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Categorical equivalence of algebras with a majority term

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The main theorem is used to get a different sort of characterization of categorical equivalence for algebras generating an arithmetical variety. We also consider clones of co-height at most two. In addition, we provide new proofs of several characteriza- tions in the literature, including quasi-primal, lattice-primal and congruence-primal algebras.

CATEGORICAL EQUIVALENCE OF ALGEBRAS WITH A MAJORITY TERM

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Abstract. Let $\mathbf{A}$ be a finite algebra with a majority term. We characterize those algebras categorically equivalent to $\mathbf{A}$. The description is in terms of a derived structure with universe consisting of all subalgebras of $\mathbf{A} \times \mathbf{A}$, and with operations of composition, converse and intersection.

The main theorem is used to get a different sort of characterization of categorical equivalence for algebras generating an arithmetical variety. We also consider clones of co-height at most two. In addition, we provide new proofs of several characterizations in the literature, including quasi-primal, lattice-primal and congruence-primal algebras.

Majority operations have long held a special place in universal algebra. It has been known for quite some time that any variety of algebras possessing a majority term is congruence distributive. In 1975, Baker and Pixley discovered that for a finite algebra $\mathbf{A}$ with a majority term, the set of subalgebras of $\mathbf{A}^2$ completely determines the term operations on $\mathbf{A}$. In addition, every subalgebra of $\mathbf{A}^k$ (with $k \geq 2$) is completely determined by all of its 2-fold projections. Conversely, G. Bergman proved that, under some obviously necessary consistency conditions, every family of subalgebras of $\mathbf{A}^2$ is obtained from a subalgebra of $\mathbf{A}^k$ by 2-fold projections.

By universal algebraic standards, algebras with a majority term are not rare. Any structure possessing a lattice reduct has a majority term, as does any quasiprimal algebra. More generally, any algebra that generates an arithmetical variety (i.e. both congruence distributive and congruence permutable) will have such a term.

It is customary to consider term-equivalence (of algebras or varieties) as a fundamental relationship in universal algebra. Indeed, term-equivalent varieties are usually treated as interchangeable. From this perspective, the Baker-Pixley result mentioned above tells us that if $\mathbf{A}$ is a finite algebra with a majority term, then $\mathsf{V}(\mathbf{A})$ is completely determined by the set $\mathrm{Sub}(\mathbf{A}^2)$.

However, it is also natural to consider a variety as a category of algebras, in which the arrows are exactly the homomorphisms. With this as our starting point, the central relationship between varieties becomes equivalence of categories, which

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is strictly weaker than term-equivalence. From this vantage point, algebras with a majority term have again proven to have strong properties. In 1983 Davey and Werner showed that if $\mathbf{A}$ is a finite algebra with a majority term then $\mathsf{V}(\mathbf{A})$ possesses a natural duality with a certain category of topological objects.

Recently, several new tools for the study of the categorical equivalence of varieties have been developed. Most important is a very general theorem of McKenzie's that describes all such categorical equivalences. (In fact, the result applies to categories even more general than varieties.) Another is a theorem of Lüder's that, for a finite algebra $\mathbf{A}$, allows us to describe the varieties categorically equivalent to $\mathsf{V}(\mathbf{A})$ via the isomorphism type of an object derived from all finite subpowers of $\mathbf{A}$.

In this paper, we will take Lüder's theorem as our starting point and provide characterizations of some familiar varieties up to categorical equivalence. In light of Lüder's theorem and the Baker-Pixley-Bergman results it should not be surprising that we can do this for any variety $\mathsf{V}(\mathbf{A})$, with $\mathbf{A}$ finite and having a majority term operation, by considering the isomorphism type of an object associated with $\mathrm{Sub}(\mathbf{A}^2)$. We will show that several results in the literature can be easily derived from this majority-term characterization. In Section 3 we attempt to classify (up to categorical equivalence) those finite algebras $\mathbf{A}$ in which $\mathrm{Sub}(\mathbf{A}^2)$ is quite small and manageable. Finally, we will apply the main result to finite algebras generating arithmetical varieties.

1. Krasner algebras

Let A be a set. For every positive integer n, let $\mathrm{Rel}_n(A)$ denote the set of all n-ary relations on A, and $\mathrm{Rel}(A) = \bigcup_{n=1}^{\infty} \mathrm{Rel}_n(A)$. Let $\theta \in \mathrm{Rel}_k(A)$ and $\lambda \in \mathrm{Rel}_\ell(A)$. We make the following definitions.

$$
\begin{aligned}
\zeta(\theta) &= \big\{\langle x_2, x_3, \dots, x_k, x_1\rangle : \langle x_1, x_2, \dots, x_k\rangle \in \theta\big\}\\
\tau(\theta) &= \big\{\langle x_2, x_1, x_3, \dots, x_k\rangle : \langle x_1, x_2, \dots, x_k\rangle \in \theta\big\}\\
\nu(\theta) &= \big\{\langle x_1, \dots, x_k, y\rangle : \langle x_1, \dots, x_k\rangle \in \theta,\ y \in A\big\}\\
\pi(\theta) &= \big\{\langle x_1, x_2, \dots, x_{k-1}\rangle : (\exists x_k)\ \langle x_1, \dots, x_k\rangle \in \theta\big\}\\
\delta &= \big\{\langle x, x\rangle : x \in A\big\}\\
\theta \sqcap \lambda &= \big\{\langle x_1, x_2, \dots, x_n\rangle : \langle x_1, \dots, x_k\rangle \in \theta, \langle x_1, \dots, x_\ell\rangle \in \lambda\big\}\\
&\qquad \text{where } n = \max\{k, \ell\}.
\end{aligned}
\tag{1.1}
$$

(If $k = 1$, each of ζ, τ, and π behaves like the identity map. Also, $\nu(\varnothing) = \varnothing$.)

The algebra $\langle \mathrm{Rel}(A), \zeta, \tau, \nu, \pi, \sqcap, \delta\rangle$ is called the *full Krasner algebra* on A. More generally, by a *Krasner Algebra* on A we mean any subalgebra of the full Krasner algebra.

This definition is essentially that of a "subdirect closure system" in [R2]. It differs superficially from the definition given in [PK]. Using the same θ and λ as above let

$$
\begin{aligned}
\Delta(\theta) &= \big\{\langle x_1, \dots, x_{k-1}\rangle : \langle x_1, x_1, \dots, x_{k-1}\rangle \in \theta\big\}\\
\theta \circ \lambda &= \big\{\langle x_1, \dots, x_{k+\ell-2}\rangle : (\exists y)\ \langle x_1, \dots, x_{k-1}, y\rangle \in \theta\\
&\qquad \&\ \ \langle y, x_k, \dots, x_{k+\ell-2}\rangle \in \lambda\big\}\\
\delta^{\{1;2,3\}} &= \big\{\langle x, y, y\rangle : x, y \in A\big\}.
\end{aligned}
\tag{1.2}
$$

In [PK], the basic operations of a Krasner algebra are taken to be $\zeta, \tau, \Delta, \circ$ and $\delta^{\{1;2,3\}}$. It is an interesting exercise to verify that from each definition of Krasner algebra, one can derive the operations in the other definition. Let us also observe that if $k > \ell$, then $\theta \sqcap \lambda = \theta \cap \nu^{k-\ell}(\lambda)$. Thus in practice, we can restrict our attention to intersection of relations of the same rank.

The terminology in this area is not at all standard. In addition to "subdirect closure system", the objects we are calling Krasner algebras have been named "relational algebras", "Post coalgebras", "coclones" and "relational clones". On the other hand, the term "Krasner algebra" (of the first and second kind) has been used for those sets of relations invariant under unary operations. We have chosen our terminology partly for simplicity and partly to avoid confusion with the algebras of binary relations considered by Tarski *et. al.*

Now let $\mathbf{A} = \langle A, F \rangle$ be an algebra and $\theta \in \mathrm{Rel}_n(A)$. We say that θ is *F-invariant* if for all $f \in F$ and all $\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^k \in \theta$, $\langle f(\mathbf{x}_1), \dots, f(\mathbf{x}_n) \rangle \in \theta$. Here, k denotes the rank of f and $\mathbf{x}^i$ (respectively $\mathbf{x}_j$) represents the i^{th} column (j^{th} row) of the $n \times k$ matrix $(x_{pq} : 1 \le p \le n, 1 \le q \le k)$. Of course, θ is F-invariant if and only if θ is a subuniverse of $\mathbf{A}^n$. However, we prefer to de-emphasize the character of θ as an algebra in favor of its role as a member of $\mathrm{Rel}(A)$.

It is not hard to verify that the set of all relations on A invariant under F is the universe of a Krasner algebra on A. We call this the Krasner algebra of $\mathbf{A}$, and denote it $\mathcal{K}(\mathbf{A})$. Conversely, if A is finite, every Krasner algebra on A is of the form $\mathcal{K}(\mathbf{A})$ for some algebra $\mathbf{A}$.

Most of our universal-algebraic and category-theoretic terminology is standard. See [MMT] for the former and [Mac] for the latter. In particular, the notions of a variety and a category, and of term-equivalent varieties and equivalent categories can be found in those references.

There is one small inconsistency between the conventions in the two fields. In universal algebra, the empty algebra is banned (although empty subuniverses are permitted). It is possible for two varieties, one with nullary operations symbols and one without, to be term-equivalent. Category theorists generally admit the empty algebra. Consequently, those same two varieties would not be equivalent as categories, since the latter has an "extra" object. Being the conciliatory type, we shall break the logjam by disallowing nullary operations in our official definition. Thus, we do not permit the empty algebra, but every algebra will have an empty subuniverse. Of course, working algebraists generally have a constant or two floating around, but these can always be replaced by constant unary operations without causing any difficulties.

The variety generated by an algebra $\mathbf{A}$ is denoted $\mathbf{V}(\mathbf{A})$ .We say that two algebras $\mathbf{A}$ and $\mathbf{B}$ are *categorically equivalent* (in symbols $\mathbf{A} \equiv_{\mathrm{c}} \mathbf{B}$) if there is an equivalence of categories $F \colon \mathbf{V}(\mathbf{A}) \to \mathbf{V}(\mathbf{B})$ such that $F(\mathbf{A}) = \mathbf{B}$. In [M], McKenzie gave a characterization of '$\equiv_{\mathrm{c}}$' in terms of matrix powers and invertible idempotent terms. Since we do not need this result, we do not present the definition. Recently Lüders [L1] proved the following.

Theorem 1.1. *Let $\mathbf{A}$ and $\mathbf{B}$ be finite algebras. Then $\mathbf{A} \equiv_{\mathrm{c}} \mathbf{B}$ if and only if $\mathcal{K}(\mathbf{A}) \cong \mathcal{K}(\mathbf{B})$.*

This theorem provides yet another method, in addition to McKenzie's explicit characterization and the construction of a duality, for determining whether two algebras are categorically equivalent. In this paper we will discuss some applications of Theorem 1.1.

Note that an isomorphism of Krasner algebras must preserve the rank of each relation. This is because θ is a k-ary relation if and only if k is the least positive integer ℓ such that $\pi^\ell(\theta) = \pi^{\ell-1}(\theta)$. Since we have disallowed nullary operations, $\varnothing$ is always an invariant relation. $\varnothing$ is the only relation that does not have a well-defined rank. Consequently, it must be preserved by every Krasner isomorphism.

Throughout, we assume all algebras are finite. If Θ is a set of relations on a set A, then $\mathcal{F}(\Theta)$ denotes the set of operations on A preserving all members of Θ. Similarly, for a set F of operations on A, $\mathcal{R}(F)$ denotes the set of all relations invariant under every operation in F.

