n-player bargaining based on preference intensity

Hyeon-soo Lee
Iowa State University

Follow this and additional works at: https://lib.dr.iastate.edu/rtd
Part of the Industrial Engineering Commons, and the Statistics and Probability Commons

Recommended Citation
Lee, Hyeon-soo, "n-player bargaining based on preference intensity " (1994). Retrospective Theses and Dissertations. 11283.
https://lib.dr.iastate.edu/rtd/11283

This Dissertation is brought to you for free and open access by the Iowa State University Capstones, Theses and Dissertations at Iowa State University Digital Repository. It has been accepted for inclusion in Retrospective Theses and Dissertations by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.
INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
313/761-4700  800/521-0600
n-player bargaining based on preference intensity

Lee, Hyeon-soo, Ph.D.
Iowa State University, 1994
n-player bargaining based on preference intensity

by

Hyeon-soo Lee

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Department: Industrial and Manufacturing Systems Engineering
Major: Industrial Engineering

Approved:
Signature was redacted for privacy.
In Charge of Major Work'
Signature was redacted for privacy.
For the Major Department
Signature was redacted for privacy.
For the Graduate College

Iowa State University
Ames, Iowa
1994

Copyright © Hyeon-soo Lee, 1994. All rights reserved.
# TABLE OF CONTENTS

**ACKNOWLEDGEMENTS** .............................................. vi

**CHAPTER 1. INTRODUCTION** ..................................... 1

1.1 Overview ..................................................... 1

1.2 Axiomatic Approach .......................................... 1

1.2.1 Nash solution ............................................. 2

1.2.2 Kalai-Smorodinsky solution ............................. 4

1.2.3 Egalitarian solution ................................... 5

1.2.4 Utilitarian solution ..................................... 6

1.2.5 Peters' solution ......................................... 7

1.3 Variable-Threat Approach ................................... 8

1.3.1 Nash's approach ......................................... 8

1.3.2 Owen's approach ........................................ 10

1.3.3 Harsanyi's approach .................................... 12

1.4 About This Dissertation .................................... 14

**CHAPTER 2. SPECIFIC BACKGROUND** ......................... 15

2.1 Stable Sets for n-person Cooperative Games ............... 15

2.2 Two-player Preference Intensity ............................ 19

2.2.1 Preference intensity ..................................... 19
2.2.2 Zeuthen's approach ............................................. 19
2.2.3 Akbar's approach ............................................. 21

CHAPTER 3. MY VIEW OF n-PLAYER BARGAINING ............ 23
3.1 Two-player Bargaining .......................................... 23
3.2 Offer Domination for n-player Bargaining Without Coalitions .... 25
3.3 Offer Domination for n-player Bargaining With Coalitions .... 26

CHAPTER 4. BARGAINING-STABLE SETS ......................... 30

CHAPTER 5. EXAMPLES .............................................. 32
5.1 Case of $n \geq 3$, $\alpha = 1$ and $a_i = 1$ .................. 33
5.2 Case of $n \geq 3$, $\alpha = 1$ and $a_i > 0$ ................. 37
5.3 Case of $n \geq 3$, $\alpha > 0$ and $a_i = 1$ ................. 37
5.4 Alternative Approach for the Case of $n = 3$, $\alpha = 1$ and $a_i = 1$ ... 43
5.5 Case of $\alpha \to \infty$ ........................................... 45

CHAPTER 6. COMPARISONS .......................................... 46
6.1 Von Neumann and Morgenstern Stable Set ................. 46
6.2 Shapley Value .................................................. 48
6.3 Shapley - Harsanyi Value .................................... 49

CHAPTER 7. ABSTRACT EXTERNAL BARGAINING STABILIT Y . ....... 53

CHAPTER 8. CONCLUSION ........................................... 56

BIBLIOGRAPHY ..................................................... 57
LIST OF TABLES

Table 6.1: A normal form structure for $n = 3$ .................. 51
Table 7.1: 5-offer game ............................................. 54
Table 7.2: 7-offer game ............................................. 55
LIST OF FIGURES

Figure 1.1: The Nash solution ........................................ 4
Figure 1.2: The Kalai-Smorodinsky solution ........................ 5
Figure 1.3: The Egalitarian solution ................................. 6
Figure 1.4: The Utilitarian solution .................................. 7
Figure 1.5: Nash solution with fixed threats ........................ 9
Figure 1.6: Variable-threat solution by Owen's approach ......... 11

Figure 2.1: A stable set $T$ for the three-person constant-sum game ... 18
Figure 2.2: A stable set $T_{3,k}$ for the three-person constant-sum game ... 18

Figure 3.1: Two player bargaining with $p$ number of offers ($p \geq 3$) ... 24

Figure 5.1: A bargaining-stable set for the case of $n = 3$, $\alpha = 1$ and $a_i > 0$ 38
Figure 5.2: A bargaining-stable set for the case of $n = 3$, $\alpha = 1$ and $a_i = 1$ 44
Figure 5.3: A bargaining-stable set for the case of $n = 3$, $\alpha = 2$ and $a_i = 1$ 45
Figure 5.4: A bargaining-stable set for the case of $n = 3$, $\alpha \to \infty$ . . . . 45
ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to Professor Herbert T. David for his eagerness, encouragement, enthusiasm, and untiring patience throughout the course of this research. My appreciation also goes to Professor William Meeker, Professor Roger Berger, Professor Howard Meeks, and Professor Jo Min for their help, time, and kindness in serving on my program of study committee. Finally, but most importantly, I would like to appreciate my family, Eun-jung, Sang-joon and Dong-joon, for their continuous support and encouragement during the studies in Iowa State University.
CHAPTER 1. INTRODUCTION

1.1 Overview

This introduction describes the two main traditions of research that have historically underlain work on bargaining: The axiomatic approach and the variable-threat approach. This introduction also describes this dissertation, and the approach to $n$-player bargaining that it presents.

1.2 Axiomatic Approach

The *axiomatic* approach to bargaining was first developed by Nash [17] for the case of $n = 2$ players, the case to which it largely remains focused to the present day. Nash was interested in predicting an outcome for any given bargaining situation, and proposed a set of axioms about the relationship of the predicted outcome to the set of feasible outcomes.

The axiomatic approach includes formulating axioms on how a problem can be solved and checking whether the axioms are compatible (i.e., whether there exist solutions satisfying them all). The primary task within this approach is to formulate the axioms.
1.2.1 Nash solution

According to Nash, a two-person bargaining problem is a triple \((S, d_1^*, d_2^*)\) where \(S\), the feasible set, is a subset of the two-dimensional Euclidean space, and \((d_1^*, d_2^*)\), a status quo or disagreement or threat point, taken by Nash to be the maximin pair, is a point of \(S\). Here, each point of \(S\) represents the utility levels, obtained by the two players through the choice of some joint action. If both players agree on a particular point of \(S\), then that point is what they get; however, if they fail to reach an agreement, each player gets \(d_1^*\) and \(d_2^*\), respectively. It is assumed that the set \(S\) is compact and convex and that there exists a point \(x = (x_1, x_2)\) of \(S\) strictly dominating \(d\), which means \(x_1 > d_1\) and \(x_2 > d_2\). Compactness includes closedness (contains its boundary) and boundedness (it is contained in some sphere of finite radius).

The assumption that \(S\) contains at least one point strictly dominating \(d\) is to guarantee that both players are non-trivially involved; i.e., both should have something to gain.

A bargaining solution \((d_1, d_2)\) will be a function of \((S, d_1^*, d_2^*)\):

\[
(d_1, d_2) = \varphi(S, d_1^*, d_2^*).
\]

The axioms formulated by Nash, for the two-player case, are as follows (where the symbol \(\leq\) denotes component-wise inequalities):

1. Individual Rationality: \((d_1, d_2) \geq (d_1^*, d_2^*)\).

2. Feasibility: \((d_1, d_2) \in S\).

3. Pareto-Optimality: If \((d_1, d_2) \in S\), and \((d_1, d_2) \geq (d_1, d_2)\), then \((d_1, d_2) = \)
4. Independence of Irrelevant Alternatives: If \((\tilde{d}_1, \tilde{d}_2) \in T \subset S\), and \((d_1, d_2) = \varphi(S, d_1^*, d_2^*)\), then \((\tilde{d}_1, \tilde{d}_2) = \varphi(T, d_1^*, d_2^*)\).

5. Independence of Linear Transformations: Let \(T\) be obtained from \(S\) by the linear transformation

\[
\begin{align*}
\tilde{d}_1 &= \alpha_1 \cdot d_1 + \beta_1, \\
\tilde{d}_2 &= \alpha_2 \cdot d_2 + \beta_2.
\end{align*}
\]

Then, if \(\varphi(S, d_1^*, d_2^*) = (\tilde{d}_1, \tilde{d}_2)\),

\[
\varphi(T, \alpha_1 \cdot d_1^* + \beta_1, \alpha_2 \cdot d_2^* + \beta_2) = (\alpha_1 \cdot \tilde{d}_1 + \beta_1, \alpha_2 \cdot \tilde{d}_2 + \beta_2).
\]

6. Symmetry: Suppose \(S\) is such that \((d_1, d_2) \in S \iff (d_2, d_1) \in S\).

Suppose also \(d_1^* = d_2^*\), and \(\varphi(S, d_1^*, d_2^*) = (\tilde{d}_1, \tilde{d}_2)\). Then,

\[
\tilde{d}_1 = \tilde{d}_2.
\]

Nash proposed the following solution, which will be denoted by \(\nu\):

Given \((S, d_1^*, d_2^*) \in R_2\), where \(R_2\) is two-dimensional Euclidean space, \(\nu(S, d_1^*, d_2^*)\) is the maximizer of the product \((x_1 - d_1^*) \cdot (x_2 - d_2^*)\) over the points \(x\) of \(S\) dominating \((d_1^*, d_2^*)\). According to Thomson and Lensberg, Nash's domain and his solution can directly be extended to the \(n\)-player case. The \(n\)-player solution associates with every \((S, d^*)\), \(d^* \in R_n\) the unique maximizer of the product \(\Pi(x_i - d_i^*)\) over the points \(x\) of \(S\) dominating \(d^*\).
The Nash solution for the two-player case, with each $d_i^* = 0$, is presented in Figure 1.1.

