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Clifford Bergman
Iowa State University, cbergman@iastate.edu

T. Penza
Warsaw University of Technology

A. B. Romanowska
Warsaw University of Technology

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Abstract

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Keywords

Mal'tsev product of varieties, Semilattice sums, Prolongation, Płonka sums, Lallement sums, Regular and irregular identities, Regularization and pseudo-regularization of a variety

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SEMILATTICE SUMS OF ALGEBRAS AND MAL'TSEV PRODUCT OF VARIETIES

C. BERGMAN[1] AND T. PENZA[2], A.B. ROMANOWSKA[3]

Abstract. The Mal'tsev product of two varieties of similar algebras is always a quasivariety. We consider when this quasivariety is a variety. The main result shows that if $\mathcal{V}$ is a strongly irregular variety with no nullary operations, and $\mathcal{S}$ is a variety, of the same type as $\mathcal{V}$, equivalent to the variety of semilattices, then the Mal'tsev product $\mathcal{V} \circ \mathcal{S}$ is a variety. It consists precisely of semilattice sums of algebras in $\mathcal{V}$. We derive an equational basis for the product from an equational basis for $\mathcal{V}$. However, if $\mathcal{V}$ is a regular variety, then the Mal'tsev product may not be a variety. We discuss examples of various applications of the main result, and examine some detailed representations of algebras in $\mathcal{V} \circ \mathcal{S}$.

Let $\mathcal{K}$ be a quasivariety of Ω-algebras, and $\mathcal{Q}$ and $\mathcal{R}$, two of its subquasivarieties. Assume additionally that $\mathcal{R}$-algebras are idempotent. Then the *Mal'tsev product* $\mathcal{Q} \circ_{\mathcal{K}} \mathcal{R}$ of $\mathcal{Q}$ and $\mathcal{R}$ relative to $\mathcal{K}$ consists of $\mathcal{K}$-algebras A with a congruence θ such that A/θ is in $\mathcal{R}$, and each θ-class a/θ is in $\mathcal{Q}$. Note that by the idempotency of $\mathcal{R}$, each θ-class is always a subalgebra of A. If $\mathcal{K}$ is the variety of all Ω-algebras, then the Mal'tsev product $\mathcal{Q} \circ_{\mathcal{K}} \mathcal{R}$ is called simply the *Mal'tsev product* of $\mathcal{Q}$ and $\mathcal{R}$, and is denoted by $\mathcal{Q} \circ \mathcal{R}$. It follows by Mal'tsev results [15] that Mal'tsev product $\mathcal{Q} \circ_{\mathcal{K}} \mathcal{R}$ is a quasivariety.

In this paper we are interested in Mal'tsev products $\mathcal{V} \circ \mathcal{S}$ such that $\mathcal{V}$ is a variety of Ω-algebras and $\mathcal{S}$ is the variety of the same type as $\mathcal{V}$, equivalent to the variety of semilattices. Members of $\mathcal{V} \circ \mathcal{S}$ are disjoint unions of $\mathcal{V}$-algebras over its semilattice homomorphic image, and are known as semilattice sums of $\mathcal{V}$-algebras. The product $\mathcal{V} \circ \mathcal{S}$ is a quasivariety. However, until recently it was not known under what conditions it is a variety. The main result of this paper (Theorem 5.3) shows that if $\mathcal{V}$ is a strongly irregular variety of a type with no nullary

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operations, then the Mal'tsev product $\mathcal{V} \circ \mathcal{S}$ is a variety. We also show how to build an equational basis for this variety from an equational basis of $\mathcal{V}$. Then we provide an example showing that in the case $\mathcal{V}$ is a regular variety, the Mal'tsev product does not need to be a variety.

Next we present examples of Mal'tsev products and relative Mal'tsev products (Mal'tsev product relative to some varieties) which show different aspects of such products. In fact, two special examples gave us an inspirations to investigate semilattice sums in general, one concerns certain semilattice sums of lattices and one certain semilattice sums of Steiner quasigroups. (See Section 5.)

As a converse process to the decomposition of an algebra A from $\mathcal{V} \circ \mathcal{S}$ into a semilattice sum of $\mathcal{V}$-algebras, some construction techniques are available to recover an algebra A from its summands and its semilattice quotient. Such techniques include, for example, Płonka sums, and the more general Lallement sums [24, Ch. 4]. We recall some of the basic constructions of this type, and discuss their applicability for representing algebras in the Mal'tsev product $\mathcal{V} \circ \mathcal{S}$.

Let us mention that in [15], Mal'tsev has shown that if $\mathcal{K}$ is a congruence permutable variety of algebras with certain type of idempotent elements, then the Mal'tsev product of any two subvarieties of $\mathcal{K}$ is a variety. This result was then extended by Iskander [13] to weakly congruence permutable varieties of algebras with a constant term. He has also shown how to find an equational basis for the Mal'tsev product of two subvarieties of $\mathcal{K}$ in this case.

We begin with introductory information concerning Mal'tsev products of the form $\mathcal{V}_t \circ \mathcal{S}$, where $\mathcal{V}_t$ is a strongly irregular variety satisfying a strongly irregular identity $t(x, y) = x$, and certain known examples of sub(quasi)varieties of such products (e.g. the class of Płonka sums of $\mathcal{V}_t$-algebras, which form so-called regularization of $\mathcal{V}_t$). In Section 2, a certain special set of identities true in $\mathcal{V}_t \circ \mathcal{S}$ is pointed out and used to define the variety denoted by $\mathcal{P}(\mathcal{V}_t)$, which will play an essential role in the main result of the paper. The following section discusses free $\mathcal{P}(\mathcal{V}_t)$-algebras. The main Theorem 5.3 says that the variety $\mathcal{P}(\mathcal{V}_t)$ and the Mal'tsev product $\mathcal{V}_t \circ \mathcal{S}$ coincide. The proof is divided into two parts. The first part (Section 4) concerns a variety of Ω-algebras defined by precisely one (strongly irregular) identity of the form $t(x, y) = x$, while the second part (Section 5) concerns any strongly irregular variety of the type described above. In Section 4 we also show that the Mal'tsev product $\mathcal{S} \circ \mathcal{S}$ of the variety of semilattices with itself is not a variety. Section 6 provides a number of examples of Mal'tsev products and relative Mal'tsev products, discusses some aspects of them, and different situation where they are useful to characterize algebras in some (quasi)varieties. Relative Mal'tsev products are often more interesting as they provide a kind of representation for algebras in more familiar varieties. The general Mal'tsev products lay very "far" from the factors in the sense that they "usually" do not preserve identities true in $\mathcal{V}_t$.

The paper finishes with a discussion concerning detailed representation of algebras in Mal'tsev product $\mathcal{V}_t \circ \mathcal{S}$. Here a version of a construction of algebras called Lallement sum ([23]) is used. In particular, certain very special Lallement sums provide a good detailed representation for algebras in $\mathcal{V}_t \circ \mathcal{S}$, in the case when $\mathcal{V}_t$-algebras have units.

We use notation and conventions similar to those of [23, 24]. For details and further information concerning quasivarieties and Mal'tsev product of quasivarieties we refer the reader to [15] and [16], and then also to [1] and [24, Ch. 2]; for universal algebra, see [2] and [24], in particular, for methods of constructing algebras as semilattice sums see [21] and [24, Ch. 4]; for semigroup theory, see [6, 10, 11]; for affine spaces, barycentric algebras and convex sets, we refer the reader to the monographs [23, 24].

1. Background

Let $\tau : \Omega \to \mathbb{N}$ be a similarity type of Ω-algebras with no nullary operation symbols and containing at least one non-unary operation symbol. Such similarity type is called *plural* [24]. We write $\mathcal{T}$ for the variety of all Ω-algebras.

For a positive integer n, let T_n be the set of all n-ary Ω-terms containing precisely n different variables. They may be denoted $x_1, x_2, \dots x_n$ or by some other letters like y or z. In particular, we may write $T_2(x, y)$ in place of $T_2(x_1, x_2)$. For an Ω-term w, the symbol $x_1 \dots x_n w$ means that $x_1, \dots, x_n$ are precisely the variables of w.

Suppose that $t(x, y)$ is a binary Ω-term containing both x and y. We write $\mathcal{T}_t$ for the subvariety of $\mathcal{T}$ defined by the identity $t(x, y) = x$. Such an identity is called *strongly irregular*. We often find it convenient to write $x \cdot y$ (or $x \cdot_t y$ if necessary) in place of $t(x, y)$.

Another subvariety of $\mathcal{T}$ that is of interest, is the variety $\mathcal{S}$ of Ω-semilattices, i.e. Ω-algebras equivalent to semilattices. If S belongs to the subvariety $\mathcal{S}$, then for any $x \cdot y = x \cdot_t y = t(x, y) \in T_2$ and n-ary operation symbol $\omega \in \Omega$,

$$x_1 \dots x_n \omega = x_1 \cdot \dots \cdot x_n \tag{1.1}$$

holds in S. Note that if S is an Ω-semilattice, then we can recover the original semilattice operation as $x \cdot_t y$ no matter which term t has been selected. In particular, the operation $\cdot$ is idempotent, commutative and associative

$$x \cdot x = x,\ x \cdot y = y \cdot x,\ \text{and } xy \cdot z = x \cdot yz,$$

and will be interpreted as a meet-semilattice. Let us note that the variety $\mathcal{S}$ satisfies all the regular identities of the type τ (i.e. the identities with the same sets of variables on both sides).

Definition 1.1. Let $\mathcal{V}$ be a subvariety of the variety $\mathcal{T}$. An Ω-algebra A is a *semilattice sum* of $\mathcal{V}$-algebras, if A has a congruence ϱ such that the quotient $S = A/\varrho$ is an Ω-semilattice (or briefly a semilattice) and the congruence blocks a/ϱ are $\mathcal{V}$-algebras.

Note that since the quotient A/ϱ is a semilattice, it is idempotent. Hence the congruence blocks of ϱ are always subalgebras, and A is a disjoint sum of these subalgebras. We will denote such a sum by $\bigsqcup_{s\in S} A_s$, where A_s are the congruence blocks of ϱ and S is (isomorphic to) the Ω-semilattice quotient. The class of semilattice sums of $\mathcal{V}$-algebras forms the Mal'tsev product $\mathcal{V} \circ \mathcal{S}$ of the varieties $\mathcal{V}$ and $\mathcal{S}$, and is known to be a quasivariety. (See Mal'tsev [15] and [16] as well as Bergman [1] and also Romanowska, Smith [24, Ch. 3].) Whenever A is a semilattice sum of $\mathcal{V}$-algebras, it suffices to take as ϱ the semilattice replica congruence of A, i.e. the smallest congruence of A with the corresponding quotient a semilattice. Furthermore, in case when $\mathcal{V}$ is a strongly irregular variety, the semilattice replica congruence of A is the unique congruence ϱ providing a decomposition of A into the semilattice sum of $\mathcal{V}$-subalgebras (see [24, S. 3.3]).