2. $\mathcal{S}_2$-STRUCTURES

The first case one might consider is one in which the Krasner algebra is generated by unary relations. This is equivalent, for a finite algebra $\mathbf{A}$, to the condition: $\mathrm{Clo}(\mathbf{A}) = \mathcal{F}(\mathrm{Sub}(\mathbf{A}))$. In [BB], such an algebra was called subalgebra-primal. (The more traditional terminology is "semiprimal".) That paper contains a proof of the following theorem.

Theorem 2.1. *Let* $\mathbf{A}$ *be a finite, subalgebra-primal algebra. Then* $\mathbf{A} \equiv_c \mathbf{B}$ *if and only if* $\mathbf{B}$ *is finite, subalgebra-primal and*

$$\langle\, \mathrm{Sub}(\mathbf{A}), \cap, \mathrm{E}(\mathbf{A})\,\rangle \cong \langle\, \mathrm{Sub}(\mathbf{B}), \cap, \mathrm{E}(\mathbf{B})\,\rangle.$$

Here, $\mathrm{E}(\mathbf{A})$ *is the set of singleton subuniverses of* $\mathbf{A}$.

As G. Gierz has observed, Theorem 2.1 can be derived from Corollary 2.5 below.

We now turn to binary relations. While we do not have a complete solution, we can analyze an important subcase. Recall that a ternary term m is called a *majority term* on an algebra $\mathbf{A}$ if $\mathbf{A}$ satisfies the identities

$$m(x, x, y) = m(x, y, x) = m(y, x, x) = x.$$

The importance of a majority term was explicated by Baker-Pixley and G. Bergman in [BP] and [B]. Let A be a set and suppose $i, j \le k$ are positive integers. We have projection mappings $p^k_{i,j} \colon A^k \to A^2$ taking $\langle x_1, x_2, \ldots, x_k\rangle$ to $\langle x_i, x_j\rangle$. The inverse image under $p^k_{i,j}$ will be denoted $\overleftarrow{p}^{\,k}_{i,j}$. Thus, for $\lambda \subseteq A^2$,

$$\overleftarrow{p}^{\,k}_{i,j}(\lambda) = \{\, \langle x_1, \ldots, x_k\rangle : \langle x_i, x_j\rangle \in \lambda \,\}\,.$$

If θ is binary, it is customary to write $\theta^{\smile}$ in place of $\tau(\theta)$.

Theorem 2.2. *Let* $\mathbf{A}$ *be a finite algebra with a majority term.*

(1) $\mathrm{Clo}(\mathbf{A}) = \mathcal{F}(\mathrm{Sub}(\mathbf{A}^2))$. *Equivalently,* $\mathcal{K}(\mathbf{A})$ *is generated by its binary members.*

(2) *Let* $\theta, \psi \in \mathrm{Sub}(\mathbf{A}^k)$. *Suppose that for all* $i, j \le k$, $p^k_{i,j}(\theta) = p^k_{i,j}(\psi)$. *Then* $\theta = \psi$.

(3) *Conversely, let* $k \ge 2$ *and, for all* $i, j \le k$, *let* $\theta_{ij} \in \mathrm{Sub}(\mathbf{A}^2)$. *Then there is* $\theta \in \mathrm{Sub}(\mathbf{A}^k)$ *such that* $p^k_{i,j}(\theta) = \theta_{ij}$, *for all* i, j, *if and only if for all* $i, j, \ell \le k$:

$$\theta_{i\ell} \subseteq \theta_{ij} \circ \theta_{j\ell}, \qquad \theta_{ij} = \theta_{ji}^{\smile}, \qquad \theta_{ii} \subseteq \delta. \tag{2.1}$$

Following G. Bergman, we shall call a system $\langle \theta_{ij} : 1 \leq i, j \leq k \rangle$ of relations *consistent* if it satisfies the conditions in (2.1).

We define, for an algebra $\mathbf{A}$, the structure

$$\mathcal{S}_2(\mathbf{A}) = \langle \operatorname{Sub}(\mathbf{A}^2), \cap, \circ, \smile, \delta, A^2 \rangle.$$

Note that, unlike the situation for arbitrary relations, if θ and λ are both binary relations, then $\theta \circ \lambda$ is also binary. If Θ is a collection of binary relations on A closed under the above operations then Θ will be called an $\mathcal{S}_2$-structure on A. We wish to prove the following theorem.

Theorem 2.3. *Let* $\mathbf{A}$ *be a finite algebra with a majority term. Then* $\mathbf{A} \equiv_{\mathrm{c}} \mathbf{B}$ *if and only if* $\mathbf{B}$ *is finite, has a majority term and* $\mathcal{S}_2(\mathbf{A}) \cong \mathcal{S}_2(\mathbf{B})$.

Proof. First suppose $\mathbf{A} \equiv_{\mathrm{c}} \mathbf{B}$. Since the axioms defining a majority operation form a linear Maltsev condition, the fact that $\mathbf{B}$ is finite and has a majority term follows from [DW] (or see [M, 3.1 and 6.10]). The isomorphism of the $\mathcal{S}_2$-structures is obtained from Theorem 1.1.

Now assume that $\mathbf{B}$ is finite, has a majority term and $\Phi\colon \mathcal{S}_2(\mathbf{A}) \to \mathcal{S}_2(\mathbf{B})$ is an isomorphism. We wish to extend Φ to an isomorphism $\overline{\Phi}$ of $\mathcal{K}(\mathbf{A})$ with $\mathcal{K}(\mathbf{B})$.

By assumption, $\Phi(A^2) = B^2$ and $\Phi(\delta_A) = \delta_B$. To define $\overline{\Phi}$ on unary relations, we proceed as follows. For $C \in \operatorname{Sub}(\mathbf{A})$, let $\delta_C = \{\, \langle x, x \rangle : x \in C \,\}$. Since Φ induces an order-isomorphism of the downsets of $\mathcal{S}_2(\mathbf{A})$ and $\mathcal{S}_2(\mathbf{B})$ below δ, we define $\overline{\Phi}(C) = D$, where $\Phi(\delta_C) = \delta_D$.

We can now restrict our attention to relations of rank greater than 1. We introduce the following intermediate structure. For any $k \geq 2$ let $\operatorname{Consis}_k(\mathbf{A})$ consist of those $k \times k$ matrices $\langle \theta_{ij} : 1 \leq i, j \leq k \rangle$ of invariant binary relations that satisfy the consistency conditions in equations (2.1). Observe that $\operatorname{Consis}_k(\mathbf{A})$ can be partially ordered componentwise.

Claim. *For* $k \geq 2$, *the ordered sets* $\operatorname{Sub}(\mathbf{A}^k)$ *and* $\operatorname{Consis}_k(\mathbf{A})$ *are isomorphic via the mappings*

$$\theta \overset{f}{\longmapsto} \langle p^k_{i,j}(\theta) : 1 \leq i, j \leq k \rangle \quad \textit{and} \quad \langle \theta_{ij} : 1 \leq i, j \leq k \rangle \overset{g}{\longmapsto} \bigcap_{i,j=1}^{k} \overleftarrow{p}^{\,k}_{i,j}(\theta_{ij}).$$

Proof of Claim. It is implicit in the proof of Theorem 1 of [B] that the mappings f and g are mutually inverse. Since the direct and inverse image (under any function) always preserves set-theoretic inclusions, both f and g will be order-preserving.

Since Φ is an $\mathcal{S}_2$-isomorphism, it is order-preserving and preserves the consistency of families of binary relations. Therefore, we have the following order-preserving bijections:

$$\begin{array}{ccc} \operatorname{Sub}(\mathbf{A}^k) & & \operatorname{Sub}(\mathbf{B}^k) \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\uparrow \\ \operatorname{Consis}_k(\mathbf{A}) & \xrightarrow{\Phi^{(k\times k)}} & \operatorname{Consis}_k(\mathbf{B}) \end{array}$$

where $\Phi^{(k\times k)}$ is the coordinatewise application of Φ.

We now define $\overline{\Phi}$ to be the composition of these three maps. Explicitly, for every k-ary member θ of $\mathcal{K}(\mathbf{A})$ (with $k \geq 2$), we define

$$\overline{\Phi}(\theta) = \bigcap \Big\{ \overleftarrow{p}^{\,k}_{i,j}\big(\Phi(p^k_{i,j}(\theta))\big) : 1 \leq i,j \leq k \Big\}. \tag{2.2}$$

Note that $p^k_{i,j}$ is operating on A-relations, while $\overleftarrow{p}^{\,k}_{i,j}$ is operating on B-relations.

From the discussion so far, we conclude that $\overline{\Phi}$ is an order-preserving bijection of $\mathcal{K}(\mathbf{A})$ with $\mathcal{K}(\mathbf{B})$. It remains to verify that $\overline{\Phi}$ is a Krasner homomorphism. It is easy to verify that $\overline{\Phi}$ preserves ζ, τ and δ. Since $\overline{\Phi}$ is an order-isomorphism, it preserves intersection of relations of the same rank. So as we indicated earlier, once we verify that $\overline{\Phi}$ preserves ν, it will follow that it preserves '$\sqcap$' as well.

Let $\theta \in \mathrm{Sub}(\mathbf{A}^k)$ and $\psi = \overline{\Phi}(\theta)$. We show that $\overline{\Phi}(\pi(\theta)) = \pi(\psi)$. If we think of $f(\theta)$ as a $k \times k$ matrix, then it is easy to see that $f(\pi(\theta))$ is the $(k-1) \times (k-1)$ submatrix obtained by deleting the last row and column. The same relationship holds between the matrices $f(\psi)$ and $f(\pi(\psi))$. Since the mapping $\Phi^{k \times k}$ operates componentwise and maps $f(\theta)$ to $f(\psi)$, it must carry the submatrix of $f(\theta)$ to that of $f(\psi)$.

Finally, we verify the preservation of ν. Let $\theta \in \mathrm{Sub}(\mathbf{A}^k)$. Notice that $\nu(\theta)$ is essentially $\theta \times A$. For $i,j \leq k$, $p^{k+1}_{i,j}(\nu(\theta)) = p^k_{i,j}(\theta)$ while $p^{k+1}_{i,k+1}(\nu(\theta)) = \theta_i \times A \supseteq p^{k+1}_{i,1}(\nu(\theta))$ (where θ_i is the projection of θ on its ith coordinate). Also, $p^{k+1}_{k+1,k+1}(\nu(\theta)) = A \times A \supseteq p^{k+1}_{i,1}(\nu(\theta))$. Therefore

$$\begin{aligned}
\overline{\Phi}(\nu(\theta)) &= \bigcap_{i,j=1}^{k+1} \overleftarrow{p}^{\,k+1}_{i,j}\big(\Phi(p^{k+1}_{i,j}(\nu(\theta)))\big) \\
&= \bigcap_{i,j=1}^{k} \overleftarrow{p}^{\,k+1}_{i,j}\big(\Phi(p^{k+1}_{i,j}(\nu(\theta)))\big) \cap \bigcap_{i=1}^{k+1} \overleftarrow{p}^{\,k+1}_{i,k+1}\big(\Phi(p^{k+1}_{i,k+1}(\nu(\theta)))\big) \\
&= \bigcap_{i,j=1}^{k} \overleftarrow{p}^{\,k+1}_{i,j}\big(\Phi(p^{k}_{i,j}(\theta))\big) = \bigcap_{i,j=1}^{k} \overleftarrow{p}^{\,k}_{i,j}\big(\Phi(p^{k}_{i,j}(\theta))\big) \times B \\
&= \nu(\overline{\Phi}(\theta)).
\end{aligned}$$

□

This theorem includes several similar results in the literature. We discuss three examples. An algebra $\mathbf{A}$ is called *congruence-primal* if $\mathrm{Clo}(\mathbf{A}) = \mathcal{F}(\mathrm{Con}(\mathbf{A}))$.

