Numerical modeling of tight (stress corrosion) cracks for the development of new electromagnetic NDT pipeline tools

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Numerical modeling of tight (stress corrosion) cracks for the development of new electromagnetic NDT pipeline tools

by

Gregory Kobidze

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of

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Major: Electrical Engineering (Electromagnetics)

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ABSTRACT

The dissertation is focused on the problem of thin (stress corrosion) crack modeling in the numerical electromagnetic NDT simulations for gas transmission pipelines. Axially oriented cracks are known as the most difficult for electromagnetic detection and a number of methods have been designed to achieve the necessary sensitivity of the output signal. Two recently developed methods, the velocity induced current perturbation (VICP) method and the rotating magnetic field (RMF) method, are considered as possible ways to improve the quality of inspection. In the VICP method the material in the inspected region is magnetized to saturation, so that the changes in output signal are mostly caused by perturbations due to the velocity induced currents. Due to the axial symmetry in geometry, the velocity induced currents are circumferentially oriented, and hence, are normal to the plane of axially oriented cracks. In the RMF method, saturation magnetization is not used, and the main contribution in the changes of output signal is from the magnetic flux perturbations in the pipe, caused by cracks. In this case the magnetic flux lines are normal to the plane of axially oriented cracks.

Conventional modeling requires dense mesh discretization around the crack. In the proposed approach a crack is modeled in such a way, that the material properties are allowed to vary arbitrarily within a single element, which significantly simplifies the mesh. A number of tight crack models have been developed and integrated in the numerical codes, simulating the electromagnetic field redistribution in a pipe with VICP and RMF test bed vehicles. It has been found that better accuracy is achieved if, for each model, a specific shape function is used. 1-D, 2-D (axisymmetric and polar) and 3-D codes were implemented and studied. Initial experimental studies have been made to calibrate the codes.
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CHAPTER 1. INTRODUCTION

The maintenance of engineering structures subject to aging in harsh environments demands the highest level of confidence in nondestructive evaluation (NDE). One of the most critical issues in the inspection of gas transmission pipelines remains the detection and characterization of stress corrosion cracking (SCC) [1].

SCC results from the combined action of stress, a cracking (electrochemical) environment and temperature to cause cracks to initiate and grow in a susceptible steel pipeline. When a suitable combination of conditions exists at the pipe surface, the cathodic current imposed on a line can help cause the formation of an electrochemical environment that can lead to external stress corrosion cracking. If not found over a period of time and quickly remedied, the cracks may grow and eventually lead to pipeline rupture [2].

Although SCC has been a recognized failure mechanism in other industries, it is relatively new to the natural gas industry [3]. The first documented failure attributed to SCC occurred in Nachitoches, Louisiana, in 1965. That natural soil environments would be capable of producing SCC in the relatively resistant mild steel pipelines was surprising to many experts in the field. Since 1965 more than 80 service leaks or breaks have been attributed to SCC. It became apparent soon after the Nachitoches failure that, while the frequency of SCC failures was quite low, they could occur from time to time and that research to investigate the cause and to find solutions was warranted. Since then, much has been learned about the nature of SCC and the influencing factors. There have been significant results from both the field and laboratory. Studies have been initiated to evaluate the feasibility of specific preventive measures. Even though substantial progress has been made,
it is apparent that the problem of SCC in buried pipelines is quite complex and more work will be required before specific solutions can be recommended.

Stress corrosion cracks have several unique features that are not associated with any other failure mechanism [4]; the cracks are branched, they are usually found in clusters, and they usually contain a corrosion deposit. They may be intergranular or transgranular, depending on the environment and the alloy system. The environments that can cause cracking are usually those that are mildly corrosive, consequently, severe corrosion usually is not associated with SCC. Although there may be evidence of pitting corrosion around stress-corrosion cracks in some pipes, there has been little or no pitting or general corrosion on most of the SCC failures that have been studied. Figure 1 shows an example of stress corrosion cracking found on the outside pipe surface after the coating has been removed. The set of cracks had been on a pipeline in service but had not failed. Stress corrosion cracks are generally oriented perpendicular to the maximum stress and parallel to the pipe axis. Groups

Figure 1. Sparse colony of SCC [2].
of cracks usually occur in colonies. A colony is considered "sparse" if the cracks are far apart in the circumferential direction and "dense" if the cracks are circumferentially close together, as in Figure 2.

Individual cracks can range all the way from shallow to very deep. Many cracks in the middle of dense colonies are less than 10% of the wall thickness, whereas in sparse colonies the cracks can grow in a stable manner until they reach nearly through the wall. These deeper cracks are of primary concern in inspections to evaluate pipeline integrity.

The circumferential distance between the adjacent cracks in a colony can range from very small to larger than the wall thickness of the pipe, and the axial spacing between the individual cracks in a colony, as measured on pipes with service failures, vary from overlapped to many times the wall thickness. The shape of the stress corrosion cracking colony often depends on the coating conditions. Figure 3 shows an example of

Figure 2. Dense colony of SCC [2].
a colony that is oriented along the helical angle of the tape overlap. A limited number of occurrences of circumferentially oriented stress corrosion cracks near girth welds have also been reported.

Two forms of SCC are high pH and low pH. High pH stress corrosion cracks are typically intergranular, with a cracking path along the grain boundaries of the material. High pH cracks can also grow in a transgranular mode when they become relatively deep.

High pH cracks are generally filled with oxide with little or no separation or opening between the crack faces, and are usually very tight [5]. Low pH stress corrosion cracking is often transgranular. The crack sides suffer significant lateral dissolution, with appreciable amounts of loosely adherent corrosion products forming in the crack enclaves [6]. Transgranular fracture surfaces are smoother than intergranular. Figure 4 shows examples of the two types of SCC.
Electromagnetic inspection tools move at high speeds under the pressure of gas through the pipe, while an array of sensors monitors the changes in electromagnetic properties of steel around cracks and other possible imperfections. Specifics of the geometry, its axisymmetry, causes difficulties in the detection of outer surface cracks oriented parallel to the axis of the pipe. A number of new inspection techniques, employing modern technology, have been developed to overcome these difficulties.

