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Notes on Quasivarieties and Maltsev Products

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Notes on Quasivarieties and Maltsev Products

Abstract
These notes were prepared for my research group in 2014. These results are not new. Most appear in similar form in [4]. Since neither [4] nor its English translation, [5] is easily accessible, it seemed worthwhile to make these notes available to a wider audience. The notation and terminology follows that of my book, [1].

Disciplines
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NOTES ON QUASIVARIETIES
AND MALTSEV PRODUCTS

CLIFFORD BERGMAN

These notes were prepared for my research group in 2014. These results are not new. Most appear in similar form in [4]. Since neither [4] nor its English translation, [5] is easily accessible, it seemed worthwhile to make these notes available to a wider audience. The notation and terminology follows that of my book, [1].

Definition 1. A quasivariety is a class of algebras closed under subalgebra, product, and ultraproduct. Equivalently (see [1, thm 5.4]) a class is a quasivariety iff closed under subalgebra and reduced product.

It is easy to see that the intersection of a family of quasivarieties is again a quasivariety. Thus we can talk about the quasivariety generated by a class of algebras.

Proposition 2. Let $\mathcal{K}$ be a class of algebras. The quasivariety generated by $\mathcal{K}$ is $\text{SPP}_u(\mathcal{K}) = \text{SP}_r(\mathcal{K})$.

A proof can be found in [2, thm. V.2.23] or [1, thm. 5.4].

Corollary 3. Let $A$ be a finite algebra. The quasivariety generated by $A$ is $\text{SP}(A)$.

Definition 4. A quasiidentity is a formula of the form

$$(p_1(x) \approx q_1(x)) \land (p_2(x) \approx q_2(x)) \land \cdots \land (p_k(x) \approx q_k(x)) \rightarrow s(x) \approx t(x)$$

When $k = 0$ we have the identity $s(x) \approx t(x)$, so every identity is a quasiidentity.

Theorem 5. A class of algebras is a quasivariety if and only if it is defined by a set of quasiidentities.

For a proof of Theorem 5 see [2, V.2.25].

Congruence classes play a double role in the context of Maltsev products: as elements of a quotient algebra and as (potential) subalgebras. In order to help keep things straight, let us write $[a]_\theta$ for a congruence class being treated as a subset, and continue to write $a/\theta$ for the corresponding element of the quotient algebra.

An element, $a$, of an algebra, $A$, is called idempotent if $\{a\}$ forms a subuniverse of $A$. Put another way, for every basic operation, $f$, we have

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The algebra $A$ is idempotent if every element is idempotent. A class, $\mathcal{K}$, of algebras is idempotent if every member algebra is idempotent.

**Definition 6.** Let $A$ and $B$ be quasivarieties. Then

$$A \circ B = \{R : (\exists \theta \in \text{Con}(R)) \ R/\theta \in B \text{ and } (\forall r \in R) [r]_\theta \in \text{Sub}(R) \implies [r]_\theta \in A\}.$$  

The class $A \circ B$ is called the Maltsev product of $A$ and $B$. If $C$ is another quasivariety containing both $A$ and $B$, we write $A \circ C = (A \circ B) \cap C$. For the extent of this paper, by an $A,B$-pivot (or just a pivot if the context is clear) we mean a congruence $\theta$ satisfying the conditions of Definition 6.

Let $h : A \to B$ be a homomorphism with kernel $\alpha$. Let $r \in A$. Then $[r]_\alpha$ is a subalgebra of $A$ if and only if $h(r)$ is idempotent in $B$. This is an immediate consequence of the fact that a direct image of a subalgebra is a subalgebra and the inverse image of a subalgebra is a subalgebra. Another way to say this is

(1) $[r]_\alpha \in \text{Sub}(A) \iff f(r, r, \ldots, r) \alpha r$ for every basic operation $f$.

Now suppose that $\alpha \leq \beta \in \text{Con}(A)$. Then

(2) $[r]_\alpha \in \text{Sub}(A) \implies (\forall f) f(r, r, \ldots, r) \alpha r \implies (\forall f) f(r, r, \ldots, r) \beta r \implies [r]_\beta \in \text{Sub}(A)$

in which the quantifier on $f$ ranges over all basic operations of $A$.

Here is another observation.

