7-10-2019

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Disciplines

Discrete Mathematics and Combinatorics

Comments

This is a pre-print made available through arxiv: https://arxiv.org/abs/1907.04066.

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Abstract

For a plane near-triangulation G with the outer face bounded by a cycle C, let $n^{\star}_G$ denote the function that to each 4-coloring ψ of C assigns the number of ways ψ extends to a 4-coloring of G. The block-count reducibility argument (which has been developed in connection with attempted proofs of the Four Color Theorem) is equivalent to the statement that the function $n^{\star}_G$ belongs to a certain cone in the space of all functions from 4-colorings of C to real numbers. We investigate the properties of this cone for $|C| = 5$, formulate a conjecture strengthening the Four Color Theorem, and present evidence supporting this conjecture.

By the Four Color Theorem [1, 2, 5], every planar graph is 4-colorable. Nevertheless, many natural followup questions regarding 4-colorability of planar graphs are wide open. Even very basic precoloring extension questions, such as the one given in the following problem, are unresolved (a *near-triangulation* is a connected plane graph in which all faces except for the outer one have length three).

Problem 1. *Does there exists a polynomial-time algorithm which, given a near-triangulation G with the outer face bounded by a 4-cycle C and a 4-coloring ψ of C, correctly decides whether ψ extends to a 4-coloring of G?*

Note that there exist infinitely many near-triangulations G with the outer face bounded by a 4-cycle C such that not every precoloring of C extends to a 4-coloring of G; and we do not have any good guess at how the near-triangulations with this property could be described.

Nevertheless, we do have some information about the precoloring extension properties of plane near-triangulations. For a plane near-triangulation G with the outer face bounded by a cycle C, let $n^{\star}_G$ denote the function that to each 4-coloring ψ of C assigns the number of ways ψ extends to a 4-coloring of G; hence, ψ extends to a 4-coloring of G if and only if $n^{\star}_G(\psi) \neq 0$. Suppose

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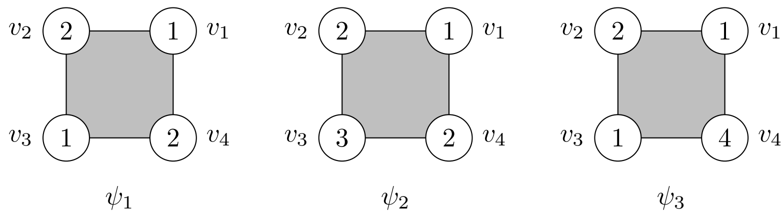

Figure 1: Precolorings ψ_1, ψ_2, and ψ_3 of a 4-cycle.

$C = v_1v_2v_3v_4$ is a 4-cycle and ψ_1, ψ_2 and ψ_3 are its 4-colorings such that $\psi_i(v_j) = j$ for $i \in \{1,2,3\}$ and $j \in \{1,2\}$, $\psi_1(v_3) = \psi_3(v_3) = 1$, $\psi_2(v_3) = 3$, $\psi_1(v_4) = \psi_2(v_4) = 2$, and $\psi_3(v_4) = 4$; see Figure 1. A standard Kempe chain argument shows that if $n^\star_G(\psi_1) \neq 0$, then $n^\star_G(\psi_2) \neq 0$ or $n^\star_G(\psi_3) \neq 0$.

Actually, much more information can be obtained along these lines, using the idea of *Block-count reducibility* [3, 4] developed in connection with the attempts to prove the Four Color Theorem: Certain inequalities between linear combinations of $n^\star_G(\psi_1)$, $n^\star_G(\psi_2)$, and $n^\star_G(\psi_3)$ are satisfied for all near-triangulations G, or equivalently, the vector $(n^\star_G(\psi_1), n^\star_G(\psi_2), n^\star_G(\psi_3))$ is contained in a certain cone in $\mathbb{R}^3$. The main goal of this note is to present and motivate a conjecture regarding this cone in the case of near-triangulations with the outer face bounded by a 5-cycle; this conjecture strengthens the Four Color Theorem. We also provide evidence supporting this conjecture.

1 Definitions

In order to describe the cone we alluded to in the introduction, we need a number of definitions, which we introduce in this section. It is easier to state the idea in the dual setting of 3-edge-colorings of cubic plane graphs, which is well-known to be equivalent to 4-coloring of plane triangulations [6].

1.1 Near-cubic graphs and their edge-colorings

Let G be a connected graph and let v be a vertex of G. We consider each edge of G as consisting of two half-edges. Let ν be a bijection between the half-edges incident with v and $\{0,\ldots,\deg(v)-1\}$ (so, if v is incident with a loop, each half of the loop is assigned a different number by ν). If all vertices of G other than v have degree three, we say that $\tilde{G} = (G, v, \nu)$ is a *near-cubic graph.* We say that $\tilde{G}$ is a *plane near-cubic graph* if G is a plane graph and the half-edges incident with v are drawn around it in the clockwise cyclic order $\nu^{-1}(0)$, ..., $\nu^{-1}(\deg(v)-1)$. We define $d(\tilde{G}) = \deg(v)$.

A 3-*edge-coloring* of $\tilde{G}$ is an assignment of colors 1, 2, and 3 to edges of G such that any two edges incident with a common vertex other than v have different colors. For an integer $d \geq 2$, a function $\psi : \{0,\ldots,d-1\} \to \{1,2,3\}$ is a *d-precoloring* if $|\psi^{-1}(1)| \equiv |\psi^{-1}(2)| \equiv |\psi^{-1}(3)| \equiv d \pmod 2$. We say that a 3-edge-coloring φ of $\tilde{G}$ *extends* a $d(\tilde{G})$-precoloring ψ if for any edge e incident with v and a half-edge h of e incident with v, we have $\varphi(e) = \psi(\nu(h))$. Let $n_{\tilde{G}}(\psi)$ denote the number of 3-edge-colorings of $\tilde{G}$ which extend ψ. Via the theory of nowhere-zero flows [7], it is easy to establish the following correspondence between 4-colorings of near-triangulations and 3-edge-colorings in their duals.

Observation 2. *Let $\tilde{G} = (G, v, \nu)$ be a plane near-cubic graph, and let $G^\star$ be the dual of G drawn so that the outer face of $G^\star$ corresponds to v. Suppose the outer face of $G^\star$ is bounded by a cycle C. Then there exists a mapping f from 4-colorings of C to $d(\tilde{G})$-precolorings such that*

- *f maps exactly four 4-colorings of C to each $d(\tilde{G})$-precoloring, and*
- *every 4-coloring ψ of C satisfies $n^\star_{G^\star}(\psi) = n_{\tilde{G}}(f(\psi))$.*

Given two near-cubic graphs $\tilde{G}_1 = (G_1, v_1, \nu_1)$ and $\tilde{G}_2 = (G_2, v_2, \nu_2)$ with $\deg(v_1) = \deg(v_2)$, let $\tilde{G}_1 \oplus \tilde{G}_2$ denote the graph obtained from G_1 and G_2 by, for $0 \le i \le \deg(v_1) - 1$, removing the half-edges $\nu_1^{-1}(i)$ and $\nu_2^{-1}(i)$ and connecting the other halfs of the edges. Note that $\tilde{G}_1 \oplus \tilde{G}_2$ is a cubic graph, and if $\tilde{G}_1$ and $\tilde{G}_2$ are plane near-cubic graphs, then $\tilde{G}_1 \oplus \tilde{G}_2$ is a planar graph. Observe that the number of 3-edge-colorings of $\tilde{G}_1 \oplus \tilde{G}_2$ is

$$\sum_{\psi} n_{\tilde{G}_1}(\psi) n_{\tilde{G}_2}(\psi),$$

where the sum goes over all $\deg(v_1)$-precolorings ψ. For any integer $n \ge 3$, let $\tilde{C}_n$ denote the plane near-cubic graph (W_n, v, ν), where W_n is the wheel with the central vertex v adjacent to all vertices of an n-cycle.