In this paper we are especially interested in semilattice sums of $\mathcal{T}_t$-algebras. The class of semilattice sums of $\mathcal{T}_t$-algebras forms the Mal'tsev product $\mathcal{T}_t \circ \mathcal{S}$ of the varieties $\mathcal{T}_t$ and $\mathcal{S}$. Since $\mathcal{T}_t$ is a strongly irregular variety, each algebra A in $\mathcal{T}_t \circ \mathcal{S}$ decomposes into a semilattice sum $\bigsqcup_{s\in S} A_s$ of $\mathcal{T}_t$-algebras A_s in a unique way given by the semilattice replica congruence ϱ of A.

The variety $\mathcal{T}_t$ contains other irregular varieties. In particular, if $\mathcal{V}_t$ is a subvariety of the variety $\mathcal{T}_t$, then it is known that $\mathcal{V}_t$ is defined by a set Σ_t consisting of a set Γ_t of regular identities and the identity $t(x, y) = x$. (See e.g. [21] and [24, Ch. 4].) Then the Mal'tsev product $\mathcal{V}_t \circ \mathcal{S}$ of $\mathcal{V}_t$ and $\mathcal{S}$ consists of algebras which are semilattice sums of $\mathcal{V}_t$-subalgebras, moreover it contains $\mathcal{V}_t$ and $\mathcal{S}$ as subvarieties, and is contained in the quasivariety $\mathcal{T}_t \circ \mathcal{S}$. The smallest variety $\mathsf{V}(\mathcal{V}_t, \mathcal{S})$ containing both $\mathcal{V}_t$ and $\mathcal{S}$ is called the *regularization* of $\mathcal{V}_t$ and is denoted by $\widetilde{\mathcal{V}_t}$. (See e.g. [24, Ch. 4].) The smallest quasivariety $\mathsf{Q}(\mathcal{V}_t, \mathcal{S})$ containing both $\mathcal{V}_t$ and $\mathcal{S}$ is called the *quasi-regularization* of $\mathcal{V}_t$ and is denoted by $\widetilde{\mathcal{V}}^q_t$. (See

[4].) Note that $\mathsf{V}(\mathcal{V}_t, \mathcal{S})$ and $\mathsf{Q}(\mathcal{V}_t, \mathcal{S})$ are not necessarily equal. For each $t(x, y) \in T_2$, we have the following chains of quasivarieties

$$\mathcal{V}_t, \mathcal{S} \subseteq \widetilde{\mathcal{V}}_t^q \subseteq \widetilde{\mathcal{V}}_t \subseteq \mathcal{V}_t \circ \mathcal{S} \subseteq \mathcal{T}_t \circ \mathcal{S}.$$

In particular, if the set Γ_t of regular identities is empty, then $\mathcal{V}_t$ coincides with $\mathcal{T}_t$.

2. Some varieties related to the quasivariety $\mathcal{T}_t \circ \mathcal{S}$

Recall that, for $x \cdot y = t(x, y) \in T_2$, the subvariety $\mathcal{V}_t$ of $\mathcal{T}_t$ is defined by a set Σ_t consisting of the identity $x \cdot y = x$ and a set of regular identities Γ_t. The best known subvariety of $\mathcal{V}_t \circ \mathcal{S}$, different from $\mathcal{V}$ and $\mathcal{S}$, is the regularization $\widetilde{\mathcal{V}}_t$ of the variety $\mathcal{V}_t$. We will briefly recall the basic facts concerning the regularization. First note that a semilattice S and the variety $\mathcal{T}$ may be considered as categories. Suppose that there is a covariant functor

$$F : S \to \mathcal{T}; (s \geq t) \mapsto (\varphi_{s,t} : A_s \to A_t),$$

then the disjoint sum $\bigsqcup_{s \in S} A_s$ may be viewed as the semilattice ordered system

$$(A_s; S; \{\varphi_{s,t}\} | s, t \in S, s \geq t),$$

where the Ω-operations on the sum are defined by

$$\omega : A_{s_1} \times \cdots \times A_{s_n} \to A_s; (a_{s_1}, \ldots, a_{s_n}) \mapsto a_{s_1}\varphi_{s_1,s} \ldots a_{s_n}\varphi_{s_n,s}\omega$$

for each (n-ary) $\omega \in \Omega$, where each $s_i \in S$ and $s = s_1 \cdot \ldots \cdot s_n$. The resulting semilattice sum is called the *Płonka sum* of algebras A_s, and is denoted by $\sum_{s \in S} A_s$. (See [19], [21], [24, Ch. 4] .)

It is a remarkable fact that the class of Płonka sums of $\mathcal{V}_t$-algebras forms the *regularization* $\widetilde{\mathcal{V}}_t$ of the variety $\mathcal{V}_t$. The class $\widetilde{\mathcal{V}}_t$ is the smallest subvariety of $\mathcal{T}$ containing both $\mathcal{V}_t$ and $\mathcal{S}$, thus it satisfies precisely the regular identities true in $\mathcal{V}_t$. Its axiomatization follows from Płonka's theorem (see [21], [24, Ch. 4]). The variety $\widetilde{\mathcal{V}}_t$ is defined by the regular identities Γ_t of Σ_t and the following identities (P1) – (P5):

(P1) $x \cdot x = x$,
(P2) $x \cdot yz = xy \cdot z$,
(P3) $x \cdot yz = x \cdot zy$,
(P4) $y \cdot x_1 \ldots x_n\omega = y \cdot x_1 \cdot \ldots \cdot x_n$,
(P5) $x_1 \ldots x_n\omega \cdot y = (x_1 \cdot y) \ldots (x_n \cdot y)\omega$,

where ω ranges through all symbols in Ω. Let us note that we have employed our shortcut $x \cdot y = t(x, y)$, and written $x \cdot yz$ for $x \cdot (y \cdot z)$ and $xy \cdot z$ for $(x \cdot y) \cdot z$. Let us note as well that if Γ_t is empty, then the regularization $\widetilde{\mathcal{V}_t}$ becomes the regularization $\widetilde{\mathcal{T}_t}$, and is defined by the identities (P1) – (P5).

By results of [4], it is known that the quasi-regularization $\widetilde{\mathcal{V}_t}^q$ of $\mathcal{V}_t$ consists of Płonka sums of $\mathcal{V}_t$-algebras with injective homomorphisms $\varphi_{s,t}$. It coincides with the class $\mathsf{SP}(\mathcal{V}_t \cup \mathcal{S})$ of subalgebras of products of algebras in $\mathcal{V}_t \cup \mathcal{S}$, and is defined by the quasi-identity

$$(x \cdot y = x \,\&\; y \cdot x = y \,\&\; x \cdot z = z \cdot x = z \,\&\; y \cdot z = z \cdot y = z) \to (x = y).$$

The congruence ϱ of a $\widetilde{\mathcal{V}_t}$-algebra A providing its decomposition into the Płonka sum of its $\mathcal{V}_t$-summands can be obtained explicitely by

$$\varrho = \{(a, b) \mid a \cdot b = a \text{ and } b \cdot a = b\}. \tag{2.1}$$

Now, let $\overline{\mathcal{T}_t}$ denote the subvariety of $\mathcal{T}$-algebras defined by the identities (P1) – (P4) above. It follows immediately that $\widetilde{\mathcal{T}_t} \subseteq \overline{\mathcal{T}_t}$. Then for a subvariety $\mathcal{V}_t$ of $\mathcal{T}_t$, let $\overline{\mathcal{V}_t}$ be defined by Γ_t and (P1) – (P4). As observed by C. Bergman and D. Failing [3], $\overline{\mathcal{V}_t} \subseteq \mathcal{V}_t \circ \mathcal{S}$. That is to say, each algebra A in $\overline{\mathcal{V}_t}$ is a semilattice sum of its $\mathcal{V}_t$-subalgebras. Moreover, the decomposition is given by the same formula (2.1) as is the case for the regularization. However, the $\overline{\mathcal{V}_t}$-algebras are not necessarily Płonka sums of $\mathcal{V}_t$-algebras, though the reduct $(A, \cdot) = (A, \cdot_t)$ of each $\overline{\mathcal{V}_t}$-algebra $A = \bigsqcup_{s \in S} A_s$ is a left-normal band and the Płonka sum $\sum_{s \in S} A_s$ of the left-zero band reducts $(A_s, \cdot)$. We will call the variety $\overline{\mathcal{V}_t}$ the *pseudo-regularization* of the variety $\mathcal{V}_t$. If the operation $x \cdot y$ of an Ω-algebra (A, Ω) satisfies the conditions (P1) – (P4), then it is called a *pseudopartition operation*, and in the case it satisfies all (P1) – (P5) a *partition operation*.

The third variety we are interested in will play an essential role in our main results.

Definition 2.1. Consider any subvariety $\mathcal{V}$ of the variety $\mathcal{T}$. Let Σ be an equational basis for $\mathcal{V}$ and let σ be an identity $u = v$ from Σ of the form

$$y_1 \dots y_j y_{j+1} \dots y_k u = y_1 \dots y_j y_{k+1} \dots y_n v,$$

where the set $\{y_1, \dots, y_j\}$ may be empty or one or both of $\{y_{j+1}, \dots, y_k\}$ and $\{y_{k+1}, \dots, y_n\}$ may be empty. For each $m > 0$, define the set σ^p_m of identities

(2.2)

$$(x_1 \dots x_m r_1) \dots (x_1 \dots x_m r_j)(x_1 \dots x_m r_{j+1}) \dots (x_1 \dots x_m r_k)\, u =$$
$$(x_1 \dots x_m r_1) \dots (x_1 \dots x_m r_j)(x_1 \dots x_m r_{k+1}) \dots (x_1 \dots x_m r_n)\, v$$

obtained from σ by substituting $x_1 \dots x_m r_i$ for y_i, where $r_1, \dots, r_n$ are terms of T_m. Let σ^p be the union of all σ^p_m for $m > 0$. Finally, let Σ^p denote the union of all sets σ^p obtained from the identities σ of Σ.