Classical solution concepts subsequent to Nash's do not explicitly involve the point $(d_1^*, d_2^*)$.

![Figure 1.1: The Nash solution](image)

1.2.2 Kalai-Smorodinsky solution

According to Thomson and Lensberg, a new impetus to the axiomatic theory of bargaining was given by Kalai and Smorodinsky [13].

- The **Kalai-Smorodinsky solution** $K$ is defined by setting, for all $S \in R_2$, $K(S)$ to be the maximal point of $S$ on the segment connecting the origin to $a(S)$, the ideal point of $S$, defined by $a_i(S) = \max\{x_i \mid x \in S\}$ for each $i$.

This solution has been studied primarily for the two-player case, as shown in Figure 1.2. A distinguishing feature between the Nash solution and the Kalai-
Smorodinsky solution is that the latter responds more satisfactorily to expansions and contractions of the feasible set. In addition to the axioms listed in the previous section, except that of independence of irrelevant alternatives, it satisfies the following axiom.

- **Individual monotonicity:** Consider $S$ and $S'$ in $R_2$, with $S \subset S'$ such that $a_2(S) = a_2(S')$ (respectively, $a_1(S) = a_1(S')$). Then $\varphi_1(S) \leq \varphi_1(S')$ (respectively, $\varphi_2(S) \leq \varphi_2(S')$).

![Figure 1.2: The Kalai-Smorodinsky solution](image)

1.2.3 Egalitarian solution

A third solution, namely the Egalitarian solution, has as its main feature distinguishing it from the previous two solutions the feature that it involves interpersonal comparisons of utility. This solution was suggested by Kalai, and is illustrated in Figure 1.3.
• The *Egalitarian solution* \( E \) is defined by setting, for all \( S \in R_2 \), \( E(S) \) to be the maximal point of \( S \) of equal coordinates.

The distinguishing feature of this solution is that it satisfies the following monotonicity condition, which involves no restrictions on the expansions that take \( S \) into \( S' \).

- **Strong monotonicity**: For all \( S, S' \in R_2 \), if \( S \subset S' \), then \( \varphi(S) \leq \varphi(S') \).

![Diagram of the Egalitarian solution](image)

**Figure 1.3**: The Egalitarian solution

### 1.2.4 Utilitarian solution

A fourth solution, namely the Utilitarian solution characterized by Myerson [16], has played a basic role in the theory of social choice but a marginal role in bargaining theory.
The Utilitarian solution, illustrated in Figure 1.4, is achieved by maximizing the sum of utilities over the feasible set. In the two-person case, when there exist more than one point maximizing the sum of utilities, selecting the midpoint of the maximizers may be a natural choice. However, for the case of \( n > 2 \), there is no equally natural choice. Also, the Utilitarian solution does not satisfy the axiom of independence of irrelevant alternatives.

\begin{figure}
  \centering
  \includegraphics[width=0.5\textwidth]{utilitarian_solution.png}
  \caption{The Utilitarian solution}
\end{figure}

1.2.5 Peters’ solution

Peters [21] is concerned almost entirely with the case of two players. He considered two other approaches to bargaining: namely, multisolution (multivalued bargaining solution) and probabilistic solution. The multisolution assigns a set of outcomes, not one outcome, to a bargaining game, and are considered with an independence of irrelevant alternatives property and with restricted monotonicity property. The
probabilistic solution assigns a probability measure, instead of one fixed outcome, to the subsets of outcomes of a bargaining game. The multisolution concept is taken up in this dissertation to treat the case of \( n \) players, albeit in a sense different from Peters'.

### 1.3 Variable-Threat Approach

The variable-threat approach also was pioneered by Nash [18]. Nash extended his previous work on *The Bargaining Problem* [17] to a situation in which threats can play a role, the so-called variable-threat approach. In this section, we will briefly describe Nash’s version of the variable-threat approach, followed by our description of Owen’s [20] additional contributions, followed in turn by our description of Harsanyi’s [10] further enhancement of the theory.

#### 1.3.1 Nash’s approach

Nash was interested in situations involving two players whose interests are neither completely coincident nor completely opposed.

1.3.1.1 Fixed-threat approach The fixed threat approach is relatively simple, and does not require an underlying bi-matrix normal form. This approach, illustrated in Figure 1.5, deals with a compact “prospect space” \( B \) that need not be convex, but is usually taken to be such, and a given “threat pair” \( (t_1, t_2) \). In this context, Nash suggests a certain behavioral considerations that leads to a bargaining payoff vector \( u = (u_1, u_2) \in B \) by a “hyperbolic construction”. That construction identifies \( u \) as that point in the boundary of \( B \) that maximizing the function (\( u_1 – \)
Figure 1.5: Nash solution with fixed threats

\[ t_1 \left( u_2 - u_2 \right), \text{subject to } u_i > t_i. \]

1.3.1.2 Variable-threat approach  Unlike the fixed threat approach, this approach requires an underlying bi-matrix normal form, and is now illustrated for the 2x2 case. To begin with, one constructs \( B \) as the convex hull \( (CH) \) of the four points \((K_{1,lm}, K_{2,lm})\):

\[
CH : \{(x, y) | (x, y) = \theta_{11} \cdot (K_{1,11}, K_{2,11}) + \theta_{12} \cdot (K_{1,12}, K_{2,12})
+ \theta_{21} \cdot (K_{1,21}, K_{2,21}) + \theta_{22} \cdot (K_{1,22}, K_{2,22})\}
\]

where the four \( \theta_{lm} \) constitute an arbitrary set of four convex weights, and where \( K_{i,lm} \) denotes the element in \( l \)th row and \( m \)th column of the 2x2 payoff matrix for player \( i \). The rationale for \( CH \) is that we are modeling a cooperative game, where mixtures are chosen jointly.
The possible threat pairs \((t_1, t_2)\) are imagined to be those points of \(CH\) that can be generated by independent mixing. The set of such possible threat pairs is here denoted by \(C\):

\[
C : \{ (t_1, t_2) \mid (t_1, t_2) = (t_1(\theta^1, \theta^2), t_2(\theta^1, \theta^2)) \}
= \theta_1^1 \cdot \theta_1^2 \cdot (K_{11,11}, K_{21,21}) + \theta_1^1 \cdot \theta_2^2 \cdot (K_{11,12}, K_{21,22})
+ \theta_2^1 \cdot \theta_1^2 \cdot (K_{12,11}, K_{22,21}) + \theta_2^1 \cdot \theta_2^2 \cdot (K_{12,12}, K_{22,22})
\]

where \((\theta^1, \theta^2)\) are called a threat strategy pair.

\(C\) typically is strictly contained in \(B\), and may not be convex. Of course, for every possible threat strategy pair \((\theta^1, \theta^2)\) in \(C\) we can develop a payoff pair \((U_1(\theta^1, \theta^2), U_2(\theta^1, \theta^2))\), by applying the fixed threat hyperbolic solution to the threat pair \((t_1(\theta^1, \theta^2), t_2(\theta^1, \theta^2))\).

We now treat the parameterized pair \((U_1(\theta^1, \theta^2), U_2(\theta^1, \theta^2))\) as a bi-kernel game. Nash argues that bi-kernel game is "almost" like an ordinary extension of a matrix game, in that all of its equilibrium points are "maximin" points of equal payoff.

1.3.2 Owen's approach

Consider then the bi-kernel game \((U_1(\theta^1, \theta^2), U_2(\theta^1, \theta^2))\). According to Owen [20], computing optimal threat strategies generally is complicated since the arbitration value corresponding to a pair of threat strategies depends not only on \(t_1(\theta^1, \theta^2)\) and \(t_2(\theta^1, \theta^2)\), but also on the form of the Pareto-optimal (northeastern) boundary of the set \(B\). Owen argued that since all that is required of \(B\) is that it be convex, there is, in general, no obvious method of solution.

However, Owen pointed out that, the problem becomes simple if the northeastern
boundary of $CH$ is linear; i.e., has equation
\[ H(x, y) = a_1 \cdot x + a_2 \cdot y - k = 0, \quad (1.1) \]
as shown in Figure 1.6.

If we let $(u_1, u_2)$ be the solution point, the slope of the line segment connecting $(t_1, t_2)$ and $(u_1, u_2)$ should be the negative of the slope of the relation (1.1), hence
\[ \frac{y - t_2}{x - t_1} = \frac{a_1}{a_2} \]
which can be rewritten as
\[ a_1 \cdot (x - t_1) = a_2 \cdot (y - t_2). \quad (1.2) \]

Then, we are able to find $(u_1, u_2)$ from the relations (1.1) and (1.2), yielding

\[ a_1 \cdot x + a_2 \cdot y - k = 0 \]

Figure 1.6: Variable-threat solution by Owen's approach
\[ u_1 = \frac{t_1 - \frac{a_2}{a_1} \cdot t_2 + k}{2} \]

and

\[ u_2 = \frac{t_2 - \frac{a_1}{a_2} \cdot t_1 + k}{2} \]

So, in effect, player I's payoffs, as functions of the players' threat strategies, in effect are given by

\[
M_1(\theta^1, \theta^2) = \theta^1_1 \cdot \theta^2_1 \cdot (K_{1,11} - \frac{a_2}{a_1} \cdot K_{2,11}) \\
+ \theta^1_1 \cdot \theta^2_2 \cdot (K_{1,12} - \frac{a_2}{a_1} \cdot K_{2,12}) \\
+ \theta^1_2 \cdot \theta^2_1 \cdot (K_{1,21} - \frac{a_2}{a_1} \cdot K_{2,21}) \\
+ \theta^1_2 \cdot \theta^2_2 \cdot (K_{1,22} - \frac{a_2}{a_1} \cdot K_{2,22}), \tag{1.3}
\]

and player II's payoffs are in effect given by the negative of (1.3), multiplied by \( \frac{a_1}{a_2} \).

