INTRODUCTION

Ultrasonic transmission and scattering in the high frequency regime are important problems in ultrasonic NDE, relating to both scattering from large flaws, and the transmission behavior of an ultrasonic beam in a complex geometry component. Ultrasonic modeling efforts are often confronted with problems for which asymptotic methods such as geometrical diffraction theory (GTD) prove inadequate due to insufficiently high frequency or lack of an appropriate canonical solution, but which are too high in frequency (too large in \( ka \)) for conventional numerical methods such as boundary element methods (BEM). The inefficiency of BEM at high frequency arises from the need to represent a rapidly oscillating wavefield, resulting in a fine BEM mesh. It is observed in this paper that if the phase characteristics of the wavefield are known \textit{a priori}, the spatial frequency bandwidth of the unknowns to be determined via numerical methods can be significantly reduced, resulting in a significant reduction in the number of basis functions (boundary elements) required for solution. It is fortunate that GTD generally yields good asymptotic phase information with few complications: it is in the determination of field amplitudes where GTD most often runs into difficulties, such as with singularities in the transport equations (caustics), and the determination of diffraction coefficients. A solution method described here which transforms the boundary integral equation (BIE) using an ansatz similar to that of GTD, in which the field phase is prescribed \textit{a priori} via GTD analysis, and field amplitudes are determined via a BEM-type numerical scheme. An example of application to critical angle beam transmission at a curved fluid-solid interface is presented.

ASYMPTOTIC PHASE TRANSFORMATION

The boundary integral equation governing ultrasonic phenomena at a smooth (continuously varying outward normal) fluid-solid interface can be written

\[
\int_B v_j(s) K_{ij}(s | s') \, ds = v_i^m(s') \quad i, j = 0, 1, 2, 3
\]  

\[
v_0(s) = p(x(s)) \quad i = 0 \tag{2}
\]

\[
v_i(s) = u_i(x(s)) \quad i = 1, 2, 3 \tag{3}
\]

\[
x(s) = (x_1(s_1, s_2), x_2(s_1, s_2), x_3(s_1, s_2)) \quad \in B \tag{4}
\]
\( K_{00} = \frac{1}{2} \delta(s-s') - \rho \phi_j(x(s)|x(s')) n_j(s) J(s) \) \hspace{1cm} (5)

\( K_{0j} = \rho \omega^2 \phi G_j(x(s)|x(s')) n_j(s) J(s) \quad j = 1, 2, 3 \) \hspace{1cm} (6)

\( K_{10} = u \phi G_j(x(s)|x(s')) n_j(s) J(s) \quad i = 1, 2, 3 \) \hspace{1cm} (7)

\( K_{ij} = \frac{1}{2} \delta_{ij} \delta(s-s') + \tau_k \phi G_j(x(s)|x(s')) n_k(s) J(s) \quad i, j = 1, 2, 3 \) \hspace{1cm} (8)

\[
\tau_{ij} = \delta_{ij} \lambda u_k + \mu (u_{i,j} + u_{j,i})
\]

\( p(x) = -\pi_{11}(x) = -\pi_{22}(x) = -\pi_{33}(x) \quad \text{in fluid} \) \hspace{1cm} (9)

where \( \phi_i(x), \tau_{ij}(x) \) are displacements and stresses in the solid, \( p(x) \) is the pressure in the fluid, \( \rho_f, \rho \) are the densities in the fluid and solid, respectively, \( \lambda, \mu \) are solid elastic constants, \( \lambda_f \) is the fluid elastic constant, and \( \omega \) is time harmonic frequency. The interface surface \( B \) is parameterized by \( x(s) \), for which \( n_i \) is the outward unit normal. The Jacobian of the mapping is denoted \( J(s) \). The superscript "in" denotes an incident wavefield, and the superscripts "G" denote the Green states determined by

\[
\tau_{ij,j:k}(x|x') + \rho \omega^2 \phi G_j(x|x') = \delta_{ik} \delta(x-x')
\]

\( p^{G_{ij}}(x|x') + \rho_f \omega^2 \lambda_f^{-1} p G(x|x') = \delta(x-x') \) \hspace{1cm} (11

Eq.(1) represents four coupled integral equations for the quantities \( \phi_i(s) \).

