from \( v \) by \( C' \) and let \( Y(G') \) denote the set of faces of \( G' \) separated from \( v \) by \( C \) but not by \( C' \). For \( i \in \{1, \ldots, 7\} \), we say that a face \( f \in X(G') \) is \( P_i \) if there exists a face \( f' \in Y(G') \) such that \( f' \) is incident with an edge of \( P_i \) and the boundaries of \( f \) and \( f' \) share at least one edge.

If for some \( i \in \{1, \ldots, 7\} \), some face of \( X(G) \) saw \( P_i, P_{i+2}, \) and \( P_{i+4} \) (with indices taken cyclically) then \( \tilde{G} = \gamma_4(r_2(\gamma_2(\tilde{G}_1, \tilde{G}_2)), \tilde{G}_3) \) for plane near-cubic graphs \( \tilde{G}_1, \tilde{G}_2, \) and \( \tilde{G}_3 \) with \( d(\tilde{G}_1) = d(G_2) = 6 \) and \( d(G_3) = 7 \) (see Figure 4). Lemma 13 would imply \( n_{\tilde{G}_j} \in K_{d(\tilde{G}_j)} \) for \( j \in \{1, 2, 3\} \), and by Lemma 11(h) and (i), we would have \( n_{\tilde{G}} \in K_7 \), which is a contradiction. Hence,

\[
\text{no face of } X(G) \text{ sees } P_i, P_{i+2}, \text{ and } P_{i+4}. \tag{2}
\]

Let \( b_1 \) be the number of edges of \( G \) with one end in \( C \) and the other end in \( C' \), let \( b_2 \) be the number of chords of \( C' \), let \( b_3 \) be the number of edges with one end in \( C' \) and the other end in \( V(G) \setminus V(C \cup C') \), and let \( b_4 \) be the number of edges of \( G - v - V(C \cup C') \). Note that \( b_1 \geq 7, b_3 \) is at least three times the number of components of \( G - v - V(C \cup C') \), \( |E(C)| = 7 + b_1, |E(C')| = b_1 + 2b_2 + b_3, \) and \(|E(G)| = 7 + (7 + b_1 + b_1 + (b_1 + 2b_2 + b_3) + b_2 + b_3 + b_4 = 14 + 3b_1 + 3b_2 + 2b_3 + b_4.\)

A case analysis shows that since (2) holds, one of the following conditions holds:

- \( b_1 \geq 8 \) and \( b_2 \geq 2 \), or
- \( b_1 \geq 8 \) and \( b_3 \geq 3 \), or
- \( b_3 \geq 6 \), or
- \( b_3 \geq 4 \) and \( b_4 \geq 1 \).
Hence $3b_1 + 3b_2 + 2b_3 + b_4 \geq 30$, and thus $G$ has at least 44 edges. Consequently, 
$|V(G)| \geq (2|E(G)| - 4)/3 \geq 28$.

As a consequence, this verifies Conjecture 7 for small graphs.

**Corollary 19.** Conjecture 7 holds for all plane near-cubic graphs with less than 30 vertices.

**Proof.** Let $\tilde{G} = (G, v, \nu)$ be a counterexample to Conjecture 7 and in particular $n_{\tilde{G}} \notin B'_5 = K_5$. By Lemma 12 there exists a plane near-cubic graph $G_0 = (G_0, v_0, \nu_0)$ such that $d(\tilde{G}_0) = 7$, $n_{\tilde{G}_0} \notin K_7$, and $|V(G_0)| \leq |V(\tilde{G})| - 2$. Hence, there exists an extremal plane near-cubic graph $\tilde{G}_1 = (G_1, v_1, \nu_1)$ such that $|V(G_1)| \leq |V(G_0)|$. By Lemma 18 we have $|V(G_1)| \geq 28$, and thus $|V(G)| \geq 30$.

Note that the analysis at the end of the proof of Lemma 18 can be improved. By a computer-assisted enumeration, one can show that to ensure that (2) holds, $G - v$ must contain one of 38 specific graphs as a minor; the smallest are depicted in Figure 7. Hence, every counterexample to Conjecture 7 must contain one of these 38 as a minor. The list of these 38 graphs is available at [http://lidicky.name/pub/4cone/](http://lidicky.name/pub/4cone/)

**References**

Figure 7: The smallest minors.


