ULTRASONIC SCATTERING FROM SPHERICALLY ORTHOTROPIC SHELLS

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INTRODUCTION

Concerns over the detectability of embrittlement in high strength alloys has led to studying a simple anisotropic shell model [1] for grain boundaries decorated by precipitates, or otherwise enriched by segregated inhomogeneities. In this model the shell is presumed to be "spherically orthotropic," having five independent elastic constants and symmetry about the origin of a spherical coordinate system. This structure is analogous to transversely isotropic materials in a Cartesian coordinate system. By studying ultrasonic scattering from such shells (embedded in an isotropic host, and surrounding an isotropic core), we hope to learn whether their presence could be detected, and differentiated from scattering due to the inherent anisotropy of single metal crystals [2,3].

The problem of scattering from a single isotropic spherical obstacle in an isotropic elastic continuum was solved by Ying and Truell [4], and can be extended in a straight-forward manner to concentric spherical shells. The critical feature allowing this extension is that the differential equations for displacement (or its potential) separate, and that the portion describing the dependence on polar angle is independent of material properties. Boundary conditions can therefore be matched at shell surfaces, which are constant values of radius. Conceptually, one can envision a shell composed of numerous concentric layers of isotropic materials; even in this case the solution in every layer is separable, and has the same dependence on polar angle. In the limit as layer thickness goes to zero and the number of layers increases correspondingly, we arrive at a homogeneous shell with spherically orthotropic elastic constants.

Pursuing the shell composed of concentric isotropic layers a bit further, we see that if the layers are alternately one isotropic material, then another, the elastic properties of the composite shell can only depend on the two Lamé constants for each material, and the ratio of thicknesses attributed to each of the isotropic materials. Thus the five independent elastic constants associated with the spherically orthotropic material can be related to four Lamé constants and a ratio of thicknesses. Postma [5] presents this derivation in the case of plane layers.
The starting point for the present study are exact differential equations for displacements in a spherically orthotropic material [1].

\[ 0 = P_m(x) \left\{ C_{11} F_m + \frac{2}{r} C_{11} F_m + \left( \rho \omega^2 + \frac{2}{r^2} (C_{12} - C_{22} - C_{23}) - \frac{1}{r^2} m(m+1)C_{66} \right) F_m \right\} \]

\[ -\frac{1}{r} m(m+1)(C_{12} + C_{66}) G_m + \frac{1}{r^2} m(m+1)(C_{22} + C_{23} + C_{66} - C_{12}) G_m \} \]  
(1a)

\[ 0 = \frac{\partial}{\partial \phi} P_m(x) \left\{ C_{66} G_m^+ + \frac{2}{r} C_{66} G_m^+ + \left( \rho \omega^2 + \frac{1}{r^2} ((1-m(m+1))C_{22} - C_{23} - 2C_{66}) \right) G_m \right\} + \frac{1}{r} (C_{12} + C_{66}) F_m^+ + \frac{1}{r^2} (C_{22} + C_{23} + 2C_{66}) F_m \} \]  
(1b)

In these equations symmetry about the \( \hat{x}_3 \) is assumed, as is appropriate for an incident plane L-wave arriving along the \( \hat{x}_3 \) axis (see Figure 1). The solution's dependence on polar angle is contained in the Legendre polynomials, \( P_m(x) \), where \( m \) is the separation variable (an integer), \( x = \cos \phi \). The solution's dependence on radial position is contained in the functions \( F_m(r) \) and \( G_m(r) \), where:

\[ u_r(r, \phi) = \sum_{m=0}^{\infty} F_m(r) P_m(\cos \phi) \]  
(2a)

\[ u_\phi(r, \phi) = \sum_{m=0}^{\infty} G_m(r) \frac{\partial}{\partial \phi} P_m(\cos \phi) \]  
(2b)

AN EXACT SOLUTION TO THE ANISOTROPIC FIELD EQUATIONS

Temporarily suppressing subscripts indicating order and superscripts indicating region, the "\( r \)" dependent parts of the anisotropic field equations (1a) and (1b) may be written:

\[ 0 = r^2 F'' + A_{11} r F' + A_{10} F + K_1^2 r^2 F + B_{11} r G' + B_{10} G \]  
(3a)

\[ 0 = A_{21} r F' + A_{20} F + K_2^2 r^2 G + r^2 G'' + B_{21} r G' + B_{20} G \]  
(3b)
where values of the coefficients are given in Appendix A.