**Lemma 7.** Let $A$ be an algebra, $\alpha < \beta$ congruences on $A$ and $r \in A$.

(1) $[r/\alpha]_\beta/\alpha = ([r]_\beta)/\alpha$.

(2) $[r]_\beta \in \text{Sub}(A) \iff [r/\alpha]_\beta/\alpha \in \text{Sub}(A/\alpha)$.

**Proof.** For (1), $x/\alpha \in [r/\alpha]_\beta/\alpha \iff x/\alpha \equiv r/\alpha \text{ (mod } \beta/\alpha) \iff x \equiv r \text{ (mod } \beta) \iff x \in [r]_\beta \iff x/\alpha \in [r]_\beta/\alpha$.

The second claim follows from equivalence (1) since

$[r]_\beta \in \text{Sub}(A) \iff f(r, r, \ldots, r) \beta r \iff f(r, r, \ldots, r) \beta (r/\alpha)$.

\[ \square \]
Let $\mathcal{B}$ be a quasivariety and $\mathbf{R}$ an algebra of the same similarity type as $\mathcal{B}$. Define

$$
\Lambda^R_B = \{ \theta \in \text{Con}(\mathbf{R}) : \mathbf{R}/\theta \in \mathcal{B} \}
$$

$$
\lambda^R_B = \bigcap \Lambda^R_B.
$$

The congruence $\lambda^R_B$ is called the **verbal congruence on $\mathbf{R}$ induced by $\mathcal{B}$**. We leave off the sub- and superscript when the context is clear. Notice that $1_R \in \Lambda$ since $\mathcal{B}$ contains a trivial algebra. Observe also that

$$
\mathbf{R}/\lambda \leq \prod_{\theta \in \Lambda} \mathbf{R}/\theta \in \text{SP}(\mathcal{B}) = \mathcal{B}.
$$

Thus $\lambda \in \Lambda$. In fact the verbal congruence is the smallest congruence on $\mathbf{R}$ whose induced quotient falls into the quasivariety $\mathcal{B}$.

Now suppose that $\mathbf{R} \in \mathcal{A} \circ \mathcal{B}$. Let $\theta$ be any $\mathcal{A}, \mathcal{B}$-pivot congruence on $\mathbf{R}$. Since $\mathbf{R}/\theta \in \mathcal{B}$ we have $\lambda^R_B \leq \theta$. Consequently, for every $r \in R$, $[r]_\lambda \subseteq [r]_\theta$. Suppose that $[r]_\lambda \in \text{Sub}(\mathbf{R})$. By implication (2) $[r]_\theta \in \text{Sub}(\mathbf{R})$ hence $[r]_\lambda \subseteq [r]_\theta \in \mathcal{A}$ which implies $[r]_\lambda \in \mathcal{A}$. Thus, in Definition 6, we can always take the $\mathcal{A}, \mathcal{B}$-pivot to be $\lambda^R_B$.

**Lemma 8.** Let $\mathcal{A}$ and $\mathcal{B}$ be any two quasivarieties. Then $\mathcal{A} \circ \mathcal{B}$ is closed under subalgebra.

**Proof.** Let $\mathbf{R} \in \mathcal{A} \circ \mathcal{B}$ and let $\theta$ be an $\mathcal{A}, \mathcal{B}$-pivot on $\mathbf{R}$. Let $\mathbf{S}$ be a subalgebra of $\mathbf{R}$. We must show $\mathbf{S} \in \mathcal{A} \circ \mathcal{B}$. Define $\psi = \theta|_S$. Then $\psi$ is a congruence on $\mathbf{S}$ and $\mathbf{S}/\psi \leq \mathbf{R}/\theta$, since $\mathcal{B}$ is closed under subs, $\mathbf{S}/\psi \in \mathcal{B}$.

Now let $t \in \mathbf{S}$ and assume $[t]_\psi \in \text{Sub}(\mathbf{S})$. We claim that $[t]_\theta \in \text{Sub}(\mathbf{R})$.

By equivalence (1)

$$
[t]_\psi \in \text{Sub}(\mathbf{S}) \implies f(t, \ldots, t) \psi t \implies f(t, \ldots, t) \theta t \implies [t]_\theta \in \text{Sub}(\mathbf{R}).
$$

Finally, since $[t]_\theta \in \text{Sub}(\mathbf{R})$, $[t]_\theta \in \mathcal{A}$. But $\mathcal{A}$ is closed under subs and $[t]_\psi \subseteq [t]_\theta$, so $[t]_\psi \in \mathcal{A}$ as desired. \qed

**Lemma 9.** Let $\mathcal{A}$ and $\mathcal{B}$ be any two quasivarieties of finite similarity type. Then $\mathcal{A} \circ \mathcal{B}$ is closed under reduced products. If $\mathcal{B}$ is idempotent, the requirement of finite similarity type can be dropped.