One way is to utilize a high level of magnetization, which is achieved in the magnetic flux leakage (MFL) method (using extremely strong permanent magnets, based on materials doped with rare earth elements). An in-depth study [7] of all available inspection techniques revealed that MFL testing was most capable of achieving the required performance for metal-
loss inspection in a pipeline environment. Physical design provides a major advantage over other methods (see Figure 5). Magnetic circuits are mounted on a pressure vessel body, which serves to protect electronic instrumentation from the pipeline product. The circuits are in contact with the pipe wall via flexible bristles that allow the magnetic assemblies to conform to varying internal diameters. The MFL tool magnetizes the pipe wall to near saturation flux density and records the flux leakage anomalies that occur inside the pipe where there is internal or external metal loss [8]. The general shapes of residual and active leakage fields around defects in ferromagnetic materials are well known and have been widely reported in the literature [9], [10]. For general metal loss due to corrosion, mechanical damage, etc., the magnetic flux leakage method of inspection has emerged as a system capable of high performance under the extreme conditions imposed by both the product and the pipeline itself. Although it is feasible for very large circumferential cracks to produce a sensor response, it is pointed out [7] that the static MFL technique is not suitable for detection of cracks that are aligned (axially) with the applied magnetic field, because in this case the crack walls do not intersect axially oriented flux lines of the field. The high speed of tool motion complicates the picture of electromagnetic field distribution, introducing

Figure 5. Magnetic flux leakage inspection pig.
significant motionally induced currents in the conducting pipe wall. Experience suggests that, for typical pipeline MFL tools, the time constant for the magnetic flux to diffuse through the pipe wall can be comparable to the transit time. Since line-pipe steel is a conducting ferromagnetic material, the changing magnetic fluxes occurring in the pipe wall during the passage of an MFL inspection tool generate induced currents. These retard the diffusion of magnetic flux through the pipe wall. Consequently, significant changes in the anomalous MFL patterns induced by defects, particularly for outer diameter ones, have been expected. The question of whether velocity effects are significant depends on the ratio of magnetic diffusion time constant to detector transit time. Efforts are being made to test moving assemblies with a damaged pipe coupon mounted above the test assembly in the laboratory; it is found that motion dramatically affects performance. Detailed measurements show that pipeline in-line inspection tool speed can cause significant reduction in defect-induced axial and radial MFL signals. On the other hand the motionally induced currents are oriented circumferentially and they are intersected by axially oriented crack walls. Detection of the interaction of a crack with the induced current constitutes the principle of the current perturbation method.

Another way of detecting axially oriented cracks is to exploit a different magnetic flux orientation in space and direction of flux changes in time. The rotating magnetic field (RMF) method is an implementation of this approach. The advantages of utilization of the rotating magnetic field are known in nondestructive testing (NDT) [12], [13]. The principle is used, for example, in detecting the far side defects in planar conductive plates [14], [15]. One assembly consists of a ferrite core, on which two exciting coils are wound with axes
perpendicular to each other. The currents are driven at a frequency of 10 Hz and are also 90° apart, so that these two phases generate a rotating magnetic field. A row of pick-up coils serves as flux density sensors. A classical solution, where three phases, 120° apart, generate the rotating magnetic field, has been used since the late 19th century in AC electric power generators and motors. Literally almost every aspect of the design, performance, magnetic flux density distribution and control in these devices has been analyzed theoretically and thoroughly studied through an infinite number of experiments [16]. However, it was realized only recently that it can be employed as a possible electromagnetic testing device for axial SCC detection in gas transmission pipe lines [17].

Numerical simulations using the finite element method (FEM) have proven to be efficient and an inexpensive means to study the interaction and redistribution of electromagnetic fields [18]-[20]. They allow a fair comparison of the effectiveness of NDT methods. Moreover, FEM simulations help in design and development of new instrumentation, decreasing the number and changing the goals of expensive experiments. It is known [11], that tremendous complications arise when solution methods are considered. These are due to the nonlinear, hysteretic behavior of pipeline steel, to the changing geometry resulting from the relative motion between the defect and detector, and to the need to consider three-dimensional modeling for realistic defects. The result is that there is very little hope of obtaining analytic solutions without oversimplifying the problem. It was stated further, that the prospects for numerical solutions using, for example, modern finite element calculation techniques are not much better. The detailed computation of such three-dimensional, nonlinear, transient effects is somewhat beyond the capabilities of both current
state-of-the-art computer hardware and software. For a complicated geometry, minimal
density of the mesh is desired to implement a code with reasonable requirements of computer
memory and computation time. On the other hand, simulations need to be accurate enough to
be experimentally validated.

Traditional FEM assumes, that all the properties of materials are constants within
each element. An approach with conductivity \( \sigma \) and permeability \( \mu \), varying within elements
around a crack, has been taken and developed in this work. The description of the crack,
corresponding test functions and shape functions, are chosen to achieve better accuracy of
simulations without increasing the discretization of the mesh.

The scope of this work is limited by the issues related to FEM simulations of the
above two methods (MFL and RMF), and to incorporation of the tight crack model into these
simulations.

The dissertation consists from eight chapters and appendix. Besides the introduction
(chapter 1), the dissertation describes the geometry of the finite element mesh and the
governing equations for the MFL and RMF tools (chapter 2), discusses the feasibility of the
existing time stepping schemes to compute the transient response (chapter 3) from the
moving defects, and the problems arising from the nonlinear nature of the magnetic
properties of the steel (chapter 4). Chapter 5 is concentrated on the rotating field utilization to
generate circumferentially oriented flux and on the possibilities to use the periodic boundary
conditions. Chapter 6 introduces and develops different models of the tight cracks, and shows
how to incorporate these models into the FEM codes for some coordinate systems. Chapter 7
shows the experimental results used for calibration of the 3-D code and chapter 8 summarizes
the results of the work. Appendix presents a FORTRAN code, implementing two ways of improving the efficiency of memory usage in 3-D finite element electromagnetic NDT simulations for long cylindrical geometry: utilization of periodic boundary conditions, and decoupling the resulting global system of linear equations.
CHAPTER 2. GEOMETRY AND GOVERNING EQUATIONS

The two-dimensional geometry of the MFL testing configuration is shown in Figure 6, and a part of the corresponding axisymmetric mesh in Figure 7. A detailed description and explanation of both the MFL inspection vehicle and the choice of mesh and material properties are given in [21]. Main characteristics of the mesh are as follows:

- **coordinate system** - cylindrical, with z-axis as axis of symmetry, and r-axis in radial direction;
- **number of nodes** = 181 (z-axis) x 77 (in r);
- **stepsize** $h$ in z-direction = 0.0191 m (constant);
- **stepsize in radial direction** - various;
- **size of the mesh** = 3.43 m (z-axis) x 4.02 m (in r).