1.2 Signatures and Kempe chains

For an integer $d \ge 2$, a *d-signature* is a set S of pairs (m, s), where m is an unordered pair of integers in $\{0, \ldots, d-1\}$ and $s \in \{-1, 1\}$, satisfying the following conditions:

(i) for any distinct $(m_1, s_1), (m_2, s_2) \in S$ we have $m_1 \cap m_2 = \emptyset$, and

(ii) S does not contain elements $(\{a, b\}, s_1)$ and $(\{c, d\}, s_2)$ such that $a < c < b < d$.

A d-precoloring ψ is *compatible* in (distinct) colors $i, j \in \{1, 2, 3\}$ with a d-signature S if

- $\psi^{-1}(\{i, j\}) = \bigcup_{(m,s)\in S} m$, and
- for each $(\{a_1, a_2\}, s) \in S$, $\psi(a_1) = \psi(a_2)$ holds if and only if $s = -1$.

Now, consider a 3-edge-coloring φ of a near-cubic graph $\tilde{G} = (G, v, \nu)$. Each vertex other than v is incident with edges of all three colors. Hence, for any distinct $i, j \in \{1, 2, 3\}$, the subgraph G_{ij} of G consisting of edges of colors i or j is a union of pairwise edge-disjoint cycles, vertex-disjoint except for possible intersections in v. An *ij-Kempe chain* of φ is a cycle C in G_{ij} containing v; the sign $\sigma(C)$ of the ij-Kempe chain C is 1 if the length of C is even and -1 if the length of C is odd. If h_1 and h_2 are the half-edges in C incident with v, we let $\mu(C) = \{\nu(h_1), \nu(h_2)\}$. The *$ij$-Kempe chain signature* $\sigma_{ij}(\varphi)$ of φ is defined as

$$\{(\mu(C), \sigma(C)) : C \text{ is an } ij\text{-Kempe chain of } \varphi\}.$$

Note that if $\tilde{G}$ is plane, then the ij-Kempe chains do not cross and the ij-Kempe chain signature of φ satisfies the condition (ii); and thus $\sigma_{ij}(\varphi)$ is a $d(\tilde{G})$-signature.

2 Coloring count cones

Let $\tilde{G} = (G, v, \nu)$ be a plane near-cubic graph and let ψ be a $d(\tilde{G})$-precoloring. Suppose that ψ is compatible (in colors $i, j \in \{1, 2, 3\}$) with a $d(\tilde{G})$-signature S. We define $n_{\tilde{G},S}(\psi)$ as the number of 3-edge-colorings φ of $\tilde{G}$ extending ψ such that $\sigma_{ij}(\varphi) = S$. Note that swapping the colors i and j on any set of ij-Kempe chains of φ results in another 3-edge-coloring with the same ij-Kempe chain signature. Furthermore, clearly for any permutation π of colors, we have $n_{\tilde{G},S}(\psi \circ \pi) = n_{\tilde{G},S}(\psi)$. This establishes bijections implying the following.

Observation 3. *Let $\tilde{G}$ be a plane near-cubic graph and let S be a $d(\tilde{G})$-signature. Any $d(\tilde{G})$-precolorings ψ_1 and ψ_2 compatible with S satisfy*

$$n_{\tilde{G},S}(\psi_1) = n_{\tilde{G},S}(\psi_2).$$

Hence, we can define an integer $n_{\tilde{G},S}$ to be equal to $n_{\tilde{G},S}(\psi)$ for an arbitrarily chosen $d(\tilde{G})$-precoloring ψ compatible with S.

Let $d \geq 2$ be an integer and let $i, j \in \{1, 2, 3\}$ be distinct colors. For a d-precoloring ψ, let us define $\mathcal{S}_{\psi,ij}$ as the set of d-signatures compatible with ψ in colors ij. Since every 3-edge-coloring of $\tilde{G}$ has an ij-Kempe chain signature, we have

$$n_{\tilde{G}}(\psi) = \sum_{S \in \mathcal{S}_{\psi,ij}} n_{\tilde{G},S}(\psi) = \sum_{S \in \mathcal{S}_{\psi,ij}} n_{\tilde{G},S}. \tag{1}$$

Let $\mathcal{P}_d$ denote the set of all d-precolorings and $\mathcal{S}_d$ the set of all d-signatures. We will work in the vector spaces $\mathbb{R}^{\mathcal{P}_d}$ and $\mathbb{R}^{\mathcal{S}_d}$ with coordinates corresponding to the d-precolorings and to the d-signatures, respectively. For each integer $d \geq 2$, the *coloring count cone* B_d is the set of all $x \in \mathbb{R}^{\mathcal{P}_d}$ such that

- $x(\psi) \geq 0$ for every d-precoloring ψ, and
- there exists $y \in \mathbb{R}^{\mathcal{S}_d}$ such that

– $y(S) \geq 0$ for every d-signature S, and
– $x(\psi) = \sum_{S \in \mathcal{S}_{\psi,ij}} y(S)$ for every d-precoloring ψ and distinct colors $i, j \in \{1,2,3\}$.

Note that B_d is indeed a cone, i.e., an unbounded polytope closed under linear combinations with non-negative coefficients. By (1), the vector of precoloring extension counts for any plane near-cubic graph belongs to the corresponding coloring count cone.

Theorem 4. *For each plane near-cubic graph $\tilde{G}$, we have*

$$n_{\tilde{G}} \in B_{d(\tilde{G})}.$$

Each cone is uniquely determined as the set of non-negative linear combinations of its rays. For $d \in \{2,3,4,5\}$, the rays of B_d are easy to enumerate by hand or using polytope-manipulation software such as Sage Math or the Parma Polyhedra Library (a program doing so for $d = 5$ can be found at `http://lidicky.name/pub/4cone/`). For a near-cubic graph $\tilde{G}$ such that $n_{\tilde{G}}$ is not the zero function, let $\mathrm{ray}(\tilde{G})$ denote the set of all non-negative multiples of $n_{\tilde{G}}$.