**Proof.** Let $\mathbf{R}_i \in \mathcal{A} \circ \mathcal{B}$, for $i \in I$, and let $\mathcal{F}$ be a filter on $I$. We must show $\prod_I \mathbf{R}_i/\eta_\mathcal{F} \in \mathcal{A} \circ \mathcal{B}$. By assumption, for each $i \in I$ we have a pivot congruence, $\theta_i$ on $\mathbf{R}_i$. Let us write $\mathbf{R} = \prod_I \mathbf{R}_i$.

For every $\mathbf{a}, \mathbf{b} \in \mathbf{R}$ define $J(\mathbf{a}, \mathbf{b}) = \{ i \in I : (a_i, b_i) \in \theta_i \}$. Note that $J(\mathbf{a}, \mathbf{b}) \supseteq [\mathbf{a} = \mathbf{b}]$. Let $\psi = \{ (\mathbf{a}, \mathbf{b}) \in R^2 : J(\mathbf{a}, \mathbf{b}) \in \mathcal{F} \}$. It is easy to check that $\psi \in \text{Con}(\mathbf{R})$ and that $\eta_\mathcal{F} \leq \psi$. By the correspondence theorem we have $\mathbf{R}/\psi \cong (\mathbf{R}/\eta_\mathcal{F})/(\psi/\eta_\mathcal{F})$.

Let us write $\overline{\mathbf{R}}$ in place of $\mathbf{R}/\eta_\mathcal{F}$, $\overline{\psi}$ for $\psi/\eta_\mathcal{F}$ and $\overline{\mathbf{r}}$ in place of $\mathbf{r}/\eta_\mathcal{F}$. Then the isomorphism in the previous paragraph can be rewritten as $\mathbf{R}/\psi \cong \overline{\mathbf{R}}/\overline{\psi}$. Our task is to show that $\overline{\mathbf{R}} \in \mathcal{A} \circ \mathcal{B}$. $\overline{\psi}$ will be the pivot congruence on $\overline{\mathbf{R}}$ that makes this happen.
Let \( h \) be the composite of the natural maps \( R \to \prod (R_i/\theta_i) \to \prod (R_i/\theta_i)/\eta_F \). Then \( h \) is surjective and unwinding the definition shows that \( \ker(h) = \psi \). Thus

\[
\mathbf{R}/\overline{\psi} \cong \mathbf{R}/\psi \cong \prod (R_i/\theta_i)/\eta_F \in \mathcal{B}
\]
since \( \mathcal{B} \) is closed under reduced products.

Now let \( \bar{r} \in \mathbf{R} \) and suppose that \( \bar{r}/\overline{\psi} \) is a subuniverse of \( \mathbf{R} \). We must show that \( [\bar{r}]_{\overline{\psi}} \in \mathcal{A} \). Let \( \bar{r} \) be an element of \( R \) such that \( \bar{r}/\eta_F = \bar{r} \). Note that \( \bar{r} \) is not unique. By Lemma 7, \( [\bar{r}]_{\overline{\psi}} = [\bar{r}]_{\psi}/\eta_F \) and \( [\bar{r}]_{\psi} \leq \mathbf{R} \).

**Claim:** Let \( K = \{ i \in I : [r_i]_{\theta_i} \in \text{Sub}(R_i) \} \). Then \( K \in \mathcal{F} \).

**Proof:** First, if \( \mathcal{B} \) is idempotent then \( K = I \) which is automatically a member of \( \mathcal{F} \). Now assume that the similarity type consists of finitely many operation symbols \( f_1, \ldots, f_m \). Then for any \( i \in I \), the condition that \([r_i]_{\theta_i} \) be a subuniverse is equivalent to

\[
(f_1(r, r, \ldots, r) \theta_i r) & (f_2(r, r, \ldots, r) \theta_i r) & \cdots & (f_m(r, r, \ldots, r) \theta_i r)
\]

which in turn is equivalent to

\[
i \in J(f_1(r, \ldots, r), r) \cap J(f_2(r, \ldots, r), r) \cap \cdots \cap J(f_m(r, \ldots, r), r).
\]

