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## Sign patterns that require or allow power-positivity

### Abstract

A matrix  $A$  is power-positive if some positive integer power of  $A$  is entrywise positive. A sign pattern  $A$  is shown to require power-positivity if and only if either  $A$  or  $-A$  is nonnegative and has a primitive digraph, or equivalently, either  $A$  or  $-A$  requires eventual positivity. A sign pattern  $A$  is shown to be potentially power-positive if and only if  $A$  or  $-A$  is potentially eventually positive.

### Keywords

Power-positive matrix, Eventually positive matrix, Requires power-positivity, Potentially power-positive, Potentially eventually positive, Sign pattern

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## SIGN PATTERNS THAT REQUIRE OR ALLOW POWER-POSITIVITY\*

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**Abstract.** A matrix  $A$  is power-positive if some positive integer power of  $A$  is entrywise positive. A sign pattern  $\mathcal{A}$  is shown to require power-positivity if and only if either  $\mathcal{A}$  or  $-\mathcal{A}$  is nonnegative and has a primitive digraph, or equivalently, either  $\mathcal{A}$  or  $-\mathcal{A}$  requires eventual positivity. A sign pattern  $\mathcal{A}$  is shown to be potentially power-positive if and only if  $\mathcal{A}$  or  $-\mathcal{A}$  is potentially eventually positive.

**Key words.** Power-positive matrix, Eventually positive matrix, Requires power-positivity, Potentially power-positive, Potentially eventually positive, Sign pattern.

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**1. Introduction.** A matrix  $A \in \mathbb{R}^{n \times n}$  is called *power-positive* [2, 10] if there is a positive integer  $k$  such that  $A^k$  is entrywise positive ( $A^k > 0$ ). Note that if  $A$  is a power-positive matrix, then  $-A$  is also power-positive, because  $A^k > 0$  implies  $(-A)^{2k} > 0$ . If there is an odd positive integer  $k$  such that  $A^k > 0$ , then  $A$  is called *power-positive of odd exponent*. Power-positive matrices have applications to the study of stability of competitive systems in economics; see, e.g., [7, 8, 9]. A real square matrix  $A$  is *eventually positive* if there exists a positive integer  $k_0$  such that  $A^k > 0$  for all  $k \geq k_0$ . An eventually positive matrix and its negative are both obviously power-positive.

A *sign pattern matrix* (or *sign pattern*) is a matrix having entries in  $\{+, -, 0\}$ . For a real matrix  $A$ ,  $\text{sgn}(A)$  is the sign pattern having entries that are the signs of the corresponding entries in  $A$ . If  $\mathcal{A}$  is an  $n \times n$  sign pattern, the *sign pattern class* (or *qualitative class*) of  $\mathcal{A}$ , denoted  $\mathcal{Q}(\mathcal{A})$ , is the set of all  $A \in \mathbb{R}^{n \times n}$  such that

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$\text{sgn}(A) = \mathcal{A}$ .

If  $\mathcal{P}$  is a property of a real matrix, then a sign pattern  $\mathcal{A}$  *requires*  $\mathcal{P}$  if every real matrix  $A \in \mathcal{Q}(\mathcal{A})$  has property  $\mathcal{P}$ , and  $\mathcal{A}$  *allows*  $\mathcal{P}$  or is *potentially*  $\mathcal{P}$  if there is some  $A \in \mathcal{Q}(\mathcal{A})$  that has property  $\mathcal{P}$ . Sign patterns that require eventual positivity have been characterized in [4], and sign patterns that allow eventual positivity have been studied in [1]. Here we characterize patterns that require power-positivity (Theorem 2.6 and Corollary 2.7) and show that a sign pattern  $\mathcal{A}$  allows power-positivity if and only if  $\mathcal{A}$  or  $-\mathcal{A}$  allows eventual positivity (Theorem 3.1).