Lemma 5. *Refering to graphs in Figure 2:*

- *the cone B_2 has exactly one ray equal to* $\mathrm{ray}(\tilde{R}_{2,1})$*;*
- *the cone B_3 has exactly one ray equal to* $\mathrm{ray}(\tilde{R}_{3,1})$*;*
- *the cone B_4 has exactly four rays equal to* $\mathrm{ray}(\tilde{R}_{4,1}), \ldots, \mathrm{ray}(\tilde{R}_{4,4})$*; and*
- *the cone B_5 has exactly* 12 *rays equal to* $\mathrm{ray}(\tilde{R}_{5,1}), \ldots, \mathrm{ray}(\tilde{R}_{5,12})$.

Let us remark that B_6 has 208 rays; the direct method we employ is too slow to enumerate all rays for $d \geq 7$ on current workstations.

3 The cone B_5 and the conjecture

Note that while $\tilde{R}_{5,1}, \ldots, \tilde{R}_{5,11}$ are plane, $\tilde{R}_{5,12}$ is not. Indeed, the following holds.

Lemma 6. *The following claims are equivalent.*

(a) *Every planar cubic* 2*-edge-connected graph is* 3*-edge-colorable.*

(b) *For every plane near-cubic graph $\tilde{G}$ with $d(\tilde{G}) = 5$, if $n_{\tilde{G}} \in \mathrm{ray}(\tilde{R}_{5,12})$, then $n_{\tilde{G}}$ is the zero function.*

Proof. Let us first prove that (a) implies (b). Consider a plane near-cubic graph $\tilde{G} = (G, v, \nu)$ such that $n_{\tilde{G}} \in \mathrm{ray}(\tilde{R}_{5,12})$, and thus for some constant $c \geq 0$, we have $n_{\tilde{G}}(\psi) = c \cdot n_{\tilde{R}_{5,12}}(\psi)$ for every 5-precoloring ψ. Observe that

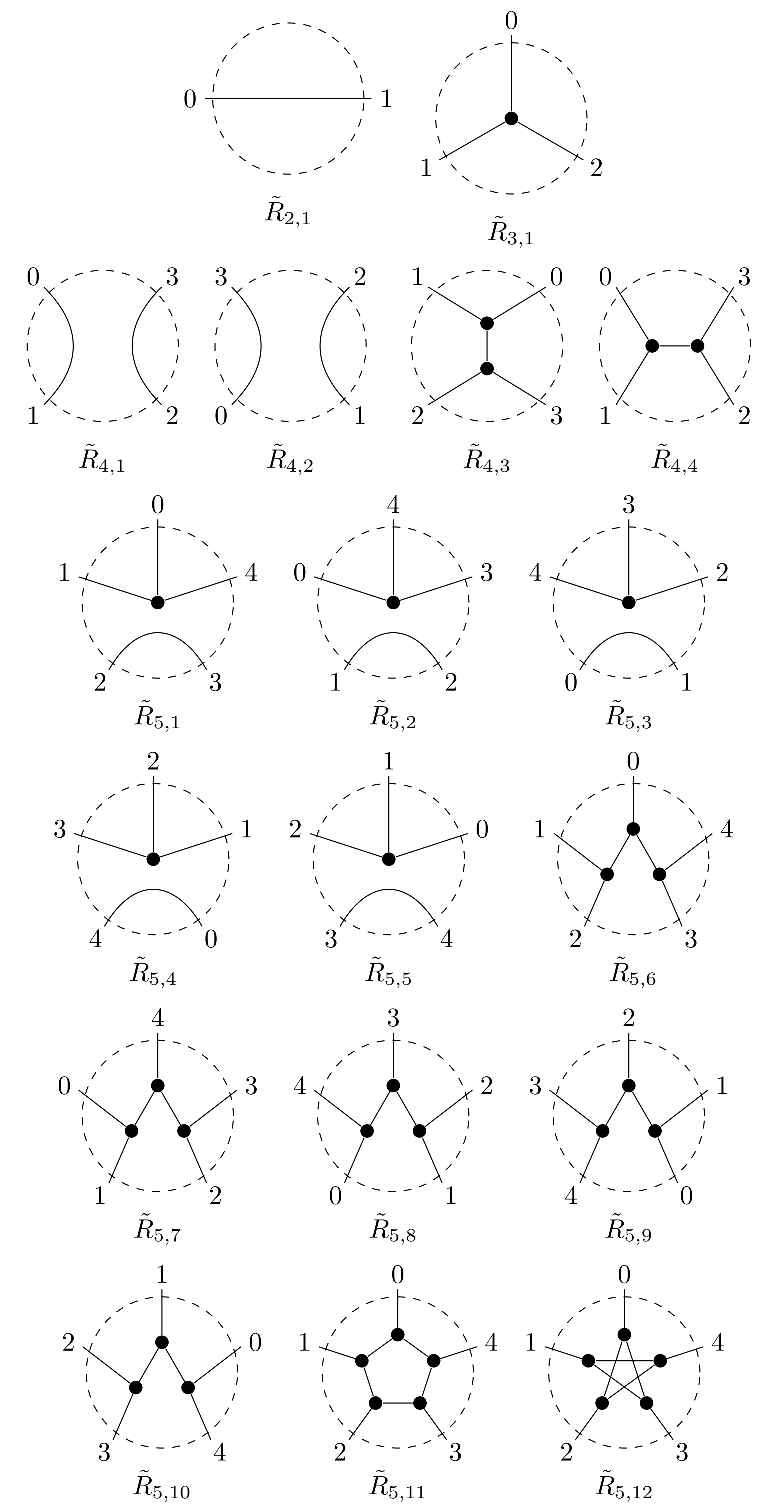

Figure 2: Graphs $\tilde{R}_{2,1}, \dots, \tilde{R}_{5,12}$. The dashed circle intersects the half-edges incident with the vertex v, which is not depicted; the values of ν are written at the respective half-edges.

$n_{\tilde{R}_{5,12}}(\psi)n_{\tilde{C}_5}(\psi) = 0$ for every 5-precoloring ψ (since $\tilde{R}_{5,12} \oplus \tilde{C}_5$ is the Petersen graph, which is not 3-edge-colorable), and thus the number of 3-edge-colorings of $\tilde{G} \oplus \tilde{C}_5$ is

$$\sum_{\psi} n_{\tilde{G}} n_{\tilde{C}_5}(\psi) = c \sum_{\psi} n_{\tilde{R}_{5,12}} n_{\tilde{C}_5}(\psi) = 0.$$

Hence, the planar cubic graph $\tilde{G} \oplus \tilde{C}_5$ is not 3-edge-colorable. By (a), $\tilde{G} \oplus \tilde{C}_5$ has a bridge, and thus G has a bridge. But then a standard parity argument implies that $\tilde{G}$ has no 3-edge-coloring, and thus $n_{\tilde{G}}$ is the zero function.