Corollary 2.4. (*Bergman-Berman* [BB]) *Let* $\mathbf{A}$ *be finite, congruence-primal and arithmetical. Then* $\mathbf{A} \equiv_{\mathrm{c}} \mathbf{B}$ *if and only if* $\mathbf{B}$ *is finite, congruence-primal, arithmetical and* $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{B})$.

Proof. Pixley [P2] proved that $\mathbf{A}$ will generate an arithmetical variety. Consequently, it will have a majority term. Notice that by congruence-primality, $\mathrm{Clo}(\mathbf{A})$ will contain every constant operation. Therefore, every nonempty member of $\mathrm{Sub}(\mathbf{A}^2)$ will contain δ. And therefore by congruence-permutability, $\mathrm{Sub}(\mathbf{A}^2) = \mathrm{Con}(\mathbf{A}) \cup \{\varnothing\}$ (see [BB, 2.4]). It follows that the structure $\mathcal{S}_2(\mathbf{A})$ and the lattice $\mathrm{Con}(\mathbf{A})$ are term-equivalent. The corollary now follows from Theorem 2.3. □

One of the most important notions in universal algebra is that of a quasi-primal algebra. Let $\mathrm{Iso}(\mathbf{A})$ denote the set of isomorphisms between subalgebras of $\mathbf{A}$. We consider Iso($\mathbf{A}$) to be a set of binary relations on A. This set forms an inverse semigroup, $\mathbf{Iso}(\mathbf{A})$, under composition and converse. $\mathbf{A}$ is called *quasi-primal* if A is finite and $\mathrm{Clo}(\mathbf{A}) = \mathcal{F}(\mathrm{Iso}(\mathbf{A}))$.

Corollary 2.5. (*Gierz* [G]) *Let* $\mathbf{A}$ *be quasi-primal. Then* $\mathbf{A} \equiv_{\mathrm{c}} \mathbf{B}$ *if and only if* $\mathbf{B}$ *is quasi-primal and* $\langle\, \mathbf{Iso}(\mathbf{A}), \mathrm{E}(\mathbf{A}^2)\,\rangle \cong \langle\, \mathbf{Iso}(\mathbf{B}), \mathrm{E}(\mathbf{B}^2)\,\rangle$.

Proof. Let $\Theta\colon \langle\, \mathbf{Iso}(\mathbf{A}), \mathrm{E}(\mathbf{A}^2)\,\rangle \to \langle\, \mathbf{Iso}(\mathbf{B}), \mathrm{E}(\mathbf{B}^2)\,\rangle$ be an isomorphism. Since $\mathbf{A}$ is quasi-primal, it generates an arithmetical variety and is hereditarily simple. Consequently, it has a majority term and, by Fleischer's Lemma, every subalgebra of $\mathbf{A}^2$ is either a member of Iso($\mathbf{A}$) or is of the form $C_1 \times C_2$ for $C_1, C_2 \in \mathrm{Sub}(\mathbf{A})$.

Now the idempotent members of $\mathbf{Iso}(\mathbf{A})$ are precisely the identity maps on the various subalgebras of $\mathbf{A}$. Using the notation from Theorem 2.3, the identity map on C is just δ_C. Notice that $C_1 \subseteq C_2 \iff \delta_{C_1} \circ \delta_{C_2} = \delta_{C_1}$. Thus Θ induces a lattice isomorphism of Sub($\mathbf{A}$) with Sub($\mathbf{B}$). Consequently, we can define a map $\bar\Theta\colon \mathrm{Sub}(\mathbf{A}^2) \to \mathrm{Sub}(\mathbf{B}^2)$ given by

$$\begin{aligned} \bar\Theta(C_1 \times C_2) &= D_1 \times D_2 \qquad && \text{where } \Theta(\delta_{C_i}) = \delta_{D_i}, \text{ for } i = 1, 2;\\ \bar\Theta(\alpha) &= \Theta(\alpha) && \text{for } \alpha \in \mathrm{Iso}(\mathbf{A}). \end{aligned}$$

That $\bar\Theta$ is a bijection relies on the fact that Θ maps $\mathrm{E}(\mathbf{A}^2)$ onto $\mathrm{E}(\mathbf{B}^2)$. It is now a straightforward matter to check that $\bar\Theta$ is an $\mathcal{S}_2$-homomorphism by checking several cases corresponding to the various possibilities for the shapes of the subalgebras of $\mathbf{A}^2$. For example, since $(C_1 \times C_2) \circ (D_1 \times D_2)$ is equal to $C_1 \times D_2$, if $C_2 \cap D_1 \neq \varnothing$ and is empty otherwise, we have $\bar\Theta((C_1 \times C_2) \circ (D_1 \times D_2)) = \bar\Theta(C_1 \times C_2) \circ \bar\Theta(D_1 \times D_2)$. □

Let D denote the variety of bounded distributive lattices. This variety is generated by the algebra $\mathbf{2} = \big\langle \{0,1\}, \wedge, \vee, 0, 1 \big\rangle$. Since it is a lattice, $\mathbf{2}$ has a majority term. It is easy to check that Sub($\mathbf{2}^2$) consists of five elements: $\varnothing$, δ, β, $\beta^{\smile}$, 2^2 where $\beta = \big\{(0,0),\ (0,1)\,(1,1)\big\}$. Notice that β is nothing but the lattice-ordering of $\mathbf{2}$.

An algebra $\mathbf{A}$ is called *lattice-primal* if there is a lattice ordering α on A such that $\mathrm{Clo}(\mathbf{A}) = \mathcal{F}(\alpha)$. The following corollary has been proved several times by different means. See [DR], [DW], [L2], [M] and [Q].

Corollary 2.6. *An algebra* $\mathbf{A}$ *is categorically equivalent to* $\mathbf{2}$ *if and only if* $\mathbf{A}$ *is finite and lattice-primal. A variety is categorically equivalent to* D *if and only if it is generated by a finite, lattice-primal algebra.*

Proof. The second statement follows by definition from the first. Suppose that $\mathbf{A} \equiv_{\mathrm{c}} \mathbf{2}$. By Theorem 2.3, $\mathbf{A}$ is finite, has a majority term and $\mathcal{S}_2(\mathbf{A}) \cong \mathcal{S}_2(\mathbf{2})$. Let α be the relation on $\mathbf{A}$ that maps to β under the isomorphism. Then from the corresponding facts for β we have

$$\alpha \cap \alpha^{\smile} = \delta, \qquad \alpha \circ \alpha = \alpha, \qquad \alpha \circ \alpha^{\smile} = \alpha^{\smile} \circ \alpha = A^2. \tag{2.3}$$

From the first two of these we deduce that α is a partial ordering of A, while the third implies that for any pair of elements x, y of A, the set $\{x, y\}$ has both an upper and a lower bound under the ordering α. Let us write $x\ \alpha\ z$ in place of $(x, z) \in \alpha$.

To show that α is a lattice-ordering, it suffices to show that if both u and v are upper bounds of $\{x, y\}$ then there is a point $w \in A$ such that $x, y \ \alpha \ w \ \alpha \ u, v$. Let m denote the majority term of $\mathbf{A}$, and set $w = m(x, y, u)$. Then

$$\begin{aligned} x = m(x,y,x) \ \alpha \ m(x,y,u) \ \alpha \ m(u,u,u) &= u \\ y = m(x,y,y) \ \alpha \ m(x,y,u) \ \alpha \ m(v,v,u) &= v. \end{aligned}$$

Finally, since $\mathcal{S}_2(\mathbf{A})$ is generated by α and $\mathcal{K}(\mathbf{A})$ is generated by the members of $\mathcal{S}_2(\mathbf{A})$, we conclude that $\mathbf{A}$ is lattice-primal.

Conversely, suppose that $\mathbf{A}$ is finite and lattice-primal. Then by assumption, there is a lattice-ordering α on A such that $\mathrm{Clo}(\mathbf{A}) = \mathcal{F}(\alpha)$. Let '$\wedge$' and '$\vee$' denote the meet and join operations on A associated with α. Since both of these operations preserve α, $\mathbf{A}$ will have a majority operation in its clone.

Now, for any pairs $\langle a, b \rangle$ and $\langle c, d \rangle$ in A^2, define the operation f by

$$f(x) = \begin{cases} c \wedge d & \text{if } \langle x, a \wedge b \rangle \in \alpha \\ c & \text{if } \langle x, a \rangle \in \alpha \text{ and } \langle x, a \wedge b \rangle \notin \alpha \\ d & \text{if } \langle x, b \rangle \in \alpha \text{ and } \langle x, a \wedge b \rangle \notin \alpha \\ 1 & \text{otherwise} \end{cases}$$

(where 1 is the α-largest element of A). One easily checks that $f \in \mathcal{F}(\alpha)$ and from this it follows that the only subalgebras of $\mathbf{A}^2$ are $\varnothing, \delta, \alpha, \alpha^{\smile}$ and A^2. Thus $\mathcal{S}_2(\mathbf{A})$ is generated by α and the relationships in display (2.3) hold. Therefore, $\mathcal{S}_2(\mathbf{A}) \cong \mathcal{S}_2(\mathbf{2})$, so by Theorem 2.3, $\mathbf{A} \equiv_{\mathrm{c}} \mathbf{2}$. $\square$

Remarks: 1. In [L2], Lüders showed that in Corollary 2.6, the fact that α is a lattice-order does not depend on the existence of a majority term.

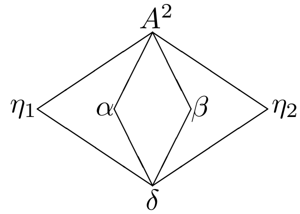
2. Theorem 2.2 has a generalization to algebras with a $(k+1)$-ary near-unanimity term. (The case $k = 2$ being that of a majority term.) J. Snow [S] has found an analogous generalization of Theorem 2.3.

Let us return for another look at Theorem 2.1. Suppose that A is a finite set and $\Theta \subseteq \mathrm{Rel}_1(A)$. Let $\mathbf{A} = \langle A, \mathcal{F}(\Theta) \rangle$. Then $\mathbf{A}$ is subalgebra-primal, so according to Theorems 1.1 and 2.1, $\mathcal{K}(\mathbf{A})$ is completely determined by the structure $\mathcal{S}_1(\mathbf{A}) = \langle \mathrm{Sub}(\mathbf{A}), \cap, E(\mathbf{A}) \rangle$. And of course, $\mathcal{K}(\mathbf{A})$ is the Krasner algebra generated by Θ. On the other hand, it follows from the Birkhoff-Frink Theorem (see [BF]) that $\mathrm{Sub}(\mathbf{A})$ is precisely the closure of Θ under arbitrary intersection. Put another way, the set of unary members of the Krasner algebra generated by Θ is equal to the $\mathcal{S}_1$-algebra generated by Θ.

Theorem 2.3 would seem to suggest an analogous situation for binary relations. Suppose now that $\Theta \subseteq \mathrm{Rel}_2(A)$ and that $\mathcal{F}(\Theta)$ contains a majority term. The Krasner algebra on $\mathbf{A} = \langle A, \mathcal{F}(\Theta) \rangle$ is generated by Θ, and is determined, up to isomorphism, by the $\mathcal{S}_2$-structure on $\mathrm{Sub}(\mathbf{A}^2)$. Is it true that $\mathrm{Sub}(\mathbf{A}^2)$ is equal to the $\mathcal{S}_2$-algebra generated by Θ? The answer is 'yes', as was shown by J. Snow [S] and, independently by L. Zadori. We can state this as the following "two-dimensional" version of Birkhoff-Frink.