Hence, the optimal threat strategies simply are derivable by treating (1.3) as an ordinary extension of the matrix game with matrix

\[ K_{l,m} = \frac{a_2}{a_1} \cdot K_{l,m} \quad \text{for} \quad l, m = 1, 2. \]

1.3.3 Harsanyi's approach

We have seen so far that Nash invented the idea that the variable-threat game is "almost" a probabilistic extension of a zero-sum game, and that Owen showed that, with a linear northeast boundary of \( CH \), the variable-threat game is solvable as the probabilistic extension of a zero-sum game. Harsanyi's contribution is that the
variable-threat game in the case of a general convex compact $CH$ can be iteratively solved, essentially, by solving a sequence of extended zero-sum games. So let the northeast boundary of $CH$ be given by

$$H(x, y) = 0,$$

and let $H_1$ and $H_2$ be the first derivatives of the function $H$ with respect to $x$ and $y$, respectively. Harsanyi proposed a solution that is implementable recursively in the following way, beginning with a candidate optimal payoff pair $(u_1, u_2)$:

1. Pick a point $(u_1, u_2)$ on the northeast boundary.
2. Compute $(a_1, a_2)$, where $a_i = H_i(u_1, u_2)$.
3. Solve the zero-sum game with kernel $M_1(\theta^1, \theta^2)$, obtaining $(\theta_0^1, \theta_0^2)$.
4. Compute $t_1(\theta_0^1, \theta_0^2)$ and $t_2(\theta_0^1, \theta_0^2)$.
5. Check whether

$$a_1 \cdot (u_1 - t_1(\theta_0^1, \theta_0^2)) - a_2 \cdot (u_2 - t_2(\theta_0^1, \theta_0^2)) = 0. \quad (1.4)$$

6. If (1.4) holds, then $u_1$ and $u_2$ constitute the solution. If not, try another point $(u_1, u_2)$, in keeping with the sign of the left hand side of (1.4).

Harsanyi also developed a complex model for the $n$-player cooperative game based on the two-player case, reducing the $n$-player cooperative game to combinations of two-player subgames between coalitions.
1.4 About This Dissertation

The organization of this dissertation is as follows:

In Chapter 2, ordinary stable sets for n-person cooperative games will be described.

Chapter 3 deals with my view of an n-player bargaining with and without coalitions, based on the concept of preference intensity.

Chapter 4 describes the definition of bargaining-stable sets.

Chapter 5 explores examples of bargaining-stable sets for various cases.

Chapter 6 compares my bargaining-stable sets with von Neumann - Morgenstern stable sets, the Shapley value and the Shapley - Harsanyi value.

Chapter 7 discusses certain aspects of external bargaining-stability.

Chapter 8 presents conclusions.

As indicated in this introduction, there is little consensus on whether and how bargaining theory is naturally extendable to the case of n players. In sum, this dissertation proposes one further approach to n-player bargaining, combining the classical idea of a stable set with the idea of preference intensity, and verifies its implementability.
CHAPTER 2. SPECIFIC BACKGROUND

2.1 Stable Sets for n-person Cooperative Games

An important solution concept for n-player cooperative games involves a set of payoff vectors \( x = (x_1, x_2, \ldots, x_n) \) that might possibly result as a consequence of playing the game. Suggested "solutions" to the game are small subsets of such a set, or even particular compelling members of the set.

The classical concept for the set of possible payoff vectors is von Neumann and Morgenstern's concept of a set of imputations, and among their classical solution ideas is the notion of a stable set. Both concepts assume the possibilities of side-payments among coalition members, so that it is meaningful to talk about coalition guarantees - indeed, about cumulative (i.e., summed over coalition members) coalition payoffs.

According to von Neumann and Morgenstern's definition, an imputation, for an n-person cooperative game, is a vector \( x = (x_1, x_2, \ldots, x_n) \), where \( x_i \) is the amount imputed to player \( i \). The imputations will satisfy the following two conditions:

Firstly, no player will agree to receive less than he can obtain individually by playing independently of the other players. Hence, it is required that

\[
x_i \geq v(i) \quad \text{for all } i \in N,
\]

where \( v(i) \) is the amount that player \( i \) can guarantee for him/herself, and where \( N \)
denotes the set of all players.

Secondly, it is required that

$$\sum_{i \in N} x_i = v(N),$$

where $v(N)$ is the cumulative payoff (summed over all players) that the coalition of all players can guarantee for itself.

Let $x$ and $y$ be two imputations in a game, and suppose that the players are faced with a choice between $x$ and $y$. Then it is interesting to find a criterion for comparing $x$ to $y$. Unless $x = y$, it is clear that some players will prefer $x$ to $y$ while others prefer $y$ to $x$. An important consideration, then, is the question whether the players preferring $x$ to $y$ are strong enough, as a coalition, to enforce the choice of $x$.

If we let $x$ and $y$ be two imputations and $C$ be a coalition, then it is said that $x$ dominates $y$ through $C$, $C \subseteq N$,

$$\sum_{i \in C} x_i \leq v(C)$$

and

$$x_i > y_i \text{ for all } i \in C$$

where $v(C)$ is the total amount that the members of $C$ can guarantee for themselves, whatever the remaining players may do.

Condition (2.1), called the effectiveness condition, expresses the fact that the coalition $C$ is truly capable of obtaining what the imputation $x$ gives it collectively. And condition (2.2), called preferability condition, expresses the fact that all members of coalition $C$ prefer $x$ to $y$. Finally, $x$ is said to dominate $y$ if $x$ dominates $y$ through some $C$. 
With the definitions mentioned above, a set $T$ of imputations is defined to be *stable* if

1. For $x, y \in T$, $x$ does not dominate $y$.

2. For any $x$ not in $T$, there is a $y \in T$ such that $y$ dominates $x$.

The first condition says that no imputation in a stable set $T$ dominates another imputation in $T$; this is known as *internal stability*. The second condition says that any imputation which is not in the stable set $T$ is dominated by some imputation in $T$; this is known as *external stability*.

The following example, taken from Owen [20], shows two specific kinds of stable sets for a three player game.

(Example) Considering the constant-sum three person game in $(0,1)$ normalization, for which the imputations form the simplex $x_1 + x_2 + x_3 = 1$, $x_i \geq 0$, with $u(1)$, $v(2)$, $v(3)$ all equal to zero, and $u(\{1,2\})$, $v(\{1,3\})$, $v(\{2,3\})$ and $u(\{1,2,3\})$ all equal to one. Then, the set

$$ T = \left\{ \left( \frac{1}{2}, \frac{1}{2}, 0 \right), \left( \frac{1}{2}, 0, \frac{1}{2} \right), \left( 0, \frac{1}{2}, \frac{1}{2} \right) \right\} \quad (2.3) $$

and the sets

$$ T_{1,k} = \{(k, x_2, 1-k-x_2) \mid 0 \leq x_2 \leq 1-k\}, \quad \text{for } k \in [0, \frac{1}{2}), \quad \text{and} $$

$$ T_{2,k} = \{(1-k-x_3, k, x_3) \mid 0 \leq x_3 \leq 1-k\}, \quad \text{for } k \in [0, \frac{1}{2}), \quad \text{and} $$

$$ T_{3,k} = \{(x_1, 1-k-x_1, k) \mid 0 \leq x_1 \leq 1-k\}, \quad \text{for } k \in [0, \frac{1}{2}), $$

shown in Figure 2.1 and Figure 2.2, are stable sets.
Figure 2.1: A stable set $T$ for the three-person constant-sum game

Figure 2.2: A stable set $T_{3,k}$ for the three-person constant-sum game
2.2 Two-player Preference Intensity

2.2.1 Preference intensity

The concept of the two-player preference intensity was first introduced by Zeuthen [28]. Zeuthen's approach is based on a direct analysis of the process of collective bargaining on the labor market. However, his approach has general validity for any kind of bargaining situation. Akbar [1] introduced an algebraic version of preference intensity that is the basis of the n-player formulation in this dissertation.

2.2.2 Zeuthen's approach

Zeuthen's concept, which essentially was formulated using zero threat values, may be discussed as follows:

If both players bargain for the best terms, they may be expected to have their own favorable terms that could be accepted or rejected by their opponents.

Suppose player I would like to gain $A_1$, but has been offered the less favorable terms $A_2$ by player II. The question here is: should he/she insist on obtaining $A_1$ or should he/she accept $A_2$? Let $U_1(A_1)$ and $U_1(A_2)$ be the net utility gains over the conflict situation that player I would derive from the terms $A_1$ and $A_2$, respectively. Then we see, by assumption, that $U_1(A_1) > U_1(A_2)$. And let $p_2$ be the probability that player II would reject the term $A_1$. Then, if player I accepts $A_2$, he will obtain $U_1(A_2)$, while if he rejects $A_2$ and insists on the better terms $A_1$ he will have the probability $(1 - p_2)$ of obtaining the higher utility $U_1(A_1)$ and the probability $p_2$ of obtaining nothing.

By assumption, player I tries to maximize his expected utility; hence, he will
accept the term $A_2$ if

$$U_1(A_2) > (1 - p_2) \cdot U_1(A_1);$$

that is, if

$$\frac{U_1(A_1) - U_1(A_2)}{U_1(A_1)} < p_2$$

or, simply

$$\frac{\Delta U_1}{U_1} < p_2$$

and will reject $A_2$ and insist on the better terms $A_1$ in the opposite case. The utility quotient $\Delta U_1/U_1$ represents the maximum risk that player I is prepared to face in order to secure the better terms $A_1$ instead of the less favorable term $A_2$.

