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The Inverse Eigenvalue Problem of a Graph

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Abstract

Historically, matrix theory and combinatorics have enjoyed a powerful, mutually beneficial relationship. Examples include:

- Perron-Frobenius theory describes the relationship between the combinatorial arrangement of the entries of a nonnegative matrix and the properties of its eigenvalues and eigenvectors (see [53, Chapter 8]).
- The theory of vibrations (e.g., of a system of masses connected by strings) provides many inverse problems (e.g., can the stiffness of the springs be prescribed to achieve a system with a given set of fundamental vibrations?) whose resolution intimately depends upon the families of matrices with a common graph (see [46, Chapter 7]).

The Inverse Eigenvalue Problem of a graph (IEP-G), which is the focus of this chapter, is another such example of this relationship. The IEP-G is rooted in the 1960s work of Gantmacher, Krein, Parter and Fielder, but new concepts and techniques introduced in the last decade have advanced the subject significantly and catalyzed several mathematically rich lines of inquiry and application. We hope that this chapter will highlight these new ideas, while serving as a tutorial for those desiring to contribute to this expanding area.

Disciplines

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Comments

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Draft of Chapter 13: The Inverse Eigenvalue Problem of a Graph in **50 Years of Combinatorics, Graph Theory, and Computing**

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1 Introduction

Historically, matrix theory and combinatorics have enjoyed a powerful, mutually beneficial relationship. Examples include:

- Perron–Frobenius theory describes the relationship between the combinatorial arrangement of the entries of a nonnegative matrix and the properties of its eigenvalues and eigenvectors (see [53, Chapter 8]).
- The theory of vibrations (e.g., of a system of masses connected by strings) provides many inverse problems (e.g., can the stiffness of the springs be prescribed to achieve a system with a given set of fundamental vibrations?) whose resolution intimately depends upon the families of matrices with a common graph (see [46, Chapter 7]).

The Inverse Eigenvalue Problem of a graph (IEP-G), which is the focus of this chapter, is another such example of this relationship. The IEP-G is rooted in the 1960s work of Gantmacher, Krein, Parter and Fielder, but new concepts and techniques introduced in the last decade have advanced the subject significantly and catalyzed several mathematically rich lines of inquiry and application. We hope that this chapter will highlight these new ideas, while serving as a tutorial for those desiring to contribute to this expanding area.

Throughout, unless otherwise stated, all matrices have real entries, and all graphs are simple graphs, undirected, and finite. For a graph $G = (V(G), E(G))$, we use $|G|$ to denote $|V(G)|$. We refer the reader to [53] (respectively, [34]) for matrix (respectively, graph) theoretic results and concepts.

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Let $A = [a_{ij}]$ be an $n \times n$ symmetric matrix; the ij-entry of A is also denoted by $(A)_{ij}$. The *spectrum* of A is the multiset of eigenvalues of A and is denoted by $\operatorname{spec}(A)$. The *graph of* A, denoted $\mathcal{G}(A)$, has vertex set $\{1, 2, \ldots, n\}$ and edge ij joining i and j provided $i \neq j$ and $a_{ij} \neq 0$. Given a graph G with vertices $\{1, 2, \ldots, n\}$, the class $\mathcal{S}(G)$ consists of the symmetric matrices whose graph is G. For example, the $n \times n$ matrix A is irreducible and tridiagonal if and only if $\mathcal{G}(A)$ is the path on vertices $1, 2, \ldots, n$ (see Figure 1). The *IEP-G* asks: For a given graph G, what are the possible spectra of matrices in $\mathcal{S}(G)$?

$$A = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix}$$

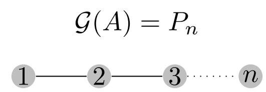

Figure 1: An irreducible $n \times n$ tridiagonal matrix and its graph

The IEP-G dates back at least as far as Gantmacher and Krein's work on tridiagonal matrices [43], and arguably even back to Stieltjes' work on continued fractions [69]. In many cases, a numerical linear algebra algorithm consists of a process to reduce the computation to one on a tridiagonal matrix followed by an optimized algorithm for tridiagonal matrices. In particular, early numerical methods for finding the spectrum of a matrix fit this paradigm. Tridiagonal matrices are flawed in the sense that no irreducible tridiagonal matrix has multiple eigenvalues. To see this, suppose that λ is an eigenvalue of A, and $\mathbf{x} = [x_i]$ is a null vector of $A - \lambda I$ with $x_1 = 0$. By examining the first entries of $(A - \lambda I)\mathbf{x} = \mathbf{0}$, we see that $x_2 = 0$, which similarly implies that $x_3 = 0$, etc. Hence, λ is a simple eigenvalue of A.

This naturally leads to the question: Which sets of n distinct real numbers can be the spectra of irreducible, tridiagonal $n \times n$ matrices? Or in modern language: What is the solution to the IEP-P_n, where P_n is a path on n vertices? Stieltjes [69], and later Gantmacher ad Krein [43], resolved the IEP-P_n by showing that every set of n distinct real numbers is the spectrum of some irreducible tridiagonal $n \times n$ matrix.

Motivated by the work of Gantmacher and Krein, Parter initiated the study of the spectral properties of matrices whose zero-nonzero entries are "highly structured." He implicitly defined the graph of a matrix and initiated the study of the IEP-T for trees T, with the goal of finding a generalization of tridiagonal matrices allowing irreducible matrices with non-simple eigenvalues. Parter proved the following surprising result [67]: If T is a tree, $A \in \mathcal{S}(T)$, and λ is an eigenvalue of A of multiplicity $m = 2$, then there exists an index i such that λ is an eigenvalue of $A(i)$ of multiplicity 3 (here $A(i)$ denotes the principal submatrix of A obtained by deleting row and column i of A). Wiener [71] later generalized this result to any multiplicity $m \geq 2$.

Other notable contributions to the IEP-G pertinent to our discussion and prior to the mid-2000s are given in the next theorem.

Theorem 1.

1. [40] *Every matrix in $\mathcal{S}(G)$ has distinct eigenvalues if and only if G is a path.*
2. [65] If G is a graph on n vertices and Λ is a set of n distinct real numbers, then there is a matrix $A \in \mathcal{S}(G)$ with spectrum Λ.
3. [55, 66] *If T is a tree, then the maximum multiplicity of an eigenvalue of a matrix in $\mathcal{S}(T)$ equals the smallest number of vertex disjoint induced paths that cover the vertices of T.*
4. [41] *If T is a tree, then the sign pattern of an eigenvector $\mathbf{v}$ corresponding to an eigenvalue λ of $A \in \mathcal{S}(T)$ and the sign-pattern of A determine the ordinal position of λ among the eigenvalues of A.*

We note that Theorem 1.4 is a seminal result on nodal domains for graph Laplacians, and is a key tool in graph partitioning, graph coloring, and other combinatorial optimization problems. More results concerning the maximum multiplicity and the minimum number of distinct eigenvalues will be discussed in Section 2.1.

This chapter focuses on two new approaches to the IEP-G that were introduced in the mid 2000s. The first, which we refer to as strong properties, is the topic of Section 3. The matrix A has the *strong Arnold property* (or the *SAP*) if $X = O$ is the only real symmetric matrix that satisfies $A \circ X = O$, $I \circ X = O$, and $AX = O$, where $\circ$ is the entrywise product. A graph parameter defined using matrices where the matrices are required to satisfy the SAP is called a *Colin de Verdière type parameter*. For example, $\xi(G)$ is the maximum nullity over every matrix in $\mathcal{S}(G)$ with the SAP. The utility of Colin de Verdière type parameters is based on the observation that the IEP-G (and its variants) is fundamentally a problem about the intersection of two manifolds. More precisely, given a multiset Λ of n real numbers, the set of all symmetric matrices with spectrum Λ is denoted by $\mathcal{E}_\Lambda$. It is known that $\mathcal{E}_\Lambda$ is a manifold [4]. The set $\mathcal{S}(G)$ is also a manifold, and there is a matrix $A \in \mathcal{S}(G)$ with spectrum Λ if and only if $\mathcal{E}(E) \cap \mathcal{S}(G) \neq \emptyset$. The transverse intersection theorem for manifolds, Theorem 11 below, provides a sufficient condition under which small perturbations of the manifolds will still intersect. For the IEP-G and its variants, the sufficient conditions can be nicely phrased as concrete conditions on a given matrix A in the intersection of the pertinent manifolds (see Table 1). In Section 3 we will see that determining whether or not a matrix satisfies the conditions leads to easily stated and accessible combinatorial and matrix theory problems. There are two major consequences of finding a matrix A with these special properties. First, one is able to argue that every supergraph of $\mathcal{G}(A)$ has a matrix with the same properties of interest as A (see Theorem 6). Second, one can define minor monotone graphical invariants; that is, a graph invariant τ such that $\tau(G) \leq \tau(H)$ whenever G is a minor of H. We note that this approach was informed by Colin de Verdière's work on discrete Schödinger operators and its implications to planarity of graphs [29], and research related to spectrally arbitrary patterns [35].