According to the method of Frobenius [6] normally used for obtaining series solutions to second-order linear differential equations, we may assume solutions of the form:

\[ F = \sum_{i=0}^{\infty} f_i r^{i+p} \quad \text{and} \quad G = \sum_{i=0}^{\infty} g_i r^{i+p} \]

and differentiate term by term. The resulting series, substituted into equations (3a) and (3b) give rise to a pair of algebraic equations for each power of \( r \) in order to satisfy the equations at all radii. The lowest power, obtained with \( i = 0 \), gives:

\[ 0 = f_0 r^p (p(p-1) + 2p + A_{10}) + g_0 r^p (B_{11} + B_{10}) \]

\[ 0 = f_0 r^p (A_{21} + A_{20}) + g_0 r^p (p(p-1) + 2p + B_{20}) \]

which may be written in matrix form, for \( r \neq 0 \):

\[
\begin{pmatrix}
(A_{21} + A_{20}) & (p^2 + p + A_{10}) \\
B_{11} + B_{10} & (B_{11} + B_{10})
\end{pmatrix}
\begin{pmatrix}
f_0 \\
g_0
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\tag{4c}
\]

For non-trivial solutions we deduce the indicial equation by setting the determinant of the left-hand side matrix to zero:

\[ 0 = p^4 + 2p^3 + p^2(A_{10} + B_{20} - A_{21} B_{11} + 1) + p(A_{21} B_{10} - A_{20} B_{11} + (A_{10} B_{20} - A_{20} B_{10}) \]

\[ \tag{5} \]

The four roots to this equation:

\[ p = -1 \pm \sqrt{1 + 2 \left( A_{31} B_{11} - A_{10} B_{20} \right) \pm \sqrt{(A_{31} B_{11} - A_{10} B_{20})^2 - 4(A_{10} B_{20} - A_{20} B_{10})}} \]

are related in pairs:

\[ p = p_1, -(p_1 + 1), p_2, \text{ or } -(p_2 + 1) \]

For isotropic elastic constants the roots are \( m + 1, -(m + 2), m - 1, \text{ and } -m \); these values lead to power series representations of the spherical Bessel and Hankel functions that appear in the Ying and Truell solution.

Associated with each root are values of \( f_0 \) and \( g_0 \) which may be determined (to within a multiplicative constant) by considering either equation (4a) or (4b). Once these are determined, other terms in the series are calculated by the iterative procedure described next.

For each order, \( m \), of the indicial equation, we define

\[ M(x) = \begin{pmatrix}
(x^2 + x + A_{10}) & (B_{11} x + B_{10}) \\
(A_{21} x + A_{20}) & (x^2 + x + B_{20})
\end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix}
K_1^2 & 0 \\
0 & K_2^2
\end{pmatrix} \tag{6} \]

The field equations may now be written.
\[ \sum_{i=0}^{\infty} r^{-i+p}(M(i+p)+Kr^2) \left( \begin{array}{c} f_i \\ g_i \\ p \end{array} \right) = (0) \] (7)

Requiring that this hold for all values of radius means that each power of \( r \) may be treated separately; indeed, the indicial equation corresponds to \( i = 0 \). For \( i = 1 \), and roots not differing by exactly 1, \( \text{Det}(M(p+1)) \neq 0 \); to satisfy equation (7) we must set \( f_i \) and \( g_i \) equal to zero.

For \( i \geq 2 \) and \( r \neq 0 \), values of \( f_i r^i \) and \( g_i r^i \) are calculated iteratively:

\[ \left( \begin{array}{c} f_i r^i \\ g_i r^i \\ p \end{array} \right) = -M^{-1}(i+p)Kr^2 \left( \begin{array}{c} f_i -2r^{i-2} \\ g_i -2r^{i-2} \\ p \end{array} \right) \] (8)

In the special case of isotropic elastic constants, for \( p_1 = (m-1) \) or \( p_1 = -(m+2) \), one may show that:

\[ \left( \begin{array}{c} f_2 \\ g_2 \\ p_1 \end{array} \right) = \lim_{x \to p_1^+} \left( -M^{-1}(x)K \left( \begin{array}{c} f_0 \\ g_0 \end{array} \right) \right) \] (9)

exists even though \( M^{-1}(p_1+2) \) is indeterminate (explicit expressions are given in Appendix A). It should also be pointed out that for the roots \( p_2 = (m+1) \) and \( p_2 = -(m) \):

\[ M(p_2) \left( \begin{array}{c} f_0 \\ g_0 \end{array} \right)_{p_2} = (0) \] (10a)

while for the roots \( p_1 = (m-1) \) and \( p_1 = -(m+2) \):

\[ M(p_1+2) \left( \begin{array}{c} f_2 \\ g_2 \end{array} \right)_{p_1} = -K \left( \begin{array}{c} f_0 \\ g_0 \end{array} \right)_{p_1} \] (10b)

Therefore, when \( p_2 = p_1 + 2 \), the values calculated by equation (9) for \( \left( \begin{array}{c} f_2 \\ g_2 \end{array} \right)_{p_1} \) may contain an additive multiple of \( \left( \begin{array}{c} f_0 \\ g_0 \end{array} \right)_{p_2} \). When boundary conditions are matched, however, the coefficients of each series will account for this uncertainty.