The governing equation describing the electromagnetic field around the MFL tool in vector form is written as

$$\nabla \times \left( \frac{l}{\mu} \nabla \times \mathbf{A} \right) = \mathbf{J} + \sigma \frac{\partial \mathbf{A}}{\partial t} - \sigma \nabla \times \nabla \times \mathbf{A}$$

Figure 6. Two-dimensional MFL inspection tool (axisymmetric along main axis).
Figure 7. A part of the two-dimensional axisymmetric mesh, corresponding to the geometry in Figure 1.

where $\overline{A}$ is the vector magnetic potential (oriented normally to the 2-D plane), $\overline{J}_s$ - equivalent current density in the model of the permanent magnets, $\mu$ - magnetic permeability, $\sigma$ - electrical conductivity of material, $\overline{V}$ - velocity of vehicle's motion along the pipe. In cylindrical coordinates, the assumption of constant $\mu$ and $\sigma$ within each element leads to

$$\frac{l}{\mu} \left( \frac{A}{r^2} - \frac{l}{r} \frac{\partial A}{\partial r} - \frac{\partial^2 A}{\partial r^2} - \frac{\partial^2 A}{\partial \zeta^2} \right) + \sigma \frac{\partial A}{\partial t} - \sigma \overline{V} \frac{\partial A}{\partial \zeta} + J_s = 0 \quad (2).$$
It is shown, that letting $\mu$ and $\sigma$ vary within elements, adjacent to cracks, allows better flexibility in crack modeling. In this case (2) undergoes certain changes.

Generation of the rotating magnetic field is illustrated by Figure 8, where three currents $I_1$, $I_2$, and $I_3$ have the same amplitude, but they are electrically shifted in phase $120^\circ$ apart each other (as shown in Figure 8a), and physically also oriented $120^\circ$ apart in cylindrical coordinates (as can be seen from Figure 8b). Flux lines of the resulting magnetic field at different time instants are shown in Figure 8b, which illustrates a real rotation of the field. Flux lines have maximum density in the pipe wall, having components in the circumferential direction. Hence, axially oriented cracks may cause perturbations in the surrounding electromagnetic field.

A picture of a rotor, generating the RMF, is shown in Figure 9. A 2-D mesh approximating the central cross section of the rotor in a pipe, is shown in Figure 10. It is seen from Figure 9, that the core laminations in the rotor are assembled in such a way, that the winding slots are skewed with respect to the main axis. Hence, a twisted cylindrical 3-D mesh was built. Figure 11 shows one cylindrical surface of the two meshes, where Figure 11a corresponds to the regular mesh, and Figure 11b to the twisted rotor region.

The 3-D visualization is better understood from the corresponding cross sections. The central cross section resembles the corresponding 2-D mesh: a cross section at the rotor edge, showing the closures of 3-phase current loops, is given in Figure 12. It is seen from Figures 9-12, that the rotation of the rotor $60^\circ$ around its main ($z$) axis does not change the mesh, and, taking into account electrical $60^\circ$ phase difference, periodic boundary conditions could be
Figure 8. Excitation currents (a) and the magnetic flux lines (b), generated by the RMF in a pipe.

Figure 9. Rotor, generating the rotating magnetic field.
imposed, thus six times reducing the number of elements in the mesh. It was discovered that the distribution of the simulated electromagnetic field depends on the description of the excitation current density in the conductor corners. An approach realizing smooth distribution of the current density in elements has been implemented in simulations. Experiments have been done in order to verify the numerical solution, so that the dimensions in the model were taken from the available experimental setup.

The study of an RMF tool can be started with the quasi-static case. The governing equation, describing the field, is the Poisson's equation

$$\nabla \times \frac{1}{\mu} \nabla \times A = J_s - j\omega\sigma A$$

(3).
Figure 11. A cylindrical surface in a regular (a) and twisted (b) meshes.

where $\omega$ is the excitation angular frequency. $A$ and $J$ are phasors.

At least two regions are usually distinguished. The first one is the *near field* region around the central cross section of the rotor, and the second one is the *far field* region, somewhere away from the rotor. A 2-D mesh is used for study of the near field, and a 3-D mesh is necessary for the far field region.
Figure 12. A cross section at the edge of the rotor's 3-D model.
CHAPTER 3. VELOCITY EFFECT

The finite element method of solving a partial differential equation (PDE) implies a discretization of space $\Omega$ into elementary elements. It is assumed that within each of these elements, a function, approximating the PDE solution, takes a prescribed, apriori chosen form (or shape) with a fixed number of local degrees of freedom.

For example, if a linear form is chosen in the 1-D case, an approximating function, which is a line segment, has two local degrees of freedom, i.e. it is defined by the values on its two ends. Linear (or polylinear for 2-D and 3-D cases) shape functions are the most commonly used in practice for their simplicity.

The FEM solution, $\mathbf{A}^C$, is seen as a vector in a very specific space - a space $C_{\varphi}^N$ of all functions that can be represented as a linear combination $\sum_{i=1}^{N} A_i \varphi_i$ of chosen shape functions $\varphi_i$, with $N$ - global number of degrees of freedom (for 1-D linear shape functions. $C_{\varphi}^N$ is a space of all possible piecewise linear functions, which may have slope changes only at the grid points, called the mesh nodes, and $N$ is the total number of nodes in the mesh). We want to have the vector $\mathbf{A}^C$, a numerical solution, as close to the real solution, as possible.

The procedure of choosing the best approximation $\{A_i\}_{i=1}^{N}$ consists of i) substitution of $\mathbf{A}^C$ into the PDE, ii) multiplication of both sides of the PDE by test functions $\psi_i$ (usually $\psi_i = \varphi_i$), iii) integration of the products numerically, and iv) solving the resulting linear system with $N$ unknowns $A_i$. The step (iii) results in a system, which can be written in matrix form as
\[ [SK] \times \dot{A} + [C] \times \frac{\partial A}{\partial t} + Q = 0, \] where \([SK] = [S] + [SV]\), \([S]\) is a stiffness matrix, dependent on material properties, \([SV]\) - convection matrix, \([C]\) - dynamic matrix, \(Q\) - forcing vector term.

and \(A\) stands short for vector \(\{A_i\}_{i=1}^N\).