Next, let us prove that (b) implies (a). Suppose for a contradiction that (b) holds, but there exists a plane cubic 2-edge-connected graph that is not 3-edge-colorable, and let H be one with the smallest number of vertices. By Euler's formula, H has a face f of length $d \leq 5$; hence, we can write $H = \tilde{G} \oplus \tilde{C}_d$ for a plane near-cubic graph $\tilde{G}$. By Theorem 4, we have $n_{\tilde{G}} \in B_d$, and by Lemma 5, there exist non-negative real numbers c_i such that

$$n_{\tilde{G}} = \sum_i c_i n_{\tilde{R}_{d,i}}.$$

Observe there exists a plane near-cubic graph $\tilde{P}$ with $d-1$ vertices such that $\tilde{G} \oplus \tilde{P}$ is 2-edge-connected. By the minimality of H, $\tilde{G} \oplus \tilde{P}$ is 3-edge-colorable, and in particular $n_{\tilde{G}}$ is not the zero function. By (b), $n_{\tilde{G}}$ is not a positive multiple of $n_{\tilde{R}_{5,12}}$, and thus there exists an index $k \leq 11$ such that $c_k > 0$. Observe that $\tilde{R}_{d,k} \oplus \tilde{C}_d$ is 3-edge-colorable, and thus there exists a d-precoloring ψ_0 such that $n_{\tilde{R}_{d,k}}(\psi_0)n_{\tilde{C}_d}(\psi_0) > 0$. However, then the number of 3-edge-colorings of H is

$$\sum_{\psi} n_{\tilde{G}}(\psi)n_{\tilde{C}_d}(\psi) \geq c_k \sum_{\psi} n_{\tilde{R}_{d,k}}(\psi)n_{\tilde{C}_d}(\psi) \geq c_k n_{\tilde{R}_{d,k}}(\psi_0)n_{\tilde{C}_d}(\psi_0) > 0.$$

This contradicts the assumption that H is not 3-edge-colorable. $\square$

Note that (a) is well-known to be equivalent to the Four Color Theorem [6], and thus indeed there is no plane near-cubic graph $\tilde{G}$ with $d(\tilde{G}) = 5$ such that $n_{\tilde{G}}$ is not the zero function and $n_{\tilde{G}} \in \text{ray}(\tilde{R}_{5,12})$; and furthermore, a direct proof of this fact would imply the Four Color Theorem. Motivated by this observation (and experimental evidence), we propose the following conjecture, a strengthening of the Four Color Theorem. Let B'_5 denote the cone in $\mathbb{R}^{\mathcal{P}_d}$ with rays $\text{ray}(\tilde{R}_{5,1})$, ..., $\text{ray}(\tilde{R}_{5,11})$.

Conjecture 7. *Every plane near-cubic graph $\tilde{G}$ with $d(\tilde{G}) = 5$ satisfies $n_{\tilde{G}} \in B'_5$.*

For $i \in \{0, \dots, 4\}$, let $\psi_i^{5,a}$ and $\psi_i^{5,b}$ denote the 5-precolorings whose values at $j \in \{0, \dots, 4\}$ are defined by the following table; see also Figure 3.

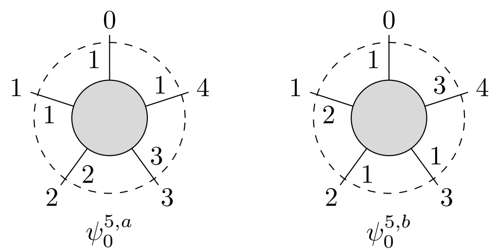

Figure 3: Precolorings $\psi_0^{5,a}$ and $\psi_0^{5,b}$.

$(j-i) \bmod 5$	$\psi_i^{5,a}(j)$	$\psi_i^{5,b}(j)$
0	1	1
1	1	2
2	2	1
3	3	1
4	1	3

Note that each 5-precoloring is obtained from one of these ten by a permutation of colors. The cone B_5' has exactly one facet which is not also a facet of B_5, giving an equivalent formulation of Conjecture 7.

Conjecture 8. *Every plane near-cubic graph $\tilde{G}$ with $d(\tilde{G}) = 5$ satisfies*

$$3\sum_{i=0}^{4} n_{\tilde{G}}(\psi_i^{5,a}) \geq \sum_{i=0}^{4} n_{\tilde{G}}(\psi_i^{5,b}).$$

In the rest of the note, we provide some evidence supporting Conjecture 7; in particular, we show there are no counterexamples to the conjecture with less than 30 vertices.

4 Evidence

Before we present the experimental evidence for the validity of Conjecture 7, we need a few more definitions. A vector $x \in \mathcal{P}_d$ is *invariant with respect to permutation of colors* if all d-precolorings ψ and ψ' that only differ by a permutation of colors satisfy $x(\psi) = x(\psi')$.

See Figure 4 for an illustration of the following definitions. The *rotation by t* of a d-precoloring ψ is the d-precoloring $r_t(\psi)$ such that $r_t(\psi)((i+t) \bmod d) = \psi(i)$ for $i \in \{0, \ldots, d-1\}$. The *flip* of a d-precoloring ψ is the d-precoloring $f(\psi)$ such that $f(\psi)(i) = \psi(d-1-i)$ for $i \in \{0, \ldots, d-1\}$. For $x \in \mathbb{R}^{\mathcal{P}_d}$, let $r_t(x)$ be defined as $y \in \mathbb{R}^{\mathcal{P}_d}$ such that $y(r_t(\psi)) = x(\psi)$ for every d-precoloring ψ, and let $f(x)$ be defined as $z \in \mathbb{R}^{\mathcal{P}_d}$ such that $z(f(\psi)) = x(\psi)$ for every d-precoloring ψ. A set $K \subseteq \mathbb{R}^{\mathcal{P}_d}$ is *closed under rotations and flips* if we have $x \in K$ if and only

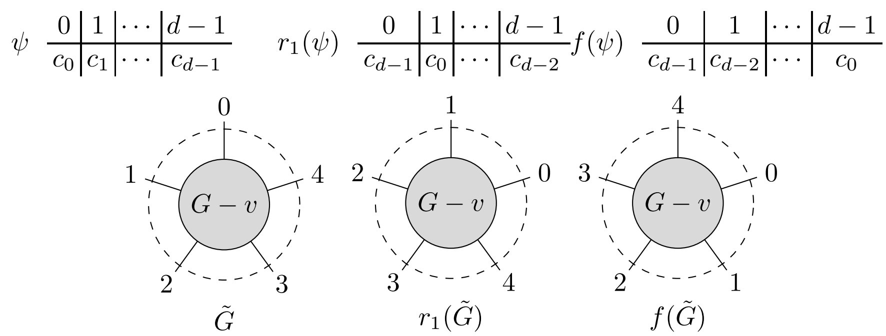

Figure 4: Rotation and flip operations. Colors are denoted by $c_0, \dots, c_{d-1}$.

if $f(x) \in K$ and $r_t(x) \in K$ for all $t \in \{0, 1, \dots, d-1\}$. For a near-cubic graph $\tilde{G} = (G, v, \nu)$ with $\deg(v) = d$, let $r_t(\tilde{G})$ denote the near-cubic graph (G, v, ν_1), where $\nu_1^{-1}((i+t) \bmod d) = \nu^{-1}(i)$ for $i \in \{0, \dots, d-1\}$, and let $f(\tilde{G})$ denote the near-cubic graph (G, v, ν_2), where $\nu_2^{-1}(i) = \nu_2^{-1}(d-1-i)$ for $i \in \{0, \dots, d-1\}$.