Meanwhile, player II is faced with the analogous situation, and will accept the terms $A_1$ if

$$U_2(A_1) > (1 - p_1) \cdot U_2(A_2);$$

that is, if

$$\frac{U_2(A_2) - U_2(A_1)}{U_2(A_2)} < p_1$$

or, simply

$$\frac{\Delta U_2}{U_2} < p_1.$$

Thus the two utility quotients, $\Delta U_1/U_1$ and $\Delta U_2/U_2$, decide the strength of each players’ determination to insist on their respective favorable terms.
Zeuthen introduced the further assumption that player I will always make a concession to player II if

\[
\frac{\Delta U_1}{U_1} < \frac{\Delta U_2}{U_2}
\]  

(2.4)

Meanwhile, player II will make a concession in the opposite case.

In effect, inequality (2.4) can also be written in the form:

\[
U_1(A_1) \cdot U_2(A_1) < U_1(A_2) \cdot U_2(A_2)
\]

Thus, the two players will arrive at terms maximizing the value of \( U_1 \cdot U_2 \), which is exactly Nash's solution.

### 2.2.3 Akbar's approach

As shown above, most approaches (J.R. Hicks [11], F. Zeuthen [28], John C. Harsanyi [7], John Nash [17]) to the two-player bargaining problem point to the following "hyperbolic" solution: Of two possible payoff pairs \( A_1 \) and \( A_2 \), with \( U_1(A_i) \) for player I, and \( U_2(A_i) \) for player II, that pair will obtain, following bargaining, for which the product \( (U_1(A_i) - v_1)(U_2(A_i) - v_2) \) is the larger. In other words, \( A_1 \) will obtain whenever

\[
(U_1(A_1) - v_1)(U_2(A_1) - v_2) > (U_1(A_2) - v_1)(U_2(A_2) - v_2). 
\]

(2.5)

Here \( v_1 \) is a "guarantee" or "threat" for player I, and similarly, \( v_2 \) for player II. While the above references do include other behavioral justifications of this hyperbolic solution, Akbar [1] pointed out that it is a consequence of the proposition that that player will prevail in the bargaining who sees \( A_1 \) and \( A_2 \) as more disparate from his
own perspective. That is, for example, if

\[ \frac{U_1(A_1) - v_1}{U_1(A_2) - v_1} > \frac{U_2(A_2) - v_2}{U_2(A_1) - v_2} > 1, \]

then player I (reacting to the \( U_1(A_i) \)) will wish for the outcome \( A_1 \), and player II (reacting to the \( U_2(A_i) \)) will wish for the outcome \( A_2 \), with player I's preference for \( A_1 \) over \( A_2 \) stronger than player II's preference for \( A_2 \) over \( A_1 \), so that \( A_1 \) will obtain.

Akbar's behavioral interpretation of circumstances under which \( A_1 \) will obtain generally has the algebraic translation

\[
\text{max} \left\{ \frac{U_1(A_1) - v_1}{U_1(A_2) - v_1}, \frac{U_2(A_1) - v_2}{U_2(A_2) - v_2} \right\} > \text{max} \left\{ \frac{U_1(A_2) - v_1}{U_1(A_1) - v_1}, \frac{U_2(A_2) - v_2}{U_2(A_1) - v_2} \right\},
\]

which is indeed equivalent to (2.5).
CHAPTER 3. MY VIEW OF n-PLAYER BARGAINING

3.1 Two-player Bargaining

Adopting a notation natural for later extension to the n-player case, suppose that two offers \((U_1, U_2)\) and \((V_1, V_2)\) are on the table, where \(U_1\) and \(U_2\) are the terms offered to player I and player II, respectively, under the first offer, and \(V_1\) and \(V_2\) are the terms offered to player I and player II, respectively, under the second offer.

In player I's point of view, assuming the threats equal zero, player I will consider \(\frac{U_1}{V_1}\) or \(\frac{V_1}{U_1}\), and we can say, player I will like \(U\) better than \(V\) if \(\frac{U_1}{V_1} > 1\). Similarly, in player II’s point of view, we can say, player II will like \(U\) better than \(V\) if \(\frac{U_2}{V_2} > 1\). Due to the difference in preference between player I and II, there exists four different possibilities we can encounter as follows:

i) \(\frac{U_1}{V_1} > 1\) and \(\frac{U_2}{V_2} > 1\)

ii) \(\frac{U_1}{V_1} > 1\) and \(\frac{U_2}{V_2} < 1\)

iii) \(\frac{U_1}{V_1} < 1\) and \(\frac{U_2}{V_2} > 1\)

iv) \(\frac{U_1}{V_1} < 1\) and \(\frac{U_2}{V_2} < 1\)

In case i) and case iv) above, since both players prefer same offer in each case, it is quite natural that payoff \(U\) and \(V\), respectively, to be the outcome for each case. However, in case ii) and case iii), we can not determine the outcome so quickly as
we did in case i) and case iv) since each player prefers different offer. For instance, in case ii), where player I prefers $U$ while player II prefers $V$, we say, $U$ will be the outcome if $\frac{U_1}{V_1} > \frac{V_2}{U_2}$ which means player I likes $U$ more than player II likes $V$. In other words, offer $U$ will be the outcome if

$$U_1 \cdot U_2 > V_1 \cdot V_2 \quad (3.1)$$

which is exactly same as Nash’s solution.

Now, suppose a two player bargaining with $p$ number of offers where $p \geq 3$. Clearly, by a same token, the outcome will be an offer with pair that maximizes its products. As shown in Figure 3.1, the outcome will always be a point that touches the hyperbola $XY = n$, where $n$ is any real number, for largest $n$.

Also, we see the transitivity in here, meaning that if an offer $U$ is more likely
than $V$ and an offer $V$ is more likely than $W$, then we say an offer $U$ is more likely than $W$. Accordingly, if offers $U$, $V$, $W$, ... are available, then the offers can be ordered like $U > V > W > \ldots$.

Hence, for two players, our bargaining concept is in line with the standard bargaining theory approach.

### 3.2 Offer Domination for n-player Bargaining Without Coalitions

In this section, we extend the two player bargaining concept to the general n-player bargaining case.

Suppose there are four players ($n = 4$) confronting two different offers $U$ and $V$. Then the possible payoff vectors could be written as:

- $U : (U_1, U_2, U_3, U_4)$
- $V : (V_1, V_2, V_3, V_4)$

As one can expect, the question here is what should be the outcome from those two offers $U$ and $V$ with four different players. One clear case for an offer $U$ to be the outcome is

- No player prefers $V$ to $U$.

However, we have already recognized that the players have different preferences in general, we need to have a condition to decide the outcome from those two offers. Hence, we posit that the offer $U$ is more likely to be the outcome when

- The strongest preference among players preferring offer $U$ is stronger than the strongest preference among players preferring offer $V$. 
For instance, from above example, if \( \frac{U_1}{V_1} > \frac{U_2}{V_2} > 1 \) and \( \frac{V_3}{U_3} > \frac{V_4}{U_4} > 1 \), then offer \( U \) is more likely to be the outcome if \( \frac{U_1}{V_1} > \frac{V_3}{U_3} \), that is, if

\[
U_1 \cdot U_3 > V_1 \cdot V_3
\]

which constitutes a natural extension of the two player bargaining case. Generally, for two payoff \( n \)-tuples \( U = (U_1, U_2, \cdots, U_n) \) and \( V = (V_1, V_2, \cdots, V_n) \), we posit that \( U \) will be more likely to obtain than \( V \) if the player with the largest stake in the outcome favors \( U \); this amounts to

\[
\max \left\{ \frac{U_i}{V_i} \right\} > \max \left\{ \frac{V_i}{U_i} \right\},
\]

which we denote by

\[
U \succ V.
\]

Excepting ties, (2.5), (2.6) or equivalently (3.1), provides a complete ordering of payoff pairs in the case of \( n = 2 \) players. Not so for relation (3.2) and (3.3) in the case of \( n \) players. Indeed, when we move forward to \( p \) number of offers \( (p \geq 3) \), transitivity does not hold any more. That is, considering \( n \)-player bargaining with 3 offers \( U, V \) and \( W \), an offer \( U \) being more likely than \( V \) and an offer \( V \) being more likely than \( W \) does not guarantee an offer \( U \) being more likely than \( W \). That discourages us from seeking a dominating offer.

### 3.3 Offer Domination for \( n \)-player Bargaining With Coalitions

Based on the discussion in section 3.2, we may expand the concept of bargaining to include consideration of coalitions.
Suppose there are two payoff \( n \)-tuples \( U = (U_1, U_2, \cdots, U_n) \) and \( V = (V_1, V_2, \cdots, V_n) \), and any coalition \( c \) with \( c \subseteq N \).

**Definition 3.3.1** \( U \) dominates \( V \), denoted by

\[
U \succ V,
\]

if

\[
\max_c \left\{ \frac{\sum U_i}{\sum V_i} \right\} > \max_c \left\{ \frac{\sum V_i}{\sum U_i} \right\}
\]

where the maxima are taken over all coalitions, including singletons.

**Definition 3.3.2** The value of the left hand side of (3.4) is called the preference intensity for \( U \) over \( V \).

**Definition 3.3.3** \( U \) dominates \( V \) through a coalition \( C \), denoted by

\[
U \succ_C V,
\]

if

\[
\frac{\sum U_i}{\sum V_i} = \max_c \left\{ \frac{\sum U_i}{\sum V_i} \right\} > \max_c \left\{ \frac{\sum V_i}{\sum U_i} \right\}.
\]

**Definition 3.3.4** \( U \) strongly dominates \( V \) through \( C \) if the \( C \) of Definition 3.3.3 is unique.

**Definition 3.3.5** \( U \) and \( V \) are equally likely, denoted by

\[
U \sim V,
\]
if
\[
\max_{c} \left\{ \frac{\sum_{c} U_i}{\sum_{c} V_i} \right\} = \max_{c} \left\{ \frac{\sum_{c} V_i}{\sum_{c} U_i} \right\}. \tag{3.6}
\]

**Note:** Suppose a four player bargaining situation with
\[
U = (3/15, 6/15, 2/15, 4/15) \quad \text{and} \quad V = (1/15, 2/15, 4/15, 8/15).
\]

This \(U\) and \(V\) satisfies the condition given in (3.5) with preference intensity of 3.