The transient nature of the crack's NDT using MFL or RMF modes requires utilization of numerical methods with discretization both in space and time domains.

Accuracy and stability are important issues, determining the validity of simulations. It was suggested [22], that for electromagnetic NDT applications the Leismann-Frind method [23] is the most powerful. For 2-D axisymmetric formulation it is written in the form of a finite difference equation

\[
\sigma \frac{A^{n+1} - A^n}{\Delta t} + \left[ \frac{l}{\mu} + \frac{\theta}{2} \left( \frac{A^{n+1}}{r^2} - \frac{\partial^2 A^{n+1}}{\partial z^2} - \frac{\partial^2 A^{n+1}}{\partial r^2} - \frac{l}{r} \frac{\partial A^{n+1}}{\partial r} \right) \right] + \\
+ \frac{\theta}{2} \left( \frac{A^n}{r^2} - \frac{\partial^2 A^n}{\partial z^2} - \frac{\partial^2 A^n}{\partial r^2} - \frac{l}{r} \frac{\partial A^n}{\partial r} \right) - \sigma V \frac{\partial A^n}{\partial z} = 0 \tag{4}
\]

with \(\theta = \frac{V^2 \Delta t \sigma}{2} = \frac{Vh\sigma}{2}\) - an artificial reluctance term. Here \(A^n\) and \(A^{n-1}\) are values of vector magnetic potential \(A\) at two successive moments in time.

In the FEM formulation, equation (4) is written in the form

\[
([S] + [C] + [S\Theta])A^{n+1} = Q + ([C] - [S\Theta] - [SV])A^n \tag{5}
\]

where \([S\Theta]\) is the artificial reluctance matrix.

Stability of the Leismann-Frind method has been examined for 1-D case in [22], whereas the accuracy, as will be shown, is overestimated. In fact, the Leismann-Frind method
is selectively accurate. This means that it is first-order accurate for resting parts, and higher order accurate for moving parts of the geometry.

Consider the 1-D case with uniform mesh of stepsiz$h$ in the $z$ direction, and time step $\Delta t = h/V$, where $V$ is the velocity of motion (also in the $z$ direction). We use midpoints $t = t^n + \Delta t / 2 = (t^n + t^{n+1}) / 2$ for time derivative evaluations (and try to get high accuracy in $t$), so that

$$A = A(t) = A(t^n + \Delta t / 2), \quad \text{and}$$

$$\frac{\partial A}{\partial t} = \frac{\partial A(t^n + \Delta t / 2)}{\partial t} = \frac{A^{n+1} - A^n}{\Delta t}.$$

Taylor series expansion gives

$$A^{n+1} = A + \frac{\Delta t}{2} \frac{\partial A}{\partial t} + \frac{\Delta t^2}{8} \frac{\partial^2 A}{\partial t^2} + \frac{\Delta t^3}{48} \frac{\partial^3 A}{\partial t^3} + ...$$  \hspace{1cm} (6)

$$A^n = A - \frac{\Delta t}{2} \frac{\partial A}{\partial t} + \frac{\Delta t^2}{8} \frac{\partial^2 A}{\partial t^2} - \frac{\Delta t^3}{48} \frac{\partial^3 A}{\partial t^3} + ...$$  \hspace{1cm} (7)

Consider a PDE for conducting moving elements without current sources

$$\frac{l}{\mu} \frac{\partial^2 A}{\partial z^2} = \sigma \frac{\partial A}{\partial t} - V \sigma \frac{\partial A}{\partial z}, \quad \text{or}$$

$$\sigma \frac{\partial A}{\partial t} = \frac{l}{\mu} \frac{\partial^2 A}{\partial z^2} + V \sigma \frac{\partial A}{\partial z}$$ \hspace{1cm} (8)

with (8) replaced by:

$$\sigma \frac{A^{n+1} - A^n}{\Delta t} = \frac{l}{\mu} \frac{\partial^2 A^{n+1}}{\partial z^2} + V \sigma \frac{\partial A^n}{\partial z} + \theta \left( \frac{l}{\mu} \frac{\partial^2 A^{n+1}}{\partial z^2} + \frac{l}{\mu} \frac{\partial^2 A^n}{\partial z^2} \right),$$ \hspace{1cm} (9)
where \( \theta = \frac{V^2 \Delta t \sigma}{2} = \frac{V h \sigma}{2} \) an artificial reluctance term. From (6) and (7) we have

\[
\frac{l}{\mu} \frac{\partial^2 A^{n+1}}{\partial z^2} = \frac{l}{\mu} \left( \frac{\partial^2 A}{\partial z^2} + \frac{\Delta t}{2} \frac{\partial^2}{\partial t} \frac{\partial A}{\partial t} + \frac{\Delta t^2}{8} \frac{\partial^2}{\partial z^2} \frac{\partial^2 A}{\partial t^2} \right) + O(\Delta t^3)
\]

\[
\sigma \frac{A^{n+1} - A^n}{\Delta t} = \sigma \frac{\partial A}{\partial t} + \frac{\Delta t^2}{24} \frac{\partial^3 A}{\partial t^3} + O(\Delta t^4)
\]  

(10)

\[
V \sigma \frac{\partial A^n}{\partial z} = V \sigma \left( \frac{\partial A}{\partial z} - \frac{\partial}{\partial z} \frac{\partial A}{\partial t} + \frac{\partial}{\partial z} \frac{\partial^2 A}{\partial t^2} \right) + O(\Delta t^3)
\]  