But \( [\bar{r}]_{\overline{\psi}} \) is a subuniverse, so for each \( j \leq m \), \( J(f_j(r, \ldots, r), r) \in \mathcal{F} \). Hence

\[
K = \bigcap_{j=1}^{m} J(f_j(r, \ldots, r), r) \in \mathcal{F}.
\]

Let \( \mathcal{F}' = \{ X \cap K : X \in \mathcal{F} \} \). Then one easily checks that \( \mathcal{F}' \) is a filter on \( K \). We shall show that

\[
[r]_{\psi}/\eta_F \cong \prod_{k \in K} [r_k]_{\theta_k}/\eta_{F'}
\]

This will finish the proof since for \( k \in K \), \( [r_k]_{\theta_k} \in \text{Sub}(R_k) \), hence by assumption, \( [r_k]_{\theta_k} \in \mathcal{A} \). Thus \( [\bar{r}]_{\overline{\psi}} = [\bar{r}]_{\psi}/\eta_F \in \mathcal{P}_r(\mathcal{A}) \subseteq \mathcal{A} \).

Recall that if \( x \in [\bar{r}]_{\overline{\psi}} \) then \( J(x, r) \in \mathcal{F} \), hence \( J(x, r) \cap K \in \mathcal{F}' \). For such an \( x \), define, for each \( k \in K \)

\[
\bar{x}_k = \begin{cases} x_k & \text{if } k \in J(x, r), \\ r_k & \text{otherwise.} \end{cases}
\]

Notice that \( \bar{x} \in \prod_{k \in K} [r_k]_{\theta_k} \) and \( \bar{x} \) agrees with \( x \) in “almost all” components.

Now define the map \( g: [\bar{r}]_{\overline{\psi}} \to \prod_{k \in K} [r_k]_{\theta_k}/\eta_{F'} \) by

\[
g(x) = \bar{x}/\eta_{F'}.
\]

\( g \) is easily seen to be a surjective homomorphism. We can finish the verification of (4) by showing that \( \ker(g) = \eta_F \) on \( [\bar{r}]_{\overline{\psi}} \). So let \( x, y \in [\bar{r}]_{\overline{\psi}} \). Then \( x \psi y \) implies \( J(x, y) \in \mathcal{F} \). Let \( Z = \{ k \in K : \bar{x}_k = \bar{y}_k \} \). Then

\[
g(x) = g(y) \iff Z \in \mathcal{F}' \iff K \cap J(x, y) \cap Z \in \mathcal{F}.
\]

But \( K \cap J(x, y) \cap Z \subseteq [x = y] \), so \( [x = y] \in \mathcal{F} \), hence \( (x, y) \in \eta_F \) as desired. \( \square \)
Theorem 10. The Maltsev product of two quasivarieties of finite type is again a quasivariety. (If the second quasivariety is idempotent, the assumption of finite type can be dropped.)


Lemma 11. If $A$ and $B$ are idempotent quasivarieties, then $A \circ B$ is idempotent.

Proof. Let $R \in A \circ B$ and $r \in R$. We must show that $r$ is idempotent. Let $\theta$ be a pivot congruence on $R$. Since $B$ is idempotent, $r/\theta$ is an idempotent element of $R/\theta \in B$, so $[r]_{\theta}$ is a subuniverse of $R$. Hence $[r]_{\theta} \in A$. Since all members of $A$ are idempotent and $r \in [r]_{\theta}$, $r$ is an idempotent element. 

The noteworthy thing about idempotence is that every congruence class is a subuniverse. Thus when both $A$ and $B$ are idempotent, we can ignore the clause $"[r]_{\theta} \in \Sub(R)"$ in the definition of Maltsev product.

Assume that $A$ and $B$ have finite similarity type, or that $B$ is idempotent. Then $C = A \circ B$ is a quasivariety, by Theorem 10. Let $F = F_C(X)$ be a free $C$-algebra over a set $X$. Then $F/\lambda_{_{\beta}} \cong F_B(X)$, the free $B$-algebra on $X$, [1, thm. 4.28]. Since $\lambda_{_{\beta}}$ can always serve as a pivot, we must have $[r]_{\lambda} \in \Sub(F) \implies [r]_{\lambda} \in A$. Unfortunately, there does not seem to be a natural way to view the algebra $[r]_{\lambda}$ as a homomorphic image of a free algebra on $A$.