Observation 9. *Let $\tilde{G}$ be a near-cubic graph, $d = d(\tilde{G})$ and $t \in \{0, \dots, d-1\}$. Then $n_{r_t(\tilde{G})} = r_t(n_{\tilde{G}})$ and $n_{f(\tilde{G})} = f(n_{\tilde{G}})$.*

Let ψ_1 be a d_1-precoloring and ψ_2 a d_2-precoloring. For an integer $k \leq \min(d_1, d_2)$, we say that ψ_1 *k-matches* ψ_2 if $\psi_1(d_1 - k + i) = \psi_2(d_2 - 1 - i)$ for $i \in \{0, 1, \dots, k-1\}$. By $\gamma_k(\psi_1, \psi_2)$, we denote the $(d_1 + d_2 - 2k)$-precoloring γ such that $\gamma(i) = \psi_1(i)$ for $i \in \{0, \dots, d_1 - k - 1\}$ and $\gamma(i) = \psi_2(i - (d_1 - k))$ for $i \in \{d_1 - k, \dots, d_1 + d_2 - 2k - 1\}$. For $x_1 \in \mathbb{R}^{\mathcal{P}_{d_1}}$ and $x_2 \in \mathbb{R}^{\mathcal{P}_{d_2}}$, we define $\gamma_k(x_1, x_2)$ as the vector $y \in \mathbb{R}^{\mathcal{P}_{d_1+d_2-2k}}$ such that

$$y(\psi) = \sum_{\psi_1, \psi_2 : \gamma_k(\psi_1, \psi_2) = \psi} x_1(\psi_1) x_2(\psi_2),$$

where the sum is over all k-matching d_1-precolorings ψ_1 and d_2-precolorings ψ_2. For near-cubic graphs $\tilde{G}_1 = (G_1, v_1, \nu_1)$ with $\deg(v_1) = d_1$ and $\tilde{G}_2 = (G_2, v_2, \nu_2)$ with $\deg(v_2) = d_2$, let $\gamma_k(\tilde{G}_1, \tilde{G}_2)$ denote the near-cubic graph (G, v, ν), where G is obtained from G_1 and G_2 by identifying v_1 with v_2 to a single vertex v and for $i \in \{0, 1, \dots, k-1\}$ removing the half-edges $\nu_1^{-1}(d_1 - k + i)$ and $\nu_2^{-1}(d_2 - 1 - i)$ and connecting the other halfs of the edges; and $\nu^{-1}(i) = \nu_1^{-1}(i)$ for $i \in \{0, \dots, d_1 - k - 1\}$ and $\nu^{-1}(i) = \nu_2^{-1}(i - (d_1 - k))$ for $i \in \{d_1 - k, \dots, d_1 + d_2 - 2k - 1\}$. See Figure 5 for an illustration.

Observation 10. *Let $\tilde{G}_1$ and $\tilde{G}_2$ be near-cubic graphs. For every integer $k \in \{0, \dots, \min(d(\tilde{G}_1), d(\tilde{G}_2))\}$, we have $n_{\gamma_k(\tilde{G}_1, \tilde{G}_2)} = \gamma_k(n_{\tilde{G}_1}, n_{\tilde{G}_2})$.*

By a computer-assisted enumeration, we verified the following claim.

Lemma 11. *There exists cones $K_d \subseteq \mathbb{R}^{\mathcal{P}_d}$ for $d = 2, \dots, 8$ such that the following claims hold.*

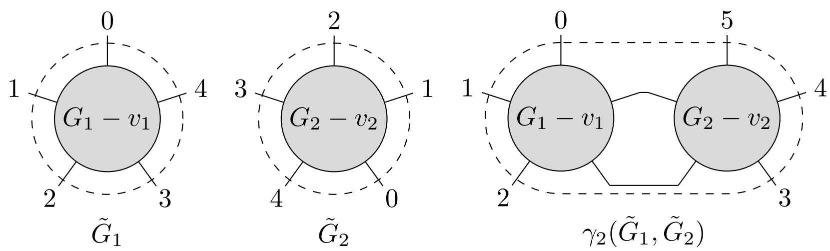

Figure 5: $\gamma_k(\tilde{G}_1, \tilde{G}_2)$

(a) *$K_d = B_d$ when $d \le 4$ and $K_5 = B'_5$.*

(b) *For all $d \in \{2, \dots, 8\}$, the elements of K_d are invariant with respect to permutation of colors.*

(c) *For $d \in \{2, \dots, 7\}$, the cone K_d is closed under rotations and flips.*

(d) *If $2 \le d_1 \le d_2$ and $d_1 + d_2 \le 7$, then for all $x_1 \in K_{d_1}$ and $x_2 \in K_{d_2}$ we have $\gamma_0(x_1, x_2) \in K_{d_1+d_2}$.*

(e) *If $2 \le d \le 5$, then for all $x \in K_d$ we have $\gamma_1(n_{\tilde{R}_{3,1}}, x) \in K_{d+1}$.*

(f) *If $3 \le d \le 7$, then for all $x \in K_d$ we have $\gamma_2(n_{\tilde{R}_{3,1}}, x) \in K_{d-1}$.*

(g) *If $2 \le d_1 \le 6$ and $1 \le c \le d_1/2$, then for all $x_1 \in K_{d_1}$ and $x_2 \in K_{7+2c-d_1}$, we have $\gamma_c(x_1, x_2) \in K_7$.*

(h) *For every $x_1 \in K_8$ and $x_2 \in K_7$, we have $\gamma_4(x_1, x_2) \in K_7$.*

(i) *For every $x_1, x_2 \in K_6$, we have $r_2(\gamma_2(x_1, x_2)) \in K_8$.*

Proof. The proof and the program to verify the proof can be found at `http://lidicky.name/pub/4cone/`. The cones are described by their rays, enumerated in the file. Cone K_6 has 102 rays, K_7 has 22605 rays, and K_8 has 4330 rays. It suffices to verify all the claims for x, x_1, x_2 being the rays of the cones specified in the claims; the inclusion of the resulting vectors in the appropriate cone is certified by expressing them as a linear non-negative combination of the rays of the cone. $\square$

Parts (e) and (f) of Lemma 11 have the following corollary.

Lemma 12. *Let $\tilde{G} = (G, v, \nu)$ be a plane near-cubic graph and let $d = d(\tilde{G})$. If $d \in \{2, \dots, 7\}$ and $n_{\tilde{G}} \notin K_d$, then there exists a plane near-cubic graph $\tilde{G}_0 = (G_0, v_0, \nu_0)$ such that $d(\tilde{G}_0) = 7$, $n_{\tilde{G}_0} \notin K_7$, $G_0 - v_0$ is an induced subgraph of $G - v$, and $|V(G_0)| \le |V(G)| - (7 - d)$.*

Proof. We prove the claim by induction on the number of vertices of G. When $d \le 4$, the claim is vacuously true by Theorem 4, since $K_d = B_d$. When $d = 7$, we can set $\tilde{G}_0 = \tilde{G}$. Hence, suppose that $d \in \{5, 6\}$. Since $n_{\tilde{G}} \notin K_d$, the function $n_{\tilde{G}}$ is not identically zero.