(11)

\[
\theta \left( \frac{l}{2} \frac{\partial^2 A^{n+1}}{\partial z^2} + \frac{l}{2} \frac{\partial^2 A^n}{\partial z^2} \right) = \theta \left( \frac{\partial^2 A}{\partial z^2} + \frac{\Delta t^2}{8} \frac{\partial^2}{\partial z^2} \frac{\partial^2 A}{\partial t^2} \right) + O(\Delta t^4)
\]

and making substitutions in (9),

\[
\sigma \frac{\partial A}{\partial t} = \frac{l}{\mu} \frac{\partial^2 A}{\partial z^2} + \frac{\Delta t}{2\mu} \frac{\partial^2}{\partial z^2} \frac{\partial A}{\partial t} + \frac{\Delta t^2}{8\mu} \frac{\partial^2}{\partial z^2} \frac{\partial^2 A}{\partial t^2} - \frac{\Delta t^2}{24} \sigma \frac{\partial^3 A}{\partial t^3} +
\]

\[+V \sigma \left( \frac{\partial A}{\partial z} - \frac{\partial}{\partial z} \frac{\partial A}{\partial t} + \frac{\partial}{\partial z} \frac{\partial^2 A}{\partial t^2} \right) + \theta \left( \frac{\partial^2 A}{\partial z^2} + \frac{\Delta t^2}{8} \frac{\partial^2}{\partial z^2} \frac{\partial^2 A}{\partial t^2} \right) + O(\Delta t^3)
\]

so that, due to (8), the truncation error is

\[
T = \frac{\Delta t}{2\mu} \frac{\partial^2 A}{\partial z^2} + \frac{\Delta t^2}{8\mu} \frac{\partial^2}{\partial z^2} \frac{\partial^2 A}{\partial t^2} - V \sigma \left( \frac{\partial}{\partial z} \frac{\partial A}{\partial t} - \frac{\partial}{\partial z} \frac{\partial^2 A}{\partial t^2} \right)
\]

\[ - \frac{\Delta t^2}{24} \sigma \frac{\partial^3 A}{\partial t^3} + \theta \left( \frac{\partial^2 A}{\partial z^2} + \frac{\Delta t^2}{8} \frac{\partial^2}{\partial z^2} \frac{\partial^2 A}{\partial t^2} \right) + O(\Delta t^3)
\]

Since from (8)

\[
\frac{\partial}{\partial z} \frac{\partial A}{\partial t} = \frac{l}{\mu \sigma} \frac{\partial^3 A}{\partial z^3} + V \frac{\partial^2 A}{\partial z^2}
\]
\[
\frac{\partial^2 \partial A}{\partial t^2} = \frac{l}{\mu \sigma} \frac{\partial^4 A}{\partial x^4} + V \frac{\partial^3 A}{\partial x^3}.
\]

\[
\frac{\partial^2 A}{\partial t^2} = \frac{l}{\mu \sigma} \frac{\partial^2 A}{\partial x^2} + V \frac{\partial A}{\partial t} = \frac{l}{\mu^2 \sigma^2} \frac{\partial^4 A}{\partial x^4} + 2V \frac{\partial^3 A}{\partial x^3} + V^2 \frac{\partial^2 A}{\partial x^2}.
\]

\[
\frac{\partial^3 A}{\partial t^3} = \frac{l}{\mu^2 \sigma^2} \frac{\partial^4 A}{\partial x^4} + V \frac{\partial^3 A}{\partial x^3} + V^2 \frac{\partial^2 A}{\partial x^2} = \frac{l}{\mu^2 \sigma^2} \frac{\partial^6 A}{\partial x^6} + 3V \frac{\partial^5 A}{\partial x^5} + \frac{3V^2}{\mu} \frac{\partial^4 A}{\partial x^4} + V^3 \frac{\partial^3 A}{\partial x^3}
\]

we have

\[
T = \frac{\Delta t}{2\mu} \left( \frac{l}{\mu \sigma} \frac{\partial^4 A}{\partial x^4} + V \frac{\partial^3 A}{\partial x^3} \right) + \frac{\Delta t^2}{8\mu} \left( \frac{l}{\mu^2 \sigma^2} \frac{\partial^6 A}{\partial x^6} + 2V \frac{\partial^5 A}{\partial x^5} + V^2 \frac{\partial^4 A}{\partial x^4} \right) - V \sigma \left\{ \frac{\Delta t}{2} \left( \frac{l}{\mu \sigma} \frac{\partial^3 A}{\partial x^3} + V \frac{\partial^2 A}{\partial x^2} \right) - \frac{\Delta t^2}{8} \left( \frac{l}{\mu^2 \sigma^2} \frac{\partial^4 A}{\partial x^4} + 2V \frac{\partial^3 A}{\partial x^3} + V^2 \frac{\partial^2 A}{\partial x^2} \right) \right\} - \frac{\Delta t^2}{24} \left( \frac{l}{\mu^2 \sigma^2} \frac{\partial^6 A}{\partial x^6} + 3V \frac{\partial^5 A}{\partial x^5} + \frac{3V^2}{\mu} \frac{\partial^4 A}{\partial x^4} + V^3 \frac{\partial^3 A}{\partial x^3} \right) + \theta \left\{ \frac{\partial^2 A}{\partial x^2} + \frac{\Delta t^2}{8} \left( \frac{l}{\mu^2 \sigma^2} \frac{\partial^6 A}{\partial x^6} + 2V \frac{\partial^5 A}{\partial x^5} + V^2 \frac{\partial^4 A}{\partial x^4} \right) \right\} + O(\Delta t^3) =
\]

\[
= \frac{\Delta t}{2} \left( \frac{l}{\mu^2 \sigma^2} \frac{\partial^4 A}{\partial x^4} - V^2 \frac{\partial^2 A}{\partial x^2} \right) + \Delta t^2 \left( \frac{l}{12} \frac{\partial^6 A}{\partial x^6} + \frac{l}{4} \frac{V}{\mu^2 \sigma^2} \frac{\partial^5 A}{\partial x^5} + \frac{l}{4} \frac{V^2}{\mu} \frac{\partial^4 A}{\partial x^4} + \frac{l}{12} V^3 \frac{\partial^3 A}{\partial x^3} \right).
\]
\[ + \theta \left[ \frac{\partial^2 A}{\partial x^2} + \frac{\Delta t^2}{8} \left( \frac{l}{\mu^2 \sigma^2} \frac{\partial^6 A}{\partial x^6} + \frac{2V \partial^5 A}{\mu \sigma} + \frac{V^2 \partial^4 A}{\partial x^4} \right) \right] + O(\Delta t^3) = \]

\[ = \frac{\Delta t}{2} \frac{l}{\mu^2 \sigma} \frac{\partial^4 A}{\partial x^4} + \]

\[ + \frac{\Delta t^2}{12} \frac{l}{\mu^2 \sigma^2} \frac{\partial^6 A}{\partial x^6} + \frac{3V \partial^5 A}{\mu \sigma} + \frac{3V^2 \partial^4 A}{\partial x^4} + \frac{V^3 \sigma \mu}{\partial x^3} \]

\[ + \frac{\Delta t^2}{8} \left( \frac{l}{\mu^2 \sigma^2} \frac{\partial^6 A}{\partial x^6} + \frac{2V \partial^5 A}{\mu \sigma} + \frac{V^2 \partial^4 A}{\partial x^4} \right) + O(\Delta t^3) \]

For high velocities the artificial reluctance \( \theta \) term dominates over \( l/\mu \), so that \( \theta \mu >> l \).