The second new approach, known as *zero forcing*, is a graph theoretical approach to providing an upper bound on the maximum multiplicity of an eigenvalue of a matrix in $\mathcal{S}(G)$. It is based on the following combinatorial observations about null vectors of a symmetric matrix [2]:

- If $A \in \mathcal{S}(G)$ is an $n \times n$ matrix with nullity $k \geq 2$, then for each subset of $k-1$ indices of $\{1, 2, \ldots, n\}$, there is a nonzero null vector with 0 in those $k-1$ positions.
- If $\mathbf{v}$ is a null vector of A and i is a vertex of G for which $v_i = 0$, then it is not the case that $v_\ell \neq 0$ for exactly one neighbor ℓ of i.

These observations lead to various graph propagation problems and parameters that provide bounds on the maximum multiplicity of an eigenvalue of any matrix in $\mathcal{S}(G)$. Additionally, the notion of zero forcing is closely related to other graph coloring and searching games that have been studied in various applications. Zero forcing and its variants are surveyed in Section 4.

One way to attack the IEP-G is to study various simpler invariants. These are discussed in the next section, as are variants.

2 Ancillary Problems

For a graph G the *minimum rank* (respectively, *maximum nullity*) of a matrix in $\mathcal{S}(G)$ is denoted by $\operatorname{mr}(G)$ (respectively, $\operatorname{M}(G)$). Clearly $\operatorname{mr}(G) + \operatorname{M}(G) = |G|$. As $A - \lambda I \in \mathcal{S}(G)$ whenever $A \in \mathcal{S}(G)$ and λ is real, $\operatorname{M}(G)$ equals the largest multiplicity of an eigenvalue of G. The number of distinct eigenvalues of A is denoted $q(A)$, and the *minimum number of distinct eigenvalues of* G, denoted $q(G)$, is the smallest value of $q(A)$ for $A \in \mathcal{S}(G)$.

The *ordered multiplicity list* of A is denoted by $\mathbf{m}(A)$ and defined by $\mathbf{m}(A) = (m_1, \ldots, m_{q(A)})$, where m_i is the multiplicity of the ith smallest eigenvalue of A. The IEP-T was solved for several families of trees T by determining feasible ordered multiplicity lists and showing that any ordered list of real numbers worked for each feasible ordered multiplicity list (see [37] and the references therein). However, Barioli and Fallat [7] gave an example of a tree that has restrictions on which real numbers can be used as the eigenvalues for one feasible ordered multiplicity list.

The *unordered multiplicity list of* A is the non-increasing sequence $\hat{m}_1, \ldots, \hat{m}_{q(A)}$ representing the multiplicities of the distinct eigenvalues of A. If an unordered multiplicity list is associated to a Ferrer's diagram, then $\operatorname{M}(G)$ (respectively, $q(G)$) denotes the greatest width (respectively, least height) of a Ferrer's diagram of the unordered multiplicity list of a matrix $A \in \mathcal{S}(G)$. Thus, $\operatorname{M}(G)$ and $q(G)$ provide fundamental constraints on the IEP-G.

2.1 Maximum Nullity and Minimum Rank

The paper [37] provides a useful survey of the minimum rank–maximum nullity problem. The book chapter [38] updates the survey, and provides initial results on zero forcing and implications for minimum rank; zero forcing is described in Section 4. Notable results on minimum rank–maximum nullity that do not rely on zero forcing or Colin de Verdière type properties are given in the next theorem ($\dot\cup$ denotes disjoint union).

Theorem 2.

1. [71] For each tree T, and each $A \in \mathcal{S}(T)$, the multiplicity of the smallest (respectively, largest) eigenvalue of A is one.
2. [38, Facts 46.1.15-16] Deletion of a vertex or edge changes the maximum nullity of a graph by at most one.
3. [38, Fact 46.1.20] $\mathrm{M}(G) = 1$ if and only if G is a path.
4. [38, Fact 46.1.21] If G is a connected graph on n vertices, then $\mathrm{M}(G) = n - 1$ if and only if $G = K_n$.
5. [13] A graph G satisfies $\mathrm{mr}(G) \leq 2$ if and only if G does not contain any of (P_4, dart, $P_3 \dot\cup K_2$, $3K_2$, $K_{3,3,3}$) as an induced subgraph.
6. [47] For n sufficiently large, the average minimum rank $\mathrm{amr}(n)$, of a graph on n vertices satisfies
$$.146907n < \mathrm{amr}(n) < .5n + \sqrt{7n \ln n}.$$

A catalog of known values or bounds on $\mathrm{M}(G)$ for various families of graphs can be found in [49]. Two well-known problems related to minimum rank are:

- **The Delta Conjecture:** If G is a graph with minimum degree $\delta(G)$, then $\delta(G) \leq \mathrm{M}(G)$.
  It is believed that the Delta Conjecture is true. A proof has been presented but is not yet published. It is known [63] that $\kappa(G) \leq \mathrm{M}(G)$ where $\kappa(G)$ is the vertex connectivity.
- **The Graph Complement Conjecture:** For each graph G on n vertices, $\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) \leq n + 2$.
  This Nordhaus–Gaddum type problem is very much an open problem, although it is known to be true for many specific families of graphs.

Since 2005, there have been many results about the maximum multiplicity that utilize Colin de Verdière type parameter techniques. We note the minor monotonicity of ξ can be used to give a simple proof of Theorem 1.1. One can verify that the adjacency matrix of $K_{1,3}$ satisfies the SAP and has nullity 2, and the 3×3 all ones matrix $J_3 \in \mathcal{S}(K_3)$ has nullity 2 and trivially has the SAP. Hence, every graph G containing a $K_{1,3}$ or K_3 minor, which is every connected graph other than a path, has $\mathrm{M}(G) \geq 2$. The minor monotonicity of ξ and strong properties are used to establish the next result.

Theorem 3.

1. [8] *If K_p is a minor of G, then $\mathrm{M}(G) \geq p - 1$. If $K_{p,q}$ with $p \leq q$ and $q \geq 3$ is a minor of G, then $\mathrm{M}(G) \geq p + 1$.*
2. [50] *A forbidden minor characterization of the graphs G for which $\xi(G) \leq 2$ is given.*
3. [11] *A forbidden minor characterization of the graphs G for which no matrix $A \in \mathcal{S}(G)$ has two multiple eigenvalues is given.*
4. [11] *A forbidden minor characterization of the graphs G for which no matrix $A \in \mathcal{S}(G)$ has consecutive multiple eigenvalues is given.*

2.2 Variants of Maximum Nullity and Minimum Rank

The ubiquity of positive semidefinite (PSD) matrices in applications and the relationship of PSD matrices to geometry have led to the study of eigenvalues of positive semidefinite matrices with off-diagonal nonzero pattern described by the edges of G. We denote this class of matrices by $\mathcal{S}_+(G)$. The *minimum positive semidefinite rank* of a graph G is denoted by $\mathrm{mr}_+(G)$ and is defined to be the smallest rank of a positive semidefinite matrix whose graph is G. We denote the *maximum positive semidefinite nullity* of matrices with graph G by $\mathrm{M}_+(G)$. The next theorem lists selected results for M_+ that utilize Colin de Verdière type parameters and arguments; results using more elementary techniques can be found in [38].

Theorem 4.