We are therefore led to a complete solution for displacements in the anisotropic shell formed by a linear combination of the four series:

\[ \left( \begin{array}{c} F_m^S \\ G_m^S \end{array} \right) = \sum_{k=1}^{4} C_{mk}^* r^k \sum_{i=0}^{\infty} \left( \begin{array}{c} f_i \\ g_i \end{array} \right)_{p_{mk}} r^i \] (11)

where the superscript "S" (denoting the shell) and subscripts have been fully restored ("m" denotes the order of the equation and "k" indicates which root of the indicial equation is involved in each series). This solution may now be differentiated term by term, as needed, to calculate stresses. The eight available boundary condition equations (matching two stresses and two displacements at each shell surface) are used to solve for the four coefficients in equation (11) and the four coefficients associated with the L-waves and T-waves in the isotropic host and core.
By studying scattering from anisotropic shells it is hoped that it will eventually be possible to differentiate backscatter due to decorated grain boundaries from that produced by metal grains with ordinary grain boundaries. The geometry shown in Figure 2 is being studied numerically, with several distinct types of scatters. Scattering from single crystals, randomly rotated, and embedded in an effective medium [7,8], has been studied through the Independent Scattering Model [9,10]. We extend this work to numerical studies of scattering from anisotropic shells, where shell properties have been chosen to represent the possible effects of precipitates or microvoids on the grain boundaries. In these numerical studies, the contribution of each scatterer to the displacement at each field point is determined by multiplying the incident field at that scatterer by the farfield scattering amplitude at the appropriate polar angle. We account for the phase of the incident field, which varies with the scatterer’s position relative to the origin \( \vec{R} - \vec{r} \), and polar angle, which varies with the scatterer’s position relative to the field point \( \vec{r} \cdot \hat{x}_3 \).

In all studies, the concentration of scatterers was kept rather dilute, and multiple scattering was not considered.

Calculations made so far have embedded anisotropic shells in an aluminum matrix. Shell properties are variations on the elastic constants for silicon carbide, with only \( C_{444} \), which is equal to \( C_{111} \), modified (values are given in reference [1]). Typical results for scatterers with a radius slightly larger than the longitudinal wavelength \( (k\alpha = 10) \) and a thin shell (thickness of the shell 1% of radius), are shown in Figures 3 and 4. Figure 3 shows results for strongly anisotropic shells, while Figure 4 is for weakly anisotropic shells. These figures are presented as plots of \( \hat{x}_3 \) direction displacements on a grid of field points covering \( \pm 15^\circ \) around the \( \hat{x}_3 \) axis. We observe that displacements appear to increase roughly as the square root of the number of scatterers, as one might expect if the contribution from each scatterer is viewed as one step in a two-dimensional random walk process. (We recognize that the argument for this analogy is currently very loose, but feel that it is nonetheless physically significant). Some basic features of the single scatterer response, such as the angular distribution of scattered energy, appear to be preserved in the case of numerous individual scatterers, and these may be useful in experimentally determining grain boundary properties from ultrasonic observations.
SUMMARY

An exact solution for scattering from spherically orthotropic shells has been developed and numerically verified. Plots of the scattered displacement field from a distribution of such scatterers suggest that anisotropy may be reflected in the magnitude and angular distribution of scattered energy.

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APPENDIX A

In equation (4):

\[ A_{11} = 2 \quad \quad A_{10} = \frac{1}{C_{11}}(2(C_{12} - C_{22} - C_{33} - m(m + 1)C_{46}) \]

\[ A_{23} = \frac{1}{C_{66}}(C_{12} + C_{46}) \quad \quad A_{20} = \frac{1}{C_{66}}(C_{22} + C_{23} + 2C_{46}) \]

\[ B_{11} = -\frac{1}{C_{11}} m(m + 1)(C_{12} + C_{46}) \quad \quad B_{20} = \frac{1}{C_{11}} m(m + 1)(C_{22} + C_{23} + C_{46} - C_{12}) \]

\[ B_{23} = 2 \quad \quad B_{20} = \frac{1}{C_{66}} (C_{22} - C_{23} - 2C_{46} - m(m + 1)C_{23}) \]

\[ K_1^2 = \frac{1}{C_{11}} \rho \alpha^2 \]

\[ K_2^2 = \frac{1}{C_{66}} \rho \alpha^2 \]

In equation (9):

\[ \left( \begin{array}{c} f_0 \\ g_{0,1} \\ \end{array} \right) = \left( \begin{array}{c} m \\ 1 \\ \end{array} \right) \quad \quad \left( \begin{array}{c} f_1 \\ g_{1,1} \\ \end{array} \right) = \left( \begin{array}{c} -K_2^2 \\ 2(1 + 2m)(3 + 2m) \end{array} \right) \left( \begin{array}{c} m((2 + m) + \nu(1 + m)) \\ m + \nu(3 + m) \end{array} \right) \]

\[ \left( \begin{array}{c} f_0 \\ g_{0,2} \\ \end{array} \right) = \left( \begin{array}{c} -(1 + m) \\ 1 \\ \end{array} \right) \quad \quad \left( \begin{array}{c} f_1 \\ g_{1,2} \\ \end{array} \right) = \left( \begin{array}{c} -K_2^2 \\ 2(1 + 2m)(1 - 2m) \end{array} \right) \left( \begin{array}{c} (1 + m)((1 - m) - \nu m) \\ (1 + m) - \nu(2 - m) \end{array} \right) \]

where \( \nu = C_{11}/C_{66} = K_2^2/K_1^2 \)