Lemma 2.2. *The quasivariety $\mathcal{V} \circ \mathcal{S}$ satisfies the identities of Σ^p.*

Proof. Let A be in $\mathcal{V} \circ \mathcal{S}$. We want to show that A satisfies the identities of Σ^p. Consider any of the identities (2.2). Let $a_1, \dots, a_m \in A$. Since the congruence ϱ of A is the semilattice replica congruence, it follows that for any two $i, j \in \{1, \dots, n\}$

$$(a_1 \dots a_m r_i)/\varrho = (a_1/\varrho) \dots (a_m/\varrho) r_i = (a_1/\varrho) \cdot \dots \cdot (a_m/\varrho) =$$
$$(a_1/\varrho) \dots (a_m/\varrho) r_j = (a_1 \dots a_m r_j)/\varrho.$$

Hence

$$(a_1 \dots a_m r_i,\, a_1 \dots a_m r_j) \in \varrho.$$

Since each block of the congruence ϱ satisfies the identities of Σ, it follows that if $u = v$ belongs to Σ, then

$$(a_1 \dots a_m r_1) \dots (a_1 \dots a_m r_j)(a_1 \dots a_m r_{j+1}) \dots (a_1 \dots a_m r_k)\, u =$$
$$(a_1 \dots a_m r_1) \dots (a_1 \dots a_m r_j)(a_1 \dots a_m r_{k+1}) \dots (a_1 \dots a_m r_n)\, v.$$

Hence the algebra A satisfies the identities (2.2). □

For a given variety $\mathcal{V}$, let $\mathcal{P}(\mathcal{V})$ denote the class of all Ω-algebras satisfying the identities in Σ^p. The next proposition is a restatement of Lemma 2.2.

Proposition 2.3. *For any subvariety $\mathcal{V}$ of $\mathcal{T}$,*

$$\mathcal{V} \circ \mathcal{S} \subseteq \mathcal{P}(\mathcal{V}). \tag{2.3}$$

Now consider again the subvariety $\mathcal{V}_t$ of $\mathcal{T}_t$ defined by $\Sigma_t = \{t(x, y) = x \cdot y = x\} \cup \Gamma_t$, where Γ_t is a given set of regular identities. For all identities σ in Σ_t, define the set of identities Σ^p_t as the union of all sets σ^p, similarly as in Definition 2.1.

Corollary 2.4. *Let $\mathcal{V}_t$ be the subvariety of $\mathcal{T}_t$ defined by $\Sigma_t = \{x \cdot y = x\} \cup \Gamma_t$. Then Σ^p_t consist of the set Σ°_t of the identities of the form*

$$(x_1 \dots x_m r_1) \cdot (x_1 \dots x_m r_2) \;=\; x_1 \dots x_m r_1 \tag{2.4}$$

for every $m > 0$ and every pair of terms $r_1, r_2 \in T_m$, and the set Γ^p_t of all identities of the form

$$(x_1 \dots x_m r_1) \dots (x_1 \dots x_m r_n)\, u \;=\; (x_1 \dots x_m r_1) \dots (x_1 \dots x_m r_n)\, v \tag{2.5}$$

obtained from every identity $y_1 \dots y_n u = y_1 \dots y_n v$ of Γ_t by substituting $x_1 \dots x_m r_i$ for y_i, where $r_1, \dots, r_n$ are terms of T_m, for all $m > 0$. Then the quasivariety $\mathcal{V}_t \circ \mathcal{S}$ satisfies the identities of Σ^p_t.

Proof. It is sufficient to note that the identities (2.4) are obtained from the identity $u = v$, where $u = x \cdot y$ and $v = x$. □

Let $\mathcal{P}(\mathcal{V}_t)$ be the subvariety of $\mathcal{T}$ defined by the identities Σ_t^p. Then Proposition 2.3 and Corollary 2.4 imply the following.

Proposition 2.5. *For any subvariety $\mathcal{V}_t$ of $\mathcal{T}_t$,*

$$\mathcal{V}_t \circ \mathcal{S} \subseteq \mathcal{P}(\mathcal{V}_t). \tag{2.6}$$

If Γ_t is empty, then the variety $\mathcal{V}_t$ is just the variety $\mathcal{T}_t$ and the set Σ_t reduces to the identity $x \cdot y = x$, while Σ_t^p coincides with Σ_t°. In this case we will denote the variety $\mathcal{P}(\mathcal{V}_t) = \mathcal{P}(\mathcal{T}_t)$ briefly by $\mathcal{P}_t$.

Let us note that among the identities of Σ_t° there are the following:

(a) $x \cdot x = x$,
(b) $xy \cdot yx = xy$,
(c) $(xy \cdot z) \cdot (x \cdot yz) = xy \cdot z \quad \text{and} \quad (x \cdot yz) \cdot (xy \cdot z) = x \cdot yz$,
(d) $(x_{1\pi} \dots x_{n\pi}\, \omega) \cdot (x_{1\sigma} \dots x_{n\sigma}\, \omega) = x_{1\pi} \dots x_{n\pi}\, \omega$
for each (n-ary) $\omega \in \Omega$ and for any two mappings π and σ from $\mathbf{n} = \{1, \dots, n\}$ into $\mathbf{n}$ such that $\pi(\mathbf{n}) = \sigma(\mathbf{n})$.

3. Free algebras in varieties $\mathcal{P}(\mathcal{V})$

As before, $\mathcal{V}$ is a subvariety of the variety $\mathcal{T}$. Basic properties of free algebras in sub(quasi)varieties of $\mathcal{P}(\mathcal{V})$ follow from basic properties of free algebras in prevarieties of Ω-algebras (i.e. classes of similar algebras closed under subalgebras and direct products) as given for example in [24, S. 3.3]. Recall that any Ω-algebra A has a (uniquely defined) $\mathcal{S}$-replica (i.e. the semilattice replica) given by the quotient A/ϱ, where ϱ is the semilattice replica congruence of A (i.e. ϱ is the intersection of all semilattice congruences of A). For any subquasivariety $\mathcal{Q}$ of $\mathcal{T}$, the free $\mathcal{Q}$-algebra $X\mathcal{Q}$ over X is isomorphic to the $\mathcal{Q}$-replica of the absolutely free Ω-algebra $X\mathcal{T}$ over X by the least $\mathcal{Q}$-congruence of $X\mathcal{T}$ [24, Prop. 3.3.5]. Moreover, if $\mathcal{Q}$ and $\mathcal{Q}'$ are two subquasivarieties of $\mathcal{T}$ with $\mathcal{Q}' \leq \mathcal{Q}$, then the free $\mathcal{Q}'$-algebra $X\mathcal{Q}'$ over X is also isomorphic to the $\mathcal{Q}'$-replica of the algebra $X\mathcal{Q}$. (See also [2, Thm. 4.28].) In particular, if $\mathcal{V}$ is a regular subvariety of $\mathcal{T}$, then the semilattice replica $X\mathcal{V}/\varrho$ of $X\mathcal{V}$ is isomorphic to the free semilattice $X\mathcal{S}$:

$$X\mathcal{V}/\varrho \cong X\mathcal{S}. \tag{3.1}$$

Recall also that the free semilattice $X\mathcal{S}$ over X is isomorphic to the semilattice $P_f(X)$ of all finite non-empty subsets of X under union.

In what follows, the symbol $x_1 \dots x_n w$, where w is an Ω-term, will always mean that w contains precisely the variables $x_1, \dots, x_n$. In other words, $x_1 \dots x_n w$ is a member of T_n. Then $\mathsf{var}(w)$ will denote the set $\{x_1, \dots, x_n\}$ of variables of w. Let us note that if $\mathcal{V}$ is a regular subvariety of $\mathcal{T}$, then each element of $X\mathcal{V}$ is represented by a term of T_m for a fixed $m > 0$ depending only on this element.

Lemma 3.1. *Let $\mathcal{V}$ be a regular subvariety of the variety $\mathcal{T}$ and let ϱ be the semilattice replica congruence of the free algebra $X\mathcal{V}$ over X. Then for any two Ω-terms w, v*

$$(w, v) \in \varrho \iff \mathsf{var}(w) = \mathsf{var}(v).$$

Proof. Let θ be a binary relation on $A = X\mathcal{V}$ defined as follows. For any two Ω-terms w, v

$$(w, v) \in \theta \iff \mathsf{var}(w) = \mathsf{var}(v).$$

It is easy to see that the relation θ is a congruence relation, and is the kernel of the homomorphism

$$h \colon X\mathcal{V} \to P_f(X);\ x_1 \dots x_n u \mapsto \mathsf{var}(u)$$

onto the semilattice of all finite subsets of X. Now by universality of replication (see [24, Lemma 3.3.1]), it follows that there is a unique homomorphism $\overline{h} : X\mathcal{V}/\varrho \to P_f(X)$ such that the composition $(\mathsf{nat}\varrho)\overline{h}$ equals h. In particular, under this composition $u \mapsto u/\varrho \mapsto \mathsf{var}(u)$. By (3.1), $\overline{h}$ is an isomorphism. Hence $\theta = \varrho$. □

Let us note that, in particular, the semilattice replica $X\mathcal{P}(\mathcal{V})/\varrho$ of $X\mathcal{P}(\mathcal{V})$ as well as the semilattice replica $X\mathcal{P}(\mathcal{V}_t)/\varrho$ of $X\mathcal{P}(\mathcal{V}_t)$ are isomorphic to $X\mathcal{S}$:

$$X\mathcal{P}(\mathcal{V})/\varrho \cong X\mathcal{S} \text{ and } X\mathcal{P}(\mathcal{V}_t)/\varrho \cong X\mathcal{S}. \tag{3.2}$$

Proposition 3.2. *For any subvariety $\mathcal{V}$ of $\mathcal{T}$, the free $\mathcal{P}(\mathcal{V})$-algebra $A = X\mathcal{P}(\mathcal{V})$ is a semilattice of $\mathcal{V}$-algebras. In particular, $A = \bigsqcup_{s \in S} A_s$, where $S = X\mathcal{S}$ is the free semilattice over X and all A_s are members of $\mathcal{V}$. Hence*

$$A = X\mathcal{P}(\mathcal{V}) \in \mathcal{V} \circ \mathcal{S}.$$

Proof. As before, we assume that Σ is an equational basis for $\mathcal{V}$ and $\mathcal{P}(\mathcal{V})$ is defined by the identities Σ^p of Definition 2.1. By (3.2), we already know that A/ϱ is isomorphic to $S = X\mathcal{S}$. We complete the proof by showing that each A_s satisfies the identities of Σ. Consider any identity $y_1 \dots y_j y_{j+1} \dots y_k u = y_1 \dots y_j y_{k+1} \dots y_n v$ of Σ. Let $a_1, \dots, a_n$ be elements of A_s. Since A_s is a ϱ-class, it follows that all $a_1, \dots, a_n$ are ϱ-equivalent. This means that for a fixed $m > 0$, each a_i may be represented as a term $x_1 \dots x_m r_i \in T_m$. Thus

$$\begin{aligned}
&a_1 \dots a_j a_{j+1} \dots a_k u = \\
&(x_1 \dots x_m r_1) \dots (x_1 \dots x_m r_j)(x_1 \dots x_m r_{j+1}) \dots (x_1 \dots x_m r_k) u = \\
&(x_1 \dots x_m r_1) \dots (x_1 \dots x_m r_j)(x_1 \dots x_m r_{k+1}) \dots (x_1 \dots x_m r_n) v = \\
&a_1 \dots a_j a_{k+1} \dots a_n v,
\end{aligned}$$

as desired. □

Corollary 3.3. *The variety $\mathcal{P}(\mathcal{V})$ is generated by the quasivariety $\mathcal{V} \circ \mathcal{S}$.*

In next two sections we will show that if $\mathcal{V}$ is a strongly irregular variety, then the variety $\mathcal{P}(\mathcal{V})$ coincides with the Mal'tsev product $\mathcal{V} \circ \mathcal{S}$.