If $G - v$ is disconnected, we can by symmetry assume that $\tilde{G} = \gamma_0(\tilde{G}_1, \tilde{G}_2)$ for plane near-cubic graphs $\tilde{G}_1$ and $\tilde{G}_2$ such that $d = d(\tilde{G}_1) + d(\tilde{G}_2)$ and $d(\tilde{G}_1) \le d(\tilde{G}_2)$. Since $n_{\tilde{G}}$ is not the zero function, $n_{\tilde{G}_1}$ is not the zero function either, and thus $d(\tilde{G}_1) \ne 1$. Hence $d(\tilde{G}_1) \ge 2$, and thus $d(\tilde{G}_2) \le 4$. Hence, $n_{\tilde{G}_1} \in K_{d(\tilde{G}_1)}$ and $n_{\tilde{G}_2} \in K_{d(\tilde{G}_2)}$, and $n_{\tilde{G}} \in K_d$ by Lemma 11(d), which is a contradiction.

Hence, $G - v$ is connected (and the same argument shows that no loop is incident with v). Consequently, v is not incident with a triple edge. If v is incident with a double edge, then we can by symmetry assume that $\tilde{G} = \gamma_1(\tilde{R}_{3,1}, \tilde{G}_1)$ for a plane near-cubic graph $\tilde{G}_1 = (G_1, v_1, \nu_1)$ with $d(\tilde{G}_1) = d - 1 \le 5$. By Lemma 11(e), since $n_{\tilde{G}} \notin K_d$, we have $n_{\tilde{G}_1} \notin K_{d-1}$. By the induction hypothesis, there exists a plane near-cubic graph $\tilde{G}_0 = (G_0, v_0, \nu_0)$ with $d(\tilde{G}_0) = 7$, such that $n_{\tilde{G}_0} \notin K_7$, $G_0 - v_0$ is an induced subgraph of $G_1 - v_1$, and thus also of $G - v$, and $|V(G_0)| \le |V(G_1)| - (7 - (d-1)) < |V(G)| - (7 - d)$, as required.

Hence, we can assume v is not incident with a double edge. Consequently, we can by symmetry assume that $\tilde{G} = \gamma_2(\tilde{R}_{3,1}, \tilde{G}_1)$ for a plane near-cubic graph $\tilde{G}_1 = (G_1, v_1, \nu_1)$ with $d(\tilde{G}_1) = d + 1$. By Lemma 11(f), since $n_{\tilde{G}} \notin K_d$, we have $n_{\tilde{G}_1} \notin K_{d+1}$. By the induction hypothesis, there exists a plane near-cubic graph $\tilde{G}_0 = (G_0, v_0, \nu_0)$ with $d(\tilde{G}_0) = 7$, such that $n_{\tilde{G}_0} \notin K_7$, $G_0 - v_0$ is an induced subgraph of $G_1 - v_1$, and $|V(G_0)| \le |V(G_1)| - (7 - (d+1)) = |V(G)| - (7 - d)$. Hence, the claim of the lemma follows. □

We will say that a plane near-cubic graph $\tilde{G} = (G, v, \nu)$ is *extremal* if $d(\tilde{G}) = 7$, $n_{\tilde{G}} \notin K_7$, and there does not exist any plane near-cubic graph $\tilde{G}_0 = (G_0, v_0, \nu)$ with $d(\tilde{G}_0) = 7$ such that $n_{\tilde{G}_0} \notin K_7$ and $G_0 - v_0$ is a proper minor of $G - v$.

Lemma 13. *If $\tilde{G} = (G, v, \nu)$ is an extremal plane near-cubic graph and $\tilde{G}' = (G', v', \nu')$ is a plane near-cubic graph with $d(\tilde{G}') \le 7$ such that $G' - v'$ is a proper minor of $G - v$, then $n_{\tilde{G}'} \in K_{d(\tilde{G}')}$.*

Proof. If $n_{\tilde{G}'} \notin K_{d(\tilde{G}')}$, then by Lemma 12 there would exist a plane near-cubic graph $\tilde{G}_0 = (G_0, v_0, \nu_0)$ such that $d(\tilde{G}_0) = 7$, $n_{\tilde{G}_0} \notin K_7$ and $G_0 - v_0$ is an induced subgraph of $G' - v'$. However, then $G_0 - v_0$ would be a proper minor of $G - v$, contradicting the assumption that $\tilde{G}$ is extremal. □

Next, let us explore consequences of part (g) of Lemma 11.

Lemma 14. *If $\tilde{G} = (G, v, \nu)$ is an extremal plane near-cubic graph, then v is not incident with loops or parallel edges and $G - v$ is 2-edge-connected.*

Proof. Analogously to the proof of Lemma 12, if v were incident with a loop or a parallel edge or if $G - v$ were not 2-edge-connected, we would have $\tilde{G} =$

$\gamma_c(\tilde{G}_1, \tilde{G}_2)$ for plane near-cubic graphs $\tilde{G}_1$ and $\tilde{G}_2$ such that $2 \le d(\tilde{G}_1) \le d(\tilde{G}_2)$, $d(\tilde{G}_1) + d(\tilde{G}_2) = 7 + 2c$, and $c \le 1$; in particular, $d(\tilde{G}_2) \le 7$ and $d(\tilde{G}_1) \le \lfloor (7+2c)/2 \rfloor \le 4$. By Lemma 13, we have $n_{\tilde{G}_i} \in K_{d(\tilde{G}_i)}$ for $i \in \{1, 2\}$. By Lemma 11(g), we conclude $n_{\tilde{G}} \in K_7$, which is a contradiction. □

Suppose A and B form a partition of the vertex set of a graph H, and let S be the set of edges of H with one end in A and the other end in B. In this situation, we say S is an *edge cut* of H with sides A and B.

Lemma 15. *If $\tilde{G} = (G, v, \nu)$ is an extremal plane near-cubic graph, then $G - v$ does not contain an edge cut S such that v has at least $|S|$ neighbors in each side of the cut.*

Proof. Suppose for a contradiction $G - v$ contains such an edge cut S of size c, and thus $\tilde{G} = \gamma_c(\tilde{G}_1, \tilde{G}_2)$ for plane near-cubic graphs $\tilde{G}_1$ and $\tilde{G}_2$ such that $2c \le d(\tilde{G}_1) \le d(\tilde{G}_2)$ and $d(\tilde{G}_1) + d(\tilde{G}_2) = 7 + 2c$. Since v has 7 neighbors and at least c of them are contained in each of the sides of the cut, we have $c \le 3$. Note that $d(\tilde{G}_2) \le 7$ and $d(\tilde{G}_1) \le \lfloor (7 + 2c)/2 \rfloor \le 6$. By Lemma 13, we have $n_{\tilde{G}_i} \in K_{d(\tilde{G}_i)}$ for $i \in \{1, 2\}$. By Lemma 11(g), we conclude $n_{\tilde{G}} \in K_7$, which is a contradiction. □

An edge cut S of size at most five in a near-cubic graph $\tilde{G} = (G, v, \nu)$ is *essential* if the side of S containing v contains at least one other vertex and the other side B of S induces neither a tree nor a 5-cycle.