Conventional numerical BIE methods operate by projecting the solution onto a discrete basis \( b_{\alpha}(s), \alpha=1,\ldots,N \)

\[
\phi_i(s) = \sum_{\alpha} \phi_{i\alpha} b_{\alpha}(s)
\]

which, when substituted into the BIE, yields the linear system

\[
\sum_{\beta} \phi_{i\beta} M_{i\alpha\beta} = \phi_{i\alpha}^{in}
\]

\[
M_{i\alpha\beta} = \int_B b_{\alpha}'(s) K_{ij}(s|x') ds
\]

\[
\phi_{i\alpha}^{in} = \phi_{i\alpha}^{in}(x(s_{\alpha}))
\]

The performance of the solution method depends upon the appropriateness of the chosen basis set. This method of solution is generically referred to here as BEM, where the definition of "boundary element" is extended to include elements which might overlap, such as the use of a sinc function basis, some type of wavelet set, or in the extreme, a finite Fourier sinusoid series.

Interest in this paper is in problems whose solutions are effectively expressed in the form

\[
\phi_i(s) = \sum_{m=1}^{M} E_{i}^{m}(s) \exp( i f_m(s) )
\]

where \( f_m(s) \) represent field phase functions, \( E_{i}^{m}(s) \) are slowly varying field envelope (amplitude) functions, and the index \( m \) denotes various field components such as diffracted L and T waves, surface waves, etc., where the total number of such components \( M \) is a relatively small number. The numerical method proposed here treats eq.(17) as a variable transformation, where it is assumed that the phase functions \( f_m(s) \) are known \textit{a priori}. The phase functions are obtained via high-frequency asymptotics, hence the transformation is referred to as an \textit{asymptotic phase transformation}. Substitution of eq.(17) into the BIE yields
\[
\sum_{m=1}^{M} \int_{B} E_{j}^{m}(s) \exp(i f_{m}(s)) K_{ij}(s \mid s') \, ds = \nu_{i}^{in}(s')
\]

where the \(E_{j}^{m}(s)\) are now the unknown functions. Note that eq.(18) is a set of 4 integral equations in \(4M\) unknown functions, so, assuming eq.(1) has a unique solution, any solutions to eq.(18) clearly will not be unique. The problem is discretized by projecting the \(E_{j}^{m}(s)\) onto a basis set

\[
E_{j}^{m}(s) = \sum_{\alpha} E_{j,\alpha} b_{\alpha}(s)
\]

Substitution in eq.(18) yields

\[
\sum_{m=1}^{M} \sum_{\beta=1}^{N} E_{j,\beta}^{m} K_{ij,\alpha \beta} = \nu_{i}^{in}
\]

(19)

\[
K_{ij,\alpha \beta} = \int_{B} b_{\beta}(s) \exp(i f_{m}(s)) K_{ij}(s \mid s'_{\alpha}) \, ds
\]

(20)

(21)

Note that the product of the basis function and the phase factor in eq.(21) can be viewed as simply defining a new basis function. Therefore, assuming eq.(1) has a unique solution, it is readily seen that the linear system eq.(21) will have a unique solution if the functions formed by the product

\[
b_{\beta}(s) \exp(i f_{m}(s))
\]

are linearly independent for all \(\beta\) and \(m\). The efficiency of this solution method arises from the fact that the envelope functions \(E_{j}^{m}(s)\) are slowly varying functions, and hence should be adequately approximated using a relatively small number of basis functions.

The phase functions \(f_{m}(s)\) are to be determined \(a \text{ priori}\) by any possible means. The fact that the ansatz of GTD is essentially the same form as eq.(17) naturally suggests the use of GTD analysis for determining the solution form (number and type of \(m\) components) and approximation of the field phase. The ansatz of GTD is obtained by assuming a frequency dependence of the phase function in the form \(f_{m}(s, \omega) = \omega \psi_{m}(s)\). Substitution of this ansatz into the partial differential field equations, and equating the leading order terms in \(\omega\) to zero, yields a first order non-linear PDE for the determination of \(\psi_{m}(s)\) (eikonal equation). The solution of this PDE is generally straight-forward, involving simple analytic expressions (straight rays) or simple ray tracing computations (curved rays). The initial conditions for this PDE are extracted from consideration of appropriate canonical problems.