Theorem 2.7. *Let Θ be an $\mathcal{S}_2$-structure on a finite set A, and assume that $\mathcal{F}(\Theta)$ contains a majority operation. Then there is an algebra $\mathbf{A} = \langle A, F \rangle$ such that $\mathrm{Sub}(\mathbf{A}^2) = \Theta$.*

Here is an example to demonstrate the necessity of a majority operation in the above theorem. Let $\mathbf{B}$ be a three-element group with universe $\{0,1,2\}$, and let $\mathbf{A} = \mathbf{B} \times \mathbf{B}$. Then $\mathrm{Clo}(\mathbf{A})$ contains a Maltsev term and $\mathrm{Con}(\mathbf{A})$ looks like



Each of the four intermediate congruences is the kernel of a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. η_i is the kernel of $(x_1, x_2) \mapsto x_i$ (for $i = 1, 2$), α is the kernel of $(x_1, x_2) \mapsto x_1 - x_2$ and β the kernel of $(x_1, x_2) \mapsto x_1 - 2x_2$.

Let $\Theta = \{\varnothing, \delta, \eta_1, \alpha, \eta_2, A^2\}$. (Note that $\beta \notin \Theta$.) Then Θ is an $\mathcal{S}_2$-structure on A. We will show that there is no algebra $\mathbf{A}' = \langle A, F\rangle$ such that $\mathrm{Sub}(\mathbf{A}'^2) = \Theta$.

Let $F = \mathcal{F}(\Theta)$ and $\mathbf{A}' = \langle A, F\rangle$. It suffices to show that $\beta \in \mathrm{Sub}(\mathbf{A}'^2)$. Since $\Theta \subseteq \mathcal{K}(\mathbf{A})$, we have $F \supseteq \mathrm{Clo}(\mathbf{A})$ and $\Theta \subseteq \mathcal{K}(\mathbf{A}') \subseteq \mathcal{K}(\mathbf{A})$. Consequently, F contains a Maltsev term and $\mathrm{Con}(\mathbf{A}')$ is either equal to $\mathrm{Con}(\mathbf{A})$ or to $\{\delta, \eta_1, \alpha, \eta_2, A^2\}$. In either case, $\mathrm{Con}(\mathbf{A}')$ is not distributive, so F certainly does not contain a majority term.

On the other hand, since $\mathrm{Con}(\mathbf{A}')$ does contain a "spanning $\mathbf{M}_3$", $\mathbf{A}'$ is an Abelian algebra in a Maltsev variety (see [MMT, 4.15.3]). Furthermore, $\mathbf{A}' \cong \mathbf{A}'/\eta_1 \times \mathbf{A}'/\eta_2$. Let $\mathbf{B}' = \mathbf{A}'/\eta_1 \cong \mathbf{A}'/\eta_2$. Then $\mathbf{B}'$ is also Abelian and furthermore, $\mathbf{B}'$ can be considered to be an expansion of $\mathbf{B} \cong \mathbf{A}/\eta_1$. Therefore, $\mathrm{Pol}_1(\mathbf{B}) \subseteq \mathrm{Pol}_1(\mathbf{B}')$, where Pol_1 denotes the set of unary polynomial operations.

It is well-known that $\mathrm{Pol}_1(\mathbf{B})$ has 9 elements. Let $f \in \mathrm{Pol}_1(\mathbf{B}')$ and let $g(x) = f(x) - f(0)$. From the Abelianness of $\mathbf{B}'$ we have $g(x+y) = g(x) + g(y)$. It follows that g is determined by the value of $g(1)$. Therefore f is determined by the values of $f(1)$ and $f(0)$, so $|\mathrm{Pol}_1(\mathbf{B}')| \leq 9$. We conclude that $\mathrm{Pol}_1(\mathbf{B}) = \mathrm{Pol}_1(\mathbf{B}')$, and therefore $\mathrm{Pol}_1(\mathbf{A}) = \mathrm{Pol}_1(\mathbf{A}')$. Since the congruences of $\mathbf{A}'$ are precisely those equivalence relations on A that lie in $\mathcal{R}(\mathrm{Pol}_1(\mathbf{A}'))$ (see [MMT, 4.18]), we have $\beta \in \mathrm{Con}(\mathbf{A}') \subseteq \mathrm{Sub}(\mathbf{A}'^2)$.

3. A modest catalog

In this section we provide a catalog of finite algebras $\mathbf{A}$ with a majority term and a very small $\mathcal{S}_2$-structure. By "small" we mean that $\mathcal{S}_2(\mathbf{A})$ has cardinality at most 5 and is generated (as an $\mathcal{S}_2$-structure) by a single relation ρ. According to Theorem 2.3, classifying $\mathbf{A}$ up to categorical equivalence is equivalent to classifying $\mathcal{S}_2(\mathbf{A})$ up to isomorphism.

We begin with some general observations. Then the analysis splits into several cases.

Lemma 3.1. *Let A_1 and A_2 be nonempty, disjoint sets, and let ρ_i be a k-ary relation on A_i containing at least one reflexive element, for $i = 1, 2$. If ρ_1 and ρ_2 each admit a majority operation, then so does $\rho_1 \cup \rho_2$.*

Proof. Let m_i be a majority operation on A_i compatible with ρ_i, for $i = 1, 2$. Fix an element $r_i \in A_i$ such that $\langle r_i, r_i, \ldots, r_i\rangle \in \rho_i$. We must define a majority operation on $A = A_1 \cup A_2$ that preserves $\rho = \rho_1 \cup \rho_2$. Do this as follows. Let $(a, b, c) \in A^3$.

At least two of these three elements must come from the same component of the partition of A. Without loss of generality, say that $a, b \in A_1$. Then define

$$m(a,b,c) = \begin{cases} m_1(a,b,c) & \text{if } c \in A_1 \\ m_1(a,b,r_1) & \text{if } c \in A_2. \end{cases}$$

Notice that if $a = b$, then $m(a,b,c) = m_1(a,a,x) = a$ (for x equal to c or to r_1). So it should be clear that m will be a majority operation. To see that m preserves ρ, let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be k-tuples in ρ. At least two of these, say $\mathbf{a}$ and $\mathbf{b}$, must come from (say) ρ_1. Then $\langle\, m(a_1,b_1,c_1),\dots,m(a_k,b_k,c_k)\,\rangle = \langle\, m_1(a_1,b_1,x_1),\dots,m_1(a_k,b_k,x_k)\,\rangle$, where $\mathbf{x} = \mathbf{c}$ if $\mathbf{c} \in \rho_1$, and $\mathbf{x} = \langle r_1,\dots,r_1\rangle$ otherwise. In either case, $\mathbf{x} \in \rho_1$. Since m_1 preserves ρ_1, $\langle\, m_1(a_1,b_1,x_1),\dots,m_1(a_k,b_k,x_k)\,\rangle \in \rho_1 \subseteq \rho$. $\square$

Definition 3.2. A proper binary relation θ on a finite set A is called *central* if it is reflexive, symmetric and has a nonempty center. By the center of θ we mean the set

$$Z(\theta) = \{\, z \in A : (\forall x \in A)\ \ \langle z,x\rangle,\, \langle x,z\rangle \in \theta \,\}.$$

By "θ is a proper relation on A" we mean that $\theta \subsetneq A^2$. Otherwise, θ is called *total.* The notion of a central relation can be defined for any rank, not just 2, see [R1]. These relations play an important role in the study of the lattice of clones on a finite set. The next lemma seems to have been discovered several times by several different people.

Lemma 3.3. *Let θ be a proper binary relation on a finite set A. Then θ is central if and only if θ admits a majority term, $\delta \subseteq \theta = \theta^{\smile}$ and $\theta \circ \theta = A^2$.*

Proof. That every central relation satisfies these conditions is easy to see. For the converse, the condition $\delta \subseteq \theta = \theta^{\smile}$ says that θ is reflexive and symmetric. We need to show that the center of θ is nonempty.

Let $|A| = n$ and let $\mathbf{A}$ be the algebra $\langle A, \mathcal{F}(\theta)\rangle$. Define

$$\psi = \{\, \langle x_1,\dots,x_n\rangle : (\exists z \in A)\ \langle x_i, z\rangle \in \theta \text{ for } i = 1,2,\dots,n \,\}.$$

It is easy to check that $\psi \in \mathcal{K}(\mathbf{A})$. (Either show directly from the operations in (1.1) that ψ is in the Krasner algebra generated by θ, or check that $\psi \in \mathcal{RF}(\theta)$.) Let $a, b \in A$. Since $\theta \circ \theta = A^2$, there is $z \in A$ such that $a\ \theta\ z\ \theta\ b$. Therefore $\langle a,b,b,b,\dots,b\rangle \in \psi$. Since a and b were arbitrary, we conclude that $p^n_{1,2}(\psi) = A^2$. By moving the 'a' to any position in the n-tuple we see that for every $1 \le i < j \le n$ we have $p^n_{i,j}(\psi) = A^2 = p^n_{i,j}(A^n)$. But by assumption, $\mathbf{A}$ has a majority term, so by Theorem 2.2(2), $\psi = A^n$. Finally, let $A = \{a_1,a_2,\dots,a_n\}$. Then $\langle a_1,a_2,\dots,a_n\rangle \in \psi$, so $Z(\theta) \neq \varnothing$. $\square$

In light of all of the results discussed in this paper, it is obviously desirable to know which binary relations admit a majority operation. In general, this seems to be quite difficult. However, two important cases have been fully analyzed, apparently several times.

Theorem 3.4. *Let ρ be a connected, symmetric, binary relation on a finite set A. If ρ is reflexive (irreflexive) then ρ admits a majority operation if and only if the relational structure $\langle A,\rho\rangle$ is a retract of a finite product of reflexive (irreflexive) paths.*

The characterization in the reflexive case was first obtained by Jawhari, Pouzet and Misane in [JPM]. For irreflexive relations, see Bandelt, [Ba]. Larose has a comprehensive treatment of the subject in [La].

Lemma 3.5. *Let ρ be an irreflexive, symmetric binary relation on a finite set A. If ρ contains a triangle (i.e. distinct points a, b, c such that $a\ \rho\ b\ \rho\ c\ \rho\ a$) then ρ does not admit a majority operation.*

Proof. One could easily derive this from Theorem 3.4, but we give a direct proof. Suppose that m is a majority term that preserves ρ. Let $m(a,b,c) = x$. Then

$$\begin{aligned} x = m(a,b,c)\ \rho\ m(b,a,a) &= a \\ x = m(a,b,c)\ \rho\ m(b,c,b)\ &= b \\ x = m(a,b,c)\ \rho\ m(c,c,a) &= c \end{aligned}$$

from which it follows that $x = m(a,b,c)\ \rho\ m(x,x,x) = x$, contradicting the fact that ρ is irreflexive. □

For the remainder of this section, let us assume that $\mathbf{A}$ is a finite nontrivial algebra with a majority term and that $|\mathcal{S}_2(\mathbf{A})| \leq 5$. Recall that $\{\varnothing, \delta, A^2\} \subseteq \mathcal{S}_2(\mathbf{A})$. We make a couple of simple observations on cardinality grounds.

Lemma 3.6. (1) *There is no $\psi \in \mathcal{S}_2(\mathbf{A})$ with $\varnothing \subset \psi \subset \delta$.*

(2) *If $\psi \in \mathcal{S}_2(\mathbf{A}) - \{\varnothing\}$, then $\psi \circ A^2 = A^2 \circ \psi = A^2$. Consequently, the domain and range of ψ are both equal to A.*

Proof. If $\varnothing \subset \psi \subset \delta$, then ψ is of the form δ_D for some subalgebra D of $\mathbf{A}$. But then ψ, $D \times A$ and $A \times D$ are members of $\mathcal{S}_2(\mathbf{A})$, which pushes its cardinality up to at least six.