**1.1. Definitions and notation.** Let  $\mathcal{A} = [\alpha_{ij}]$  and  $\hat{\mathcal{A}} = [\hat{\alpha}_{ij}]$  be sign patterns. If  $\alpha_{ij} \neq 0$  implies  $\alpha_{ij} = \hat{\alpha}_{ij}$ , then  $\mathcal{A}$  is a *subpattern* of  $\hat{\mathcal{A}}$ . For a sign pattern  $\mathcal{A} = [\alpha_{ij}]$ , the *positive part* of  $\mathcal{A}$  is  $\mathcal{A}^+ = [\alpha_{ij}^+]$  where  $\alpha_{ij}^+$  is  $+$  if  $\alpha_{ij} = +$  and  $0$  if  $\alpha_{ij} = 0$  or  $\alpha_{ij} = -$ ; the *negative part* of  $\mathcal{A}$  is defined analogously (see [1]). Note that  $\mathcal{A}^- = (-\mathcal{A})^+$ . We use  $[-]$  (respectively,  $[+]$ ) to denote a (rectangular) sign pattern consisting entirely of negative (respectively, positive) entries. The *characteristic matrix*  $C_{\mathcal{A}}$  of the sign pattern  $\mathcal{A}$  is the  $(0, 1, -1)$ -matrix obtained from  $\mathcal{A}$  by replacing  $+$  by  $1$  and  $-$  by  $-1$ .

For an  $n \times n$  sign pattern  $\mathcal{A} = [\alpha_{ij}]$ , the *signed digraph* of  $\mathcal{A}$  is

$$\Gamma(\mathcal{A}) = (\{1, \dots, n\}, \{(i, j) : \alpha_{ij} \neq 0\})$$

where an arc  $(i, j)$  is positive (respectively, negative) if  $\alpha_{ij} = +$  (respectively,  $-$ ). Conversely, for a signed digraph  $\Gamma$  on the vertices  $\{1, \dots, n\}$ , the *sign pattern* of  $\Gamma$  is  $\text{sgn}(\Gamma) = [s_{ij}]$  where  $s_{ij} = +$  (respectively,  $-$ ) if there is a positive (respectively, negative) arc from vertex  $i$  to vertex  $j$ , and  $s_{ij} = 0$  otherwise. There is a one-to-one correspondence between  $n \times n$  sign patterns and signed digraphs on the vertices  $\{1, \dots, n\}$  and we adopt some sign pattern notation for signed digraphs. For example,  $\mathcal{Q}(\Gamma) = \mathcal{Q}(\text{sgn}(\Gamma))$  and  $C_{\Gamma} = C_{\text{sgn}(\Gamma)}$ .

A signed digraph  $\Gamma$  is called *primitive* if it is strongly connected and the greatest common divisor of the lengths of its cycles is  $1$ . This definition applies the standard definition of “primitive” for a digraph that is not signed to a signed digraph by ignoring the signs. Clearly for a sign pattern  $\mathcal{A}$ ,  $\Gamma(\mathcal{A})$  is primitive if and only if  $\Gamma(-\mathcal{A})$  is primitive.

A *signed subdigraph* of a signed digraph is a subdigraph in which the arcs retain the signs of the original signed digraph. Let  $\Gamma'$  be a signed digraph on  $n$  vertices, and let  $\Gamma$  be a signed subdigraph of  $\Gamma'$  on  $k$  vertices. Without loss of generality (by relabeling the vertices of  $\Gamma'$ ) assume that the vertices of  $\Gamma$  are  $\{1, \dots, k\}$ . For  $A = [a_{ij}] \in \mathcal{Q}(\Gamma)$ , define the  $n \times n$  matrix  $B = [b_{ij}]$  by  $b_{ij} = a_{ij}$  if  $(i, j) \in \Gamma'$ , and  $0$  otherwise. Then we call  $B$  the  $\Gamma'$ -*embedding* of  $A$ . Note that the sign pattern  $\mathcal{B} = \text{sgn}(B)$  is a subpattern of  $\text{sgn}(\Gamma')$ . When a  $\Gamma'$ -embedding is used in Section

2,  $\Gamma'$  is the signed digraph  $\Gamma(\mathcal{A})$  of a sign pattern  $\mathcal{A}$ , and we assume the necessary relabeling has been done.

**1.2. Power-positive and eventually positive matrices.** This subsection contains some known results about power-positive matrices and their applications. Any matrix in the sign pattern class of the sign pattern in Example 3.4 below illustrates the well known fact that there exist power-positive matrices that are not eventually positive.