Lemma 16. *If $\tilde{G} = (G, v, \nu)$ is an extremal plane near-cubic graph, then $\tilde{G}$ does not contain an essential edge cut S of size at most five.*

Proof. Suppose for a contradiction that $\tilde{G}$ contains an essential edge-cut S of size $k \le 5$, and choose one with minimum k, and subject to that one for which the side B not containing v is minimal. We claim $G[B]$ is 2-edge-connected. Otherwise, B is a disjoint union of non-empty sets B_1 and B_2, where G contains $r \le 1$ edges with one end in B_1 and the other end in B_2. For $i \in \{1, 2\}$, let S_i denote the set of edges of G with exactly one end in B_i. Since $\tilde{G}$ is extremal, $n_{\tilde{G}} \notin K_7$ is not identically zero, and thus G is 2-edge-connected, implying $|S_i| \ge 2$. Hence, $|S_i| = k + 2r - |S_{3-i}| \le k$. By the minimality of B, we conclude that B_i induces a tree or a 5-cycle, and thus $|S_i| \ge 3$. Hence $5 \ge k = |S_1| + |S_2| - 2r \ge 6 - 2r$, and thus $r = 1$ and $|S_1|, |S_2| \le 4$. This implies that neither B_1 nor B_2 induces a 5-cycle, and thus both of them induce trees; and G contains an edge between them, implying that B induces a tree, contrary to the assumption that S is an essential edge cut.

Since $G[B]$ is 2-edge-connected and subcubic, each face of $G[B]$ is bounded by a cycle. Let C_S denote the cycle bounding the face f of $G[B]$ whose interior contains v. Observe that all edges of S are drawn inside f. Otherwise, the set S' of edges of S drawn inside C forms an edge cut of order smaller than k and by the minimality of k, its side $B' \supsetneq B$ induces a tree or a 5-cycle; this is not possible, since $G[B]$ is 2-edge connected and not a tree.

Let $\tilde{G}_c$ be the plane near-cubic graph obtained from G by contracting the side of the cut containing v to a single vertex. By Lemma 13, we have $n_{\tilde{G}_c} \in K_k$. Since $K_d = B_d$ for $d \le 4$ and $K_5 = B'_5$,

$$n_{\tilde{G}_c} = \sum_i c_i n_{\tilde{R}_{k,i}},$$

where $i \le 11$ if $k = 5$ and the coefficients c_i are non-negative. Let $\tilde{G}_i = (G_i, v_i, \nu_i)$ denote the plane near-cubic graph obtained from $\tilde{G}$ by replacing the side of the cut S not containing v by $\tilde{R}_{k,i}$. Note that $n_{\tilde{G}} = \sum_i c_i n_{\tilde{G}_i}$, and since K_7 is a cone and $n_{\tilde{G}} \notin K_7$, there exists i such that $n_{\tilde{G}_i} \notin K_7$. Because B contains the cycle C_S and all edges of S are incident with vertices of C_S, we see $G_i - v_i$ is a proper minor of $G - v$, contradicting the extremality of $\tilde{G}$. □

In Lemma 14, we argued that if $\tilde{G} = (G, v, \nu)$ is an extremal plane near-cubic graph, then $G - v$ is 2-edge-connected, and thus its face containing v is bounded by a cycle C. Let us now argue that the graph stays 2-edge-connected after removing $V(C)$ as well.

Lemma 17. *Let $\tilde{G} = (G, v, \nu)$ be an extremal plane near-cubic graph and let C be the cycle bounding the face of $G - v$ containing v. The cycle C is induced, no two neighbors of v in C are adjacent, and the graph $G - (V(C) \cup \{v\})$ is 2-edge-connected and has more than one vertex.*

Proof. Consider a simple closed curve c in the plane intersecting G in two edges of C, $b \le 4$ edges incident with v, and $r \le 1$ edges of $E(G - v) \setminus E(C)$, where each edge is intersected at most once. The curve c separates the plane into two parts; let A and B be the corresponding partition of vertices of G, where $v \in A$, and let S be the edge cut in G consisting of the edges with one end in A and the other end in B. By Lemma 15 applied to the edge cut in $G - v$ obtained from S by removing the edges incident with v, it follows that $b \le r + 1$, and thus $|S| \le 3 + 2r \le 5$. By Lemma 16 we conclude that the edge cut satisfies one of the following conditions.

- $r = 0$, $b = 1$, $|S| = 3$, and B consists of a single vertex of C, or
- $r = 1$ and $G[B]$ is a subpath of C, or
- $r = 1$, $b = 2$, and $G[B]$ is a 5-cycle containing exactly one vertex not in $V(C)$.

If C had a chord e, this would give a contradiction by considering a curve c (with $r = 0$) drawn next to the chord so that $e \in E(G[B])$ and $b \le 3$; hence, C is an induced cycle. If two neighbors of v in C were adjacent, we would obtain a contradiction by considering a curve c (with $r = 0$ and $b = 2$) drawn around them. If the graph $G - (V(C) \cup \{v\})$ were not connected, we would obtain a contradiction by considering a curve c (with $r = 0$ and $b \le 3$) chosen so that both A and B contain a vertex of $G - (V(C) \cup \{v\})$. Finally, if the graph $G - (V(C) \cup \{v\})$ were not 2-edge-connected, then we could choose c so that $r = 1$, $b \leq 3$, and B contains a vertex of $G - (V(C) \cup \{v\})$. But then $G[B]$ would be a 5-cycle containing exactly one vertex not in $V(C)$ and consequently two adjacent vertices of C would be neighbors of v, which is a contradiction.

Therefore, the graph $G - (V(C) \cup \{v\})$ is 2-edge-connected. Since no two neighbors of v in C are adjacent, G contains at least 7 edges between $V(C)$ and $V(G) \setminus (V(C) \cup \{v\})$, and thus $G - (V(C) \cup \{v\})$ has more than one vertex. $\square$

Finally, let us apply the parts (h) and (i) of Lemma 11.

Lemma 18. *If $\tilde{G} = (G, v, \nu)$ is an extremal plane near-cubic graph, then G has at least 28 vertices.*

Proof. By Lemma 14, the face of $G - v$ containing v is bounded by a cycle C. Let v_1, …, v_7 be the neighbors of v in C in order. For $i \in \{1, \ldots, 7\}$, let P_i denote the subpath of C from v_i to v_{i+1} (where $v_8 = v_1$).