4. The Mal'tsev product $\mathcal{T}_t \circ \mathcal{S}$

We start with the case when $\mathcal{V}$ is the variety $\mathcal{T}_t$, and consider the quasivariety $\mathcal{T}_t \circ \mathcal{S}$ and the variety $\mathcal{P}_t = \mathcal{P}(\mathcal{T}_t)$.

Following Proposition 3.2, we assume that the free $\mathcal{P}_t$-algebra $A = X\mathcal{P}_t$ over X is the semilattice sum $A = \bigsqcup_{s \in S} A_s$, where S, the semilattice replica of A, is the free semilattice over X, and the summands A_s are in $\mathcal{T}_t$. The elements of A_s will be denoted by small letters $a_s, b_s, ...$ with the same index as in A_s, and we will write xy for $x \cdot y$.

Lemma 4.1. *Let θ be a congruence of the free $\mathcal{P}_t$-algebra $A = X\mathcal{P}_t$. Assume that there is a pair (a'_r, b'_s) of elements of A, where $r, s \in S$, such that $(a'_r, b'_s) \in \theta$. Then for any $a_r \in A_r$ and any $b_s \in A_s$*

$$(a_r, a_r b_s) \in \theta \quad \textit{and} \quad (b_s, b_s a_r) \in \theta.$$

Proof. First note that each congruence θ of the Ω-algebra A is also a congruence of the reduct $(A, \cdot)$. Therefore, for any $a_r \in A_r$ and $b_s \in A_s$

$$\begin{aligned}
a'_r \;\theta\; b'_s &\implies a_r a'_r \;\theta\; a_r b'_s \\
b'_s \;\theta\; a'_r &\implies b_s b'_s \;\theta\; b_s a'_r.
\end{aligned}$$

Since $a_r,\ a'_r \in A_r$, and $A_r \in \mathcal{T}_t$, we have $a_r a'_r = a_r$. Similarly, $b_s b'_s = b_s$. Thus

$$a_r \;\theta\; a_r b'_s \quad \text{and} \quad b_s \;\theta\; b_s a'_r. \tag{4.1}$$

Recall that elements of A are represented by Ω-terms, and note that $\mathsf{var}(b_s a'_r) = \mathsf{var}(a_r b'_s) = \mathsf{var}(a_r) \cup \mathsf{var}(b_s)$. Hence both $b_s a'_r$ and $a_r b'_s$ lie in $A_{rs} \in \mathcal{T}_t$. Therefore $(b_s a'_r)(a_r b'_s) = b_s a'_r$. Combining this equality with (4.1) we obtain

$$b_s a_r \;\theta\; (b_s a'_r)(a_r b'_s) = b_s a'_r \;\theta\; b_s.$$

A similar argument yields $a_r \;\theta\; a_r b_s$. □

Remark 4.2. In particular, if $(a'_r, b'_s) \in \theta$, then each a_r and each b_s is θ-related to some element of A_{rs}. Moreover, if additionally $q < rs < r$ and also $(c'_{rs}, d'_q) \in \theta$ for some $c'_{rs} \in A_{rs}$ and $d'_q \in A_q$, then each element of A_{rs} is θ-related to some element of A_q, and in particular, $a_r \,\theta\, a_r c_{rs} \,\theta\, a_r c_{rs} \cdot d_q$ for any a_r, c_{rs}, d_q.

Next lemma follows from [17, L. 4.66].

Lemma 4.3. *Let A be an Ω-algebra and let α and β be congruences of A. Then the following conditions are equivalent.*

(a) $\alpha \circ \beta \circ \alpha = \beta \circ \alpha \circ \beta$,
(b) $\beta \circ \alpha \circ \beta \subseteq \alpha \circ \beta \circ \alpha$,
(c) $\alpha \vee \beta = \alpha \circ \beta \circ \alpha$.

Lemma 4.4. *Let θ be a congruence of the free $\mathcal{P}_t$-algebra $A = X\mathcal{P}_t$, and let ϱ be the semilattice replica congruence of A. Then*

$$\theta \vee \varrho = \theta \circ \varrho \circ \theta = \varrho \circ \theta \circ \varrho. \tag{4.2}$$

Proof. By Lemma 4.1 and (4.1), if

$$a_r \;\varrho\; a'_r \;\theta\; b'_s \;\varrho\; b_s,$$

then

$$a_r \;\theta\; a_r b'_s \;\varrho\; b_s a'_r \;\theta\; b_s.$$

Hence

$$\varrho \circ \theta \circ \varrho \;\subseteq\; \theta \circ \varrho \circ \theta.$$

By Lemma 4.3, it follows that

$$\varrho \vee \theta \;=\; \varrho \circ \theta \circ \varrho \;=\; \theta \circ \varrho \circ \theta.$$

□

Let θ be a congruence of a $\mathcal{P}_t$-algebra A, and ϱ the semilattice replica congruence of A. Let ψ be the join of the congruences θ and ϱ. It is well known that $(A/\theta)/(\psi/\theta) \cong A/\psi$, and since $\psi \geq \varrho$, it follows that A/ψ is a member of the variety $\mathcal{S}$ of Ω-semilattices.

Lemma 4.5. *Let θ be a congruence of a $\mathcal{P}_t$-algebra A, and let $\psi = \theta \vee \varrho$. Then a congruence class $(a_r/\theta)/(\psi/\theta)$ of A/θ satisfies the identity $x \cdot y = x$ if and only if for any $b_s \in A$*

$$(a_r, b_s) \in \psi \implies (a_r b_s, a_r) \in \theta. \tag{4.3}$$

Proof. Let a_r/θ and b_s/θ be members of A/θ. Then

$$(a_r/\theta, b_s/\theta) \in \psi/\theta \iff (a_r, b_s) \in \psi.$$

On the other hand

$$(a_r/\theta) \cdot (b_s/\theta) = a_r/\theta \iff (a_r b_s, a_r) \in \theta.$$

Hence the assertion that $(a_r/\theta)/(\psi/\theta)$ satisfies $x \cdot y = x$ is equivalent to (4.3). □

Proposition 4.6. *Let θ be a congruence of the free $\mathcal{P}_t$-algebra $A = X\mathcal{P}_t$ over X. Then the quotient $B = A/\theta$ of A is a semilattice sum of $\mathcal{T}_t$-subalgebras. In particular, ψ/θ is the semilattice replica congruence of A/θ and the quotient B is the semilattice sum of ψ/θ-classes $(a_r/\theta)/(\psi/\theta)$.*

Proof. Since $\psi > \varrho$, it follows that $A/\psi \in \mathcal{S}$. Thus we only need to show that for each $a_r/\theta \in A/\theta$, the congruence class $(a_r/\theta)/(\psi/\theta)$ satisfies $x \cdot y = x$. By Lemma 4.5, this is equivalent to the condition (4.3).

So our aim now is to verify the implication (4.3). For $a_r \in A_r$ and $b_s \in A_s$, let $(a_r, b_s) \in \psi$. By Lemma 4.4, we know that

$$\psi = \theta \vee \varrho = \varrho \circ \theta \circ \varrho = \theta \circ \varrho \circ \theta.$$

Thus there are elements $a'_r \in A_r$ and $b'_s \in A_s$ such that

$$a_r \ \varrho \ a'_r \ \theta \ b'_s \ \varrho \ b_s.$$

By Lemma 4.1

$$(a_r, a_r b_s) \in \theta,$$

and hence (4.3) holds.

Consequently, it is clear that B is the semilatice sum of the congruence classes $(a/\theta)/(\psi/\theta)$ of A/θ, which are $\mathcal{T}_t$-subalgebras, over the semilattice replica $B/(\psi/\theta)$, which is isomorphic to A/ψ. □

Let us note that A/θ is isomorphic to the semilattice sum of the ψ-classes a/ψ of A over the semilattice A/ψ, where each a/ψ is the union of θ-classes b/θ such that $(a, b) \in \psi$ (or equivalently $(a/\theta, b/\theta) \in (\psi/\theta)$).

As a direct corollary of Proposition 4.6 one obtains the following theorem.

Theorem 4.7. *The quasivariety $\mathcal{T}_t \circ \mathcal{S}$ and the variety $\mathcal{P}_t$ coincide. In particular, the Mal'tsev product $\mathcal{T}_t \circ \mathcal{S}$ is a variety, and Σ_t° is its equational basis.*

Proof. We have already observed that $\mathcal{T}_t \circ \mathcal{S} \subseteq \mathcal{P}_t$. Conversely, if an algebra B is a member of $\mathcal{P}_t$, then there is a set X so that B is a homomorphic image of $X\mathcal{P}_t$. Then by Proposition 4.6, the algebra B belongs to $\mathcal{T}_t \circ \mathcal{S}$. □

4.1. **The most basic case of $\mathcal{T}_t$.** If the variety $\mathcal{T}$ is the variety of all groupoids (magmas, binars) $(G, \cdot)$, and we take $t(x, y) = x \cdot y$, then $\mathcal{T}_t$ is the variety $\mathcal{LZ}$ of left-zero semigroups and one obtains the following corollary for Mal'tsev product of the varieties $\mathcal{LZ}$ and $\mathcal{S}$.

Corollary 4.8. *The quasivariety $\mathcal{LZ} \circ \mathcal{S}$ is a variety. In particular $\mathcal{LZ} \circ \mathcal{S} = \mathcal{P}(\mathcal{LZ})$.*

Note however that members of $\mathcal{P}(\mathcal{LZ})$ are not necessarily semigroups. First observe that the equational basis of $\mathcal{P}(\mathcal{LZ})$ contains the idempotent law $x \cdot x = x$. The free $\mathcal{P}(\mathcal{LZ})$-algebra $A = X\mathcal{P}(\mathcal{LZ})$ over the two element set $X = \{x, y\}$ is the semilattice sum of three subalgebras: two one-element subalgebras $A_x = \{x\}$ and $A_y = \{y\}$ and one subalgebra $A_{x,y}$ consisting of all elements represented by terms of T_2. Obviously $xy, yx \in A_{x,y}$. To describe further elements let us introduce the following notation. For any element a of $A_{x,y}$ and a variable z, let $aR(z) = a \cdot z$ and $aL(z) = z \cdot a$. Then let $aE(z)$ be either $aR(z)$ or $aL(z)$.