An eigenvalue  $\lambda_0$  of a matrix  $A$  is *strictly dominant* if  $|\lambda_0| = \rho(A)$  and for every eigenvalue  $\lambda \neq \lambda_0$ ,  $|\lambda| < |\lambda_0|$ . Every power-positive matrix  $A$  has a unique real simple strictly dominant eigenvalue  $\lambda_0$  having positive left and right eigenvectors [10]. Furthermore, if  $A$  is power-positive of odd exponent, then  $\lambda_0 = \rho(A)$ ; otherwise,  $\lambda_0$  may be negative. For example, any negative matrix  $A$  is power-positive (with only the even powers being positive), and in this case  $\lambda_0 = -\rho(A)$ . The next theorem can be deduced from [2] and the discussions on pages 43-47 in [10].

**THEOREM 1.1.** [2, Theorem 3] *If  $A$  is a power-positive matrix, then either  $A$  or  $-A$  has a positive simple strictly dominant eigenvalue having positive left and right eigenvectors.*

**THEOREM 1.2.** [6, p. 329] *The matrix  $A$  is eventually positive if and only if  $A$  is power-positive of odd exponent.*

**THEOREM 1.3.** [6, Theorem 1] *The matrix  $A$  is eventually positive if and only if  $A$  has a positive simple strictly dominant eigenvalue having positive left and right eigenvectors.*

**COROLLARY 1.4.**  *$A$  is a power-positive matrix if and only if either  $A$  or  $-A$  is eventually positive.*

**REMARK 1.5.** *Note that Corollary 1.4 is not in general true if positive is replaced by nonnegative. For example, the matrix  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  is a power-nonnegative matrix since  $A^2 \geq 0$ , but neither  $A$  nor  $-A$  is eventually nonnegative.*

In economics, power-positive matrices arise in the context of stability of competitive systems. Let  $A = B - sI$ ,  $s > 0$ . A system of dynamic equations [9] such as

$$\frac{dx}{dt} = Ax, \quad x(0) = x_0 \quad (1.1)$$

can be interpreted as a system of price adjustment equations of competitive markets in a general equilibrium analysis.

THEOREM 1.6. [9, Theorem 1] *The competitive system (1.1) is dynamically stable if and only if  $s > \rho(B)$  and  $B$  satisfies one of the following conditions:*

1.  *$B$  is a power-positive matrix of odd exponent, or*
2.  *$B$  is a power-positive matrix and the entries of a row or of a column of  $B$  are all nonnegative.*

Note that the first of the two conditions on  $B$  given in Theorem 1.6 is equivalent to the eventual positivity of  $B$ , while the second implies that  $B$  is eventually positive. Furthermore, the matrix  $-A = sI - B$  in such a dynamically stable system is a pseudo- $M$ -matrix as defined in [6].

**2. Sign patterns that require power-positivity.** In [4] it is shown that  $\mathcal{A}$  requires eventual positivity if and only if  $\mathcal{A}$  is nonnegative and  $\Gamma(\mathcal{A})$  is primitive. In this section we use similar perturbation techniques to show that a sign pattern  $\mathcal{A}$  requires power-positivity if and only either  $\mathcal{A}$  or  $-\mathcal{A}$  is nonnegative and  $\Gamma(\mathcal{A})$  is primitive.

OBSERVATION 2.1. *Let  $\mathcal{A}$  be an  $n \times n$  sign pattern,  $\Gamma$  a signed subdigraph of  $\Gamma(\mathcal{A})$ ,  $A \in \mathcal{Q}(\Gamma)$  and  $B$  the  $\Gamma(\mathcal{A})$ -embedding of  $A$ . Then the nonzero eigenvalues of  $B$  are the nonzero eigenvalues of  $A$ , and the eigenvectors for the nonzero eigenvalues of  $B$  are the eigenvectors of the corresponding eigenvalues of  $A$ , suitably embedded.*

It is well known that for any matrix  $A \in \mathbb{R}^{n \times n}$ , the eigenvalues of  $A$  are continuous functions of the entries of  $A$ . For a simple eigenvalue, the same is true of the eigenvector (see, for example, [5, p. 323]).