1. [38, Reference Hol03] $\mathrm{M}_+(G) = 1$ *if and only if* G *is a tree.*
2. [38, Reference Hol03] *The graphs* G *with* $\mathrm{M}_+(G) \leq 2$ *are characterized.*
3. [38, Reference Hol08b] *The 3-connected graphs with* $\mathrm{M}_+(G) \leq 3$ *are characterized.*

We note that mr_+ is related to the notion of orthogonal representations defined and studied in [63]. An *orthogonal representation* of G in $\mathbb{R}^d$ is an assignment $\mathbf{u}_i$ of a vector in $\mathbb{R}^d$ to each vertex i of G such that $\mathbf{u}_i^\top \mathbf{u}_j = 0$ whenever ij is not an edge of G. If in addition, $\mathbf{u}_i^\top \mathbf{u}_j \neq 0$ when ij is an edge, the representation is *faithful*. Thus it follows that $\mathrm{mr}_+(G)$ is the minimum dimension having a faithful orthogonal representation of G. For more details on the relationship between orthogonal representations and minimum positive semidefinite rank, see [38, References BHH08 and Hog08].

The maximum multiplicity and minimum rank for not necessarily symmetric matrices [38, Reference BFH09], skew-symmetric matrices [54], and matrices over fields other than $\mathbb{R}$ have also been studied [38, References BL05 and BFH08].

2.3 The Minimum Number of Distinct Eigenvalues

The graph parameter $q(G)$ has received considerable historical as well as recent attention. The *distance* $\text{dist}(u, w)$ between vertices u and w is the length of (number of edges in) the shortest path between u and w. The *diameter* of a connected graph G is the maximum distance between two vertices and is denoted by $d(G)$. The fact that

$$q(A) \geq d(G) + 1 \tag{1}$$

when A is the adjacency matrix of the graph G, is a folklore result in algebraic graph theory; a characterization of graphs for which equality holds is still not known.

Inequality (1) is also valid for $A \in \mathcal{S}(G)$ if A is nonnegative, or if G is a tree and A is an arbitrary matrix. This can be seen by noting that if B is the principal submatrix of A whose rows are indexed by the vertices of a diametrical path, then $I, B, B^2, \ldots, B^{d(G)}$ are linearly independent.

The first examples of trees T for which $q(T) > d+1$ are given in [7]. A family of trees for which $q(T) \geq \frac{9}{8}d(T) + \frac{1}{2}$ for $d(T) \geq 8$ is given in [59]. Interestingly, for a fixed d, there exists a constant c such that $q(T) \leq c$ for each tree T of diameter d [56] (c depends on d).

Fonseca [42] and Ahmadi et al. [1] introduce the study of $q(G)$ for graphs G that are not trees. The next theorem lists fundamental results for $q(G)$.

Theorem 5. [1]

1. *$q(G) = 1$ if and only if G has no edges.*
2. *$q(G) = |G|$ if and only if G is a path.*
3. *If there is a unique shortest path of length s between two vertices i and j, then $q(G) \geq s + 1$.*
4. *$q(G) \leq \text{mr}(G) + 1$.*
5. *$q(G) = 2$ if and only if there is an orthogonal matrix in $\mathcal{S}(G)$.*
6. *If G is connected, then $q(G \vee G) = 2$, where $\vee$ denotes the join of two graphs.*
7. *The insertion of an edge into a graph can significantly decrease the minimum number of distinct eigenvalues.*
8. *The insertion of an edge into a graph can significantly increase the minimum number of distinct eigenvalues.*

The paper [1] also began the study of graphs that require many distinct eigenvalues and gave constraints on graphs G with $q(G) = |G| - 1$.

Two minor-friendly, Colin de Verdière like parameters related to $q(G)$ are developed in [11]. In particular, these are used to characterize graphs G with $q(G) \geq |G| - 2$, and to show that $q(G)$ is bounded above by twice the chromatic number of its complement. The recent paper [15] continues this line of inquiry and establishes bounds on $q(G)$ for several families of graphs.

A generalization of zero forcing, known as *partial zero forcing* has been developed in [39] where it is used to analyze possible multiplicity lists of the spectra of matrices in $\mathcal{S}(G)$.

3 Strong Properties and Minor Monotonicity

Colin de Verdière used spectral properties of discrete Schrödinger operators on a graph to characterize the topological properties of the graph [29, 30]. The *Colin de Verdière parameter*, denoted by $\mu(G)$, is the maximum nullity over the matrices A in $\mathcal{S}(G)$ such that

- each off-diagonal entry of A is non-positive,
- A has exactly one negative eigenvalue, counting the multiplicities, and
- A has the strong Arnold property.

It is known that $\mu(G) \leq 1$ if and only if G is a disjoint union of paths; $\mu(G) \leq 2$ if and only if G is an outer planar graph; and $\mu(G) \leq 3$ if and only if G is planar. There are yet more connections between $\mu(G)$ and the topological properties of G; see, e.g., the survey by van der Holst, Lovász, and Schrijver [52].

An important property of the Colin de Verdière parameter is that $\mu(G) \leq \mu(H)$ if G is a minor of H. A parameter with this property is said to be *minor monotone*. By the graph minor theorem [34], there is a finite family $\mathcal{F}$ of graphs such that $\mu(G) \leq k$ if and only if G does not contain any $G' \in \mathcal{F}$as a minor. Take $k = 3$ as an example: $\mu(G) \leq 3$ if and only if G contains neither K_5 nor $K_{3,3}$ as a minor, which is equivalent to saying G is a planar graph. This has implications for the minimum rank problem: namely, if G is non-planar then $\mathrm{M}(G) \geq \mu(G) \geq 4$.

The SAP is a key to establishing the minor monotonicity of $\mu(G)$. Several other graph parameters are defined through the SAP and proved to have the minor monotonicity; these parameters, such as ξ, are referred as *Colin de Verdière type parameters*. Since ξ is minor monotone [8], $\mathrm{M}(H) \geq \xi(H) \geq \xi(G)$ for any minor G of H, providing lower bounds for the maximum nullity. Similarly, $\nu(G)$ is the maximum nullity over every positive semidefinite matrix in $\mathcal{S}(G)$ with the SAP, and it is also minor monotone [30], so $\mathrm{M}_+(H) \geq \nu(H) \geq \nu(G)$ if G is a minor of H.

Inspired by the SAP, two other properties of a matrix are introduced in [11] for the IEP-G; these are called *strong properties*. Recall that $[A, X] = AX - XA$ is the commutator of matrices A and X.

- A real symmetric matrix A has the *strong spectral property* (or the *SSP*) if $X = O$ is the only real symmetric matrix that satisfies $A \circ X = O$, $I \circ X = O$, and $[A, X] = O$.
- A real symmetric matrix A has the *strong multiplicity property* (or the *SMP*) if $X = O$ is the only real symmetric matrix that satisfies $A \circ X = O$, $I \circ X = O$, $[A, X] = O$, and $\mathrm{tr}(A^k X) = 0$ for $k = 0, \dots, q(A) - 1$.

Let H be a supergraph of G. The existence of a matrix $A \in \mathcal{S}(G)$ with a certain spectral property typically does not guarantee the existence of a matrix $B \in \mathcal{S}(H)$ with the same spectral property. However, as described in the next theorem, the existence of an $A \in \mathcal{S}(G)$ with one of the strong properties defined above does imply the existence of such a B.

Theorem 6. [11, 29] *Let G be a graph and H a supergraph of G with the same order. Suppose $A \in \mathcal{S}(G)$ has the SSP, SMP, or SAP, respectively. Then there is a matrix $B \in \mathcal{S}(H)$ such that*

- $\operatorname{spec}(A) = \operatorname{spec}(B)$ *and B has the SSP,*
- $\mathbf{m}(A) = \mathbf{m}(B)$ *and B has the SMP, or*
- $\operatorname{rank}(A) = \operatorname{rank}(B)$ *and B has the SAP,*

respectively.