Lemma 4.9. *Each element of $A_{x,y}$ different from xy and yx may be expressed in the standard form*

$$(xy)E(z_1)\dots E(z_k) \text{ or } (yx)E(z_1)\dots E(z_k), \tag{4.4}$$

where $z_i = x$ or $z_i = y$, and $k = 1, 2, \dots$.

Proof. The proof is by induction on the length of the element. The shortest elements xy and yx are already in $A_{x,y}$. Since $A_{x,y}$ is a left-zero band, it follows that for any $u, v \in A_{x,y}$, we have $u \cdot v = u$. So longer elements of $A_{x,y}$ may be obtained from elements of the form (4.4) only by multiplying them by a variable x or a variable y from the left or from the right. □

Corollary 4.10. *The free algebra $A = X\mathcal{P}(\mathcal{LZ})$ is not a semigroup.*

Proof. By Lemma 4.9, it is easy to see that the elements $x \cdot yx$ and $xy \cdot x$ are different. □

On the other hand, the Mal'tsev product $\mathcal{LZ} \circ_{\mathcal{SG}} \mathcal{S}$ of the varieties $\mathcal{LZ}$ and $\mathcal{S}$, but taken relative to the variety $\mathcal{SG}$ of semigroups, is a variety of semigroups and may be described in a simple way. Note that $\mathcal{LZ} \circ_{\mathcal{SG}} \mathcal{S}$ is just the intersection $\mathcal{P}(\mathcal{LZ}) \cap \mathcal{SG}$, and that the members of this variety are bands (idempotent semigroups).

Proposition 4.11. *The Mal'tsev product $\mathcal{LZ} \circ_{\mathcal{SG}} \mathcal{S}$ coincides with the variety $\mathcal{LR}$ of left-regular bands defined by the identity $xyx = xy$.*

Proof. ($\Rightarrow$) Let A be a member of $\mathcal{LZ} \circ_{\mathcal{SG}} \mathcal{S}$. Then A is a band and a semilattice of left-zero bands. For $a, b \in A$, we have $ab/\varrho = a/\varrho \cdot b/\varrho = b/\varrho \cdot a/\varrho = ba/\varrho$. Hence $ab \cdot ba = ab$. On the other hand $ab \cdot ba = abba = aba$, which implies $aba = ab$.

($\Leftarrow$) Now assume that $A \in \mathcal{LR}$. By McLean's Theorem [18], each band is a semilattice of rectangular bands. Hence $A = \bigsqcup_{s \in S} A_s$, where S is the semilattice replica of A and all A_s are rectangular bands, defined by the identity $xyx = x$. This together with the identity $xyx = xy$ gives $xy = x$. Hence all A_s are left-zero bands. □

Similar (dual) results are easily obtained for the quasivariety $\mathcal{RZ} \circ \mathcal{S}$, where $\mathcal{RZ}$ is the variety of right-zero semigroups. The Mal'tsev product $\mathcal{RZ} \circ_{\mathcal{SG}} \mathcal{S}$ coincides with the variety $\mathcal{RR}$ of right-regular bands defined by the identity $xyx = yx$.

4.2. **Mal'tsev product of some regular varieties.** We now demonstrate that the assumption of strong irregularity of the variety $\mathcal{T}_t$ in Theorem 4.7 is essential.

Consider again the type τ with one binary multiplication $\cdot$ and the variety $\mathcal{T}$ of the type τ. Let $\mathcal{CG}$ be the subvariety of $\mathcal{T}$ of commutative groupoids.

Proposition 4.12. *The quasivariety $\mathcal{CG} \circ \mathcal{S}$ satisfies the quasi-identitity*

$$(zx = x \ \& \ zy = y \ \& \ xz = yz) \to (xy = yx). \tag{4.5}$$

Proof. Let $A \in \mathcal{CG} \circ \mathcal{S}$ and let ϱ be the semilattice replica congruence of A. Assume that $a, b, c \in A$ with $ca = a$, $cb = b$, and $ac = bc$. We will show that $ab = ba$. Since A/ϱ is a semilattice, it is commutative. Hence $ac \ \varrho \ ca$ and $bc \ \varrho \ cb$. Consequently

$$a = ca \ \varrho \ ac = bc \ \varrho \ cb = b.$$

This implies

$$ab = ba,$$

as desired. □

Example 4.13. Let A be the groupoid in $\mathcal{CG}$ whose multiplication table is given below.

$\cdot$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	0	2	2	0	0	2
3	0	0	2	3	0	1	2
4	0	0	0	0	4	4	4
5	0	0	0	0	4	5	4
6	0	0	2	3	4	5	6

Let us note that the elements $0, 1, 2, 3, 4$ form a subsemilattice of A (that means a subgroupoid of A which is a semilattice). Similarly the elements $0, 1, 2, 4, 5$ form a subsemilattice, and the elements $0, 1, 2, 4, 6$ form a subsemilattice. However $3 \cdot 5 = 1 \neq 0 = 5 \cdot 3$, whence A itself is not a semilattice. The equivalence ϱ of A with the 2-element classes $\{0, 1\}$, $\{2, 3\}$, $\{4, 5\}$, and one element class $\{6\}$ is a congruence of A, each ϱ-class is a semilattice and A/ϱ is a semilattice. Thus $A \in \mathcal{S} \circ \mathcal{S}$, and hence $A \in \mathcal{CG} \circ \mathcal{S}$. Moreover, ϱ is the semilattice replica congruence of A. Indeed, if A/σ is a semilattice for some congruence σ of A, then for all $a, b \in A$ we must have $(a \cdot b,\ b \cdot a) \in \sigma$. Hence

$$(0, 1) = (5 \cdot 3, 3 \cdot 5) \in \sigma,$$
$$(2, 3) = (3 \cdot 6, 6 \cdot 3) \in \sigma,$$
$$(4, 5) = (5 \cdot 6, 6 \cdot 5) \in \sigma.$$

Thus ϱ is the semilattice replica congruence.

Now let θ be the congruence on A generated by 2 and 4. One easily checks that the congruence θ has one 3-element class $\{0, 2, 4\}$ and each of the remaining classes consists of one element. Let $B = A/\theta$. By taking $x = 3/\theta = \{3\}$, $y = 5/\theta = \{5\}$ and $z = 6/\theta = \{6\}$ we see that B fails the quasi-identity (4.5). By Proposition 4.12, $B \notin \mathcal{CG} \circ \mathcal{S}$, and hence $B \notin \mathcal{S} \circ \mathcal{S}$. Consequently, neither $\mathcal{S} \circ \mathcal{S}$ nor $\mathcal{CG} \circ \mathcal{S}$ is a variety.

Summarizing Proposition 4.12 and Example 4.13 we have the following.

Corollary 4.14. *For no regular subvariety $\mathcal{V}$ of $\mathcal{CG}$ is $\mathcal{V} \circ \mathcal{S}$ a variety.*

This applies in particular to the quasivariety $\mathcal{S} \circ \mathcal{S}$.

By definition, the variety $\mathcal{P}(\mathcal{C}G)$ is defined by the set of identities Σ^p consisting of all identities of the form

$$(x_1 \dots x_m r_1) \ \cdot \ (x_1 \dots x_m r_2) = (x_1 \dots x_m r_2) \ \cdot \ (x_1 \dots x_m r_1), \tag{4.6}$$

in which $m > 0$ and r_1, r_2 are terms from T_m. Our discussion above suggests the following problem.

Problem 4.15. Is the quasivariety $\mathcal{CG}\circ\mathcal{S}$ axiomatized by the identities in (4.6) together with the single quasi-identity (4.5) of Proposition 4.12?

Presumably, there is nothing special about commutativity here. Thus we are motivated to pose the following problem.

Problem 4.16. Let $\mathcal{V}$ be a proper regular subvariety of the variety $\mathcal{T}$ of groupoids. Is it true that $\mathcal{V}\circ\mathcal{S}$ fails to be a variety?

With regard to this last problem, let us note that, trivially, $\mathcal{T}\circ\mathcal{S}=\mathcal{T}$, which is obviously a variety. Thus the requirement that $\mathcal{V}$ be a proper subvariety of $\mathcal{T}$ is necessary.

5. The Mal'tsev product $\mathcal{V}_t\circ\mathcal{S}$

Now let $\mathcal{V}_t$ be the (strongly irregular) subvariety of the variety $\mathcal{T}_t$ of Ω-algebras defined by the identity $x\cdot y=t(x,y)=x$ for some $t\in T_2$ and a set Γ_t of regular identities. Recall from Section 2 that the regularization $\widetilde{\mathcal{V}}_t$ of $\mathcal{V}_t$ is defined by Γ_t and the identities (P1) – (P5), and consists of Płonka sums of $\mathcal{V}_t$-algebras. Moreover

$$\mathcal{T}_t\wedge\widetilde{\mathcal{V}}_t=\mathcal{V}_t \text{ and } \mathcal{T}_t\vee\widetilde{\mathcal{V}}_t=\widetilde{\mathcal{T}}_t.$$

The quasivariety $\mathcal{V}_t\circ\mathcal{S}$ is obviously contained in $\mathcal{T}_t\circ\mathcal{S}$. And by Corollary 2.4 and Proposition 2.5, it satisfies the identities of $\Sigma_t^p=\Sigma_t^\circ\cup\Gamma_t^p$. The following inclusions are obvious:

$$\mathcal{V}_t\subseteq\widetilde{\mathcal{V}}_t\subseteq\mathcal{V}_t\circ\mathcal{S}\subseteq\mathcal{P}(\mathcal{V}_t)\subseteq\mathcal{T}_t\circ\mathcal{S}=\mathcal{P}_t.$$

And by Corollary 3.3, the variety $\mathcal{P}(\mathcal{V}_t)$ is the smallest variety containing the quasivariety $\mathcal{V}_t\circ\mathcal{S}$.

Recall also that, by (3.2),

$$X\mathcal{P}(\mathcal{V}_t)/\varrho\cong X\mathcal{S},$$

and for any two Ω-terms w,v

$$(w,v)\in\varrho\iff\mathsf{var}(w)=\mathsf{var}(v).$$

As a direct consequence of Proposition 3.2 one obtains the following corollary.