LEMMA 2.2. *Let  $\mathcal{A}$  be an  $n \times n$  sign pattern,  $\Gamma$  a signed subdigraph of  $\Gamma(\mathcal{A})$  and  $A \in \mathcal{Q}(\Gamma)$ .*

1. *If every nonzero eigenvalue of  $A$  is simple and  $A$  does not have a nonnegative eigenvector, then  $\mathcal{A}$  does not require power-positivity.*
2. *If  $A$  has a simple strictly dominant eigenvalue  $\rho(A)$  that does not have a nonnegative eigenvector, then  $\mathcal{A}$  does not require power-positivity.*

*Proof.* Let  $B$  be the  $\Gamma(\mathcal{A})$ -embedding of  $A$ . In either case, by Observation 2.1, the matrix  $B$  retains the property of not having a nonnegative eigenvector for the relevant eigenvalue(s). Let  $B(\varepsilon) = B + \varepsilon C_{\mathcal{A}}$ , where  $\varepsilon$  is chosen positive so that  $B(\varepsilon) \in \mathcal{Q}(\mathcal{A})$ , and sufficiently small so that for every simple eigenvalue of  $B$ , the corresponding eigenvalue and eigenvector of  $B(\varepsilon)$  are small perturbations of the eigenvalue and eigenvector of  $B$ . In case 2, the spectral radius of  $B(\varepsilon)$  is a perturbation of  $\rho(A)$  because  $\rho(A)$  is a strictly dominant eigenvalue. In either case, by continuity, the spectral radius of  $B(\varepsilon)$  is a perturbation of one of the (nonzero) simple eigenvalues of  $A$  that did not have a nonnegative eigenvector. Thus the matrix  $B(\varepsilon)$  retains the

property of not having a nonnegative eigenvector for its spectral radius, showing (by Theorem 1.1) that  $B(\varepsilon)$  is not power-positive.  $\square$

LEMMA 2.3. *Let  $\mathcal{A}$  be an  $n \times n$  sign pattern. If  $\Gamma(\mathcal{A})$  has a signed subdigraph  $\Gamma$  that is a cycle having both a positive and a negative arc, then  $\mathcal{A}$  does not require power-positivity.*

*Proof.* Suppose that the cycle  $\Gamma$  is of length  $k$  and has a positive arc  $(p, q)$  and a negative arc  $(r, s)$ . Note that the characteristic polynomial of  $C_\Gamma$  is  $p_{C_\Gamma}(x) = x^k \pm 1$ , so the eigenvalues of  $C_\Gamma$  are all nonzero and simple. Furthermore, no eigenvector can have a zero coordinate, so any nonnegative eigenvector must be positive. Suppose that  $C_\Gamma$  has a positive eigenvector  $x = [x_i]$  corresponding to an eigenvalue  $\lambda$ . Then the equation  $C_\Gamma x = \lambda x$  gives

$$x_q = \lambda x_p \quad \text{and} \quad -x_s = \lambda x_r.$$

As  $x_p, x_q > 0$ , it follows that  $\lambda > 0$ , but on the other hand,  $x_r, x_s > 0$  implies that  $\lambda < 0$ , a contradiction. Thus,  $C_\Gamma$  cannot have a nonnegative eigenvector corresponding to a nonzero eigenvalue. The result then follows from the first statement in Lemma 2.2.  $\square$

LEMMA 2.4. *Let  $\mathcal{A}$  be an  $n \times n$  sign pattern. If  $\Gamma(\mathcal{A})$  contains a figure-eight signed subdigraph  $\Gamma(s, t) = \Gamma_s \cup \Gamma_t$  (see Figure 2.1), where  $\Gamma_s$  is a cycle of length  $s \geq 2$  with all arcs signed positively and  $\Gamma_t$  is a cycle of length  $t \geq 2$  with all arcs signed negatively, and  $\Gamma_s$  and  $\Gamma_t$  intersect in a single vertex, then  $\mathcal{A}$  does not require power-positivity.*