Assume

\[ V^2 \frac{\partial^4 A}{\partial x^4} >> \frac{l}{\mu^2 \sigma^2} \frac{\partial^6 A}{\partial x^6} \] and we write

\[ T = \frac{\Delta t}{2} \frac{l}{\mu^2 \sigma} \frac{\partial^4 A}{\partial x^4} + \frac{\Delta t^2}{24} \left( 2V \sigma \frac{\partial^3 A}{\partial x^3} + \frac{3V^2 \sigma \partial^4 A}{\partial x^4} \right) + O(\Delta t^4) = \]

\[ = \frac{\Delta t}{2} \frac{l}{\mu^2 \sigma} \frac{\partial^4 A}{\partial x^4} + \frac{\Delta t^2}{12} \frac{V^3 \sigma}{\partial x^3} + O(\Delta t^4) = \]

\[ = \frac{\Delta t}{2} \frac{l}{\mu^2 \sigma} \frac{\partial^4 A}{\partial x^4} + \theta \frac{\Delta t}{6} V \frac{\partial^3 A}{\partial x^3} + O(\Delta t^4) = \theta \frac{\Delta t}{6} V \frac{\partial^3 A}{\partial x^3} + O(\Delta t^4) = \]

\[ = \frac{Vh \sigma}{6} \frac{\partial^3 A}{\partial x^3} + O(\Delta t^4) \]

From (9) we calculate exactly
\[
A(t^{n+1}) = \frac{\Delta t}{\sigma} (\text{RHS} - T) + A(t^n) , \text{ where }
\]

\[
\text{RHS} = \frac{l}{\mu} \frac{\partial^2 A(t^{n+1})}{\partial z^2} + V\sigma \frac{\partial A(t^n)}{\partial z} + \theta \left( \frac{l}{\mu} \frac{\partial^2 A(t^{n+1})}{\partial z^2} + \frac{l}{\mu} \frac{\partial^2 A(t^n)}{\partial z^2} \right).
\]

so that for high velocities \(V\) we indeed have the local error

\[
\varepsilon_L = -\frac{T\Delta t}{\sigma} = -\frac{V\sigma h^2}{12} \frac{\partial^3 A}{\partial z^3} \frac{\Delta t}{\sigma} + O(\Delta t^4) = -\frac{h^3}{12} \frac{\partial^3 A}{\partial z^3} + O(h^3 + \Delta t^4) = O(h^3)
\]

On the other hand, for very low or zero velocities \(V\), the artificial reluctance is small, \(\theta \mu < < 1\). and (8) can be approximated by

\[
\frac{\sigma}{\partial t} = \frac{l}{\mu} \frac{\partial^2 A}{\partial z^2}.
\]

whereas (9) becomes

\[
A^{n+1} = A^n + \frac{\Delta t}{\sigma \mu} \frac{\partial^2 A^{n+1}}{\partial z^2}.
\]

which is the backward Euler method (first-order accurate).

Since the perturbations in the MFL electromagnetic field (due to cracks) are not confined by the pipe wall (indeed, the magnetic flux leaks through the crack), we conclude that the overall accuracy of the Leisman-Frind method, even for 1-D case, is only of the first order. However, it preserves symmetry of the stiffness matrix and is unconditionally stable, and hence, remains at present to be the best transient method available for incorporation into finite element code.
CHAPTER 4. NONLINEARITY IN TRANSIENT CODES

High magnetization levels result in saturation of the magnetic materials. As was already mentioned, appropriate investigations, measurements and calculations, regarding the modeling of the permanent magnets, backing iron, brush, and pole pieces were made and described in [21]. Corresponding equivalent currents and $B$-$H$ curves have been determined, so that all the data for incorporating the nonlinear properties of magnetic materials into the transient code was prepared. The problem of this incorporation, however, is complicated with high nonlinearity of $B$-$H$ characteristics, especially in backing iron, as can be seen from Figure 13. Here Figure 13a shows the $B$-$H$ curves for the pipe wall ($B1$, solid line) and backing iron ($B2$, dashed line). Figure 13b shows the corresponding reluctivities $\nu$ as functions of magnetic induction, which are used in the code for successive iterations. In conventional iterations [26] some values (initial guesses) of reluctivity is taken for each element. Then the corresponding governing equation is solved using the FEM. the magnetic induction $B$ is computed, and from the known $B$-$H$ curve the magnetic field intensity $H$ and new values of reluctivity (as $\nu_i = H_i / B_i$) are determined. The next iterative step starts with the new values of the vector magnetic potential $A$ and reluctivity $\nu$. However, in this algorithm each step results in sudden changes of reluctivity, which in turn lead to divergence. To converge the iterative process, the damping coefficient $0 < C_d < 1$ is usually used, i.e.

$$\nu_{\text{new}} = \nu_{\text{old}} + C_d (\nu_i - \nu_{\text{old}}).$$

For essentially nonlinear $B$-$H$ characteristics, $C_d$ becomes very small and the convergence is extremely slow.
A different approach is used in this work. The damping coefficient is applied to the new value of $B$ rather to the new value of $v$, that is a 'damped' value of new $B$ is taken to estimate the new reluctivity,

$$B_{id} = [B_{iold} + C_d (B_i - B_{iold})].$$

(12)

the 'damped' field intensity $H_{id}$, corresponding to $B_{id}$, is found from the $B$-$H$ curve, and a new reluctivity is

$$v_{inew} = H_{id} / B_{id}$$

(13)

with assigning $B_{iold} = B_{id}$ to use $B_{iold}$ in the next step. Experience show that this algorithm

![Figure 13. (a) B-H characteristics for pipe wall (solid) and backing iron (dashed), and (b) reluctivity as a function of magnetic induction.](image)
leads to faster convergence. Often annealing is implemented to increase the convergence rate. In this case \( C_d \) is not a constant, but initialized as some number close to 1 (as a higher limit), and then gradually decreased to a certain lower limit, for example,

\[
C_{\text{new}} = C_{\text{old}} - D (C_{\text{old}} - C_{\text{min}}).
\]

with some rate \( D, 0 < D < 1 \).