Now let $\psi \in \mathcal{S}_2(\mathbf{A}) - \{\varnothing\}$. Then $\psi \circ A^2 = D \times A$ where $D = \pi(\psi)$ is the domain of ψ. If $D \neq A$, then $\delta_D \in \mathcal{S}_2(\mathbf{A})$, contradicting the previous paragraph. An analogous argument holds for the range of ψ. □

It is useful to recall several facts about the calculus of binary relations. All are easy to verify. Let ψ, λ and θ be binary relations on a set A. Then

$$\begin{aligned} &\psi \circ \delta = \delta \circ \psi = \psi \\ &\psi \circ \varnothing = \varnothing \circ \psi = \varnothing \\ &(\psi \circ \lambda)^{\smile} = \lambda^{\smile} \circ \psi^{\smile} \\ &\psi \subseteq \theta \implies \psi \circ \lambda \subseteq \theta \circ \lambda \text{ and } \psi^{\smile} \subseteq \theta^{\smile}. \end{aligned}$$

These facts together with Lemma 3.6 determine most of the structure of $\mathcal{S}_2(\mathbf{A})$ already.

We now assume further that $\mathcal{S}_2(\mathbf{A})$ is generated by the single relation ρ. It follows that $\mathrm{Clo}(\mathbf{A}) = \mathcal{F}(\rho)$. From here, the argument splits into multiple cases. We organize them first by the cardinality of $\mathcal{S}_2(\mathbf{A})$, next by the truth of the conditions '$\rho = \breve{\rho}$' and '$\delta \subset \rho$' and within that by the value of $\rho \circ \rho$.

If $|\mathcal{S}_2(\mathbf{A})| = 3$ then $\mathcal{S}_2(\mathbf{A}) = \{\varnothing, \delta, A^2\}$ and $\mathbf{A}$ is primal. It has been known for quite some time that any two primal algebras are categorically equivalent, see [H]. To be concrete, we could say that $\mathbf{A}$ is categorically equivalent to the two-element Boolean algebra.

Now suppose that $|\mathcal{S}_2(\mathbf{A})| = 4$ i.e., $\mathcal{S}_2(\mathbf{A}) = \{\varnothing, \delta, \rho, A^2\}$. Since $\varnothing, \delta, A^2$ are all symmetric, it follows that $\breve{\rho} = \rho$.

Case 4.1. $|\mathcal{S}_2(\mathbf{A})| = 4$, $\rho = \breve{\rho}$, $\delta \subset \rho$.

Then $\mathcal{S}_2(\mathbf{A})$ is a chain of length 3 and each element is symmetric. We have $\rho \circ \rho \supseteq \rho \circ \delta = \rho$, leaving two possibilities: $\rho \circ \rho = \rho$ and $\rho \circ \rho = A^2$.

Case 4.1.1. $|\mathcal{S}_2(\mathbf{A})| = 4$, $\rho = \breve{\rho}$, $\delta \subset \rho$, $\rho \circ \rho = \rho$.

In this case ρ is an equivalence relation on A. Since $\mathrm{Clo}(\mathbf{A}) = \mathcal{F}(\rho)$ and $\mathrm{Con}(\mathbf{A})$ is a chain, $\mathbf{A}$ is congruence-primal and arithmetical. The categorical equivalence of such an algebra was studied in detail in [BB]. Also, as discussed in [DL2], $\mathbf{A}$ is preprimal. $\mathbf{A}$ is categorically equivalent to the algebra $\langle \{0,1,2\}, \mathcal{F}(\psi) \rangle$, with ψ the equivalence relation generated by $\{\langle 0,1\rangle\}$. This is the smallest algebra in the categorical equivalence class. [DL1] has a different characterization of $\mathbf{A}$ up to categorical equivalence.

Case 4.1.2. $|\mathcal{S}_2(\mathbf{A})| = 4$, $\rho = \breve{\rho}$, $\delta \subset \rho$, $\rho \circ \rho = A^2$.

By Lemma 3.3, ρ is central and $\mathbf{A}$ is preprimal. Any two such algebras are categorically equivalent. The smallest representative is $\langle \{0,1,2\}, \mathcal{F}(\psi) \rangle$ in which $\psi = \{0,1,2\}^2 - \{\langle 1,2\rangle, \langle 2,1\rangle\}$. See [DL1] for a further description of this class.

Case 4.2. $|\mathcal{S}_2(\mathbf{A})| = 4$, $\rho = \breve{\rho}$, $\delta \not\subset \rho$.

Since ρ is distinct from both $\varnothing$ and δ, by Lemma 3.6 we obtain $\rho \cap \delta = \varnothing$. Thus, ρ is a symmetric relation containing pairs of distinct elements with domain and range equal to A. Therefore, $\rho \circ \rho \supseteq \delta$, leaving the possibilities that $\rho \circ \rho$ is equal to either δ or A^2.

Case 4.2.1. $|\mathcal{S}_2(\mathbf{A})| = 4$, $\rho = \breve{\rho}$, $\delta \not\subset \rho$, $\rho \circ \rho = \delta$.

The pair of equations $\rho \circ \rho = \delta$ and $\rho = \breve{\rho}$ requires that ρ be the graph of a function, in fact an involution. Since $\rho \cap \delta = \varnothing$, this involution has no fixed points. It follows that $\mathbf{A}$ is an automorphism-primal algebra, is preprimal and is categorically equivalent to the algebra $\langle \{0,1\}, \psi \rangle$, with $\psi = \{ \langle 0,1\rangle \langle 1,0\rangle \}$. See [BB] and [DL2] for a further discussion of this case.

Case 4.2.2. $|\mathcal{S}_2(\mathbf{A})| = 4$, $\rho = \breve{\rho}$, $\delta \not\subset \rho$, $\rho \circ \rho = A^2$.

Let $\langle a,c\rangle \in \rho$. Since $\rho \circ \rho = A^2$, there is $b \in A$ such that $a \mathrel{\rho} b \mathrel{\rho} c$. Thus $\{a,b,c\}$ forms a triangle. Therefore, by Lemma 3.5, ρ does not admit a majority term, so this case can not occur.

We now consider algebras $\mathbf{A}$ with $|\mathcal{S}_2(\mathbf{A})| = 5$. Suppose first that $\rho \neq \breve{\rho}$. Then $\mathcal{S}_2(\mathbf{A}) = \{\varnothing, \delta, \rho, \breve{\rho}, A^2\}$.

Case 5.1. $|\mathcal{S}_2(\mathbf{A})| = 5$, $\rho \neq \breve{\rho}$, $\delta \subset \rho$.

Then $\delta \subset \breve{\rho}$ as well, so we must have $\rho \cap \breve{\rho} = \delta$. Furthermore, $\rho \circ \breve{\rho} \supseteq \rho \circ \delta = \rho$ and $\rho \circ \breve{\rho} \supseteq \delta \circ \breve{\rho} = \breve{\rho}$. So $\rho \circ \breve{\rho} = A^2$. Similarly, $\rho \circ \rho \supseteq \rho \circ \delta = \rho$, so $\rho \circ \rho \in \{\rho, A^2\}$.

Case 5.1.1. $|\mathcal{S}_2(\mathbf{A})| = 5$, $\rho \neq \breve{\rho}$, $\delta \subset \rho$, $\rho \circ \rho = \rho$.

This is the situation of Corollary 2.6. $\mathbf{A}$ is lattice-primal and is categorically equivalent to the two-element bounded lattice.

Case 5.1.2. $|\mathcal{S}_2(\mathbf{A})| = 5$, $\rho \neq \breve{\rho}$, $\delta \subset \rho$, $\rho \circ \rho = A^2$.

Then ρ does not admit a majority term. To see this, suppose that m is a majority term compatible with ρ. Pick $\langle a, c\rangle \in \breve{\rho} - \rho$. So $a \neq c$. Since $\rho \circ \rho = A^2$, there is $b \in A$ with $a \;\rho\; b \;\rho\; c \;\rho\; a$. Let $m(a, b, c) = x$. Then

$$\begin{aligned} x &= m(a,b,c)\,\rho\, m(a,b,a) = a \\ a &= m(a,a,c)\,\rho\, m(a,b,c) = x. \end{aligned}$$

So $\langle a, x\rangle \in \rho \cap \breve{\rho} = \delta$, which means that $a = x$. But a similar argument shows that $c = x$, contradicting the fact that $a \neq c$.

Case 5.2. $|\mathcal{S}_2(\mathbf{A})| = 5$, $\rho \neq \breve{\rho}$, $\delta \not\subset \rho$.

As in Case 4.2, $\rho \cap \delta = \breve{\rho} \cap \delta = \varnothing$. Furthermore, since ρ and $\breve{\rho}$ are incomparable, we have $\rho \cap \breve{\rho} = \varnothing$. Suppose there were an element x of A such that $\langle x, x\rangle \in \rho \circ \rho$. Then for some $y \in A$, $\langle x, y\rangle \in \rho \cap \breve{\rho}$, which we have just argued is false. Therefore, $(\rho \circ \rho) \cap \delta = \varnothing$. Thus we have two possibilities: $\rho \circ \rho = \rho$ and $\rho \circ \rho = \breve{\rho}$.

Case 5.2.1. $|\mathcal{S}_2(\mathbf{A})| = 5$, $\rho \neq \breve{\rho}$, $\delta \not\subset \rho$, $\rho \circ \rho = \rho$.

The conditions $\rho \cap \delta = \rho \cap \breve{\rho} = \varnothing$ and $\rho \circ \rho \subseteq \rho$ mean that ρ is an irreflexive, antisymmetric, transitive relation on A; in other words, a strict order. However, we also have the opposite inclusion: $\rho \circ \rho \supseteq \rho$. This means that as an ordered set, $\langle A, \rho\rangle$ has no covering pairs. For if a is covered by b (i.e., $a \;\rho\; b$ and for no x do we have $a \;\rho\; x \;\rho\; b$) then $\langle a, b\rangle \in \rho - (\rho \circ \rho)$, a contradiction. However, since $\rho \neq \varnothing$, there is some pair of distinct elements $\langle a, b\rangle \in \rho$. Consequently, the interval from a to b is infinite, which is impossible.

Case 5.2.2. $|\mathcal{S}_2(\mathbf{A})| = 5$, $\rho \neq \breve{\rho}$, $\delta \not\subset \rho$, $\rho \circ \rho = \breve{\rho}$.

Then $\breve{\rho}\circ\breve{\rho} = \rho$ and $\rho\circ\breve{\rho} \supseteq \delta$, so $\rho\circ\breve{\rho} \in \{\delta, A^2\}$. But $\rho\circ(\rho\circ\breve{\rho}) = (\rho\circ\rho)\circ\breve{\rho} = \breve{\rho}\circ\breve{\rho} = \rho$, so $\rho \circ \breve{\rho} \neq A^2$. Therefore $\rho \circ \breve{\rho} = \delta = \breve{\rho} \circ \rho$. We conclude that ρ is a permutation of A with inverse $\breve{\rho} = \rho \circ \rho$, i.e., a permutation of order 3. Since $\rho \cap \delta = \varnothing$, ρ has no fixed points.

Therefore, $\mathbf{A}$ is automorphism-primal with $\mathrm{Aut}(\mathbf{A})$ cyclic of order 3 and having no fixed points. (In the terminology of [BB], $\mathbf{A}$ is Q-demi-primal.) $\mathbf{A}$ is preprimal, and is categorically equivalent to $\langle\, \{0, 1, 2\}, \mathcal{F}(f)\,\rangle$, where $f(x) = x + 1 \pmod 3$. This case is quite analogous to Case 4.2.1.