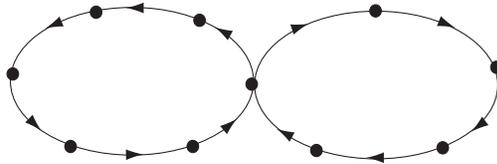


FIG. 2.1. The figure-eight  $\Gamma(5, 6)$

*Proof.* Without loss of generality, let  $2 \leq s \leq t$ . If  $s < t$  or  $s = t$  is even, the characteristic polynomial of  $C_{\Gamma(s,t)}$  is

$$p_{C_{\Gamma(s,t)}}(x) = x^{s-1}g(x), \quad \text{where } g(x) = x^t - x^{t-s} + (-1)^{t+1}.$$

Note that  $g(x)$  and  $g'(x)$  have no common roots, so every nonzero eigenvalue of  $C_{\Gamma(s,t)}$  is simple. Furthermore, as in the proof of Lemma 2.3, the cyclic nature of the digraph  $\Gamma$  prevents any zeros in an eigenvector for a nonzero eigenvalue of  $C_{\Gamma(s,t)}$ , and the opposite signs prevent a positive eigenvector for a nonzero eigenvalue. The result now follows from the first statement in Lemma 2.2.

For the case  $s = t$ , where  $s$  is odd, let  $A \in \mathcal{Q}(\Gamma(s, s))$  be obtained from  $C_{\Gamma(s, s)}$  by replacing one entry equal to 1 (in the positive cycle) by 2. Then  $p_A(x) = x^{s-1}(x^s - 1)$  and the result follows by the same argument as above.  $\square$

**COROLLARY 2.5.** *If  $\mathcal{A}$  requires power-positivity, then all off-diagonal entries are nonnegative, or all off-diagonal entries are nonpositive.*

*Proof.* If  $\mathcal{A}$  requires power-positivity, then  $\Gamma(\mathcal{A})$  is strongly connected and thus every arc in  $\Gamma(\mathcal{A})$  lies in a cycle. Suppose that  $\mathcal{A}$  has both a positive and a negative off-diagonal entry. Then  $\Gamma(\mathcal{A})$  has a positive and a negative arc that lie on the same cycle, or  $\Gamma(\mathcal{A})$  has two different cycles with arcs of opposite sign that intersect at a vertex. Lemma 2.3 or Lemma 2.4 implies that  $\mathcal{A}$  does not require power-positivity.  $\square$

**THEOREM 2.6.** *The sign pattern  $\mathcal{A}$  requires power-positivity if and only if either  $\mathcal{A}$  or  $-\mathcal{A}$  is nonnegative and  $\Gamma(\mathcal{A})$  is primitive.*

*Proof.* Assume that  $\mathcal{A}$  requires power-positivity. Then  $\Gamma(\mathcal{A})$  is strongly connected. By Corollary 2.5, the off-diagonal entries are either all nonnegative or all nonpositive. Suppose that there is a diagonal entry of opposite sign from the nonzero off-diagonal entries. Without loss of generality, suppose that the off-diagonal entries are nonpositive and that the  $(1, 1)$  entry of  $\mathcal{A}$  is  $+$ . Let  $\Gamma$  be a signed subdigraph of  $\Gamma(\mathcal{A})$  consisting of a cycle of length at least two that includes vertex 1 and the loop at vertex 1. Consider  $A = C_{\Gamma} + 2E_{11} \in \mathcal{Q}(\Gamma)$ , where  $E_{11}$  has  $(1, 1)$  entry equal to one and zeros elsewhere. By Gershgorin's Theorem applied to  $A$ , there is a unique (necessarily real) eigenvalue  $\rho$  in the unit disk centered at 3, and all other eigenvalues are in the unit disk centered at the origin, so  $\rho = \rho(A)$  is simple and strictly dominant. Furthermore, no eigenvector of  $A$  can have a zero coordinate. But the negative cycle entries do not allow a positive eigenvector for a positive eigenvalue. Thus by the second statement of Lemma 2.2,  $\mathcal{A}$  does not require power-positivity, a contradiction. Thus either  $\mathcal{A}$  or  $-\mathcal{A}$  is nonnegative, and so  $\Gamma(\mathcal{A})$  must be primitive [3, Theorem 3.4.4]. The converse is clear.  $\square$