And we say \(U\) does not strongly dominate \(V\) since the coalition \(C\) could be \(\{1\}\), \(\{2\}\) or \(\{1, 2\}\). Hence, we see that a coalition \(c\) is not necessarily a singleton.

**Lemma 3.3.1** If we let
\[
\frac{U_1}{V_1} = \max \left\{ \frac{U_1}{V_1}, \frac{U_2}{V_2}, \ldots, \frac{U_k}{V_k} \right\}
\]
then
\[
\frac{U_1}{V_1} \geq \frac{U_1 + U_2 + \cdots + U_k}{V_1 + V_2 + \cdots + V_k}.
\]

**Proof:**  The proof of this is quite obvious since
\[
\frac{U_1 + U_2 + \cdots + U_k}{V_1 + V_2 + \cdots + V_k} = \frac{U_1}{V_1} \cdot \frac{V_1}{V_1 + V_2 + \cdots + V_k} +
\]
\[
\cdots + \frac{U_k}{V_k} \cdot \frac{V_k}{V_1 + V_2 + \cdots + V_k}
\]
\[
\equiv \sum_{i} \frac{U_i}{V_i} \cdot w_i,
\]
so that the left hand side is a weighted average of the $U_i/V_i$, which clearly is no greater than the largest $\frac{U_i}{V_i}$.

**Lemma 3.3.2** *Strong domination is achieved only by singletons.*

**Proof:** Suppose that $U$ strongly dominates $V$ through the coalition $\{i, j, k\}$. Then

\[
\frac{U_i + U_j + U_k}{V_i + V_j + V_k} > \text{all other ratios. (3.7)}
\]

However, if for example

\[
\frac{U_i}{V_i} = \max \left\{ \frac{U_i}{V_i}, \frac{U_j}{V_j}, \frac{U_k}{V_k} \right\},
\]

then, by the Lemma 3.3.1,

\[
\frac{U_i}{V_i} \geq \frac{U_i + U_j + U_k}{V_i + V_j + V_k},
\]

which contradicts (3.7).
CHAPTER 4. BARGAINING-STABLE SETS

In similar circumstances, von Neumann and Morgenstern turned to the idea of stable sets in the context of "imputations", and Wald [26] to the related idea of complete classes in the context of statistical decision rules. While our concern here is with "prospect" vectors (this is a terminology due to Harsanyi), rather than with imputations, and certainly not with statistical decision rules, we nevertheless are motivated, by this prior work, to consider bargaining-stable sets of payoff n-tuples. Thus, given a collection $G$ of possible "prospect" or payoff n-tuples $U = (U_1, U_2, \cdots, U_n)$, we shall consider subsets $S$ of $G$ as follows.

- A subset $S$ is externally bargaining-stable if, for every $V \in G \setminus S$, there is a $U \in S$ such that (3.4) holds.

- A subset $S$ is strongly externally bargaining-stable if, for every $V \in G \setminus S$, there is a $U \in S$ and a coalition $C$ such that $U$ strongly dominates $V$ through.

- A subset $S$ is internally bargaining-stable if, for all pairs $(U, V)$ with $U$ and $V \in S$, (3.6) holds.

And now, we are able to give the definitions of a bargaining-stable set.

Definition 4.1 A set $S$ is bargaining-stable if

i) $S$ is externally bargaining-stable,
and

ii) $S$ is internally bargaining-stable.

**Definition 4.2** A set $S$ is strongly bargaining-stable if

i) $S$ is strongly externally bargaining-stable,

and

ii) $S$ is internally bargaining-stable.

**Lemma 4.1** If a set $S$ is strongly bargaining-stable, then all domination relations underlying its external bargaining-stability involve only singletons.

**Proof:** By Lemma 3.3.2.

The examples of bargaining-stable set for various cases will be explored in next chapter.
CHAPTER 5. EXAMPLES

In this chapter, we will explore the notion of bargaining-stable set by providing some examples for the n-player case, with the prospect space $G$ having form $\{ U : \sum_{i=1}^{n} \frac{U_i^{\alpha}}{a_i} = 1, U_i \geq 0 \}$, for $\alpha > 0$. We will first treat the case

$$n \geq 3, \quad \alpha = 1 \quad \text{and} \quad a_i = 1,$$

and this will lead easily to the case

$$n \geq 3, \quad \alpha = 1 \quad \text{and} \quad a_i > 0.$$

Next we treat the case

$$n \geq 3, \quad \alpha > 0 \quad \text{and} \quad a_i = 1.$$

The case

$$n \geq 3, \quad \alpha > 0 \quad \text{and} \quad a_i > 0$$

is not explicitly treated, since its treatment follows the treatment of the other cases in straight-forward fashion. Also, we will provide an alternative approach for the case of three players.
5.1 Case of \( n \geq 3, \ a = 1 \) and \( a_i = 1 \)

Consider \( n \)-player bargaining, with \( G \) the simplex \( \{ U : \sum_{i=1}^{n} U_i = 1, U_i \geq 0 \} \), containing the subset \( S = \{ U^{(1)}, U^{(2)}, \ldots, U^{(n)} \} \) consisting of the \( n \) payoff \( n \)-tuples.

\[
U^{(1)} : \left( \frac{1}{2}, \frac{1}{2(n-1)}, \frac{1}{2(n-1)}, \ldots, \frac{1}{2(n-1)} \right),
\]
\[
U^{(2)} : \left( \frac{1}{2(n-1)}, \frac{1}{2}, \frac{1}{2(n-1)}, \ldots, \frac{1}{2(n-1)} \right),
\]
\[
\vdots
\]
\[
U^{(n)} : \left( \frac{1}{2(n-1)}, \frac{1}{2(n-1)}, \ldots, \frac{1}{2(n-1)}, \frac{1}{2} \right).
\]

It is not hard to verify that the set \( S \) is bargaining-stable. The following lemmas are to verify this fact.

**Lemma 5.1.1** \( S = \{ U^{(1)}, U^{(2)}, \ldots, U^{(n)} \} \) is internally bargaining-stable.

**Proof:** For no \( i \) and \( j \), \( U^{(i)} \succ U^{(j)} \), since, clearly,

\[ U^{(i)} \sim U^{(j)}, \quad \text{for all } i \neq j. \]

**Lemma 5.1.2** \( S = \{ U^{(1)}, U^{(2)}, \ldots, U^{(n)} \} \) is externally bargaining-stable.

**Proof:** We shall prove Lemma 5.1.2 by means of 3 propositions.

**Proposition 5.1.1** For \( U \) with \( n \) tied minimum values, i.e., for \( U = (1/n, \ldots, 1/n) \),

\[ U^{(i)} \succ U \quad \text{for all } i. \]
**Proof:** To begin with,

\[(n - 2)^2 = n^2 - 4n + 4 > 0.\]

The above inequality can be rewritten as

\[\frac{n}{2} > \frac{2(n - 1)}{n},\]

which is equivalent to

\[n \max \left\{ \frac{1}{2(n - 1)}, \ldots, \frac{1}{2(n - 1)}, \frac{1}{2} \right\} > \frac{1}{n} \max \{2(n - 1), \ldots, 2(n - 1), 2\}. \quad (5.1)\]

Therefore, from equations (3.2), (3.3) and (5.1), we see that

\[U^{(i)} \succ U, \quad \text{for all } i.\]

**Proposition 5.1.2** For \(U\) with \((n - 1)\) tied minimum values with the tied value not at \(1/2(n - 1)\),

\[U^{(i)} \succ U \quad \text{for at least } (n - 1) \text{ values of } i.\]

**Proof:** A \(U\) with \((n - 1)\) tied minimum values with the tied value at \(1/\{2(n - 1)\}\) is one of the \(U^{(i)}\)'s. Hence, we only need to proceed for the case of the tied minimum value not at \(1/2(n - 1)\). Assume that \(U = (U_1, U_2, \ldots, U_n)\) is such that \(U_2 = U_3 = \cdots = U_n\), with \(U_1 > U_2 = U_3 = \cdots = U_n\). Then it must be that, say,

\[U_1 = 1 - (n - 1) \cdot U_n. \quad (5.2)\]

Now, from the fact that a quadratic form, \(1 - 2U_n \cdot (n - 1) \}^2 \) with \(U_n \neq 1/\{2(n - 1)\}\), always has positive value, we see that

\[
\{1 - 2U_n \cdot (n - 1) \}^2 = 1 - 4U_n \cdot (n - 1) + 4U_n^2 \cdot (n - 1)^2
\]

\[= 1 - 4U_n \cdot (n - 1)(1 - (n - 1) \cdot U_n) > 0.\]
And if we multiply $1/2U_n$, for $0 \leq U_n \leq 1/n$, to above inequality, we get

$$\frac{1}{2U_n} - 2(n-1)\{1-(n-1)\cdot U_n\} > 0$$

which reveals that

$$\frac{1}{2U_n} > 2(n-1)\{1-(n-1)\cdot U_n\}, \quad (5.3)$$

and, in view of (5.2), equation (5.3) can be rewritten as

$$\max \left\{ \frac{1}{2(n-1)\cdot U_1}, \ldots, \frac{1}{2(n-1)\cdot U_{n-1}}, \frac{1}{2U_n} \right\}$$

$$> \max \left\{ 2(n-1)\cdot U_1, \ldots, 2(n-1)\cdot U_{n-1}, 2U_n \right\},$$

Therefore, we can say that

$$U^{(n)} \succ U.$$

By the same token, we also see that

$$U^{(i)} \succ U \quad \text{for } i = 2, 3, \ldots, n.$$

**Proposition 5.1.3** *For U with k tied minimum values, for 1 \( \leq k \leq n-2 \) \( k = 1 \) denotes unique minimum value),

$$U^{(i)} \succ U \quad \text{for at least } k \text{ values of } i.$$

**Proof:** Assume that $U = (U_1, U_2, \ldots, U_n)$ has tied minima at $U_{n-(k-1)} = U_{n-(k-2)} = \cdots = U_n$. And let $U^*$ be the largest one among $U_1, U_2, \cdots, U_{n-k}$;
then

\[(n - 1) \cdot U_n + U^* < \sum_{i=1}^{n} U_i = 1\]

since \(U_n = \min(U_1, U_2, \ldots, U_n)\). And the above inequality can be rewritten as

\[1 - (n - 1) \cdot U_n > U^*. \tag{5.4}\]