3.1 Applications of the Strong Properties

Given a matrix with the SSP, Theorem 6 can be used to construct a denser matrix with the same spectrum. Suppose A and B are two real symmetric matrices with the SSP (or the SMP, respectively). Then $A \oplus B$ has the SSP (or the SMP, respectively) if and only if A and B have no common eigenvalues [11, Theorem 34]. This allows us to construct new matrices with the strong properties. For example, if $\lambda_1, \dots, \lambda_n$ are distinct, then the diagonal matrix $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ has the SSP. As a consequence, by Theorem 6, every graph on n vertices has a matrix in $\mathcal{S}(G)$ with the spectrum $\{\lambda_1, \dots, \lambda_n\}$ and the SSP. This proves statement Theorem 1.2. Also, a family of graphs G are found in [11, Figure 1] such that $q(G) \leq |G| - 2$ and the realizing matrices have the SSP. They are used to characterize graphs with $q(G) \geq |G| - 1$.

Next we focus on the SSP and introduce a more flexible tool, the Augmentation Lemma. Note that for a vector $\mathbf{x} \in \mathbb{R}^n$, the *support* $\operatorname{supp}(\mathbf{x})$ of $\mathbf{x}$ is the set of indices $i \in \{1, \dots, n\}$ such that the ith entry of $\mathbf{x}$ is nonzero.

Lemma 7 (Augmentation Lemma)**.** [10] *Let G be a graph on vertices $\{1, \dots, n\}$ and $A \in \mathcal{S}(G)$. Suppose A has the SSP and λ is an eigenvalue of A with multiplicity $k \geq 1$. Suppose that α is a subset of $\{1, \dots, n\}$ of cardinality $k+1$ with the property that for every eigenvector $\mathbf{x}$ of A corresponding to λ, $|\operatorname{supp}(\mathbf{x}) \cap \alpha| \geq 2$. Construct H from G by appending a new vertex $n+1$ adjacent exactly to the vertices in α. Then there exists a matrix $A' \in \mathcal{S}(H)$ such that A' has the SSP, the multiplicity of λ has increased from k to $k+1$, and other eigenvalues and their multiplicities are unchanged from those of A.*

We illustrate the use of the Augmentation Lemma to construct a matrix in $\mathcal{S}(C_n)$ that has the spectrum $\Lambda = \{\lambda_1, \dots, \lambda_{n-2}, \lambda_{n-1}^{(2)}\}$; C_n denotes the cycle on n vertices and $\lambda_{n-1}^{(2)}$ indicates that the multiplicity of λ_{n-1} is two. Let $A \in \mathcal{S}(P_{n-1})$ be a matrix with the spectrum $\{\lambda_1, \dots, \lambda_{n-1}\}$ and the SSP; such a matrix exists by Theorem 1.1. Let $\lambda = \lambda_{n-1}$ and $\mathbf{x}$ be an eigenvector of A with respect to the eigenvalue λ. Assume that the vertices of P_{n-1} are labeled by $\{1, \dots, n-1\}$ following the path order. Since $A - \lambda I$ is again a matrix in $\mathcal{S}(P_{n-1})$, its structure guarantees that the columns of $A - \lambda I$, except for the first column, form a linearly independent set. Therefore, if the first entry of $\mathbf{x}$ is zero, then $\mathbf{x} = \mathbf{0}$ is not an eigenvector. Equivalently, $1 \in \operatorname{supp}(\mathbf{x})$ and similarly $n-1 \in \operatorname{supp}(\mathbf{x})$. By applying the Augmentation Lemma with $\alpha = \{1, n-1\}$, there exists a matrix $A' \in \mathcal{S}(C_n)$ with the spectrum Λ and the SSP.

Note that Theorem 6 cannot be used directly with the subgraph $P_{n-1} \dot{\cup} K_1$ of C_n: Suppose we try to find a matrix $A \oplus [\lambda] \in \mathcal{S}(P_{n-1} \cup K_1)$ with the desired spectrum Λ and the SSP. Such a matrix must have $\operatorname{spec}(A) = \{\lambda_1, \dots, \lambda_{n-1}\}$ and $\lambda = \lambda_{n-1}$, since every matrix in $\mathcal{S}(P_{n-1})$ has to have all eigenvalues distinct. Thus, A and $[\lambda]$ would have a common eigenvalue, and $A \oplus [\lambda]$ will not have the SSP. The Augmentation Lemma gives us some more freedom in this case.

The Augmentation Lemma is a special case of the Matrix Liberation Lemma, which will be stated later, after we introduce the verification matrices.

The strong properties are also the key for establishing minor monotonicity. Recall that all the Colin de Verdière type parameters μ, ξ, and ν are minor monotone. That is, for example, $\mu(G) \leq \mu(H)$ if G is a minor of H. It is natural to ask whether some kind of minor monotonicity also holds for the SSP or the SMP. Indeed, there is a Minor Monotonicity Theorem in [10] that has some restrictions on how additional simple eigenvalues are added when an edge is "decontracted," as illustrated in the next result.

Theorem 8 (Decontraction Theorem for SSP)**.** [10] *Let G and H be graphs such that G is obtained from H by contracting one edge. Suppose $A \in \mathcal{S}(G)$ has the SSP. Then for every λ sufficiently large, there is an SSP matrix $A' \in \mathcal{S}(H)$ with* $\operatorname{spec}(A') = \operatorname{spec}(A) \dot{\cup} \{\lambda\}$.

Determining whether or not a given matrix has one of the strong properties reduces to finding the rank of a corresponding matrix, or can often be resolved by utilizing classic results from matrix theory. Here we offer an example of verifying the SSP. Note that lexicographic order for pairs $(i_1, j_1) \preceq (i_2, j_2)$ is defined by

$$(i_1, j_1) \preceq (i_2, j_2) \iff i_1 < i_2, \text{ or } (i_1 = i_2 \text{ and } j_1 \leq j_2).$$

Let $G = P_4$ be the path on 4 vertices and $A = [a_{ij}]$ a matrix in $\mathcal{S}(G)$. Suppose X is a real symmetric matrix that satisfies $I \circ X = O = A \circ X$. Then X can be written as

$$\begin{bmatrix} 0 & 0 & x_{1,3} & x_{1,4} \\ 0 & 0 & 0 & x_{2,4} \\ x_{1,3} & 0 & 0 & 0 \\ x_{1,4} & x_{2,4} & 0 & 0 \end{bmatrix},$$

where the x_{ij}'s are unknowns. Thus, $[A, X] = O$ is a system of 16 equations in three variables $x_{1,3}$, $x_{1,4}$, and $x_{2,4}$. However, by the skew-symmetry of $[A, X]$, 10 of the equations are redundant, and the 6 remaining equations are $([A, X])_{ij} = 0$ for $1 \leq i < j \leq 4$. Let Ψ_S be the 3×6 matrix that records the 6 equations in the columns and lists both the equations and the 3 variables in lexicographic order by index pair. Thus, the linear system becomes $\mathbf{x}\Psi_S = \mathbf{0}$ with $\mathbf{x} = \begin{bmatrix} x_{1,3} & x_{1,4} & x_{2,4} \end{bmatrix}$ and

$$\Psi_S = \begin{bmatrix} -a_{2,3} & a_{1,1} - a_{3,3} & -a_{3,4} & a_{1,2} & 0 & 0 \\ 0 & -a_{3,4} & a_{1,1} - a_{4,4} & 0 & a_{1,2} & 0 \\ 0 & 0 & a_{1,2} & -a_{3,4} & a_{2,2} - a_{4,4} & a_{2,3} \end{bmatrix}.$$

Note that the matrix Ψ_S depends on A. By definition, the given A has the SSP if and only if the corresponding Ψ_S has full row rank.

In this special case, the matrix Ψ_S for P_4 always has full row rank since $a_{2,3}$, $a_{3,4}$, and $a_{1,2}$ are nonzero for each matrix $A \in \mathcal{S}(P_4)$; that is, every matrix $A \in \mathcal{S}(P_4)$ has the SSP. In fact, this is true for paths of any length and can also be proved using some basic matrix theory. Here is the sketch of the argument: Let $G = P_n$ be a path of length n and A a matrix in $\mathcal{S}(G)$. Suppose X is a matrix with $I \circ X = O = A \circ X$. We may write $X = U + U^\top$, where U is a strictly upper triangular matrix. Thus, according to the patterns of A and U, $[A, U]$ is strictly upper triangular and $[A, U^\top]$ is strictly lower triangular. As $[A, X] = O$, this implies that $[A, U] = O$ and $[A, U^\top] = O$. However, every matrix $A \in \mathcal{S}(P_n)$ has all eigenvalues distinct and hence each matrix that commutes with A is a polynomial of A, which means U is a symmetric strictly upper triangular matrix. Therefore, both U and X are O.