Returning to the discussion preceding Case 5.1, we now assume that $\rho = \breve{\rho}$. Since the cardinality is 5, we must have $\mathcal{S}_2(\mathbf{A}) = \{\varnothing, \delta, \rho, \theta, A^2\}$, where $\theta = \rho \circ \rho$. Note that $\theta^{\smile} = \breve{\rho} \circ \breve{\rho} = \rho \circ \rho = \theta$.

Case 5.3. $|\mathcal{S}_2(\mathbf{A})| = 5$, $\rho = \breve{\rho}$, $\delta \subset \rho$.

Then $\theta = \rho \circ \rho \supseteq \rho \circ \delta = \rho$, so $\mathcal{S}_2(\mathbf{A})$ is a chain of length 4. Computing further, $\theta \circ \rho \supseteq \theta \circ \delta = \theta$, so $\theta \circ \rho = \theta$ or $\theta \circ \rho = A^2$.

Case 5.3.1. $|\mathcal{S}_2(\mathbf{A})| = 5$, $\rho = \breve{\rho}$, $\delta \subset \rho$, $\theta \circ \rho = \theta$.

Then $\theta \circ \theta = \theta$, so θ is an equivalence relation on A. Since $\theta \neq A^2$, there are at least two equivalence classes. Let E be an equivalence class of θ and let $\rho' = \rho{\restriction_E}$. Then either $\rho' = E^2$ i.e., ρ' is the total relation, or $\rho' \subsetneq \rho' \circ \rho' = E^2$, in which case

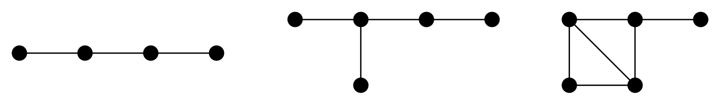


FIGURE 1

ρ' is central on E by Lemma 3.3. Finally, since $\rho \subset \theta$, there is at least one θ-class on which ρ is central.

Since these conditions seem to be new and interesting, we thought they deserved a name.

Definition 3.7. A relation ρ on a finite set A is *semi-central* if the relational structure $\langle A, \rho\rangle$ can be decomposed as a disjoint union of structures $\langle A_i, \rho_i\rangle$, for $i = 1, 2, \ldots, k$, such that for each $i \leq k$, ρ_i is central or total on A_i, and for at least one $i \leq k$, ρ_i is central on A_i.

For unary relations, "semi-central" and "central" coincide. For ranks greater than 1, a semi-central relation induces a clone of co-height 2 on its domain. If ρ is binary and semi-central on A as in the Definition, then $\rho \circ \rho = \theta$ is an equivalence relation on A and $A_1, A_2, \ldots, A_k$ are the equivalence classes. By Lemma 3.1, every semi-central relation has a compatible majority operation.

Returning to the algebra $\mathbf{A}$ of Case 5.3.1, ρ is semi-central, and $\mathbf{A}$ is categorically equivalent to an algebra $\mathbf{B}$ if and only if $\mathbf{B}$ is finite and $\mathrm{Clo}(\mathbf{B}) = \mathcal{F}(\psi)$ for some binary relation ψ that is semi-central but not central. The smallest such algebra is induced by the disjoint union of the 3-element central relation (Case 4.1.2) and a 1-element total relation.

Case 5.3.2. $|\mathcal{S}_2(\mathbf{A})| = 5$, $\rho = \breve{\rho}$, $\delta \subset \rho$, $\theta \circ \rho = A^2$.

We have $\theta \circ \theta = A^2$, so θ is central. The relational structure $\langle A, \rho\rangle$ can be thought of as a reflexive (undirected) graph of diameter 3. Such a relation does not necessarily admit a majority operation. For example, each of the relations in Figure 1 admits a majority operation, while those of Figure 2 do not. Also, neither a hexagon nor a heptagon admit a majority operation.

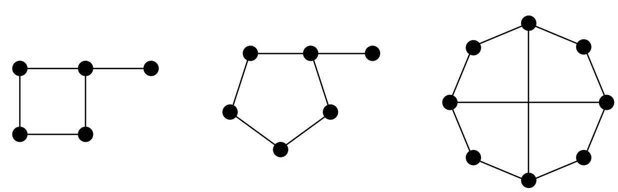


FIGURE 2

It follows from Theorem 3.4 that ρ admits a majority operation if and only if $\langle A, \rho\rangle$ is a retract of a finite power of a reflexive 4-element path. The smallest such algebra is $\mathbf{B} = \langle B, \mathcal{F}(\psi)\rangle$, where ψ is the first relation in Figure 1. Any algebra satisfying the conditions of this case and possessing a majority term will be categorically equivalent to $\mathbf{B}$. The clone of such an algebra has co-height 2.

Case 5.4. $|\mathcal{S}_2(\mathbf{A})| = 5$, $\rho = \breve{\rho}$, $\delta \not\subset \rho$.

Then $\rho \cap \delta = \varnothing$. Since $\rho = \breve{\rho}$, we get $\theta = \rho \circ \rho \supseteq \delta$. Recall that $\mathcal{S}_2(\mathbf{A})$ is assumed to consist of exactly the five relations $\varnothing$, δ, ρ, θ, A^2, and these relations are all distinct. Thus both $\rho \circ \theta$ and $\rho \cap \theta$ are in this list. Unfortunately, each of these two relations can have two possible values, leaving us with four more cases to investigate. The two possibilities for $\rho \cap \theta$ are ρ and $\varnothing$. Let us first assume that $\rho \cap \theta = \rho$, i.e., $\rho \subset \theta$. Then $\rho \circ \theta = \theta \circ \rho \supseteq \rho \circ \rho = \theta$, so $\rho \circ \theta = \theta$ or $\rho \circ \theta = A^2$.

Case 5.4.1. *$|\mathcal{S}_2(\mathbf{A})| = 5$, $\rho = \breve{\rho}$, $\delta \not\subset \rho$, $\rho \cap \theta = \rho$ and $\rho \circ \theta = \theta$.*

Thus $\rho \subset \theta$ and $\theta \circ \theta = \rho \circ \rho \circ \theta = \rho \circ \theta = \theta$, so θ is an equivalence relation. While it would be easy to attack this case directly, it might be more informative to reduce it to Case 4.2.2.

Let E be a nontrivial θ-class and $\rho' = \rho\restriction_E$. Then we have $\rho' \circ \rho' = E^2$, $(\rho')^{\smile} = \rho'$ and $\rho' \cap \delta_E = \varnothing$. If m is a majority operation on A compatible with ρ, then $m\restriction_E$ is a majority operation on E compatible with ρ'. Thus the algebra $\langle E, \mathcal{F}(\rho')\rangle$ is a witness to Case 4.2.2, which we have already determined to be impossible.

Case 5.4.2. *$|\mathcal{S}_2(\mathbf{A})| = 5$, $\rho = \breve{\rho}$, $\delta \not\subset \rho$, $\rho \cap \theta = \rho$ and $\rho \circ \theta = A^2$.*

The relation ρ is irreflexive and symmetric. Let $\langle a, c\rangle \in \rho$. Since $\rho \subseteq \theta = \rho \circ \rho$, there is $b \in A$ such that $a \; \rho \; b \; \rho \; c \; \rho \; a$. Therefore by Lemma 3.5, ρ does not admit a majority operation.

Now we suppose that $\rho \cap \theta = \varnothing$. Then $\rho \circ \theta \supseteq \rho \circ \delta = \rho$ gives us the possibilities: $\rho \circ \theta = \rho$ and $\rho \circ \theta = A^2$.

Case 5.5.1. *$|\mathcal{S}_2(\mathbf{A})| = 5$, $\rho = \breve{\rho}$, $\delta \not\subset \rho$, $\rho \cap \theta = \varnothing$ and $\rho \circ \theta = \rho$.*

We compute $\theta \circ \theta = \rho \circ (\rho \circ \theta) = \rho \circ \rho = \theta$, thus θ is an equivalence relation on A. Since $\rho \cap \theta = \varnothing$, $\rho\restriction_E = \varnothing$ for any θ-class E. Let $a \in A$. Since $\operatorname{dom}(\rho) = A$, there is some $b \in A$ such that $\langle a, b\rangle \in \rho$. Let E and F be the θ-classes of a and b respectively. Then $\theta \circ \rho \circ \theta = \rho$ implies that $E \times F \subseteq \rho$. Thus $\rho\restriction_{E \cup F}$ is a complete bipartite graph on $E \cup F$. Finally, suppose that E, F and G are three θ-classes, a, b, c are points of E, F, G respectively and $a \; \rho \; b \; \rho \; c$. Then $\langle a, c\rangle \in \rho \circ \rho = \theta$, so $E = G$. Thus $\langle A, \rho\rangle$ is the disjoint union of a collection of complete bipartite graphs.

Conversely, if $\langle A, \rho\rangle$ is a disjoint union of complete bipartite graphs, then ρ admits a majority operation. To see this, fix a representative for each θ-class. Let $\bar{x}$ denote the representative of x/θ. Then, for three distinct elements a, b, c of A define

$$m(a, b, c) = \begin{cases} \bar{b} & \text{if } b \; \theta \; c \\ \bar{a} & \text{otherwise.} \end{cases}$$

One can easily check that this can be expanded to a compatible majority operation. The smallest such algebra is $\langle \{0, 1, 2\}, \mathcal{F}(\psi) \rangle$, where ψ is the relation $\{\langle 0, 1\rangle, \langle 1, 0\rangle, \langle 0, 2\rangle, \langle 2, 0\rangle\}$.

Case 5.5.2. *$|\mathcal{S}_2(\mathbf{A})| = 5$, $\rho = \breve{\rho}$, $\delta \not\subset \rho$, $\rho \cap \theta = \varnothing$ and $\rho \circ \theta = A^2$.*

Then $\theta \circ \theta = A^2$, thus θ is central. Let $z \in Z(\theta)$ and $a \neq z$. Since z is central, $\langle a, z\rangle \in \theta = \rho \circ \rho$, so for some $b \in A$, $a \; \rho \; b \; \rho \; z$. But again, the centrality of z implies that $b \; \theta \; z$. Thus $\langle b, z\rangle \in \rho \cap \theta = \varnothing$, a contradiction.

We summarize our findings in the following theorem.

Theorem 3.8. *Let* **A** *be a finite algebra with a majority term, and suppose that* $\mathcal{S}_2(\mathbf{A})$ *is generated by the single relation* ρ*.*

(1) *If* $|\mathcal{S}_2(\mathbf{A})| = 3$ *then* **A** *is primal.* **A** *is categorically equivalent to a 2-element Boolean algebra.*
(2) *If* $|\mathcal{S}_2(\mathbf{A})| = 4$ *then one of the following holds.*
  (a) ρ *is an equivalence relation and* **A** *is categorically equivalent to a 3-element congruence-primal algebra. [Case 4.1.1]*
  (b) ρ *is central and* **A** *is categorically equivalent to a 3-element algebra of binary central type. [Case 4.1.2]*
  (c) ρ *is an involution with no fixed points and* **A** *is categorically equivalent to a 2-element automorphism-primal algebra. [Case 4.2.1]*
(3) *If* $|\mathcal{S}_2(\mathbf{A})| = 5$ *then one of the following holds.*
  (a) ρ *is a lattice order and* **A** *is categorically equivalent to a 2-element bounded lattice. [Case 5.1.1]*
  (b) ρ *is a permutation of order 3 with no fixed points, and* **A** *is categorically equivalent to a 3-element automorphism-primal algebra. [Case 5.2.2]*
  (c) ρ *is semi-central and* **A** *is categorically equivalent to a 4-element algebra induced by a relation that is semi-central but not central. [Case 5.3.1]*
  (d) ρ *is a reflexive graph of diameter 3 and* **A** *is categorically equivalent to a 4-element algebra induced by a path. [Case 5.3.2]*
  (e) ρ *is a disjoint union of complete bipartite graphs and* **A** *is categorically equivalent to a 3-element algebra induced by a complete bipartite graph.*

4. Arithmetical Varieties

A variety is called *arithmetical* if it is both congruence distributive and congruence permutable. It follows from [P1] that every arithmetical variety has a majority term. Thus Theorem 2.3 applies.