By inserting (5.4) into (5.3) which holds for any \(U_n\) with \(0 \leq U_n \leq 1/n\), we get

\[\frac{1}{2U_n} > 2(n - 1) \{1 - (n - 1) \cdot U_n\} > 2(n - 1) \cdot U^*. \tag{5.5}\]

Hence, from (5.5), we see that

\[
\max \left\{ \frac{1}{2(n - 1) \cdot U_1}, \ldots, \frac{1}{2(n - 1) \cdot U_{n-1}}, \frac{1}{2U_n} \right\} \\
> \max \{2(n - 1) \cdot U_1, \ldots, 2(n - 1) \cdot U_{n-1}, 2U_n\}
\]

since, again, \(U_1, U_2, \ldots, U_{n-k} > U_{n-(k-1)} = U_{n-(k-2)} = \cdots = U_n\). Therefore, we can say that

\[U^{(n)} \succ U.\]

Generally, by the same token,

\[U^{(i)} \succ U \quad \text{for} \quad i = n - (k - 1), n - (k - 2), \ldots, n.\]

As shown above, propositions 5.1.1, 5.1.2 and 5.1.3 imply the external bargaining stability of \(S = \{U^{(1)}, U^{(2)}, \ldots, U^{(n)}\}\). Indeed, going back over the proof, it is clear that \(S\) is strongly externally bargaining-stable.

Hence, all told, we conclude that the set \(S = \{U^{(1)}, U^{(2)}, \ldots, U^{(n)}\}\) is strongly bargaining-stable.
5.2 Case of $n \geq 3$, $\alpha = 1$ and $a_i > 0$

In this section, we will briefly consider a non-symmetric bargaining game with, say, $G$ of form
\[
\frac{U_1}{a_1} + \frac{U_2}{a_2} + \cdots + \frac{U_n}{a_n} = 1,
\]
with $U_i \geq 0$ and $a_i > 0$. In this case, we are able to find a bargaining-stable set by a simple linear transformation of the points. Indeed, a bargaining-stable set $S$ for this non-symmetric case will be $S = \{ U^{(1)}, U^{(2)}, \ldots, U^{(n)} \}$ where
\[
U^{(1)} : \left( \frac{a_1}{2}, \frac{a_2}{2(n-1)}, \frac{a_3}{2(n-1)}, \ldots, \frac{a_n}{2(n-1)} \right),
\]
\[
U^{(2)} : \left( \frac{a_1}{2(n-1)}, \frac{a_2}{2}, \frac{a_3}{2(n-1)}, \ldots, \frac{a_n}{2(n-1)} \right),
\]
\[\vdots\]
\[
U^{(n)} : \left( \frac{a_1}{2(n-1)}, \frac{a_2}{2(n-1)}, \ldots, \frac{a_{n-1}}{2(n-1)}, \frac{a_n}{2} \right).
\]

The proof of $S$ being bargaining-stable will not be given since it follows in straightforward fashion. Figure 5.1 illustrates this linear transformation for the case of $n = 3$.

5.3 Case of $n \geq 3$, $\alpha > 0$ and $a_i = 1$

In this section, we will consider $n$-player bargaining, with $G$ having form $\{ U : \sum_{i=1}^{n} U_i^\alpha = 1, U_i^\alpha \geq 0 \}$, with $\alpha > 0$, and $S = \{ U^{(1)}, U^{(2)}, \ldots, U^{(n)} \}$ the subset of $G$ consisting of the $n$ payoff $n$-tuples.
\[
U^{(1)} : \left( \frac{1}{2^{1/\alpha}}, \frac{1}{(2(n-1))^{1/\alpha}}, \frac{1}{(2(n-1))^{1/\alpha}}, \ldots, \frac{1}{(2(n-1))^{1/\alpha}} \right).
\]
Figure 5.1: A bargaining-stable set for the case of $n = 3$, $\alpha = 1$ and $a_i > 0$
The set $S$ being bargaining-stable is verified as follows:

**Lemma 5.3.1** $S = \{U^{(1)}, U^{(2)}, \ldots, U^{(n)}\}$ is internally bargaining-stable.

**Proof:** For no $i$ and $j$, $U^{(i)} \succ U^{(j)}$, since, clearly,

$$U^{(i)} \sim U^{(j)}, \quad \text{for all } i \neq j.$$

**Lemma 5.3.2** $S = \{U^{(1)}, U^{(2)}, \ldots, U^{(n)}\}$ is externally bargaining-stable.

**Proof:** We shall prove Lemma 5.3.2 by means of 3 propositions.

**Proposition 5.3.1** For $U$ with $n$ tied minima, i.e., for $U = (1/n^{1/\alpha}, \ldots, 1/n^{1/\alpha})$,

$$U^{(i)} \succ U \quad \text{for all } i.$$

**Proof:** To begin with, a quadratic form $(n - 2)^2 > 0$, or equivalently

$$\frac{n}{2} > \frac{2(n - 1)}{n}.$$

The above inequality can be rewritten as

$$\left\{ \frac{n}{2} \right\}^{1/\alpha} > \left\{ \frac{2(n - 1)}{n} \right\}^{1/\alpha},$$
which implies

\[ U^{(i)} \succ U \quad \text{for all } i. \]

**Proposition 5.3.2**  For \( U \) with \( (n - 1) \) tied minima with the tied value not at
\( 1/(2(n - 1))^{1/\alpha} \),

\[ U^{(i)} \succ U \quad \text{for at least } (n - 1) \text{ values of } i. \]

**Proof:** A \( U \) with \( (n - 1) \) tied minimum values with the tied value at \( 1/(2(n - 1))^{\alpha} \)
is one of the \( U^{(i)} \)'s. Hence, we only need to proceed for the case of the tied minimum
value not at \( 1/(2(n - 1))^{\alpha} \). Consider \( U = (U_1, \ldots, U_n) \) with \( U_1 > U_2 = \cdots = U_n \).
Then, it must be that, say,

\[ U_1^\alpha = 1 - (n - 1) \cdot U_n^\alpha, \quad (5.7) \]

Now, from the fact that a quadratic form, \( \{1 - 2U_n^\alpha \cdot (n - 1)\}^2 \) with \( U_n^\alpha \neq 1/(2(n - 1)) \), always has positive value, we see that

\[
\{1 - 2(n - 1) \cdot U_n^\alpha\}^2 = 1 - 4(n - 1) \cdot U_n^\alpha + 4(n - 1)^2 \cdot U_n^{2\alpha}
\]

\[
= 1 - 4(n - 1) \cdot U_n^\alpha \{1 - (n - 1) \cdot U_n^\alpha\} > 0.
\]

And if we multiply \( 1/(2U_n^\alpha) \) to above inequality, we get

\[
\frac{1}{2U_n^\alpha} > 2(n - 1)\{1 - (n - 1) \cdot U_n^\alpha\}, \quad (5.8)
\]

and, in view of (5.7), equation (5.8) can be rewritten as

\[
\frac{1}{2U_n^\alpha} > 2(n - 1) \cdot U_1^\alpha.
\]
or

$$\frac{1}{2^{1/\alpha} \cdot U_n} > \{2(n-1)\}^{1/\alpha} \cdot U_1.$$  \tag{5.9}

The left and right hand side of (5.9), respectively, are equivalent as denoting

$$\max\left\{ \frac{1}{(2(n-1))^{1/\alpha} \cdot U_1}, \ldots, \frac{1}{(2(n-1))^{1/\alpha} \cdot U_{n-1}}, \frac{1}{2^{1/\alpha} \cdot U_n} \right\}$$

and

$$\max\left\{ (2(n-1))^{1/\alpha} \cdot U_1, \ldots, (2(n-1))^{1/\alpha} \cdot U_{n-1}, 2^{1/\alpha} \cdot U_n \right\},$$

or

$$U^{(n)} \succ U_i$$

similarly,

$$U^{(i)} \succ U \text{ for } i = 2, 3, \ldots, n.$$

**Proposition 5.3.3** For $U$ with $k$ tied minima, for $1 \leq k \leq n-2$

($k = 1$ denotes unique minimum value),

$$U^{(i)} \succ U \text{ for at least } k \text{ values of } i.$$

**Proof:** Assume that $U = (U_1, U_2, \ldots, U_n)$ is as $U_1, \ldots, U_{n-k} > U_{n-k+1} = \cdots = U_n$. And if we let

$$U^* = \max (U_1, U_2, \ldots, U_{n-k}),$$
then
\[(n-1) \cdot U_n^\alpha + U^*_\alpha < \sum_{i=1}^{n} U_i^\alpha = 1\]

since \(U_n = \min(U_1, U_2, \ldots, U_n)\). And the above inequality can be rewritten as

\[1 - (n-1) \cdot U_n^\alpha > U^*_\alpha. \quad (5.10)\]

By inserting (5.10) into (5.8), we get

\[\frac{1}{2U_n^\alpha} > 2(n-1) \cdot \{1 - (n-1) \cdot U_n^\alpha\} > 2(n-1) \cdot U^*_\alpha.\]

Hence, from the above inequality, we get

\[\max \left\{ \frac{1}{\{2(n-1)\}^{1/\alpha} \cdot U_1}, \ldots, \frac{1}{\{2(n-1)\}^{1/\alpha} \cdot U_{n-1}}, \frac{1}{2^{1/\alpha} \cdot U_n} \right\} \]

\[> \max \left\{ \{2(n-1)\}^{1/\alpha} \cdot U_1, \ldots, \{2(n-1)\}^{1/\alpha} \cdot U_{n-1}, 2^{1/\alpha} \cdot U_n \right\} \]

or

\[U^{(n)} \succ U;\]

similarly,

\[U^{(i)} \succ U \text{ for } i = n-k+1, \ldots, n.\]

Clearly, Propositions 5.3.1, 5.3.2 and 5.3.3 imply the external bargaining-stability of \(S = \{U^{(1)}, U^{(2)}, \ldots, U^{(n)}\}\). Also, it is clear that \(S\) is strongly externally bargaining-stable.