Corollary 5.1. *For any subvariety $\mathcal{V}_t$ of $\mathcal{T}_t$, the free $\mathcal{P}(\mathcal{V}_t)$-algebra $B=X\mathcal{P}(\mathcal{V}_t)$ is a semilattice of $\mathcal{V}_t$-algebras. In particular, $B=\bigsqcup_{s\in S}B_s$, where $S=X\mathcal{S}$ is the free semilattice over X and all B_s are members of $\mathcal{V}_t$. Hence*

$$B=X\mathcal{P}(\mathcal{V}_t)\in\mathcal{V}_t\circ\mathcal{S}.$$

We intend to show that the classes $\mathcal{V}_t\circ\mathcal{S}$ and $\mathcal{P}(\mathcal{V}_t)$ coincide.

Proposition 5.2. *Let δ be a congruence of the free $\mathcal{P}(\mathcal{V}_t)$-algebra $B = X\mathcal{P}(\mathcal{V}_t)$ over X. Then the quotient $C = B/\delta$ is a semilattice sum of $\mathcal{V}_t$-algebras.*

Proof. Note that, by Theorem 4.7, the algebra C is a member of the variety $\mathcal{P}_t = \mathcal{T}_t \circ \mathcal{S}$. Hence it is a semilattice sum $\bigsqcup_{r\in R} C_r$ of $\mathcal{T}_t$-algebras C_r, where R is the semilattice replica of C by the semilattice replica congruence ϱ_C. For each $c \in C$, the ϱ_C-class c/ϱ_C satisfies the identity $x \cdot y = x$. To complete the proof we must show that every ϱ_C-class c/ϱ_C satisfies the (regular) identities of Γ_t.

First we will use the fact that C is the quotient B/δ of B to describe the decomposition of $C = B/\delta$ into the semilattice sum of $\mathcal{T}_t$-algebras in terms of the algebra B. By Corollary 5.1, $B = \bigsqcup_{s\in S} B_s$, where $S = X\mathcal{S}$ is the free semilattice over X and all B_s are members of $\mathcal{V}_t$. Hence $B = X\mathcal{P}(\mathcal{V}_t) \in \mathcal{V}_t \circ \mathcal{S}$. Let ϱ_B be the semilattice replica congruence of the algebra B, and let $\psi = \delta \vee \varrho_B$. Since $\psi \geq \varrho_B$ it follows that B/ψ is a semilattice. Moreover $(B/\delta)/(\psi/\delta) \cong B/\psi$. Since $C = B/\delta$ is in $\mathcal{P}_t = \mathcal{T}_t \circ \mathcal{S}$, it follows that the (ψ/δ)-classes $(b/\delta)/(\psi/\delta)$, for $b \in B$, belong to $\mathcal{T}_t$. Hence B/δ is the semilattice sum $\bigsqcup (b/\delta)/(\psi/\delta)$ of $\mathcal{T}_t$-algebras, and ψ/δ is the semilattice replica congruence of B/δ.

Next we will show that the (ψ/δ)-delta classes $(b/\delta)/(\psi/\delta)$ satisfy the identities of Γ_t. Let

$$x_1 \dots x_k u = x_1 \dots x_k v \tag{5.1}$$

be a typical member of Γ_t. We want to show that (5.1) holds in each ψ/δ-class. This means that for any $c^1 = b^1/\delta, \dots, c^k = b^k/\delta \in (b/\delta)/(\psi/\delta)$, with $b^i \in B_{s_i}$ and $b \in B$, we have

$$c^1 \dots c^k u = c^1 \dots c^k v, \tag{5.2}$$

which is equivalent to

$$(b^1 \dots b^k u, b^1 \dots b^k v) \in \delta. \tag{5.3}$$

Note that $(b^i/\delta, b^j/\delta) \in (b/\delta)/(\psi/\delta)$ if and only if $(b^i, b^j) \in \psi$. Since each $(b^i/\delta)/(\psi/\delta)$ satisfies the identity $x \cdot y = x$, we may use Lemma 4.5 repeatedly, to obtain the following:

$$b^i \;\delta\; b^i \cdot b^1 \;\delta\; (b^i \cdot b^1) \cdot b^2 \;\delta\; ((b^i \cdot b^1) \cdot b^2) \cdot b^3 \;\delta\; \dots$$

$$\delta\; (\dots((b^i \cdot b^1) \cdot b^2) \cdot \dots) \cdot b^k \;=:\; d^i.$$

Thus $b^i \;\delta\; d^i$ for each $1 \leq i \leq k$, and the elements $d^1, d^2, \dots, d^k$ all belong to the same congruence class $B_{s_1 \dots s_k}$ of ϱ_B. Since ϱ_B-classes satisfy the identities of Γ_t, it follows that

$$d^1 \dots d^k u = d^1 \dots d^k v.$$

Finally note that

$$(b^1 \dots b^k u, d^1 \dots d^k u) \in \delta \quad \text{and} \quad (b^1 \dots b^k v, d^1 \dots d^k v) \in \delta,$$

which implies

$$(b^1 \dots b^k u, b^1 \dots b^k v) \in \delta.$$

This verifies (5.3). □

Let us note that in particular, ψ/δ is the semilattice replica congruence of B/δ and the quotient B/δ is the semilattice sum of ψ/δ-classes $(b/\delta)/(\psi/\delta)$.

Theorem 5.3. *The Mal'tsev product $\mathcal{V}_t \circ \mathcal{S}$ is a variety, and Σ_t^p is its equational basis.*

Proof. We have already observed that $\mathcal{V}_t \circ \mathcal{S} \subseteq \mathcal{P}(\mathcal{V}_t)$. Conversely, if an algebra C is a member of $\mathcal{P}(\mathcal{V}_t)$, then there is a set X so that C is a homomorphic image of $X\mathcal{P}(\mathcal{V}_t)$. Then by Proposition 5.2, the algebra C belongs to $\mathcal{V}_t \circ \mathcal{S}$. Thus the quasivariety $\mathcal{V}_t \circ \mathcal{S}$ and the variety $\mathcal{P}(\mathcal{V}_t)$ coincide. □

Corollary 5.4. *Let $\mathcal{W}$ be any subvariety of the variety $\mathcal{T}$. Then the Mal'tsev product $\mathcal{V}_t \circ_{\mathcal{W}} \mathcal{S}$ of $\mathcal{V}_t$ and $\mathcal{S}$ relative to $\mathcal{W}$ is a variety.*

Problem 5.5. Does Theorem 5.3 hold for an irregular subvariety of $\mathcal{T}$ which is not necessarily strongly irregular?

6. Examples and counterexamples

First consider the type τ with one binary multiplication $\cdot$ and the variety $\mathcal{T}$ of the type τ, as in Sections 4.1 and 4.2. We could observe that the Mal'tsev products $\mathcal{LZ} \circ \mathcal{S}$ and $\mathcal{S} \circ \mathcal{S}$ considered in these examples do not satisfy the associative law. This is however a more general phenomenon. So let $\mathcal{T}$ be a variety of plural type τ and assume that the equational basis Σ of a subvariety $\mathcal{V}$ of $\mathcal{T}$ contains a (non-trivial) linear identity σ. (Recall that an identity is *linear* if the multiplicities of each variable in each side are at most 1.) Then the identities of σ^p obtained from σ are never linear since each variable of σ was replaced in σ^p by a term $r_i \in T_m$ with $m \geq 2$, and all of these terms contain the same sets of variables. Moreover, consequences of the identities Σ^p are never linear.

This can be summarised as follows.

Lemma 6.1. *For any non-trivial subvariety $\mathcal{V}$ of $\mathcal{T}$, the Mal'tsev product $\mathcal{V} \circ \mathcal{S}$ does not satisfy any (non-trivial) linear identity.*

In particular, for no variety $\mathcal{V}$ of semigroups is the Mal'tsev product $\mathcal{V} \circ \mathcal{S}$ a class of semigroups. A similar observation can be made for many familiar varieties, as for example groups, rings, lattices, semilattices etc.

Frequently, more interesting are subvarieties of (quasi)varieties $\mathcal{V} \circ \mathcal{S}$.

Example 6.2 (**Bands**). As already mentioned in Section 4.1, each band is a semilattice sum of rectangular bands. (See [18] and [10], [11].) Let $\mathcal{SG}$ be the variety of semigroups, $\mathcal{B}$ be the variety of bands and $\mathcal{RB}$ be the variety of rectangular bands. Then, by Corollary 5.4,

$$\mathcal{B} = \mathcal{RB} \circ_{\mathcal{SG}} \mathcal{S} = \mathcal{SG} \cap (\mathcal{RB} \circ \mathcal{S}) = \mathcal{SG} \cap \mathcal{P}(\mathcal{RB}).$$

Note that $\mathcal{P}(\mathcal{RB})$ is a variety of groupoids but is not a variety of semigroups.

In the following examples, the (pseudo-)partition operation $x \cdot y$ will be denoted by $t(x, y)$ to avoid conflict with a multiplication as a basic operation of considered algebras.

Example 6.3 (**Birkhoff systems**). Lattices may be defined as algebras $(A, +, \cdot)$ with two semilattice reducts, the (join-)semilattice $(A, +)$ and the (meet-)semilattice $(A, \cdot)$, satisfying the *Birkhoff identity* $x + xy = x(x + y)$ and the *absorption law* $t(x, y) = x + xy = x$. They form the variety $\mathcal{L}$ of lattices. By dropping the absorption law one obtains the definition of the more general variety $\mathcal{BS}$ of *Birkhoff systems*. (See [8] and [9].) An essential role in this variety is played by two 3-element bi-chains $\mathbf{3}_m$ and $\mathbf{3}_j$ built on the set $\{0, 1, 2\}$ by defining the meet-semilattice order as $0 <_. 1 <_. 2$ and the join-semilattice order as $0 <_+ 2 <_+ 1$ in $\mathbf{3}_m$ and as $1 <_+ 0 <_+ 2$ in $\mathbf{3}_j$. It was shown in [9] that the class $\mathcal{BS}(3_m, 3_j)$ of those Birkhoff systems which do not contain either $\mathbf{3}_m$ or $\mathbf{3}_j$ as a subalgebra is a variety, and is characterized as precisely the class of Birkhoff systems which are semilattice sums of lattices. It follows that

$$\mathcal{BS}(3_m, 3_j) = \mathcal{L} \circ_{\mathcal{BS}} \mathcal{S} = \mathcal{BS} \cap (\mathcal{L} \circ \mathcal{S}) = \mathcal{BS} \cap \mathcal{P}(\mathcal{L}).$$

Note that none of the basic operations of $\mathcal{P}(\mathcal{L})$-algebras is associative or commutative.

As already mentioned, for a strongly irregular variety $\mathcal{V}_t$, the best known and understood subvariety of $\mathcal{P}(\mathcal{V}_t)$ (different from $\mathcal{V}_t$ and $\mathcal{S}$) is the regularization $\widetilde{\mathcal{V}_t}$ of $\mathcal{V}_t$. Much less is known about the pseudo-regularization. It is an easy exercise to show that the regularization and the pseudo-regularization of the variety $\mathcal{LZ}$ of left-zero bands coincide.