Each of the strong properties can be verified by a matrix similar to Ψ_S. The matrix Ψ_S is known as the verification matrix, and we now define such a matrix for each of the strong properties. Let G be a graph on n vertices, $\overline{E} = E(\overline{G})$ and $p = |\overline{E}|$. Define X as an $n \times n$ symmetric matrix whose ij-entry ($i \leq j$) is a variable x_{ij} if $ij \in \overline{E}$ and zero otherwise.

Let $A \in \mathcal{S}(G)$ and $q = q(A)$. The *SSP verification matrix* $\Psi_S(A)$ of A is the $p \times \binom{n}{2}$ coefficient matrix of the linear system $\mathbf{x}\Psi_S = \mathbf{0}$ for the equations $([A, X])_{ij} = 0$ with $1 \leq i < j \leq n$. Similarly, the *SMP verification matrix* $\Psi_M(A)$ of A is the $p \times \left(\binom{n}{2} + q\right)$ coefficient matrix of the linear system $\mathbf{x}\Psi_M = \mathbf{0}$ for the equations $([A, X])_{ij} = 0$ with $1 \leq i < j \leq n$ and $\operatorname{tr}(A^k X) = 0$ for $k = 0, \ldots, q - 1$. The *SAP verification matrix* $\Psi_A(A)$ of A is the $p \times n^2$ coefficient matrix of the linear system $\mathbf{x}\Psi_A = \mathbf{0}$ for the equations $(AX)_{ij} = 0$ with $1 \leq i, j \leq n$.

Theorem 9. [10] Let A be a symmetric matrix. Then A has the SSP, the SMP, or the SAP if and only if the corresponding verification matrix has full row rank.

Sage code for computing the verification matrices and verifying the strong properties is available [62].

With the verification matrices defined, the Matrix Liberation Lemma [10] provides another tool that can be used when the matrix of interest does not have the SSP.

Lemma 10 (Matrix Liberation Lemma)**.** [10] *Let G be a graph and $A \in \mathcal{S}(G)$. Let $\Psi_S(A)$ be the SSP verification matrix. Suppose $\mathbf{x}$ is a vector in the column space of $\Psi_S(A)$ such that the complement of* $\operatorname{supp}(\mathbf{x})$ *corresponds to a linearly independent set of rows in $\Psi_S(A)$. Let H be a spanning subgraph of G whose edges correspond to* $\operatorname{supp}(x)$. *Then A can be perturbed to $A' \in S(G \cup H)$ such that A' satisfies the SSP with the same spectrum as A.*

We have seen many ways to perturb a matrix into another matrix with the desired spectral properties. These perturbations either fix the spectrum or fix the nullity, so we emphasize that if the matrix is positive semidefinite, than the resulting matrix remains positive semidefinite.

3.2 Tangent Spaces and the Implicit Function Theorem

The definitions of the strong properties come from the non-degenerate intersections between manifolds. You may imagine that manifolds are d-dimensional surfaces in $\mathbb{R}^n$; see, e.g., [60] for a formal definition. For a given point $\mathbf{x}$ on a manifold $\mathcal{M}$ in $\mathbb{R}^n$, there are various smooth, 1-dimensional paths lying on the surface and passing through the point. Each path gives a tangent vector at $\mathbf{x}$, and the span of the tangent vectors from all possible paths is the *tangent space* of $\mathcal{M}$ at $\mathbf{x}$, denoted by $\mathcal{T}_{\mathcal{M},\mathbf{x}}$. The tangent space is a linear subspace of $\mathbb{R}^n$, and its orthogonal complement is called the *normal space*, denoted by $\mathcal{N}_{\mathcal{M},\mathbf{x}}$. Now suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are manifolds in $\mathbb{R}^n$ and $\mathbf{x}$ is a point in $\mathcal{M}_1 \cap \mathcal{M}_2$. Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are said to *intersect transversally* at the point $\mathbf{x}$ if $\mathcal{N}_{\mathcal{M}_1,\mathbf{x}} \cap \mathcal{N}_{\mathcal{M}_2,\mathbf{x}} = \{\mathbf{0}\}$.

Here is an example to show all the mentioned concepts. Let $\mathcal{M}_1$ be the curve $y = x^2$ in $\mathbb{R}^2$. Let $\mathcal{M}_2$ be the line $y = mx$ in $\mathbb{R}^2$ for some slope m. Then $\mathbf{x} = (0,0)^\top$ is an intersection of $\mathcal{M}_1$ and $\mathcal{M}_2$. One may compute that $\mathcal{T}_{\mathcal{M}_1,\mathbf{x}} = \operatorname{span}\{(1,0)^\top\}$ and $\mathcal{N}_{\mathcal{M}_1,\mathbf{x}} = \operatorname{span}\{(0,1)^\top\}$. Similarly, $\mathcal{T}_{\mathcal{M}_2,\mathbf{x}} = \operatorname{span}\{(1,m)^\top\}$ and $\mathcal{N}_{\mathcal{M}_2,\mathbf{x}} = \operatorname{span}\{(-m,1)^\top\}$. Thus, $\mathcal{M}_1$ and $\mathcal{M}_2$ intersect transversally if and only if $m \neq 0$. Intuitively, a transversal intersection means the two manifolds "robustly" intersect. When $m \neq 0$ and the intersection is transversal, any small perturbation to $\mathcal{M}_1$ and $\mathcal{M}_2$ will keep an intersection nearby. (More than that, a point of the intersection can be chosen continuously with respect to the perturbations, as we will see soon.) In contrast, when $m = 0$ and the intersection is not transversal, a small perturbation to $\mathcal{M}_1$ or $\mathcal{M}_2$ may possibly cause the loss of the intersection. This intuition is formalized into Theorem 11, a version of the implicit function theorem for manifolds. The definition of a smooth family of manifolds can be found in [11] or [60].

Theorem 11. [52] *Let $\mathcal{M}_1(s)$ and $\mathcal{M}_2(t)$ be smooth families of manifolds in $\mathbb{R}^d$ for $s \in (-1,1)$ and $t \in (-1,1)$, and assume that $\mathcal{M}_1(0)$ and $\mathcal{M}_2(0)$ intersect transversally at $\mathbf{y}_0$. Then there is a neighborhood $W \subseteq \mathbb{R}^2$ of the origin and a continuous function $f : W \to \mathbb{R}^d$ with $f(0,0) = \mathbf{y}_0$ such that for each $\epsilon = (\epsilon_1, \epsilon_2) \in W$, $\mathcal{M}_1(\epsilon_1)$ and $\mathcal{M}_2(\epsilon_2)$ intersect transversally at $f(\epsilon)$.*

The set of all $n \times n$ real symmetric matrices, denoted by $S_n(\mathbb{R})$, is a linear space of dimension $\binom{n+1}{2}$ over $\mathbb{R}$. When G is a graph on n vertices, $\mathcal{S}(G)$ is a manifold in $S_n(\mathbb{R})$. Let $e = ij$ be an edge not appearing on G, and let Y be the symmetric matrix whose ij-entry and ji entry are 1, while every other entries is zero. For each $s \in \mathbb{R}$, define the manifold

$$\mathcal{S}(G,s) := \{B + sY : B \in \mathcal{S}(G)\}.$$

The family $\{\mathcal{S}(G,s)\}_{s\in(0,1)}$ is a smooth family of manifolds that is often used in studying manifolds of matrices. By definition, $\mathcal{S}(G,0) = \mathcal{S}(G)$ and $\mathcal{S}(G,s) \subset \mathcal{S}(G+e)$ for any $s \neq 0$.