Hence, we conclude that a set \(S = \{U^{(1)}, U^{(2)}, \ldots, U^{(n)}\}\) is strongly bargaining-stable.
5.4 Alternative Approach for the Case of $n = 3$, $\alpha = 1$ and $a_i = 1$

In this section, we will consider three-player ($n = 3$) bargaining in an alternative way, with the simplex $\{U : \sum_{i=1}^{3} U_i = 1, \ U_i \geq 0\}$, containing the specialization of the set $S$ of section 5.1 to the case $n = 3$; $S = \{U^{(1)}, U^{(2)}, U^{(3)}\}$ where

$$
U^{(1)} : \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right), \\
U^{(2)} : \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right), \\
U^{(3)} : \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right).
$$

The followings are to verify $S$ being bargaining-stable in alternative way.

That $S$ is internally bargaining-stable follows as in section 5.1.

The alternative argument pertains to external bargaining-stability, as follows.

From the fact that

$$(4c - 1)^2 = 16c^2 - 8c + 1 \geq 0,$$

we get, by multiplying $1/8$ the above by,

$$
\frac{1/2}{c} \geq \frac{1 - 2c}{1/4}. \quad (5.11)
$$

If we let $c$, $0 \leq c \leq 1/3$, be a fixed value of $U_3$, and consider the lower-left part of the triangle shown in Figure 5.2, we get the possible range of $U_2$ as $c \leq U_2 \leq \frac{1-c}{2}$.

From this, we get the relation,

$$
\frac{1 - 2c}{1/4} \geq \frac{1 - U_2 - c}{1/4},
$$
Figure 5.2: A bargaining-stable set for the case of $n = 3, \alpha = 1$ and $a_i = 1$

which, in view of (5.11), allows us to write

$$\frac{1/2}{c} > \frac{1-U_2-c}{1/4},$$

or

$$\max \left\{ \frac{1/4}{1-U_2-c}, \frac{1/4}{U_2}, \frac{1/2}{c} \right\} > \max \left\{ \frac{1-U_2-c}{1/4}, \frac{U_2}{1/4}, \frac{c}{1/2} \right\}$$

for $0 \leq c \leq 1/3$ and $c \leq U_2 \leq (1-c)/2$.

The above inequality is equivalent to

$$\max \left\{ \frac{1/4}{U_1}, \frac{1/4}{U_2}, \frac{1/2}{U_3} \right\} > \max \left\{ \frac{U_1}{1/4}, \frac{U_2}{1/4}, \frac{U_3}{1/2} \right\}$$

or

$$\left( \frac{1}{4}, \frac{1}{4}, \frac{1}{2} \right) \succ \left( U_1, U_2, U_3 \right)$$

which, by symmetry, reveals that $S = \{ U^{(1)}, U^{(2)}, U^{(3)} \}$ is externally bargaining-stable. Also, it is clear that $S$ is strongly externally bargaining-stable.

Hence, $S$ is strongly bargaining-stable.
5.5 Case of $\alpha \to \infty$

In the general not necessarily symmetric case, the set $G$ is the surface of a rectangular parallelootope, and the bargaining-stable set degenerates to a single point, namely the appropriate vertex of the parallelootope. This is illustrated in Figure 5.4 with Figure 5.3 given for purposes of comparison.

![Figure 5.3: A bargaining-stable set for the case of $n = 3, \alpha = 2$ and $a_i = 1$](image)

![Figure 5.4: A bargaining-stable set for the case of $n = 3, \alpha \to \infty$](image)
CHAPTER 6. COMPARISONS

In this chapter, we will compare our bargaining-type solution with three different types of solutions drawn from general theory of n-person cooperative games.

6.1 Von Neumann and Morgenstern Stable Set

It is perhaps of interest to compare our bargaining-type solution, as phrased in terms of bargaining-stable sets $S$ with the classical von Neumann and Morgenstern's solution, as phrased in terms of the usual stable sets $T$. Such a comparison faces the conceptual obstacle that a traditional stable set is a set of n-dimensional imputation vectors. While, in keeping with the traditional two-player bargaining model, a bargaining-stable set is a set of n-dimensional "prospect" vectors corresponding to a set of obtainable outcomes. Still, for one who interprets an imputation as a prospect, a comparison of corresponding solutions may be of some interest. In particular, consider the game of section 2.1, for which the set of imputations is the simplex $U_1 + U_2 + U_3 = 1; U_i \geq 0$, and consider as well the situation of section 5.1, for which the prospect space also consists of that simplex, and for which our bargaining-stable set $S$ consists of the points $(1/4, 1/4, 1/2), (1/4, 1/2, 1/4)$ and $(1/2, 1/4, 1/4)$.

It is easy to show that our bargaining-stable set $S$ is not a von Neumann and Morgenstern stable set; for that purpose, recall the von Neumann and Morgenstern
concept of an imputation $x$ dominating an imputation $y$ through a coalition $C$; for the game under consideration

- $x$ dominates $y$ through $\{1, 2\}$ if (denoted by $x \succ^{12} y$ )

  (i) $x_i > y_i$ for all $i \in \{1, 2\}$

  (ii) $\sum_{i \in \{1, 2\}} x_i \leq v(\{1, 2\}) = 1$.

And, by the same token,

- $x$ dominates $y$ through $\{1, 2, 3\}$ if (denoted by $x \succ^{123} y$ )

  (i) $x_i > y_i$ for all $i \in \{1, 2, 3\}$

  (ii) $\sum_{i \in \{1, 2, 3\}} x_i \leq v(\{1, 2, 3\}) = 1$.

- Also, $x$ dominates $y$ if $x \succ_C y$ for some $C$.

From the fact that von Neumann and Morgenstern stable sets should have both internal and external stability, we see that our bargaining-stable set $S$ is not a von Neumann and Morgenstern stable set $T$, since the set $S$ does not satisfy the external stability condition. On the other hand, $S$ does satisfy the internal stability.

With regard to internal stability, clearly there does not exist $U(i)$ that dominates $U(j)$, for any $i$ and $j$, with respect to the classical definition of domination shown above. This reveals that our bargaining-stable set satisfies ordinary internal stability.

With regard to external stability, simply note that $(1/3, 1/3, 1/3)$ is not dominated by $(1/4, 1/4, 1/2)$; which can be seen as follows:
If we let $x = (1/4, 1/4, 1/2)$ and $y = (1/3, 1/3, 1/3)$, then by the classical definition of domination, there must exist a coalition $C$ of two or more elements that allows

$$x_i > y_i \quad \text{for all } i \in C.$$  

However, we see that the coalition $C$ that satisfies the above requirement is singleton $\{3\}$, hence, we say, $(1/3, 1/3, 1/3)$ is not dominated by $(1/4, 1/4, 1/2)$.

And of course, by symmetry, neither is $(1/3, 1/3, 1/3)$ dominated by $(1/4, 1/2, 1/4)$ or $(1/2, 1/4, 1/4)$.

Hence, we conclude that our bargaining-stable set is not an ordinary stable set since our bargaining-stable set does not satisfy the ordinary external stability condition.

6.2 Shapley Value


By adopting three axioms, which are known to be the axiom of effectiveness, the axiom of symmetry and the axiom of aggregation, Shapley was able to find a value, a solution, for the $n$-player case, which consist of a single numerical payoff assignment to each player:

$$\psi_i = \sum_{Q \subseteq N} \sum_{i \in Q} \frac{(q - 1)! (n - q)!}{n!} \cdot [v(Q) - v(Q - \{i\})].$$  

(6.1)

Here $N$ stands for the set of all players, $n$ for the number of players in $N$, and $q$ for the number of players in a coalition $Q \subseteq N$. Also, the term $v(Q) - v(Q - \{i\})$ is the marginal amount which player $i$ contributes to the coalition $Q$, as payoff.
Now, consider a three-person symmetric game, for which,

\[ v(1) = v(2) = v(3) = 0, \]

\[ v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = \beta \quad \text{and} \]

\[ v(\{1, 2, 3\}) = 1 \]

where \( 0 < \beta \leq 1 \). Then, the Shapley value for each player will be

\[
\psi_i = \frac{2!}{3!} [v(\{1, 2, 3\}) - v(\{2, 3\})] + \frac{\beta}{3!} [v(\{1, 2\}) - v(2)] + \frac{\beta}{3!} [v(\{1, 3\}) - v(3)] + \frac{\beta}{3!} [v(1) - v(\phi)]
\]

\[
= \frac{1}{3}.
\]

Because of symmetry, we also get \( \psi_2 = \psi_3 = \frac{1}{3} \). This solution maintains the symmetry of our bargaining-stable set \( S \), and of the classical stable set \( T \), in singleton form.