APPLICATION TO CRITICAL ANGLE TRANSMISSION

The inversion of the BIE governing surface wave transmission at a curved interface is presented as an example of application of the asymptotic phase transformation. The problem models leaky surface wave generation on a curved surface by a time harmonic ultrasonic beam. An axially-symmetric water-aluminum interface is considered, defined by

for \(s_{1} < 0:\n\]

\[
x_{1}(s) = s_{1}
\]

(23)

\[
x_{2}(s) = r_{0} \sin(s_{2} / r_{0})
\]

(24)

\[
x_{3}(s) = r_{0} \cos(s_{2} / r_{0})
\]

(25)

for \(s_{10} > s_{1} > 0:\n\]

\[
x_{1}(s) = r_{1} \sin(s_{1} / r_{1})
\]

(26)

\[
x_{2}(s) = (r_{0} + r_{1} (1 - \cos(s_{1} / r_{1})) \sin(s_{2} / r_{0})
\]

(27)

\[
x_{3}(s) = (r_{0} + r_{1} (1 - \cos(s_{1} / r_{1})) \cos(s_{2} / r_{0})
\]

(28)

(29)

43
for $s_1 > s_{10}$:

$$x_1(s) = r_1 \sin(s_{10} / r_1) + (s_1 - s_{10}) \cos(s_{10} / r_1)$$

$$x_2(s) = (r_0 + r_1 (1 - \cos(s_{10} / r_1)) + (s_1 - s_{10}) \sin(s_{10} / r_1)) \sin(s_2 / r_0)$$

$$x_3(s) = (r_0 + r_1 (1 - \cos(s_{10} / r_1)) + (s_1 - s_{10}) \sin(s_{10} / r_1)) \cos(s_2 / r_0)$$

$$s_{10} = \frac{2\pi}{3r_1}$$

where $r_0$, $r_1$ are convex and concave radii of curvature. For simplicity, the incident pressure on the surface is assumed to have a Gaussian amplitude in the form

$$p_{in}(x(s)) = \exp\left(-\alpha_i (s_i - s_{in}) (s_j - s_{in}^j) + i q_i (s_i - s_{in}^i)\right)$$

$$\alpha_i = \gamma_1 d_i d_j + \gamma_2 t_i t_j$$

$$d_1 = \cos(\phi^{in}), \quad d_2 = \sin(\phi^{in})$$

$$t_1 = -\sin(\phi^{in}), \quad t_2 = \cos(\phi^{in})$$

$$q_1 = k_W \cos(\phi^{in}) \sin(\theta^{in}), \quad q_2 = k_W \sin(\phi^{in}) \sin(\theta^{in})$$

$$\gamma_1 = \ln(0.02) / (b_0 / \cos(\theta^{in}))^2, \quad \gamma_2 = \ln(0.02) / b_0^2$$

where $s_{in}$ is the incidence position in the s-plane, $b_0$ is the diameter of the incident beam, $k_W$ is the wavenumber in water, $\theta^{in}$ is the polar incidence angle measured from normal, and $\phi^{in}$ is the azimuthal incidence angle rotating about the surface normal, with $\phi^{in}=0$ corresponding to incidence along $x_1$ (the axis of symmetry). It is assumed that the incident field position $s_{in}$ is in the region $s_1 < 0$. A visual representation of this surface plus the real part of an incident pressure is shown in fig.(1). The surface radii are $r_0 = 40$ wavelengths (in water), $r_1 = 20$ wavelengths, beam diameter is $b_0 = 10$ wavelengths, $\theta^{in} = 30$ degrees (near the Rayleigh critical angle), and $\phi^{in} = 30$ degrees.

The form of the solution and approximation of the field phases are obtained by examining the GTD analysis of the problem. The canonical problem appropriate to this case is two-dimensional beam transmission at a plane water-aluminum interface. The solution to this problem is written as a Fourier integral in the form

$$v_i(s) = \int \tilde{v}_{ij}^{in}(k) R_{ij}(k) \exp(i k s) dk$$

where $\tilde{v}_{ij}^{in}(k)$ is the Fourier transform of the incident field $v_{ij}^{in}(s)$ (s here is one-dimensional), and $R_{ij}(k)$ are reflection/transmission coefficients. The reflection/transmission coefficients display a pole at $k = k_R$ which governs leaky Rayleigh wave generation. This pole is isolated to obtain a solution in the form