As a rule, the Maltsev product of two varieties need not be a variety (even in the idempotent case). However, if all congruences permute then we do indeed get a variety.

Theorem 12. Let $A$ and $B$ be idempotent subvarieties of a quasivariety $C$, and suppose that $C$ is congruence-permutable (see [1, pg. 122]). Then $A \circ_C B$ is a variety.

Proof. By Theorem 10, we already know that the Maltsev product is closed under subalgebra and product, so the only thing left to show is closure under homomorphic images. For this let $R \in A \circ C B$ and $\alpha \in \Con(R)$. We must show $R/\alpha \in A \circ_C B$. Let $\theta$ be an $A,B$-pivot on $R$.

Let $\hat{\theta} = \theta \lor \alpha = \theta \circ \alpha$ (by congruence-permutability). We wish to show that $\hat{\theta}$ is an $A,B$-pivot on $R/\alpha$, that is

(5) \((R/\alpha)/(\hat{\theta}/\alpha) \in B\) and

(6) \(r \in R \implies [r/\alpha]_{\hat{\theta}/\alpha} \in A\).

Note that we are tacitly appealing to idempotence in the formulation of (6). The first of these is easy. By the second isomorphism theorem [1, thm. 3.5], \((R/\alpha)/(\hat{\theta}/\alpha) \cong R/\theta \in H(R/\theta) \subset B\).

Now let $r \in R$ and set $A = [r]_{\theta}$. $A$ is a subalgebra of $R$ by idempotence and $A \in A$ by assumption. Define $A^\alpha = \bigcup_{a \in A} [a]_{\alpha}$. 

By the third isomorphism theorem [1, thm. 3.8]

\[ \mathbf{A}^\alpha/(\alpha|_{\mathbf{A}^\alpha}) \cong \mathbf{A}/\alpha|_{\mathbf{A}} \in \mathcal{A}. \]

However, \( \mathbf{A}^\alpha = [r]_\theta \) since by congruence permutability

\[ x \in \mathbf{A}^\alpha \iff (\exists a \in R) \ x \ a \ \theta \ r \iff x \bar{\theta} \ r \iff x \in [r]_{\bar{\theta}}. \]

Finally, to verify (6) we need only observe that \([r/\alpha]_{\bar{\theta}/\alpha} = [r]_{\bar{\theta}/\alpha} = \mathbf{A}^\alpha/\alpha\). \( \Box \)

**Example 13** (Li, 2017). Let \( CIB \) denote the variety of all commutative, idempotent binars, and let \( Sq \) be the variety of binars satisfying the identities

\[ x^2 \approx x, \quad xy \approx yx, \quad x(xy) \approx y. \]

This is the variety of *squags*. Let \( q(x, y, z) = y(xz) \). Then it is easy to check that \( q \) is a Maltsev term for \( Sq \) [1, thm. 4.64]. Now define the term

\[ p(x, y, z) = (x(z(xy))) \cdot (z(x(zy))). \]

Then \( p \) is a Maltsev term for \( Sq \circ Sq \).

**Proof.** Let \( A \in Sq \circ Sq \). Thus, there is \( \theta \in \text{Con}(A) \) such that \( A/\theta \in Sq \) and every \( x/\theta \in Sq \).

We shall show that \( A \vDash p(x, x, z) \approx z \), i.e., \((x(z(x^2)))(z(x(xz))) \approx z \). Let \( w = x(zx) \). Since \( A/\theta \in Sq \),

\[ w/\theta = x/\theta \cdot (z/\theta \cdot x/\theta) = z/\theta \]

thus \( w, z \in [z]_{\theta} \in Sq \). But then (working in \([z]_{\theta} \)) \( p(x, x, z) \approx w(zw) \approx z \) as desired. The other identity, \( p(x, z, z) \approx x \), is similar. \( \Box \)

Thus, by Theorem 12, \( Sq \circ Sq \) is a variety. (Take \( \mathcal{A} = \mathcal{B} = Sq \) and \( \mathcal{C} = Sq \circ Sq \).

It would be interesting to find an equational base for \( Sq \circ Sq \).

**References**


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