Let $A \in \mathcal{S}(G)$. Then the set of matrices $B \in S_n(\mathbb{R})$ with $\operatorname{rank}(A) = \operatorname{rank}(B)$ is also a manifold, denoted by $\mathcal{R}_A$. Let $\mathcal{M}_1(s) = \mathcal{S}(G,s)$ and $\mathcal{M}_2(t) = \mathcal{R}$. Suppose $\mathcal{M}_1(0)$ and $\mathcal{M}_2(0)$ intersect transversally at A, then Theorem 11

	tangent space at A	normal space at A
$\mathcal{E}_\Lambda$	$\operatorname{span}\left(\{AK_{ij}+K_{ij}A\}_{\substack{i,j\\ i<j}}\right)$	$\{X : [A,X]=O\}$
$\mathcal{U}_{\mathbf{m}}$	$\operatorname{span}\left(\{AK_{ij}+K_{ij}A\}_{\substack{i,j\\ i<j}} \cup \{A^k\}_{k=0}^{q-1}\right)$	$\left\{X : \begin{array}{l}[A,X]=O \text{ and}\\ \operatorname{tr}(A^kX)=0\\ \text{for } k=0,\ldots,q-1\end{array}\right\}$
$\mathcal{R}_r$	$\operatorname{span}(\{AE_{ij}+E_{ij}A\}_{i,j})$	$\{X : AX=O\}$
$\mathcal{S}(G)$	$\mathcal{S}_{\text{cl}}(G)$	$\{X : A\circ X = I\circ X = O\}$

Table 1: The tangent spaces and the normal spaces for manifolds associated with the strong properties

says that $\mathcal{M}_1(\epsilon)$ intersects transversally with $\mathcal{M}_2(\epsilon) = \mathcal{R}_A$ when ϵ is small enough. In particular, the intersection, called A', is a matrix in $\mathcal{S}(G+e)$ and $\operatorname{rank}(A') = \operatorname{rank}(A)$. Thus, we have implicitly constructed a matrix whose graph is a supergraph of G while preserving the rank. Next we provide more details about each of the strong properties.

For any given spectrum Λ, the *iso-spectral manifold* is

$$\mathcal{E}_\Lambda = \{B \in S_n(\mathbb{R}) : \operatorname{spec}(B) = \Lambda\}.$$

For any given ordered multiplicity list $\mathbf{m}$, the *iso-mult manifold* is

$$\mathcal{U}_{\mathbf{m}} = \{B \in S_n(\mathbb{R}) : \mathbf{m}(B) = \mathbf{m}\}.$$

For any rank r, the *iso-rank manifold* is

$$\mathcal{R}_r = \{B \in S_n(\mathbb{R}) : \operatorname{rank}(B) = r\}.$$

For convenience, we also write $\mathcal{E}_A$ for $\mathcal{E}_{\operatorname{spec}(A)}$, $\mathcal{U}_A$ for $\mathcal{U}_{\mathbf{m}(A)}$, and $\mathcal{R}_A$ for $\mathcal{R}_{\operatorname{rank}(A)}$. To verify that these sets are indeed manifolds, see [11, 31].

Table 1 lists the associated tangent spaces and normal spaces for the manifolds used to define the strong properties discussed here, so that we may discuss the transversality with ease. In Table 1, G is a graph and A is a matrix in $\mathcal{S}(G)$ with rank r, ordered multiplicity list $\mathbf{m}$, spectrum Λ, and q distinct eigenvalues. Also, E_{ij} is the $n\times n$ matrix whose ij-entry is 1 and all other entries are zeros, while $K_{ij} = E_{ij} - E_{ji}$. The notation $\mathcal{S}_{\text{cl}}(G)$ stands for the closure of $\mathcal{S}(G)$. For the details of how to find the tangent spaces and the normal spaces in Table 1, see [11].

According to Table 1, it is immediate that a symmetric matrix A has the SSP (SMP, or SAP, respectively) if and only if $\mathcal{S}(G)$ and $\mathcal{E}_A$ ($\mathcal{U}_A$, or $\mathcal{R}_A$, respectively) intersect transversally at A. Thus, a small perturbation on a matrix A with the strong property, say from $S(G,0)$ to $S(G,\epsilon)$, does not lose the intersection, so it preserves the corresponding spectral property, giving Theorem 6.

4 Zero forcing, Propagation Time, and Throttling

Zero forcing is a coloring game on a graph, where the goal is to color all the vertices blue (starting with each vertex colored blue or white). There are numerous variations and applications. A blue vertex has various interpretations in applications, such as a zero in a null vector of a matrix (see Section 1), a node in an electrical network that can be monitored, a part of a graph that has been searched, or a person who has heard a rumor in a social network. In this section, we first discuss the origin and properties of zero forcing and related parameters, then discuss "time" to complete coloring, and finally, discuss minimizing some combination of number of blue vertices and time (throttling).

4.1 Zero Forcing and Its Variants

Zero forcing and its variants are distinguished by means of their color change rules. These rules define when a vertex may color another vertex blue, i.e., perform a force. Given a color change rule X, we define the sets of interest (X-zero forcing sets) and the associated graph parameter (the X-zero forcing number or $X(G)$) for all variants with one set of definitions: A subset $B \subseteq V$ defines an initial set of blue vertices (with all vertices not in B colored white); this is called a *coloring of* G. Given a coloring B of G, a *final coloring* or X*-final coloring* for B is a set of blue vertices obtained by applying the color change rule until no more changes are possible (other terms have been used for the final coloring, including the original term *derived set* and the more recent term *closure*). A *zero forcing set* or X*-zero forcing set* for G is a subset of vertices B such that a final coloring for B is $V(G)$. The *zero forcing number*, or X*-zero forcing number*, $X(G)$ is the minimum of $|B|$ over all X-zero forcing sets $B \subseteq V(G)$. There is code available for computing the zero forcing number and its variants, using the free open-source *Sage* software [25].

The *color change rule* or Z-*color change rule* is: A blue vertex u can change the color of a white vertex w to blue if w is the unique white neighbor of u. When u can change the color of w to blue, we say u *forces* w and write $u \to w$; this terminology and notation is also applied to other color change rules. A *color change rule requires adjacency* if v and w must be adjacent for v to force w. Most color change rules, including those discussed here, require adjacency (however, minor monotone floor color change rules do not [6]).

Observation 12. *Let* G *be a graph and let* X *be a color change rule that requires adjacency. If the connected components of* G *are* $G_1, \dots, G_t$*, then*

$$X(G) = \sum_{i=1}^{t} X(G_i).$$

One of the origins of zero forcing was as an upper bound for the maximum multiplicity, $\mathrm{M}(G)$, of an eigenvalue of a matrix in $\mathcal{S}(G)$ [2]. Suppose that $A \in \mathcal{S}(G)$, S is the set of currently blue vertices, and $\mathbf{x} = [x_i] \in \ker A$. If $x_v = 0$ for every $v \in S$, u is blue, and every neighbor of u except w is blue, then the equation $A\mathbf{x} = \mathbf{0}$ implies $x_w = 0$. This observation is used to prove the next result.

Theorem 13. [2] *For every graph* G*,* $\mathrm{M}(G) \le \mathrm{Z}(G)$*.*

The bound in Theorem 13 is tight (examples of graphs G for which $\mathrm{M}(G) = \mathrm{Z}(G)$ include trees, cycles, complete graphs, complete bipartite graphs, the complete edge subdivision of any graph [12], and many others (see [2] or [49]). If $n \ge 5$ is odd, then $\mathrm{M}(C_n \circ K_1) < \mathrm{Z}(C_n \circ K_1)$ [2] (here $\circ$ denotes the corona).

Zero forcing was introduced independently by Burgarth and Giovannetti in control of quantum systems [24], where it was called *graph infection*. The same process was later rediscovered and called *fast mixed graph searching* in [72].

Basic properties of the zero forcing number are listed in the next observation.

Observation 14. *Let* G *be a graph.*

1. $1 \le \mathrm{Z}(G) \le |G|$*, and if* G *contains at least one edge,* $1 \le \mathrm{Z}(G) \le |G| - 1$*.*
2. $\delta(G) \le \mathrm{Z}(G)$*.*

Values of $\mathrm{Z}(G)$ for various families of graphs were established in [2] and other papers. A reasonably current collection of such results can be found in [49]. The next theorem collects some results about the zero forcing number.