$$v_i(s) = A_i^{spec}(s) \exp(i f^{spec}(s)) + A_i^{surf}(s) \exp(i f^{surf}(s))$$

$$f^{spec}(s) = f^{in}(s), \quad f^{surf}(s) = k_R s$$

Eq.(40) states that the solution is in the form of a specular reflection, plus a surface wave term. A form similar to eq.(40) is therefore adopted for the numerical BIE solution, where $s$ is now two-dimensional. The phase of the specular reflection term is set equal to the phase of the incident field (as also reported by Zhu [1]). A surface wave term is likewise assumed, but, rather than using eq.(41), the phase of the surface wave term is determined so as to take into account the effects of surface curvature over large distances. To this end, the ray theory
description of surface wave propagation over an arbitrarily curved surface is consulted. Surface rays are assumed to emanate perpendicular to an initial manifold determined in the s-plane by $(s_i - s_i^m) \Delta = 0$, where $\Delta$ is given by eq.(35). The surface waves follow geodesics of the surface, which are traced using a numerical ray tracing algorithm. The surface wave initial manifold and a few of the surface rays (geodesics) emanating from this manifold are depicted in fig.(2). Note the curvature of the surface rays as they traverse the surface. The initial manifold plus surface rays form a coordinate system denoted by the two-dimensional vector $g$, where $g_1$ denotes surface distance along the initial manifold, and $g_2$ denotes surface distance along the ray initiating at point $g_1$ on the initial manifold. The phase function is determined as

$$f_{\text{surf}}(s(g)) = \omega \int_0^{g_2} c_R^{-1}(s(g')) \, dg'$$

where $c_R(s)$ is the surface wave velocity as a function of surface position, and $s(g)$ represents the mapping from $g$ to $s$ surface coordinates. A rigorous development of ray methods for surface wave propagation on a curved traction-free surface is presented by Gregory [2], in which first-order corrections to the Rayleigh wave velocity due to surface curvature are obtained. It is noted that the phase perturbations introduced by surface curvature are slight, and hence, if ignored, result in a slow oscillation of the field amplitude function. (Note that the curvature-induced phase perturbation originally appeared as an oscillating amplitude factor in [2]). In the present work, it was deemed easier to specify a basis set for the amplitude function which had adequate bandwidth to represent this slight amplitude oscillation rather than explicitly evaluate and integrate the phase perturbation along the surface rays. In this case, the phase function is simply evaluated as

$$f_{\text{surf}}(s(g)) = \omega \, g_2 \, c_R^{-1}$$

where $c_R$ here denotes the Rayleigh wave velocity on a plane traction free surface. This simplification is an example of how errors in the phase approximation are corrected by the basis representing the field amplitude functions. The real part of the phase factor constructed by surface ray tracing is depicted in fig.(3).

The assumed form of solution, eq.(40), is substituted in the BIE, eq.(1), using the phase of the incident field for $f_{\text{spec}}(s)$ and the phase depicted in fig.(3) for $f_{\text{surf}}(s)$, to obtain a linear system, eq.(20), for the coefficients determining the field amplitudes. A basis of Gaussian-multiplied sinc functions was used to represent the amplitude functions

$$b_n(s_1, s_2) = \frac{\sin(\pi (s_1-s_{1n}))}{\pi (s_1-s_{1n})} \frac{\sin(\pi (s_2-s_{2n}))}{\pi (s_2-s_{2n})} e^{-\alpha_1 (s_1-s_{1n})^2-\alpha_2 (s_2-s_{2n})^2}$$

where $s_{in} = n \Delta s$; and the parameter $\alpha_i$ controls the decay of the basis function. In cases where the field phase is known exactly, the required bandwidth of the basis (point spacing $\Delta s$) is determined by the bandwidth of the field amplitude envelope. That is, the basis point spacing must be adequate to represent the envelope $E_1^m(s)$, and is independent of the frequency of the wavefield. Furthermore, if the surface curvature is not too severe, the bandwidth of the field envelope will be nearly that of the incident field envelope (in the limiting case of a plane interface, the bandwidth of the field envelope equals that of the incident field, as indicated by eq.(39)). Hence an effective practical guideline is to select a basis which accurately represents the incident field envelope. In practice, the prescribed phase functions will contain some error, so it is desirable to provide some additional bandwidth in the basis to accommodate corrections for small phase errors. In the problem at hand, a point spacing of 4.8 wavelengths in water proved adequate. The Gaussian multiplier had a value of .02 at 25 wavelengths in water. The position of discrete points for representing the field amplitude using this basis are depicted in fig.(4).
The real part of the calculated total pressure wavefield on the interface is depicted in fig. (5). The onset of curvature at large distance in the surface wave wavefront due to field diffraction is noted in the final solution, even though such diffraction effects were not explicitly built into the a priori phase functions. This observation is another example of how errors in the phase are corrected by the basis representing the field amplitude functions. The $ka$ for this problem is approximately 320, where $k$ is the wavenumber in water, and $a$ is the approximate width of the discretized surface.