### 6.3 Shapley - Harsanyi Value

Harsanyi [10] expressed Shapley's value in an alternative form:

\[
u_i = \sum_{Q \subseteq N \atop i \in Q} \frac{(q-1)! (n-q)!}{n!} [v(Q) - v(N - Q)]. \tag{6.2}
\]

Conceptually, the terms \( v(Q - \{i\}) \) given by Shapley and the term \( v(Q) - v(N - Q) \) given by Harsanyi are totally different, however, mathematically, the two terms turned out to provide the same result. This will be briefly illustrated as follows:
Consider a three-person game with \( N = \{i, j, k\} \), then all possible coalitions that can be formed from the set \( N \) is \( \{i\}, \{j\}, \{k\}, \{i, j\}, \{i, k\}, \{j, k\} \) and \( \{i, j, k\} \). Then, for each player \( i \),

\[
\sum_{Q \subseteq N} v(Q - \{i\}) = v(\{i\} - \{i\}) + v(\{i, j\} - \{i\}) + v(\{i, k\} - \{i\}) \\
+ v(\{i, j, k\} - \{i\}) \\
= v(\phi) + v(j) + v(k) + v(\{j, k\}) \\
= \sum_{Q \subseteq N} v(N - Q).
\]

This is true for any player in a coalition \( Q \), and also true for any arbitrary \( n \)-person case.

After redefining Shapley’s value in this way, Harsanyi went on to propose the following modification, particularly pertinent to bargaining. To begin with, Harsanyi goes back to “normal” form, and posits three payoff “matroids” \( K_1(i, j, k) \), \( K_2(i, j, k) \) and \( K_3(i, j, k) \), respectively for the three players, as functions of their strategy choices. In this format the quantities \( v(i) \), \( v(\{i, j\}) \) and \( v(\{i, j, k\}) \) appearing in (6.2) have the standard maxmin expressions, say,

\[
v(\{1, 2\}) = \max_{\theta_{12}} \min_{\theta_3} \left[ \sum_{i,j} \sum_k \theta_{ij}^{12} \cdot \{K_1(i, j, k) + K_2(i, j, k)\} \cdot \theta_k^3 \right].
\]

Harsanyi now goes on to replace a term such as \( v(\{1, 2\}) \) in (6.2) by a term

\[
v(\{1, 2\}) = \max_{\theta_{12}} \min_{\theta_3} \left[ \sum_{i,j} \sum_k \theta_{ij}^{12} \cdot \{K_1(i, j, k) + K_2(i, j, k) - K_3(i, j, k)\} \cdot \theta_k^3 \right].(6.3)
\]

We illustrate this concept with the zero-sum inessential symmetric game having the normal form structure shown in Table 6.1.
Table 6.1: A normal form structure for \( n = 3 \)

<table>
<thead>
<tr>
<th>I's st.</th>
<th>II's st.</th>
<th>III's st.</th>
<th>( K_1 )</th>
<th>( K_2 )</th>
<th>( K_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1/2</td>
<td>0</td>
<td>1/2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
</tr>
</tbody>
</table>

The computations are as follows:

The coalitions that player I may be involved are \( \{1\} \), \( \{1, 2\} \), \( \{1, 3\} \) and \( \{1, 2, 3\} \). At first, if we let \( Q = \{1\} \) then \( N - Q = \{2, 3\} \), then from the fact that

\[
v(1) = \max_{\theta_1} \min_{\theta_{23}} \left[ \sum_i \sum_{j,k} \theta_i^1 \cdot \left( K_1(i, j, k) - K_2(i, j, k) - K_3(i, j, k) \right) \cdot \theta_{23}^{jk} \right],
\]

and also from the normal form structure given above, we get a 2 \( \times \) 4 matrix for player I against player II and III, which can be considered as two-player (\( Q \) against \( N - Q \)) zero-sum game. The matrix for \( Q \), actually for player I in this case, is given as

\[
\begin{array}{ccccc}
\theta_{00} & \theta_{01} & \theta_{10} & \theta_{11} \\
\hline
\theta_0 & -1/3 & -1 & -1 & -1 \\
\theta_1 & 1 & 0 & 0 & -1/3 \\
\end{array}
\]

which provides \( v(1) = -1/3 \) and \( v(\{2, 3\}) = 1/3 \).

And if we let \( Q = \{1, 2\} \), then from (6.3) and also from the normal form structure given above, we get 4 \( \times \) 2 matrix for player I and II against player III, such as
which provides \( v(\{1,2\}) = 1/3 \) and \( v(3) = -1/3 \). By the same token, we get \( v(2) = -1/3 \), \( v(\{1,3\}) = 1/3 \) and \( v(\{1,2,3\}) = 1 \).

Hence, the Shapley-Harsanyi value \( u_1 \), given in (6.2), is

\[
\begin{align*}
\quad u_1 &= \frac{2!}{3!} \left[ v(\{1,2,3\}) - v(\phi) \right] + \frac{1!}{3!} \left[ v(\{1,2\}) - v(3) \right] \\
&\quad + \frac{1!}{3!} \left[ v(\{1,3\}) - v(2) \right] + \frac{0!}{3!} \left[ v(1) - v(\{2,3\}) \right] \\
&\quad = \frac{1}{3}.
\end{align*}
\]

This yields \((1/3, 1/3, 1/3)\) for Harsanyi’s modified Shapley value.

Of course, for this degenerate symmetric inessential zero-sum situation, our own bargaining-stable set also reduces to the degenerate singleton \((1/3, 1/3, 1/3)\).

For purposes of comparison with our bargaining approach, we have here at least two ways to view the prospect space \( G \):

First, we can view the prospect space as the singleton imputation set \((1/3, 1/3, 1/3)\), for which we, or for that matter any other reasonable theory of bargaining, identify the singleton solution \( S = (1/3, 1/3, 1/3) \).

Or, we can view the prospect space as the simplex \( (x_i \geq 0, \sum x_i = 1) \) derived directly from the normal form of Table 6.1, for which, once again, the bargaining-stable set \( S = \{(1/2, 1/4, 1/4), (1/4, 1/2, 1/4), (1/4, 1/4, 1/2)\} \) of chapter 5 applies.
CHAPTER 7. ABSTRACT EXTERNAL BARGAINING STABILITY

We have discussed and compared von Neumann and Morgenstern's ordinary stable sets and our new bargaining stable sets throughout the previous chapters. From those chapters we already recognized that both sets have properties of internal and external stability. In this chapter, we focus solely in an abstract pairwise domination concept, pertinent in particular to von Neumann and Morgenstern domination and our own domination concept, but equally descriptive of any anti-symmetric complete not necessarily transitive pairwise domination relation. We then note that, with respect to external stability, both von Neumann and Morgenstern and our own bargaining-stable sets are in effect complete classes with respect to their own particular domination relations, and ask, in the context of just a few compared "offers", how small, at that level of generality, a complete class can be. In particular, we ask for the number of compared offers such that a complete class of size two must exist, regardless of the nature of the set of pairwise orderings.

Consider then 5 offer game, which can be expressed as an anti-symmetric 5 x 5 matrix with 0's and 1's, as shown in Table 7.1. In Table 7.1, an offer that has 1 in a row is better than an offer which has 1 in that column, for instance, an offer $O_1$ is better than an offer $O_3$ but worse than the offers $O_2$, $O_4$ and $O_5$. One can easily
Table 7.1: 5-offer game

<table>
<thead>
<tr>
<th></th>
<th>$O_1$</th>
<th>$O_2$</th>
<th>$O_3$</th>
<th>$O_4$</th>
<th>$O_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1$</td>
<td>-</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$O_2$</td>
<td>1</td>
<td>-</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$O_3$</td>
<td>0</td>
<td>1</td>
<td>-</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$O_4$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>$O_5$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-</td>
</tr>
</tbody>
</table>

see that $O_1$ and $O_2$ form a complete class of 2 offers while $O_1$ and $O_5$ do not.

Generally, for a $p$ offer game, there are $p(p-1)/2$ number of 1's in the anti-symmetric matrix. If we denote the elements of the anti-symmetric matrix $A$ as $a_{ij}$, then we can say offer $i_1$ and $i_2$ form a complete class of 2 offers when the following condition is satisfied.

$$\sum_{j=1}^{p} (a_{i_1j} + a_{i_2j} - a_{i_1j} \cdot a_{i_2j}) = p - 1.$$ 

For up to 6 offer game, there always exist at least one pair of $i_1$ and $i_2$ which forms a complete class of 2 offers. However, for 7 or more offers, a complete class of 2 offers may not exist as shown in Table 7.2.

It is interesting to note that the domination relations of Table 7.2 actually are achievable for our particular bargaining domination, as for example, in the case of four players, when the seven offers are as follows:

- $O_1$: (350, 375, 320, 390)
- $O_2$: (360, 345, 330, 390)
- $O_3$: (330, 345, 352, 380)
Table 7.2: 7-offer game

<table>
<thead>
<tr>
<th></th>
<th>$O_1$</th>
<th>$O_2$</th>
<th>$O_3$</th>
<th>$O_4$</th>
<th>$O_5$</th>
<th>$O_6$</th>
<th>$O_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1$</td>
<td>-</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$O_2$</td>
<td>0</td>
<td>-</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$O_3$</td>
<td>1</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$O_4$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$O_5$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$O_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>$O_7$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
</tbody>
</table>

- $O_4$: (350, 350, 340, 380)
- $O_5$: (370, 370, 313, 370)
- $O_6$: (350, 350, 325, 400)
- $O_7$: (330, 375, 350, 370)
CHAPTER 8. CONCLUSION

This research proposed one further approach to n-player bargaining, based on the classical notion of stable set and the notion of preference intensity, and verified its implementability.

Various kinds of approaches for the two player case were reviewed.

With the notion of preference intensity, we developed a bargaining-stable set which does not "freeze out" any player. Identifying the "prospect space" with the set of imputations for a certain three-player game, this accommodating feature is to be viewed in the light of a harsher corresponding classical stable set that does "freeze out" players.

For ordinary stable sets, domination between imputations must involve multi-player coalitions; on the other hand, for bargaining-stable set, dominating always will involve single players, that is, for bargaining-stable sets, the coalitions will not be effective in determining the outcome of play.

Finally, the domination relation underlying classical stable sets is not anti-symmetric, a feature termed a "serious difficulty" by Owen ([20] p.148); not so for bargaining-stable sets.
BIBLIOGRAPHY


[22] Roth, Alvin E. *Axiomatic models of bargaining*, Springer-Verlag, New York, 1979