However it is possible for the regularization of a variety to be distinct from its pseudo-regularization. The first example was discovered in [3], and investigated in connection with the constraint satisfaction problem.

Example 6.4 (**Steiner quasigroups**). A *Steiner quasigroup* or *squag* is a commutative idempotent groupoid satisfying the identity $t(x, y) = xy \cdot y = x$. The variety formed by these groupoids is denoted by $\mathcal{SQ}$. As shown in [3], the regularization $\widetilde{\mathcal{SQ}}$ of $\mathcal{SQ}$ is the variety T_1 of commutative and idempotent groupoids satisfying the identity $x(x \cdot yz) = (x \cdot xy)z$. The pseudo-regularization $\overline{\mathcal{SQ}}$ of $\mathcal{SQ}$ is the variety of commutative idempotent groupoids satisfying the identities (P1) - (P4) from Section 2, where $\cdot$ is replaced by $t(x, y) = xy \cdot y$. Groupoids in the regularization are Płonka sums of squags. Groupoids in the pseudo-regularizations are semilattice sums of squags. As was shown in [3], the variety T_2 of commutative idempotent groupoids satisfying the identity $x(y \cdot yz)) = (xy \cdot y)z$ is contained in the variety $\overline{\mathcal{SQ}}$, and is different from $T_1 = \widetilde{\mathcal{SQ}}$. It follows that the three varieties $\mathcal{SQ}$, $\widetilde{\mathcal{SQ}}$ and $\overline{\mathcal{SQ}}$ are distinct:

$$\mathcal{SQ} \subset \widetilde{\mathcal{SQ}} \subset \overline{\mathcal{SQ}}.$$

However we do not know if the varieties T_2 and $\overline{\mathcal{SQ}}$ coincide, though we think it is not likely. The Mal'tsev product $\mathcal{SQ} \circ \mathcal{S}$ coincides with the variety $\mathcal{P}(\mathcal{SQ})$. Since commutativity is a linear identity, members of the variety $\mathcal{P}(\mathcal{SQ})$ are not commutative. The pseudo-regularization $\overline{\mathcal{SQ}}$ is obviously contained in but not equal to $\mathcal{P}(\mathcal{SQ})$.

In fact, the last example and the example concerning Birkhoff systems, were those which inspired us to undertake investigations of a most general class of algebras representable as semilattice sums of their subalgebras.

Examples of other varieties of algebras with distinct regularization and pseudo-regularization are provided by bisemilattices, algebras with two semilattice operations (one interpreted as a join and one as a meet, similarly as in the case of Birkhoff systems.)

Example 6.5 (**Semilattice sums of lattices**). First recall that members of the class $\mathcal{BS}(3_m, 3_j)$ of Birkhoff systems considered above are all semilattice sums of lattices. However, it is not difficult to check that this class is not the pseudo-regularization of the variety $\mathcal{L}$ of lattices. The partition operation $t(x, y) = x + xy$ of the regularization $\widetilde{\mathcal{L}}$ is not a pseudo-partition operation for $\mathcal{BS}(3_m, 3_j)$. Indeed, an easy exercise shows that the requirements for a pseudo-partition operation are not satisfied. A witness is a certain 3-element member of $\mathcal{BS}(3_m, 3_j)$. This is the bichain $\mathbf{3}_n$ built on the set $\{0,1,2\}$ with the meet-semilattice order defined as for $\mathbf{3}_m$ and $\mathbf{3}_j$ and the join-semilattice order defined by $2 <_+ 0 <_+ 1$. Note that $\mathbf{3}_n$ is the semilattice sum of the 2-element lattice $\{0,1\}$ and the 1-element lattice $\{2\}$, but it is not a Płonka sum of these lattices. (See [8] and [9].)

Now let $A = A_1$ be a (subdirectly irreducible) lattice and let A_0 be a 1-element lattice $\mathbf{1} = \{\infty\}$ disjoint from A. Then let A^∞ be the Płonka sum of A_1 and A_0 over the 2-element meet semilattice $\{0,1\}$ with $0 < 1$. Then the bisemilattice A^∞ generates the regularization of the variety $\mathsf{V}(A)$ generated by A. (See e.g. [24, Ch. 4].)

There is another interesting family of semilattice sums obtained from lattices A_1 and A_0, in the case when A_1 is a bounded lattice. Let $A = A_1$ be such a lattice with bounds 0 and 1. For an $a \in A_1$, the meet of a and ∞ is defined to be the 0 of A and the join to be the 1 of A. It is easy to see that $A_\infty = A \cup \{\infty\}$ is a bisemilattice, and the semilattice sum of the lattices A_1 and A_0 over the meet semilattice $\{0,1\}$ with $1 < 0$. A little more complicated exercise shows that the operation $t(x,y) = x + xy$ is a pseudo-partition operation in each algebra A_∞, but it is not a partition operation. It follows that each bisemilattice A_∞ generates the variety $\mathsf{V}(A_\infty)$ contained in the pseudo-regularization of $\mathcal{L}$, and different from the regularization of the variety $\mathsf{V}(A)$. We do not know if the varieties $\mathsf{V}(A_\infty)$ and $\overline{\mathsf{V}(A)}$ coincide.

Example 6.6 (Groups). Groups may be defined as algebras $(G, \cdot, ^{-1})$ satisfying the identities $xy{\cdot}z = x{\cdot}yz$, $y^{-1}y{\cdot}x = x{\cdot}yy^{-1}$, and $x{\cdot}yy^{-1} = x$. They form a strongly irregular variety with $t(x,y) = x{\cdot}yy^{-1}$, which will be denoted by $\mathcal{G}$. The regularization $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ may be defined by first two identities defining $\mathcal{G}$ and the identities $(xy)^{-1} = y^{-1}x^{-1}$, $(x^{-1})^{-1} = x$ and $xxx^{-1} = x$. (See e.g. [20] and [21].) The variety $\widetilde{\mathcal{G}}$ consists of Płonka sums of groups. In semigroup theory, Płonka sums of groups are called *strong semilattices of groups*. By results of Clifford [5], it is known that the class of Płonka sums of groups coincides with the class of semigroups which are semilattices of groups and with the class of so-called *Clifford semigroups*, semigroups with a unary operation $^{-1}$ satisfying the identities $(x^{-1})^{-1} = x, xx^{-1}x = x, xx^{-1} = x^{-1}x$ and $(xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1})$. (See also [6], [10] and [11].) The class $\widetilde{\mathcal{G}}$ is a subvariety of the variety $\mathcal{IS}$ of inverse semigroups, semigroups with one unary operation $^{-1}$ satisfying the last three identities defining $\widetilde{\mathcal{G}}$. Members of $\widetilde{\mathcal{G}}$ are precisely inverse semigroups which are Płonka sums of groups (or as semigroup theorists would say, strong semilattices of groups). (See e.g. [21] and references there.) In fact,

$$\widetilde{\mathcal{G}} = \mathcal{IS} \cap (\mathcal{G} \circ \mathcal{S}).$$

Groups defined as algebras with two basic operations $\cdot$ and $^{-1}$ as above do not contain nullary operations in its type. Hence Theorem 5.3 applies, and the Mal'tsev product $\mathcal{G} \circ \mathcal{S}$ is the variety $\mathcal{P}(\mathcal{G})$. Note that $\mathcal{P}(\mathcal{G})$ is a variety of groupoids with one unary operation, but it is not a variety of semigroups. The pseudo-regularization of $\mathcal{G}$ has not yet been investigated.

The assumption that the type of a variety $\mathcal{V}$, in a Mal'tsev product $\mathcal{V} \circ \mathcal{S}$, has no nullary operation symbols is due to the fact that semilattices have no nullary operations. However, if $\mathcal{V}$ is a variety with constant (nullary) operations, then the last assumption may be easily omitted, as it was done in the example above. Instead of a constant operation one can take a unary operation defining this constant, and consider an equivalent variety $\mathcal{V}'$ with such a unary operation replacing each constant operation. Then if $\mathcal{V}$ is strongly irregular, Theorem 5.3 applies, and the Mal'tsev product $\mathcal{V}' \circ \mathcal{S}$ is a variety.

If $\mathcal{V}$ is an irregular but not strongly irregular subvariety of the variety $\mathcal{T}$, then the class of Płonka sums of $\mathcal{V}$-algebras does not necessarily coincide with the regularization $\widetilde{\mathcal{V}}$ of $\mathcal{V}$. (See [24, Ch. 4] and [21] for more information and some references.) The following example concerns the case of semigroups.

Example 6.7 (Semigroups). It was shown by V. N. Salii [25, 26] that each semigroup in the regularization $\widetilde{\mathcal{V}}$ of an irregular variety $\mathcal{V}$ of semigroups embeds into a Płonka sum of $\mathcal{V}$-algebras. (This however is not true in general, see [7] and [22].) Hence in the case of semigroups the regularization $\widetilde{\mathcal{V}}$ is contained in the Mal'tsev product $\mathcal{V} \circ \mathcal{S}$, however not all of its members are Płonka sums of $\mathcal{V}$-algebras.

We finish this section with an example of a Mal'tsev product $\mathcal{Q} \circ \mathcal{S}$, where $\mathcal{Q}$ is a subquasivariety of $\mathcal{T}$, which is not necessarily a variety. First note that if a quasi-equational basis for $\mathcal{Q}$ is given, then one can define the quasivariety $\mathcal{P}(\mathcal{Q})$ in similar way as it was done for $\mathcal{Q} = \mathcal{V}$. Rewrite Definition 2.1 by taking a quasi-equational basis Λ of $\mathcal{Q}$ instead of the equational basis Σ of $\mathcal{V}$, then for each quasi-identity $\lambda \in \Lambda$ form the set of quasi-identities λ^p in a similar way as for identities, and then define the set Λ^p as the union of all λ^p. Now let $\mathcal{P}(\mathcal{Q})$ be the quasivariety defined by the quasi-identities Λ^p. It can be easily checked that the Mal'tsev product $\mathcal{Q} \circ \mathcal{S}$ satisfies Λ^p. Hence

$$\mathcal{Q} \circ \mathcal{S} \subseteq \mathcal{P}(\mathcal{Q}).$$

Note that each member of $\mathcal{P}(\mathcal{Q})$ can be decomposed into the semilattice sum of its subalgebras using the semilattice replica congruence ϱ, however we do not know if the corresponding summands satisfy the quasi-identities of $\mathcal{Q}$. This raises the open question as to whether the quasivarieties $\mathcal{Q} \circ \mathcal{S}$ and $\mathcal{P}(\mathcal{Q})$ coincide.