From now on, we will write function application on the right (and composition of functions will proceed left-to-right). Let **A** be an algebra, $\mathbf{A}_1$ and $\mathbf{A}_2$ subalgebras of **A**, α_i a congruence on $\mathbf{A}_i$ and $h\colon \mathbf{A}_1/\alpha_1 \to \mathbf{A}_2/\alpha_2$ an isomorphism. We set $h^{\square} = \{\, (x,y) \in A^2 : (x/\alpha_1)h = y/\alpha_2 \,\}$. Clearly $h^{\square}$ is a subalgebra of $\mathbf{A}^2$, called the *rectangular subalgebra of* $\mathbf{A}^2$ *induced by* h. It follows from Fleischer's Lemma (see [MMT, 4.74]) that if **A** lies in a Maltsev variety, then every nonempty subuniverse of $\mathbf{A}^2$ is of this form. This suggests that assertions involving $\mathcal{S}_2(\mathbf{A})$ can be reformulated in terms of isomorphisms between subquotients of **A**. Unfortunately, the notation needed to make this precise gets a bit burdensome, so we only sketch the details.

Let $f\colon \mathbf{A}_1/\alpha_1 \to \mathbf{A}_2/\alpha_2$ be an isomorphism. If **B** is a subalgebra of $\mathbf{A}_1$, then $\alpha_1{\restriction_B} = \alpha_1 \cap B^2$, and $f{\restriction_B}$ denotes the restriction of f to $B/(\alpha_1{\restriction_B})$. Also, $(B/\alpha_1{\restriction_B})^f = \{\, (b/\alpha_1)f : b \in B \,\}$. Similarly, if β_1 is a congruence on $\mathbf{A}_1$ and $\beta_1 \supseteq \alpha_1$, then

$$\beta_1^f = \big\{ \langle a,b\rangle \in A_2^2 : (\exists \langle x,y\rangle \in \beta_1)\ \langle\, (x/\alpha_1)f,\ (y/\alpha_1)f\,\rangle = \langle a/\alpha_2, b/\alpha_2\rangle \big\}.$$

Finally, $f_{\beta_1}\colon \mathbf{A}_1/\beta_1 \to \mathbf{A}_2/\beta_1^f$ is defined by $(x/\beta_1)f_{\beta_1} = y/\beta_1^f$, where $(x/\alpha_1)f = y/\alpha_2$. Note that f_{β_1} is again an isomorphism. The point of these definitions is that

$f{\restriction_B}$ and f_{β_1} are the unique maps that make the following two diagrams commute.

$$
\begin{array}{ccc}
\mathbf{A}_1/\alpha_1 & \xrightarrow{\;f\;} & \mathbf{A}_2/\alpha_2 \\
\big\uparrow & & \big\uparrow \\
\mathbf{B}/(\alpha_1{\restriction_B}) & \xrightarrow{\;f\restriction_B\;} & (\mathbf{B}/(\alpha_1{\restriction_B}))^f
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbf{A}_1/\alpha_1 & \xrightarrow{\;f\;} & \mathbf{A}_2/\alpha_2 \\
\big\downarrow & & \big\downarrow \\
\mathbf{A}_1/\beta_1 & \xrightarrow{\;f_{\beta_1}\;} & \mathbf{A}_2/\beta_1^f
\end{array}
$$

The vertical maps in these two pictures are the canonical embeddings and projections.

For the next two lemmas, let V be an arithmetical variety and $f\colon \mathbf{A}_1/\alpha_1 \to \mathbf{A}_2/\alpha_2$ and $g\colon \mathbf{B}_1/\beta_1 \to \mathbf{B}_2/\beta_2$ be isomorphisms, where $\mathbf{A}_i$ and $\mathbf{B}_i$ are subalgebras of $\mathbf{A}$, and α_i (respectively, β_i) are congruences on $\mathbf{A}_i$ (respectively, $\mathbf{B}_i$), for $i = 1, 2$.

Lemma 4.1. *$g^\square \subseteq f^\square$ if and only if $B_i \subseteq A_i$ and $\beta_i \subseteq \alpha_i{\restriction_{B_i}}$, for $i = 1, 2$; $(\alpha_1{\restriction_{B_1}})^g = \alpha_2{\restriction_{B_2}}$ and $g_{(\alpha_1\restriction_{B_1})} = f{\restriction_{B_1}}$. Put another way, $g^\square \subseteq f^\square$ if and only if there is a map that makes the following diagram commute.*

$$
\begin{array}{ccc}
\mathbf{A}_1/\alpha_1 & \xrightarrow{\;f\;} & \mathbf{A}_2/\alpha_2 \\
\big\uparrow & & \big\uparrow \\
\mathbf{B}_1/(\alpha_1{\restriction_{B_1}}) & \longrightarrow & \mathbf{B}_2/(\alpha_2{\restriction_{B_2}}) \\
\big\uparrow & & \big\uparrow \\
\mathbf{B}_1/\beta_1 & \xrightarrow{\;g\;} & \mathbf{B}_2/\beta_2
\end{array}
$$

Proof. Assume that $g^\square \subseteq f^\square$. We have $B_1 = \pi(g^\square) \subseteq \pi(f^\square) = A_1$. Let us suppose that $\langle a, b\rangle \in \beta_1$, and show $\langle a, b\rangle \in \alpha_1$. Since $a/\beta_1 = b/\beta_1$, there is some c such that $(a/\beta_1)g = (b/\beta_1)g = c/\beta_2$. Then $\langle a, c\rangle$, $\langle b, c\rangle \in g^\square \subseteq f^\square$. Hence $(a/\alpha_1)f = c/\alpha_2 = (b/\alpha_1)f$. Since f is injective, $\langle a, b\rangle \in \alpha_1$. A similar argument works for $\mathbf{B}_2$.

Let $\langle a', b'\rangle \in (\alpha_1{\restriction_{B_1}})^g$. Then $a', b' \in B_2$ and there is $\langle a, b\rangle \in \alpha_1$ so that $\langle\, (a/\beta_1)g,\, (b/\beta_1)g \,\rangle = \langle\, a'/\beta_2,\, b'/\beta_2 \,\rangle$. Thus $\langle a, a'\rangle, \langle b, b'\rangle \in g^\square \subseteq f^\square$. Therefore $a'/\alpha_2 = (a/\alpha_1)f = (b/\alpha_1)f = b'/\alpha_2$, so $\langle a', b'\rangle \in \alpha_2$. This shows that $(\alpha_1{\restriction_{B_1}})^g \subseteq \alpha_2{\restriction_{B_2}}$. For the opposite inclusion, just reverse this argument.

Finally, to see that $g_{(\alpha_1\restriction_{B_1})} = f{\restriction_{B_1}}$, let $\big(a/(\alpha_1{\restriction_B})\big)g_{(\alpha_1\restriction_{B_1})} = b/(\alpha_2{\restriction_{B_2}})$. Then $\langle a, b\rangle \in g^\square \subseteq f^\square$, and $a \in B_1$. Therefore, $(a/\alpha_1{\restriction_{B_1}})f{\restriction_{B_1}} = b/(\alpha_2{\restriction_{B_2}})$, as desired.

For the converse, it is easy to use the hypotheses to show that $g^\square \subseteq \big(g_{(\alpha_1\restriction_{B_1})}\big)^\square = (f{\restriction_{B_1}})^\square \subseteq f^\square$.

One consequence of this lemma is that every member of $\mathrm{Sub}(\mathbf{A}^2)$ has a unique representation in the form $f^\square$ for some isomorphism f.

Lemma 4.2. *Let $C = A_2 \cap B_1$ and $\gamma_2 = (\alpha_2{\restriction_C}) \vee (\beta_1{\restriction_C})$. Then $f^\square \circ g^\square = (f'_{\gamma_1} g'_{\gamma_3})^\square$, where $f' = f{\restriction_{C^{f^{-1}}}}$, $g' = g{\restriction_C}$, $\gamma_1 = {\gamma_2}^{f^{-1}}$ and $\gamma_3 = \gamma_2^g$. If $C = \varnothing$ then so is $f^\square \circ g^\square$.*

Proof. First suppose that $A_2 = B_1$. Let $\langle a, c\rangle \in f^\square \circ g^\square$. Then for some $b \in A_2$, $(a/\alpha_1)f = b/\alpha_2$ and $(b/\beta_1)g = c/\beta_2$. Note that $\gamma_2 = \alpha_2 \vee \beta_1$ is a congruence on $A_2 = B_1$, $\gamma_1 = {\gamma_2}^{f^{-1}} \supseteq \alpha_1$ and $\gamma_3 = \gamma_2^g \supseteq \beta_2$. Therefore $(a/\gamma_1)f_{\gamma_1} = b/(\gamma_2)$ and $(b/\gamma_2)g_{\gamma_2} = c/\gamma_3$. So $\langle a, c\rangle \in (f_{\gamma_1} g_{\gamma_3})^\square$. The converse is similar.

Now let A_2 and B_1 be arbitrary. It is easy to check that $f^{\Box} \circ g^{\Box} = (f')^{\Box} \circ (g')^{\Box}$. We can now apply the argument in the previous paragraph to f' and g' to derive the result. $\Box$

This lemma can also be described in terms of a commuting diagram. f'_{γ_1} and g'_{γ_3} are the unique maps that make the diagram in Figure 3 commute.

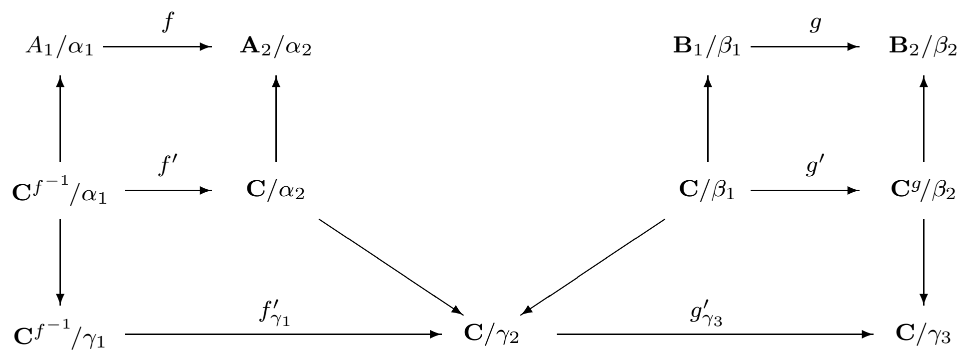


FIGURE 3

For a set X of algebras, let $\mathbf{Q}(\mathsf{X}) = \{\, \mathbf{A}/\alpha : \mathbf{A} \in \mathsf{X},\ \alpha \in \mathrm{Con}(\mathbf{A}) \,\}$. Note that in forming $\mathbf{Q}(\mathsf{X})$, we do not "close under isomorphism". Rather, we include just the actual quotients (although, of course, there may very well be isomorphic quotients). We will consider $\mathbf{Q}(\mathsf{X})$ as a full subcategory of $\mathbf{V}(\mathsf{X})$.