It is important to mention that the iterative procedures designed for the static cases are not feasible for the transient cases. In the latter case, the two different kinds of steps in computations must be distinguished. First, we have steps within the iteration process for a single time instance \( t_k \), and second, we have steps incrementing these time instances. The difference equations in time stepping procedures, used for numerical solution of the PDE, have both old and new values of the unknown function (\( A(t_k) \) and \( A(t_{k+1}) \) in our case). The iteration process for a single time instance does not allow changes in \( A(t_k) \). The same \( A(t_k) \) is used for all iterations of \( A(t_{k+1}) \), and the criterion of convergence is the proximity of \( A(t_{k+1}) \) to itself for the two consecutive iterations.

Altogether, the simplified version of the computational procedure is as follows.

1. Initialize the values of \( \nu_i \) for \( i=1...N \), where \( N \) is the number of elements in the mesh.

   Take \( A_j(t_0) = 0 \) for \( j=1...M \), where \( M \) is the number of nodes in the mesh. For the initial guess we may take the initial reluctivity (when \( B=0 \)). Do not introduce the defect on this stage.

2. Compute the values of \( A \) and \( B \) using the FEM formulation with time stepping.
3. Compare the new $A_j(t_{k+1})$ with the old ones $A_j(t_k)$. If the difference is not within the threshold, go to step 2 without changing $v_i$. Otherwise the iterations are considered as converged to a solution for initial $v_i$.

4. Compute the values of $A$ and $B$ using the FEM formulation with time stepping.

5. Compare the new $A_j(t_{k+1})$ with the old ones $A_j(t_k)$. If the difference is not within the threshold, update $v_i$ according to (12) and (13), take $k=k+1$. update $A_{jnew}(t_k) = A_{jold}(t_{k+1})$ and go to step 4. Otherwise the iterations are considered as converged to a steady state solution with $v_i$ corresponding to individual points on $B-H$ curve.

6. Introduce a defect (or, if exists, move the defect one grid step size along the $z$-direction) by setting the air properties to the element with a defect (or to a part of an element if a tight crack model is used). Take $k=k+1$. update $A_{jnew}(t_k) = A_{jold}(t_{k+1})$. Compute the values of $A_{jnew}(t_{k+1})$ and $B$ using the FEM formulation with time stepping and update $v_i$ according to (12) and (13).

7. Assign $A_{jold}(t_{k+1}) = A_{jnew}(t_{k+1})$. compute once more the values of $A_{jnew}(t_{k+1})$ and $B$ using the FEM formulation with time stepping.

8. Compare the new $A_{jnew}(t_{k+1})$ with the old ones $A_{jold}(t_{k+1})$. If the difference is not within the threshold, update $v_i$ according to (12) and (13) (but neither $k$ nor $A_j(t_k)$ can be updated due to the transient nature of solution), and go to step 7.
9. If the defect passed the test bed vehicle, then stop, otherwise go to step 6.

An example of the solution obtained by an implemented code for the 2-D axisymmetric MFL test bed vehicle moving at a speed of 2 m/s is shown in Figure 9.

Figure 9. Magnetic field distribution around a test bed vehicle moving at speed 2 m/s in a saturated pipe (a) and in a pipe without the saturation (b).
CHAPTER 5. ROTATING FIELD MODELING

The rotating magnetic field was numerically simulated using the finite element analysis. For the near field region a 2-D mesh in polar coordinates has been developed, and the FEM formulation applied. Figure 15 shows the distribution of the magnetic field. Due to relative (compared to 3-D) simplicity of the 2-D codes, the mesh reflects many fine features of the geometry, e.g. it shows all 24 winding slots in the rotor.

3-D modeling is a lot more computationally expensive, and hence, the mesh construction is very critical for the overall performance. The less elements used to represent the geometry, the more efficient computations will be. The current density specifications play a very important role in achieving a good accuracy in simulations. If the element approach is

Figure 15. 2-D mesh with the magnetic flux lines distribution for the near field simulations in polar coordinates.
taken, when the current density is considered as a constant within an element, then certain problems arise at the conductor turns and intersections. For the specifications of the current flow distribution in the conductors, a nodal approach was taken, when the current density in an element is presented as a sum of the shape functions, weighted by the values at the nodes. It allows

- smooth distribution of the current density in the element,
- continuous inter-element transition of the current density,
- avoid cases, where the current enters some element boundary, but does not leave it in the adjacent element,
- zero divergence conditions for the current.

As an illustrative example, consider a situation, shown in Figure 16, where the three conductors with uniform current densities \( J_1, J_2 \) and \( J_3 \) join at an element with the unknown density \( J_4 \). The nodal values of \( J_1 \) at all 8 nodes in the element \#1 are equal with

![Figure 16. An illustrative example of conductors intersection.](image-url)
components $J_{1z} = -J_1$, and $J_{1x} = J_{1y} = 0$. Also for element #2 $J_{2x} = J_2$, $J_{2y} = J_{2z} = 0$.

and for element #3 $J_{3x} = J_3$, $J_{3y} = J_{3z} = 0$. Then the nodal values in

the element #4 are for the top left nodes $J_{4x,tl} = J_2$, $J_{4y,tl} = 0$, $J_{4z,tl} = -J_1$, for the top

right nodes $J_{4x,tr} = J_3$, $J_{4y,tr} = 0$, $J_{4z,tr} = -J_1$, for the bottom left nodes

$J_{4x,bl} = J_2$, $J_{4y,bl} = 0$, $J_{4z,bl} = 0$, and for the bottom right nodes $J_{4x,br} = J_3$.

$J_{4y,br} = 0$, $J_{4z,br} = 0$.

Another possibility to improve the efficiency of the code is to use periodic conditions.

A mesh with 36 edges in circumferential, 47 edges in radial and 361 edges in axial directions

was used for simulations. This results in 596,160 elements, 610,812 nodes, and 1,829,436

unknowns. Without imposing periodic boundary conditions the stiffness matrix for that mesh

would be banded with $9.28 \cdot 10^9$ elements in the band. Implementation of the periodic

boundary conditions allows a 6 fold reduction of the nodes number in the circumferential

direction and a 2 fold reduction in the axial direction. This results in 50,901 nodes, 152,703

unknowns, and the number of elements in the band of the stiffness matrix becomes $128 \cdot 10^6$.

which is 72 times less than the original number.