Theorem 15. *Let* G *be a graph.*

1. [2, 55] *For any tree* T*,* $\mathrm{Z}(T) = \mathrm{M}(T)$*.*
2. [36] $\mathrm{Z}(G) + \mathrm{Z}(\overline{G}) \ge |G| - 2$*.*

The next theorem collects several results about the graphs having extreme values of the zero forcing number. The *path cover number* of G, denoted by $\mathrm{P}(G)$, is the minimum number of vertex disjoint paths occurring as induced subgraphs of G that cover all the vertices of G. A graph G is a *graph of two parallel paths* if $\mathrm{P}(G) = 2$ and the graph can be drawn in the plane in such a way that the paths are parallel and edges (drawn as segments, not curves) between the two paths do not cross. (A graph that consists of two connected components, each of which is a path, is such a graph, but a single path is not.)

Theorem 16. *Let* G *be a graph.*

1. [38, Fact 46.4.13] $\mathrm{Z}(G) = 1$ if and only if G is a path.
2. [68] $\mathrm{Z}(G) = 2$ if and only if G is a graph of two parallel paths.
3. [68] Suppose G is a connected graph of order at least two. Then $\mathrm{Z}(G) = |G| - 1$ if and only if G is a complete graph.
4. [2] $\mathrm{Z}(G) \ge |G| - 2$ if and only if G does not contain P_4, $P_3 \cup K_2$, dart, $\ltimes$, or $3K_2$ as an induced subgraph. (cf. Theorem 2.5)

Positive semidefinite zero forcing

As noted in Section 2.2, the maximum multiplicity is also studied for positive semidefinite matrices. The *PSD color change rule*, or Z_+*-color change rule*, is: Let $S \subseteq V(G)$ be the set consisting of the blue vertices. Let $W_1, \dots, W_k$ be the sets of vertices of the $k \ge 1$ components of $G - S$. Let $w \in W_i$. If $u \in S$ and w is the only white neighbor of u in $G[W_i \cup S]$, then change the color of w to blue.

Theorem 17. [5, 63] *For every graph G, $\kappa(G) \leq \mathrm{M}_+(G) \leq \mathrm{Z}_+(G) \leq \mathrm{Z}(G)$.*

Each of the inequalities in Theorem 17 is tight but can be strict. Examples of graphs G for which $\mathrm{M}_+(G) = \mathrm{Z}_+(G)$ include trees, cycles, complete graphs, complete bipartite graphs, outerplanar graphs [38, Reference BFM11], and any graph for which $\kappa(G) = \mathrm{Z}_+(G)$. The Möbius Ladder of order eight, also known as V_8, provides a contrasting example, since $\mathrm{M}_+(V_8) < \mathrm{Z}_+(V_8)$ [5, 64].

Observation 18. *For any graph G, $1 \leq \mathrm{Z}_+(G) \leq |G|$. If G contains at least one edge, $1 \leq \mathrm{Z}_+(G) \leq |G| - 1$.*

The next theorem collects several results about the PSD zero forcing number ($\mathrm{tw}(G)$ denotes the tree-width of G).

Theorem 19. *Let G be a graph.*

1. [6] $\delta(G) \leq \mathrm{tw}(G) \leq \mathrm{Z}_+(G) \leq \mathrm{Z}(G)$.
2. [36] $\mathrm{Z}_+(G) + \mathrm{Z}_+(\overline{G}) \geq |G| - 2$.

The next theorem collects several results about the graphs having extreme values of the zero forcing number.

Theorem 20. *Let G be a graph.*

1. $\mathrm{Z}_+(G) = 1$ *if and only if G is a tree.*
2. *Suppose G is connected and of order at least two. Then $\mathrm{Z}_+(G) = |G| - 1$ if and only if G is a complete graph.*
3. [36] $\mathrm{Z}_+(G) = 1, 2, |G|-2, |G|-1$ *if and only if* $\mathrm{M}_+(G) = 1, 2, |G|-2, |G|-1$, *respectively.*
4. [36] *The graphs G with $\mathrm{Z}_+(G) = \mathrm{M}_+(G) = 2$ and $\mathrm{Z}_+(G) = \mathrm{M}_+(G) = |G| - 2$ have been characterized.*

Relationships with other graph searching parameters

In the discussion above, we emphasized zero forcing and PSD zero forcing because of their close connections to the associated IEP-Gs. Zero forcing also has deep connections to other graph parameters, especially those related to graph searching. The relationship between zero forcing and tree-width and its variants, such as path-width, is studied in [6]. Connections to the graph game Cops and Robbers are discussed in Section 4.3, since the relationship between the PSD zero forcing process and the strategy cops use to clear a graph was discovered in the study of throttling.

Power domination, which arose from the need to cost-effectively monitor an electric power network, can be thought of as a domination step followed by a zero forcing process, and may be the earliest appearance of zero forcing. Power domination was defined in [48] to model the monitoring capabilities of Phase Measurement Units (PMUs). A minimum power dominating set gives a placement of PMUs that monitors the network using the minimum number of PMUs. An equivalent version of the propagation rules [23], which we use here, clarifies that power domination is a domination step followed by zero forcing. For $v \in V$, the *neighborhood* $N(v)$ of v is the set of all vertices adjacent to v. For a set S of vertices in a graph G, define $PD(S) \subseteq V(G)$ by the algorithm:

1. $PD(S) = S \cup N(S)$.
2. While there exists $v \in PD(S)$ such that $|N(v) \cap (V(G) \setminus PD(S))| = 1$: $PD(S) = PD(S) \cup N(v)$.

The *power domination number* $\gamma_P(G)$ is the minimum cardinality of a set S such that $PD(S) = V(G)$. The relationship with zero forcing was identified and applied to a specific problem in [33], and then stated generally as:

Theorem 21. [14] *Let G be a graph that has an edge. Then $\left\lceil \frac{\mathrm{Z}(G)}{\Delta(G)} \right\rceil \le \gamma_P(G)$, and this bound is tight.*

Topics not covered

The preceding discussion of zero forcing, its variants, and related parameters is far from complete. Many topics are not covered due to space limitations. Examples of such omissions include other variants such as skew zero forcing [54], connected zero forcing [22] and k-forcing [3], bounds on zero forcing number [32, 45], computation of zero forcing numbers [21], zero forcing numbers of pseudo-random graphs [57], zero forcing under restrictions [20], partial zero forcing that provides additional information for the IEP-G [39], relations to additional parameters [61], zero forcing numbers of directed graphs [38, Reference BFH09], and probabilistic zero forcing [44, 58].

4.2 Propagation Time

Propagation time is the number of time steps needed for a minimum X-forcing set to color all the vertices blue, performing all possible independent forces at each time step. In this section and Section 4.3 we follow the universal definitions of propagation and throttling introduced in [27], rather than the original definitions, in order to more efficiently discuss multiple variants. The X-color change rule (think Z or Z_+) is given. For a given zero forcing set B, we construct the X-final coloring; the set $\mathcal{F}$ of forces performed is an *X-set of forces*. Define $\mathcal{F}^{(0)} = B$, and for $t \ge 1$, $\mathcal{F}^{(t)}$ is the set of vertices w such that 1) the force $v \to w$ appears in $\mathcal{F}$, 2) $w \notin \cup_{i=0}^{t-1}\mathcal{F}^{(i)}$, and 3) when the vertices $\cup_{i=0}^{t-1}\mathcal{F}^{(i)}$ are blue, w can be X-forced by v. The *X-propagation time of $\mathcal{F}$* in G, denoted by $\mathrm{pt}_X(G, \mathcal{F})$, is the least t such that $\cup_{i=0}^{t}\mathcal{F}^{(i)} = V(G)$; if B is not an X-forcing set then $\mathrm{pt}_X(G, \mathcal{F}) = \infty$. The *$X$-propagation time* of B in G is

$$\mathrm{pt}_X(G, B) = \min\{\mathrm{pt}(G, \mathcal{F}) : \mathcal{F} \text{ is a set of forces for } B\}.$$

The *X-propagation time* of G is

$$\mathrm{pt}_X(G) = \min\{\mathrm{pt}(G, B) : B \text{ is a minimum } X\text{-forcing set}\}.$$

Here we discuss *propagation time* (also called Z-*propagation time*) $\mathrm{pt}(G)$ and *PSD propagation time* $\mathrm{pt}_+(G)$ (also called Z_+-*propagation time*). The next theorem lists a small sample of results that have been obtained for $\mathrm{pt}(G)$; see [51] for additional results.