For example, let $\mathbf{A}$ be the three-element Heyting algebra $\langle \{0, e, 1\}, \wedge, \vee, \rightarrow, 0, 1 \rangle$, with $0 < e < 1$. See [MMT, page 181] for the definition. $\mathbf{A}$ has three congruences: δ_A, α (which identifies e and 1) and A^2. Thus $\mathbf{Q}(\mathbf{A})$ has three objects: $\mathbf{A}/\delta$ (which we identify with $\mathbf{A}$), $\mathbf{A}/\alpha$ and $\mathbf{A}/A^2$. Since none of these objects has a nontrivial automorphism, the only morphisms in this category are the three identity maps and the three canonical projections.

Let's make this example more interesting. $\mathbf{A}$ has one proper subalgebra, $\mathbf{A}'$, with universe $\{0, 1\}$. Note that $\mathbf{A}'$ is isomorphic to $\mathbf{A}/\alpha$. The algebra $\mathbf{A}'$ has one proper quotient, a trivial algebra, $\mathbf{A}'/(A')^2$. Neither $\mathbf{A}'$ nor $\mathbf{A}'/(A')^2$ has a nontrivial automorphism. Thus, the category $\mathbf{Q}(\{\mathbf{A}, \mathbf{A}'\})$ consists of five objects and 19 morphisms. (There is exactly one morphism from each object to each other object, except no morphisms from the trivial objects to the nontrivial objects.) The entire category $\mathbf{Q}(\{\mathbf{A}, \mathbf{A}'\})$ can be summarized by the diagram in Figure 4 (where the various identity maps are not shown).

Recall that an *isomorphism of the categories* C and D is a functor $F\colon \mathsf{C} \to \mathsf{D}$ such that for some functor $G\colon \mathsf{D} \to \mathsf{C}$, the composite functors $F \circ G$ and $G \circ F$ are identity functors. Every isomorphism of categories is an equivalence, but in general, the former notion is much stronger.

In the following theorem, we consider $\mathrm{Sub}(\mathbf{A})$ as the set of subalgebras of $\mathbf{A}$, rather than as subuniverses. If one does not wish to admit the empty algebra, simply extend the map obtained in the proof to one that maps the empty set to itself.

Theorem 4.3. *Let* $\mathbf{A}$ *be a finite algebra generating an arithmetical variety. Then* $\mathbf{A} \equiv_{\mathrm{c}} \mathbf{B}$ *if and only if* $\mathbf{B}$ *is finite, generates an arithmetical variety and the categories* $\mathbf{Q}(\mathrm{Sub}(\mathbf{A}))$ *and* $\mathbf{Q}(\mathrm{Sub}(\mathbf{B}))$ *are isomorphic.*

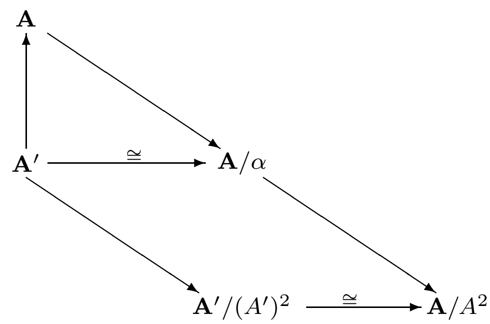


FIGURE 4

Proof. Since there is a one-to-one correspondence between the members of $\mathrm{Sub}(\mathbf{A}^2)$ and the various isomorphisms in $\mathbf{Q}(\mathrm{Sub}(\mathbf{A}))$, we have an induced bijection between $\mathrm{Sub}(\mathbf{A}^2)$ and $\mathrm{Sub}(\mathbf{B}^2)$. Since the isomorphism of categories carries commuting diagrams to commuting diagrams, Lemmas 4.1 and 4.2 imply that this bijection preserves intersection and composition of binary relations. Since $(f^{\square})^{\smile} = (f^{-1})^{\square}$, for any isomorphism f, converse is preserved as well. Thus we have an isomorphism of $\mathcal{S}_2(\mathbf{A})$ with $\mathcal{S}_2(\mathbf{B})$, so Theorem 2.3 implies $\mathbf{A} \equiv_{\mathrm{c}} \mathbf{B}$. □

Example. Let us continue with the example we began before Theorem 4.3. $\mathbf{A}$ is a three-element Heyting algebra and $\mathbf{Q}(\mathrm{Sub}(\mathbf{A}))$ is essentially described by Figure 4. Now let $\mathbf{B}$ be the complex algebra of a two-element semilattice. In other words, $\mathbf{B} = \langle \{0, a, b, 1\}, \wedge, \vee, {}', * \rangle$ is a Boolean algebra with atoms a, b and an additional binary operation that is associative, commutative, additive, and satisfies $(\forall x)\ x * 0 = 0$, $a*a = a*b = a$ and $b*b = b$ (see [Go] or [R]). Since $\mathbf{B}$ is an expansion of a Boolean algebra, it generates an arithmetical variety. $\mathbf{B}$ has exactly one proper subalgebra, $B' = \{0, 1\}$ and one proper, nontrivial congruence β (generated by $\langle 0, a\rangle$), with $\mathbf{B}/\beta \cong \mathbf{B}'$. Thus $\mathbf{Q}(\mathrm{Sub}(\mathbf{B}))$ is also characterized by Figure 4. From Theorem 4.3, we conclude that $\mathbf{A} \equiv_{\mathrm{c}} \mathbf{B}$.

As a rule, the category $\mathbf{Q}(\mathrm{Sub}(\mathbf{A}))$ is probably too complicated to analyze for a typical algebra. So we might look for additional hypotheses to make the structure more manageable. One possibility would be to assume that there are no nontrivial homomorphic images to contend with. That is, assume that $\mathbf{A}$ is finite, hereditarily simple and generates an arithmetical variety. These are precisely the quasi-primal algebras, and we wind up with yet another proof of Gierz' theorem, our Corollary 2.5. $\mathbf{Q}(\mathrm{Sub}(\mathbf{A}))$ contains nothing but the subalgebras of $\mathbf{A}$, together with a trivial algebra. The isomorphisms here form the inverse semigroup discussed in that corollary. The isomorphism of categories must preserve trivial algebras since they are the terminal objects in the category.

Instead of homomorphic images, we might try to avoid considering subalgebras. In other words, assume that $\mathbf{A}$ has no proper subalgebras. Then no quotient of $\mathbf{A}$ has a proper subalgebra either. We have the following immediate corollary.

Corollary 4.4. *Let* $\mathbf{A}$ *be finite, have no proper subalgebras and generate an arithmetical variety. Then* $\mathbf{A} \equiv_{\mathrm{c}} \mathbf{B}$ *if and only if* $\mathbf{B}$ *is finite, generates an arithmetical variety and the categories* $\mathbf{Q}(\mathbf{A})$ *and* $\mathbf{Q}(\mathbf{B})$ *are isomorphic.*

This corollary has an application to affine complete varieties. An algebra $\mathbf{A}$ is called *affine complete* if every member of $\mathcal{F}(\mathrm{Con}(\mathbf{A}))$ is induced by a polynomial of $\mathbf{A}$. A variety is affine complete if every member is affine complete. Recently, a great deal of progress has been made in the study of affine complete varieties. It is known that every affine complete variety is congruence distributive [KM]. Also, every locally finite affine complete variety has a $(k+1)$-ary near unanimity term for some k [K].

Suppose that V is affine complete, Maltsev, and of finite type. It follows from [KP1] that V is generated by a finite algebra $\mathbf{A}$ with no proper subalgebras. In order to characterize V as a category, it suffices to characterize $\mathbf{A}$ up to categorical equivalence, which we can do with Corollary 4.4. This provides an answer to Problem 4.1 of [P3].

There is another interesting aspect to this. As Proposition 4.5 shows, affine completeness is preserved by categorical equivalence. Let $\mathbf{A}$ be a finite algebra with no subalgebras and generating an arithmetical variety. Then the property of $\mathbf{A}$ being affine complete should somehow be reflected as a property of the category $\mathsf{Q}(\mathbf{A})$. It would be interesting to isolate this property explicitly.

For any algebra $\mathbf{A}$, let $\mathbf{A}^+$ denote the algebra $\mathbf{A}$ expanded to include a constant (unary) operation for each member of $\mathbf{A}$. $\mathbf{A}$ is called *functionally complete* if every operation on A is induced by a polynomial of $\mathbf{A}$.

Proposition 4.5. *Let $\mathbf{A}$ and $\mathbf{B}$ be algebras and suppose that $\mathbf{A} \equiv_c \mathbf{B}$. Then*

(1) $\mathbf{A}^+ \equiv_c \mathbf{B}^+$.
(2) *If $\mathbf{A}$ is affine complete, then so is $\mathbf{B}$.*
(3) *If $\mathbf{A}$ is functionally complete, then so is $\mathbf{B}$.*

Proof. The second and third assertions follow easily from the first. To see this, observe that $\mathbf{A}$ is affine complete if and only if $\mathbf{A}^+$ is congruence-primal. From [BB, Theorem 1.6], congruence-primality is preserved by categorical equivalence. Similarly, $\mathbf{A}$ is functionally complete if and only if it is simple and affine-complete, and both of these properties are preserved by categorical equivalence.

Now we prove (1). One way to do this is with a straightforward computation via McKenzie's theorem [M, Corollary 6.1]. We give a self-contained proof.

By assumption, there is an equivalence $F\colon \mathsf{V}(\mathbf{A}) \to \mathsf{V}(\mathbf{B})$ such that $F(\mathbf{A}) = \mathbf{B}$. Let f be a basic n-ary operation of $\mathbf{A}$, and let $a_1, a_2, \dots, a_n, b \in A$. The equality $b = f(a_1, a_2, \dots, a_n)$ can be expressed as an identity of $\mathsf{V}(\mathbf{A}^+)$. It follows that for each algebra $\mathbf{C}$ of $\mathsf{V}(\mathbf{A}^+)$ there is a unique homomorphism $h\colon \mathbf{A} \to \mathbf{C}^-$ mapping each element of A to its interpretation (as a constant operation) in $\mathbf{C}$. Here $\mathbf{C}^-$ is the reduct of $\mathbf{C}$ back to an element of $\mathsf{V}(\mathbf{A})$. Conversely, each such homomorphism $h\colon \mathbf{A} \to \mathbf{D}$ (where $\mathbf{D} \in \mathsf{V}(\mathbf{A})$) gives rise to a member of $\mathsf{V}(\mathbf{A}^+)$. In the language of category theory, we have an isomorphism of the category $\mathsf{V}(\mathbf{A}^+)$ and the category $\big(\mathbf{A} \downarrow \mathsf{V}(\mathbf{A})\big)$ of objects under $\mathbf{A}$. See [Mac, page 46]. A similar relationship holds for the categories $\mathsf{V}(\mathbf{B}^+)$ and $\big(\mathbf{B} \downarrow \mathsf{V}(\mathbf{B})\big)$.

We can now define a functor $F^+\colon \mathsf{V}(\mathbf{A}^+) \to \mathsf{V}(\mathbf{B}^+)$ as follows. Let $\mathbf{C} \in \mathsf{V}(\mathbf{A}^+)$, and identify $\mathbf{C}$ with $h\colon \mathbf{A} \to \mathbf{C}^-$. Then define $F^+(h\colon \mathbf{A} \to \mathbf{C}^-)$ to be $F(h)\colon \mathbf{B} \to F(\mathbf{C}^-)$. This describes a unique member of $\mathsf{V}(\mathbf{B}^+)$. For each homomorphism $g\colon \mathbf{C}_1 \to \mathbf{C}_2$, set $F^+(g) = F(g)$. The verification that F^+ is in fact an equivalence is left to the reader. $\square$

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