By Lemma 17, the cycle C is induced, no two neighbors of v in C are adjacent, and the graph $G - (V(C) \cup \{v\})$ is 2-edge-connected and has more than one vertex. Hence, the face of $G - (V(C) \cup \{v\})$ containing v is bounded by a cycle C'. For a subgraph $G' \subseteq G$ containing $C \cup C'$, let $X(G')$ denote the set of faces of G' separated from v by C' and let $Y(G')$ denote the set of faces of G' separated from v by C but not by C'. For $i \in \{1, \ldots, 7\}$, we say that a face $f \in X(G')$ *sees* P_i if there exists a face $f' \in Y(G')$ such that f' is incident with an edge of P_i and the boundaries of f and f' share at least one edge.

If for some $i \in \{1, \ldots, 7\}$, some face of $X(G)$ saw P_i, P_{i+2}, and P_{i+4} (with indices taken cyclically) then $\tilde{G} = \gamma_4(r_2(\gamma_2(\tilde{G}_1, \tilde{G}_2)), \tilde{G}_3)$ for plane near-cubic graphs $\tilde{G}_1$, $\tilde{G}_2$, and $\tilde{G}_3$ with $d(\tilde{G}_1) = d(\tilde{G}_2) = 6$ and $d(\tilde{G}_3) = 7$ (see Figure 6). Lemma 13 would imply $n_{\tilde{G}_j} \in K_{d(\tilde{G}_j)}$ for $j \in \{1, 2, 3\}$, and by Lemma 11(h) and (i), we would have $n_{\tilde{G}} \in K_7$, which is a contradiction. Hence,

$$\text{no face of } X(G) \text{ sees } P_i,\ P_{i+2}, \text{ and } P_{i+4}. \tag{2}$$

Let b_1 be the number of edges of G with one end in C and the other end in C', let b_2 be the number of chords of C', let b_3 be the number of edges with one end in C' and the other end in $V(G) \setminus V(C \cup C')$, and let b_4 be the number of edges of $G - v - V(C \cup C')$. Note that $b_1 \geq 7$, b_3 is at least three times the number of components of $G - v - V(C \cup C')$, $|E(C)| = 7 + b_1$, $|E(C')| = b_1 + 2b_2 + b_3$, and

$$|E(G)| = 7 + (7 + b_1) + b_1 + (b_1 + 2b_2 + b_3) + b_2 + b_3 + b_4 = 14 + 3b_1 + 3b_2 + 2b_3 + b_4.$$

A case analysis shows that since (2) holds, one of the following conditions holds:

- $b_1 \geq 8$ and $b_2 \geq 2$, or
- $b_1 \geq 8$ and $b_3 \geq 3$, or
- $b_3 \geq 6$, or
- $b_3 \geq 4$ and $b_4 \geq 1$.

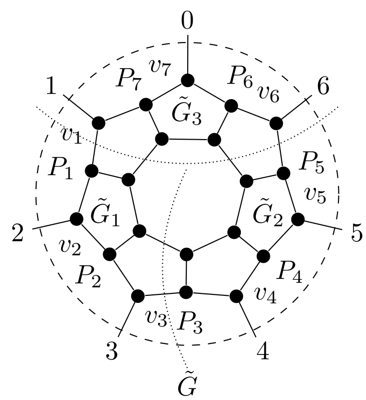

Figure 6: Graph $\tilde{G}$ from Lemma 18. Edges incident to v are crossing the dashed circle and v is not depicted. A construction of $\tilde{G}$ from $\tilde{G}_1, \tilde{G}_2$ and $\tilde{G}_3$ is indicated by the dotted lines.

Hence $3b_1 + 3b_2 + 2b_3 + b_4 \geq 30$, and thus G has at least 44 edges. Consequently, $|V(G)| \geq (2|E(G)| - 4)/3 \geq 28$. $\square$

As a consequence, this verifies Conjecture 7 for small graphs.

Corollary 19. *Conjecture 7 holds for all plane near-cubic graphs with less than* 30 *vertices.*

Proof. Let $\tilde{G} = (G, v, \nu)$ be a counterexample to Conjecture 7, and in particular $n_{\tilde{G}} \notin B'_5 = K_5$. By Lemma 12, there exists a plane near-cubic graph $\tilde{G}_0 = (G_0, v_0, \nu_0)$ such that $d(\tilde{G}_0) = 7$, $n_{\tilde{G}_0} \notin K_7$, and $|V(G_0)| \leq |V(G)| - 2$. Hence, there exists an extremal plane near-cubic graph $\tilde{G}_1 = (G_1, v_1, \nu_1)$ such that $|V(G_1)| \leq |V(G_0)|$. By Lemma 18, we have $|V(G_1)| \geq 28$, and thus $|V(G)| \geq 30$. $\square$

Note that the analysis at the end of the proof of Lemma 18 can be improved. By a computer-assisted enumeration, one can show that to ensure that (2) holds, $G - v$ must contain one of 38 specific graphs as a minor; the smallest are depicted in Figure 7. Hence, every counterexample to Conjecture 7 must contain one of these 38 as a minor. The list of these 38 graphs is available at `http://lidicky.name/pub/4cone/`.

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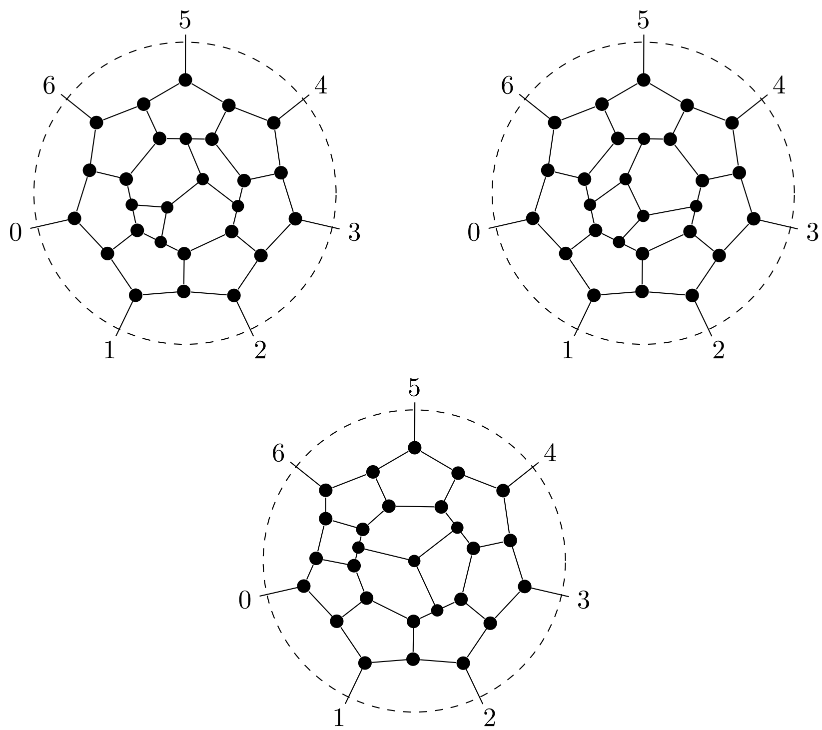

Figure 7: The smallest minors.

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