A quantitative comparison of pressure fields on flat versus curved surfaces is presented. The same surface as shown in fig. (1) is considered, but now the incident beam is aligned with the symmetry axis of the surface, $\phi^{in} = 0$, for the sake of simplicity in plotting the pressure field versus distance on the surface. The same incident beam diameter and polar incidence angle is assumed as in fig. (1). The total pressure field on the interface is calcu-
lated using the same procedure as described in reference to figs.(2 through 5), and the result is plotted in fig.(6). The same incident pressure was also applied to a planar fluid-solid interface, and the total pressure field was evaluated. A comparison between the pressure fields on the curved and planar interfaces is presented in fig.(7), which plots the real part of the total pressure down the center of the surface fields as a function of distance along the surface. The effects of the surface curvature on field phase and amplitude are evident. The surface wave amplitude decays more rapidly over the curved surface, due to coupling to bulk wave fields. This perturbation in surface wave amplitude is not predicted by first-order ray theory. The a priori specified phase function in both cases assumed a constant surface wave velocity. Note however, that the phase of the surface wave over the curved surface deviates from the phase over the plane surface. This again demonstrates how slight errors in the a priori phase are corrected by the basis functions representing the field amplitude.

Fig. 5 Real part of total pressure on surface

Fig. 6 Total field for incidence along symmetry axis

Fig. 7 Comparison of pressure in fig.6 with pressure on flat surface
A technique for numerical inversion of boundary integral equations such as govern elastodynamic transmission and scattering phenomena has been presented. It has been demonstrated how a priori knowledge of the general form and phase of the solution can be used to significantly reduce the computation required to obtain numerically exact (i.e. numerically convergent) solutions. This technique assumes a solution based on the form of the a priori information, and numerically seeks multiplicative correction factors using a BEM-type solution to the BIE which the assumed solution must satisfy. A desirable attribute of this technique is that the amount of computation required varies inversely to the amount of a priori knowledge of the solution: if the form and phase of the solution is known to a high degree of accuracy, then relatively little computation (few basis functions) is required. As errors in the assumed form increase, the number of basis functions required to correct these errors likewise increases. In the limit where no knowledge of the solution is available, the technique reduces to a conventional BEM technique.

The assumed form of the solution is essentially the ansatz of GTD. Thus the usual techniques of identifying canonical problems to extract solution forms and ray tracing techniques for determining field phases are directly applicable. However, the numerical technique demonstrated here circumvents the need to solve transport equations or evaluate diffraction coefficients, which are usually the stumbling blocks of GTD analysis. An attractive feature of this technique is that errors in the prescription of a priori phase do not necessarily result in errors in the solution: rather, they result in an increase in the amount of computation required for numerical convergence. Thus slight corrections due to diffraction effects, etc., not predicted by GTD can be obtained with little additional computation. A significant feature of this technique is that the size of the required computation does not necessarily increase with frequency: indeed, it is conceivable that computation could actually decrease with frequency as the high-frequency asymptotics become more valid.

An example of application was shown which considered critical angle transmission at a curved fluid-solid interface. By examining an appropriate canonical problem and referring to the theory of high-frequency surface wave transmission over a curved boundary, an effective approximation to the form and phase of the solution was constructed. This data was incorporated into the numerical scheme which determined the true field amplitudes and slight corrections to the field phase. The application of this technique to more complicated problems will only be limited by the quality of the a priori information available. Fortunately, for many problems of interest, a priori information regarding the field phase and form is readily available: it has been the determination of field amplitudes which has proven problematic. It is conceivable that the numerical technique described here could be applied to study a large number of important, and heretofore unsolved, problems.

REFERENCES