Example 6.8. This example comes from the geometry of affine spaces and convex sets. (See [24] and references there.) Affine spaces over a subfield F of the field $\mathbb{R}$ of reals (or affine F-spaces) may be defined as abstract algebras $(A, \underline{F})$, where $\underline{F}$ is the set of binary operations $xy\underline{p} = \underline{p}(x, y) = x(1-p)+yp$, where $p \in F$. Affine F-spaces defined in this way form a variety. Convex subsets of affine F-spaces (or convex F-sets) may be defined as abstract algebras $(C, \underline{I}^o)$, where $\underline{I}^o$ is the subset $\{\underline{p} \mid p \in I^o =]0, 1[\}$ and $]0, 1[$ is the open unit interval of F. Fix a subfield F of $\mathbb{R}$. Then convex F-subsets of affine F-spaces generate the variety $\mathcal{BA}$ of *barycentric algebras*. Convex F-sets form a subquasivariety $\mathcal{C}$ of $\mathcal{BA}$. The variety $\mathcal{BA}$ is defined by the following identities

$$\begin{aligned} &xx\underline{p} = x, \\ &xy\,\underline{p} = yx\,\underline{1-p}, \\ &xy\,\underline{p}\;z\,\underline{q} = x\;yz\,\underline{q/(p \circ q)}\;\underline{p \circ q} \end{aligned}$$

for all p, q in I^o. Here $p \circ q = p + q - pq$. (See [24, S. 5.8].) Its subquasivariety $\mathcal{C}$ is defined by the cancellation laws

$$(xy\underline{p} = xz\underline{p}) \longrightarrow (y = z), \tag{6.1}$$

which hold for all $p \in I^o$. It is known [24, S. 7.5] that each barycentric algebra is a semilattice sum of (open) convex sets. This shows that $\mathcal{BA}$ is contained in the Mal'tsev product $\mathcal{C} \circ \mathcal{S}$:

$$\mathcal{BA} \subset \mathcal{C} \circ \mathcal{S} \subset \mathcal{P}(\mathcal{C}).$$

Since the quasivariety $\mathcal{C}$ satisfies linear identities, the quasivarieties $\mathcal{BA}$ and $\mathcal{P}(\mathcal{C})$ are distinct.

In fact, a stronger result was proved, which we provide here in a little reformulated way.

Theorem 6.9. [24, S. 7.5] *Each barycentric algebra is a semilattice sum of open convex sets over its semilattice replica, and is a subalgebra of a Płonka sum of convex sets over its semilattice replica.*

The summands of the Płonka sum are certain canonical extensions of the summands of the semilattice sum.

More detailed results are possible for subquasivarieties of the variety $\mathcal{BA}$. As shown by Ignatov [12] (see also [24, S. 7.6]), irregular subquasivarieties of $\mathcal{BA}$ (i.e. those not containg the variety $\mathcal{S}$) form a chain $\mathcal{B}_1 < \mathcal{B}_2 < \ldots < \mathcal{B}_\omega$. Each $\mathcal{B}_i$ is covered by its quasi-regularizations $\widetilde{\mathcal{B}_i}^q$. The quasivarieties $\mathcal{B}_i$ and $\widetilde{\mathcal{B}_i}^q$ are all proper subquasivarieties of the variety $\mathcal{BA}$. Each of these quasivarieties for $i \neq \omega$ is generated by one finite algebra. Hence each member of such a quasivariety is a subalgebra of a power of its generator.

7. Concluding remarks and the problem of representation of algebras in Mal'tsev products

The fact that an algebra A is a semilattice sum of $\mathcal{V}_t$-subalgebras says very little about its detailed structure. An exception is of course given by algebras represented as Płonka sums. (The second best representation is given by subalgebras of Płonka sums.) So it is natural to ask how the summands of a semilattice sum are put together or how to reconstruct the algebraic structure of a semilattice sum $A = \bigsqcup_{s \in S} A_s$ from its $\mathcal{V}_t$-summands A_s and the quotient semilattice S. Such a construction exists and was introduced for general algebras in [23, § 6.2], under the name of a Lallement sum, as a generalization of a Płonka sum. (See also [14] for a basic construction introduced in the case of semigroups, and [22], [21] and [24, Ch. 4] for similar constructions in the case of general algebras.) The primary idea was to relax the requirement of functoriality in the definition of Płonka sums. There are several types of such constructions and the definition may be formulated for algebras of any plural type. However, to avoid excessive notation we will consider here only plural type with operations of arity two, and will limit ourself to the following definition.

Definition 7.1. Let $(S, \cdot)$ be a (meet) semilattice. (Here all operations of Ω are equal to $\cdot$). For each $s \in S$, let a $\mathcal{V}_t$-algebra A_s of a plural binary type $\tau : \Omega \to \{2\}$ be given, and for each operation $\star \in \Omega$ an extension $(E_s^\star, \star)$ of $(A_s, \star)$. For each pair $t \leq s$ of S, let

$$\varphi_{s,t}^\star : (A_s, \star) \to (E_t^\star, \star)$$

be a $\star$-homomorphism such that the following three conditions are satisfied:

(1) $\varphi_{s,s}^\star$ is the embedding of A_s into $E_s^\star$;
(2) $\varphi_{s,s\cdot t}^\star(A_s) \star \varphi_{t,s\cdot t}^\star(A_t) \subseteq A_{s\cdot t}$;

(3) for each $u \leq s \cdot t$ in S and $a_s \in A_s$, $b_t \in A_t$
$$\varphi^{\star}_{s\cdot t,u}(\varphi^{\star}_{s,s\cdot t}(a_s) \star \varphi^{\star}_{t,s\cdot t}(b_t)) = \varphi^{\star}_{s,u}(a_s) \star \varphi^{\star}_{t,u}(b_t).$$

An Ω-algebra structure on the disjoint sum A of all A_s is given by defining the operations $\star$ in Ω as follows:
$$a_s \star b_t = \varphi^{\star}_{s,s\cdot t}(a_s) \star \varphi^{\star}_{t,s\cdot t}(b_t).$$
Then all A_s are subalgebras of A and S is its quotient. The semilattice sum A of A_s is said to be the *semilattice sum of A_s by the mappings $\varphi^{\star}_{s,t}$*. If additionally, for each $t \in S$, one has $E^{\star}_t = \{\varphi^{\star}_{s,t}(a_s) \mid s \geq t,\ a_s \in A_s\}$, and all $E^{\star}_s$ are certain canonical extensions of A_s (see [23, § 6.1]), then this semilattice sum is called a *Lallement sum*.

By [23, Th. 624] each semilattice sum of $\mathcal{V}_t$-algebras can be reconstructed as a Lallement sum of these algebras. The usefulness of Lallement sums depends on properties of the available extensions $E^{\star}_s$. In particular, a nice situation appears if for each $s \in S$, all extensions $E^{\star}_s$ coincide with the summand A_s. Such Lallement sum is called *strict*. We next consider a special case of strict Lallement sums of $\mathcal{V}_t$-algebras, which extends a construction described for Birkhoff systems in [9]. This construction works nicely for $\mathcal{V}_t$-algebras where each operation $\star \in \Omega$ has a one-sided unit. In what follows we assume that all one-sided units are right-sided and call them simply units. We will call such algebras briefly *$\mathcal{V}_t$-algebras with units*.

Let A be a semilattice sum $\bigsqcup A_s$ of $\mathcal{V}_t$-algebras A_s with units, and let $e^{\star}_s$ be the (right-)unit element of $\star$ in A_s. Additionally assume that for all $s, t, u \in S$ with $u \leq s, t$
$$(a_s \star b_t) \star e^{\star}_u = (a_s \star e^{\star}_u) \star (b_t \star e^{\star}_u), \tag{7.1}$$
and for $t \leq s$ in S define the maps
$$\varphi^{\star}_{s,t} : A_s \to A_t; a_s \mapsto a_s \star e^{\star}_t.$$
Note that $t \leq s$ in S means $s \cdot t = t$, so these maps are well defined. It is easily seen that each $\varphi^{\star}_{s,t}$ is a $\star$-homomorphism from A_s into A_t. Moreover, by (7.1),
$$a_s \star b_t = (a_s \star b_t) \star e^{\star}_{st} = (a_s \star e^{\star}_t) \star (b_s \star e^{\star}_t) = \varphi^{\star}_{s,s\cdot t}(a_s) \star \varphi^{\star}_{t,s\cdot t}(b_t).$$
Then for each $u \leq s \cdot t$ in S
$$\varphi^{\star}_{s\cdot t,u}(\varphi^{\star}_{s,s\cdot t}(a_s) \star \varphi^{\star}_{t,s\cdot t}(b_t)) = \varphi^{\star}_{s\cdot t,u}(a_s \star b_t) =$$
$$(a_s \star b_t) \star e^{\star}_u = (a_s \star e^{\star}_u) \star (b_t \star e^{\star}_u) = \varphi^{\star}_{s,u}(a_s) \star \varphi^{\star}_{t,u}(b_t).$$
It follows that the maps $\varphi^{\star}_{s,t}$ satisfy the requirements of a strict Lallement sums. And one obtains the following theorem, a corollary of Theorem 624 of [23], and extension of Theorem 4.15 of [9].

Theorem 7.2. *Let A be a semilattice sum of $\mathcal{V}_t$-algebras A_s with units of a plural binary type. Then $A = \bigsqcup_{s \in S} A_s$ satisfies the condition* (7.1) *if and only if it is a strict Lallement sum of the subalgebras A_s over the semilattice S given by the homomorphisms $\varphi^{\star}_{s,t}$ described above.*

Examples are provided by Birkhoff systems which are semilattices of bounded lattices, and more general bisemillatices which are semilattice sums of bounded lattices. In particular, members of the pseudo-regularizations of the variety $\mathcal{L}$ of lattices provided above are of this type.

Note also that the condition (7.1) holds for all algebras A of plural binary type with self-entropic operations, that means satisfying $(x \star y) \star (z \star t) = (x \star z) \star (y \star t)$ for each $\star \in \Omega$.

One more final remark concerns the assumption about strong irregularity of the summands A_s. In this paper, we were interested in semilattices sums of algebras in a strongly irregular variety. However, both Definition 7.1 and Theorem 7.2 may be easily extended to semilattice sums of algebras in any subvariety of the variety $\mathcal{T}$.

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[1] Department of Mathematics, Iowa State University, Ames, Iowa, 50011, USA

[2,3] Faculty of Mathematics and Information Science, Warsaw University of Technology, 00-661 Warsaw, Poland

E-mail address: [1]cbergman@iastate.edu

E-mail address: [2]t.penza@mini.pw.edu.pl

E-mail address: [3]aroman@mini.pw.edu.pl