Theorem 22. *Let G be a graph.*

1. [51] $\frac{|G| - \mathrm{Z}(G)}{\mathrm{Z}(G)} \le \mathrm{pt}(G) \le |G| - \mathrm{Z}(G)$.

2. [51] $\text{pt}(G) = |G| - 1$ *if and only if* G *is a path. Graphs* G *having* $\text{pt}(G) = |G| - 2$ *are characterized.*

The study of PSD propagation time was introduced in [70]. The lower bound in Theorem 22.1 is not valid for PSD propagation time, due to the ability of one vertex to force many vertices.

Remark 23. *Let* G *be a graph and* $B \subset V(G)$*. Since any* Z*-force is a valid* Z_+*-force,* $\text{pt}_+(G, B) \leq \text{pt}(G, B)$*. However, there is no relationship between* $\text{pt}(G)$ *and* $\text{pt}_+(G)$*, because* $\text{pt}_+(G)$ *may use a smaller (minimum) forcing set.*

For all zero forcing parameters for which the color change rule requires adjacency, distance plays a fundamental role as a lower bound for propagation time and throttling. The *distance from a set* U *to a set* W of vertices is $\text{dist}(U \to W) = \max_{w \in W} \min_{u \in U} \text{dist}(u, w)$.

Observation 24. *Let* X *be a color change rule that requires adjacency. If* G *is a graph and* $B \subset V(G)$*, then* $\text{dist}(B \to V(G) \setminus B) \leq \text{pt}_X(G, B)$*.*

4.3 Throttling

Throttling addresses the question of minimizing the sum of the resources used to accomplish a task (number of blue vertices) and the time needed to complete that task (propagation time). Butler and Young [26] introduced the study of this question for standard zero forcing, Carlson et al. [28] studied throttling of PSD zero forcing, and Carlson [27] introduced the universal definition used here and studied throttling for (standard) zero forcing, PSD zero forcing, and other variants. The X-color change rule (think Z or Z_+) is given. The X*-throttling number of* B in G, is $\text{th}_X(G, B) = |B| + \text{pt}_X(G, B)$. The X*-throttling number* of G is

$$\text{th}_X(G) = \min_{B \subseteq V(G)} \{\text{th}(G, B)\}.$$

Notice that the set B that realizes the X-throttling number is not necessarily a minimum X-zero forcing set. Here we discuss the *throttling number* $\text{th}(G)$ (also called Z*-throttling number*) and the *PSD throttling number* $\text{th}_+(G)$ (also called Z_+*-throttling number*). The next two theorems list a small sample of results that have been obtained for $\text{th}(G)$ and $\text{th}_+(G)$, respectively; see [26] and [28] for additional results.

Theorem 25. [26] *If* G *is a graph of order* n*, then* $\text{th}(G) \geq \lceil 2\sqrt{n} - 1 \rceil$ *and* $\text{th}(P_n) = \lceil 2\sqrt{n} - 1 \rceil$*.*

Theorem 26. *Let* G *be a graph of order* n*.*

1. [28] *If* $\Delta(G) = 2$*, then* $\text{th}_+(G) \geq \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil$ *and* $\text{th}_+(P_n) = \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil$*.*

2. [28] *If* $\Delta(G) \geq 3$*, then*

$$\text{th}_+(G) \geq \left\lceil Z_+(G) + \log_{(\Delta(G)-1)} \left(\frac{(\Delta(G) - 2)n + 2\, Z_+(G)}{\Delta(G)\, Z_+(G)} \right) \right\rceil$$

and this bound is tight.

3. [28] $\text{th}_+(G) = n$ *if and only if* $G = K_n$. $\text{th}_+(G) = 2$ *if and only if* $G = K_{1,n-1}$ *or* $G = 2K_1$. *Graphs having* $\text{th}_+(G) = n-1$ *and* $\text{th}_+(G) = 3$ *are characterized.*

Throttling has also been studied for the game of Cops and Robbers, and results for cop throttling have implications for PSD throttling. Cops and Robbers is a two-player game on a graph G, where one player controls a team of cops, and the other controls a single robber. Initially, the cops choose a multiset of vertices to occupy, and then the robber chooses a vertex to occupy. A legal move is to remain at the current vertex or move to an adjacent vertex. In each round of the game, each cop makes a legal move, and then the robber makes a legal move. The aim for the cops is to *capture* the robber, that is, move to the same vertex that the robber currently occupies, and the aim for the robber is to avoid capture. The *cop number* $c(G)$ of G is the minimum number of cops needed to capture the robber. Cops and Robbers has been studied for more than forty years; see [17] and the references therein. The well-known *Meyniel conjecture* is that there is a constant a such that, for all n and for all graphs G of order n, $c(G) \le a\sqrt{n}$. The *k-capture time* $\text{capt}_k(G)$ is the minimum number of rounds needed for k cops to capture the robber on G over all possible games. The *cop throttling number* of a graph G is introduced in [19] and defined as $\text{th}_c(G) = \min_k\{k + \text{capt}_k(G)\}$; if $k < c(G)$, then it is assumed that the k-capture time is infinite. The *k-radius* of G is $\text{rad}_k(G) = \min_{B\subseteq V, |B|=k} \text{dist}(B \to V(G) \setminus B)$.

Theorem 27. [19] *If G is a graph and $B \subseteq V(G)$, then* $\text{capt}(G; B) \le \text{pt}_+(G; B)$, *so* $\text{th}_c(G; B) \le \text{th}_+(G; B)$. *Thus* $c(G) \le \text{Z}_+(G)$ *and* $\text{th}_c(G) \le \text{th}_+(G)$. *If T is a tree of order n then* $\text{th}_c(T) = \text{th}_+(T)$.

Theorem 28. [16] *If G is a chordal graph of order n then*

$$\text{th}_c(G) = \min_k(k + \text{rad}_k(G)) \le 2\sqrt{n}.$$

For trees, the previous result was first established in [18] ($\text{th}_c(T) = \min_k(k+\text{rad}_k(T))$) and [19] ($\text{th}_c(T) \le 2\sqrt{n}$).

5 Concluding Remarks and Open Problems

The introduction of the new strong properties has revitalized the study of the IEP-G and related mathematical topics, much as zero forcing invigorated the study of maximum multiplicity and minimum rank. Zero forcing and its related parameters, propagation time and throttling, are of independent interest. There are many further avenues to pursue, and each has combinatorial, matrix theoretic, and analytic aspects. Here we briefly comment on a few which we believe have the most potential for producing interesting mathematical results and techniques.

- The Graph Complement Conjecture and its variants
  The general goal is to obtain a good *Nordhaus–Gaddum sum* lower bound $\tau(G) + \tau(\overline{G}) \ge f(n)$ for a given graph invariant τ related to maximum nullity. Particular parameters of interest are M, M_+ and ν.

- Minimum number of distinct eigenvalues of a graph
  - Determine the asymptotic behavior of

$$\max_{\substack{\text{tree } T\\ d(T)=d}} q(T) \qquad \text{as } d \to \infty,$$

    where $d(T)$ is the diameter of T.
  - Characterize or give properties of graphs with small values of $q(G)$. Resolving $q(G) = 2$ would be a major step towards the open problem of characterizing sign-patterns of orthogonal matrices.
  - Characterize or give properties of graphs with large values (that is, near the order of G) of $q(G)$.

- Find structural properties of, or methods for constructing graphs, G for which $\mathrm{M}(G) = \mathrm{Z}(G)$.
  - There are many families of graphs G for which it has already been established that $\mathrm{M}(G) = \mathrm{Z}(G)$ (see the discussion after Theorem 13), and establishing $\mathrm{M}(G) = \mathrm{Z}(G)$ for additional families G may not be of major interest. However, finding structural properties (perhaps ones that graphs arising in applications tend to satisfy) that imply $\mathrm{M}(G) = \mathrm{Z}(G)$ (or $\mathrm{M}(G) < Z(G)$) would be of interest.
  - Find a readily computable upper bound on $\mathrm{M}(G)$ that significantly improves $\mathrm{M}(G) \leq Z(G)$. (It is known that $\mathrm{Z}(G) - \mathrm{M}(G) \geq 0.14n$ for almost all graphs for n sufficiently large [36, 47].)
  - Determine properties of graphs G for which $\mathrm{Z}(G) - \mathrm{M}(G)$ is small.

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