**COROLLARY 2.7.** *The sign pattern  $\mathcal{A}$  requires power-positivity if and only if either  $\mathcal{A}$  or  $-\mathcal{A}$  requires eventual positivity.*

*Proof.* The necessity follows from Theorem 2.6 and [4, Theorem 2.3] and the sufficiency is clear.  $\square$

**3. Sign patterns that allow power-positivity.** A square sign pattern  $\mathcal{A}$  is called *potentially power-positive* (PPP) if there exists an  $A \in \mathcal{Q}(\mathcal{A})$  that is power-positive. If  $A \in \mathcal{Q}(\mathcal{A})$  exists such that  $A$  is eventually positive, then the sign pattern  $\mathcal{A}$  is called *potentially eventually positive* (PEP) [1]. Note that  $\mathcal{A}$  is PPP if and only if  $-\mathcal{A}$  is PPP. The following characterization of PPP sign patterns follows from Corollary 1.4.

**THEOREM 3.1.** *The sign pattern  $\mathcal{A}$  is potentially power-positive if and only if  $\mathcal{A}$  or  $-\mathcal{A}$  is potentially eventually positive.*

Recall that  $\mathcal{A}^+$  is the positive part of  $\mathcal{A}$ . Theorem 2.1 of [1] and Theorem 3.1 above give the following result.

**THEOREM 3.2.** *If  $\Gamma(\mathcal{A}^+)$  or  $\Gamma(\mathcal{A}^-)$  is primitive, then  $\mathcal{A}$  is potentially power-positive.*

We next provide examples, including a sign pattern  $\mathcal{A}$  such that both  $\mathcal{A}$  and  $-\mathcal{A}$  are PPP, sign patterns  $\mathcal{A}$  that are PPP but not PEP, and an irreducible sign pattern that is not PPP.

**EXAMPLE 3.3.** The sign pattern  $\mathcal{A} = \begin{bmatrix} + & + & - \\ - & 0 & + \\ + & - & - \end{bmatrix}$  is PPP, as is  $-\mathcal{A}$ , because both  $\Gamma(\mathcal{A}^+)$  and  $\Gamma(\mathcal{A}^-)$  are primitive.

**EXAMPLE 3.4.** The block sign pattern

$$\mathcal{A} = \begin{bmatrix} [-] & [-] \\ [-] & [+] \end{bmatrix},$$

(where the diagonal blocks are square and the diagonal  $[-]$  block is nonempty) is PPP, because  $\Gamma(\mathcal{A}^-)$  is primitive. However,  $\mathcal{A}$  is clearly not PEP because the first row does not have a  $+$  [6, p. 327].

**EXAMPLE 3.5.** A square sign pattern  $\mathcal{A} = [\alpha_{ij}]$  is a *Z sign pattern* if  $\alpha_{ij} \neq +$  for all  $i \neq j$ . An  $n \times n$  Z sign pattern  $\mathcal{A}$  with  $n \geq 2$  cannot be PEP [1, Theorem 5.1], but if  $\Gamma(\mathcal{A}^-)$  is primitive, then by Corollary 3.2,  $\mathcal{A}$  is PPP. For  $n \geq 3$ , if  $\mathcal{A}$  is an  $n \times n$  Z sign pattern having every off-diagonal entry nonzero, then  $\Gamma(\mathcal{A}^-)$  is primitive and thus  $\mathcal{A}$  is PPP.

Note that when  $n = 2$ ,  $\mathcal{A} = \begin{bmatrix} + & - \\ - & + \end{bmatrix}$  is not PPP, as in the next example, where Theorem 3.1 is used to show that a generalization of this sign pattern is not PPP.

**EXAMPLE 3.6.** Let

$$\mathcal{A} = \begin{bmatrix} [+] & [-] & [+] & \dots \\ [-] & [+] & [-] & \dots \\ [+] & [-] & [+] & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where the diagonal blocks are square and there are at least 2 diagonal blocks. Then no subpattern of  $\mathcal{A}$  is PPP because by [1, Theorems 5.3 and 3.1], no subpattern of  $\mathcal{A}$  or  $-\mathcal{A}$  is PEP. Thus by Theorem 3.1, no subpattern of  $\mathcal{A}$  is PPP.

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