A simplified algorithm for incorporation of the periodic boundary in the

circumferential direction conditions is as follows

1. Take $M+1$ elements in the circumferential direction, where $M=M_1/6$, $M_1$ is the total

    number of edges in this direction in the original mesh.
2. Apply the test functions only at internal nodes, thus excluding the periodic boundary nodes. The excluded nodes are lying on two planes: one plane (call it \( P_1 \)) touches the elements, having the lowest bookkeeping numbers, another plane (call it \( P_2 \)) touches the elements with the highest bookkeeping numbers. Take the clockwise direction as the direction of node counting.

3. Substitute the nodal values of \( A \), corresponding to the boundary nodes, with the nodal values at the corresponding periodic nodes, i.e. if a node lies on the \( P_1 \) plane, then substitute it with the node lying \( M \) steps in the clockwise direction. If a node lies on the \( P_2 \) plane, then substitute it with the node lying \( M \) steps in the counterclockwise direction.

4. Multiply by \( \exp(i\pi/3) \) all the nodes substituting the nodes from the \( P_1 \) plane and by \( \exp(-i\pi/3) \) all the nodes substituting the nodes from the \( P_2 \) plane.

5. If the components of \( A \) are presented not in cylindrical coordinates, then apply the corresponding transformations to nodes, substituting the boundary nodes. For example, if the components of \( A \) are in Cartesian coordinates, then apply the 60° rotational transforms in the corresponding direction.

Also a program, allowing considerable reduction in the required computer memory size and allowing utilization of high performance multiprocessor clusters with parallel code execution, was developed for solving banded matrix equations. A description of the used method, along with a version of the code, are placed in appendix.
CHAPTER 6. TIGHT CRACKS

As was pointed out, the necessity of modeling the stress corrosion type cracks with reasonable accuracy and mesh density led to the notion of the tight cracks. The nodal description approach (as opposed to the element), in which the material properties (reluctivity and conductivity) may vary within an element, was taken [27]. It is shown in this work that this is equivalent to the first order (or linear) properties model. Other models are developed in section 6.1 of this work, and results for 1-D and 2-D cases are presented in sections 6.2 - 6.5.

6.1. Analytical Solution for 1-D

If we allow material properties to vary within an element, then the governing equation for the magnetic vector potential (MVP) \( A \).

\[
\nabla \times \frac{1}{\mu} \nabla \times \bar{A} = 0
\]

becomes

\[
v \nabla \times \nabla \times \bar{A} + \nabla v \times \nabla \times \bar{A} = 0.
\]

where \( v = l/\mu, \mu - \) magnetic permeability, \( v - \) reluctance. For the 2-D axisymmetric case we use cylindrical coordinates with \( \hat{r}, \hat{\phi}, \hat{z} - \) unit vectors in radial, circumferential, and axial directions respectively. In this case, assuming \( \frac{\partial v}{\partial \phi} = 0 \).

\[
\nabla v = \hat{r} \frac{\partial v}{\partial r} + \hat{z} \frac{\partial v}{\partial z},
\]
\[ \overline{A} = A \hat{\phi} \cdot \nabla \times \overline{A} = - \hat{r} \frac{\partial A}{\partial z} + \hat{z} \left( \frac{A}{r} + \frac{\partial A}{\partial r} \right). \]

\[ \nu \nabla \times (\nabla \times \overline{A}) = \nu \phi \left( \frac{A}{r^2} - \frac{l}{r} \frac{\partial A}{\partial r} - \frac{\partial^2 A}{\partial r^2} - \frac{\partial^2 A}{\partial z^2} \right) \]

\[ \nabla \nu \times \nabla \times \overline{A} = - \phi \left[ \left( \frac{A}{r} + \frac{\partial A}{\partial r} \right) \frac{\partial \nu}{\partial r} + \frac{\partial A}{\partial z} \frac{\partial \nu}{\partial z} \right] \]

and the governing equation is written as

\[ \nu \left( \frac{A}{r^2} - \frac{l}{r} \frac{\partial A}{\partial r} - \frac{\partial^2 A}{\partial r^2} - \frac{\partial^2 A}{\partial z^2} \right) = - \frac{\partial}{\partial r} \left( \frac{A}{r} + \frac{\partial A}{\partial r} \right) \frac{\partial \nu}{\partial r} - \frac{\partial A}{\partial z} \frac{\partial \nu}{\partial z} = 0. \quad (15) \]

For simplicity, consider a 1-D differential equation

\[ \frac{d^2 A}{dz^2} + \nu \frac{d A}{dz} = 0 \quad (16). \]

Define \( B = dA/dz \) - magnetic induction. Then \( \frac{d\nu}{dz} B = - \nu \frac{dB}{dz} \), or \( B(z) = B(z_k) \frac{\nu(z_k)}{\nu(z)} \) for \( z_k \)

- some fixed point. Hence, \( A(z) = A(z_k) + B(z_k) \nu(z_k) \frac{\zeta}{\nu(\zeta)} \). \quad (17)

Let us use a piecewise constant version of \( \nu \):

\[ \nu = \begin{cases} \nu_o, & \text{if } z_{k-1} < z < z_k, \\ \nu_m, & \text{otherwise} \end{cases} \quad (18) \]

as is shown in Figure 17a. Also take \( z_{j+1} - z_j = h \) for \( j = 1 \ldots N-2 \).

\[ z_1 - z_0 = z_N - z_{N-1} = h/2, \] with \( N \) - positive odd integer, \( h \) - grid stepsize. Then analytical solution for (16) is
\[ A(z_0) + \frac{dA(z_0)}{dz}(z-z_0), \text{ if } z < z_{k-1}. \]
\[ A(z_0) + \frac{dA(z_0)}{dz} h \left( k - \frac{3}{2} \right), \text{ if } z = z_{k-1}. \]
\[ A(z_0) + \frac{dA(z_0)}{dz} h \left( k - \frac{3}{2} \right) + \frac{dA(z_0)}{dz} v_m (z-z_{k-1}), \text{ if } z_{k-1} < z < z_k. \]
\[ A(z_0) + \frac{dA(z_0)}{dz} h \left( k - \frac{3}{2} \right) + \frac{dA(z_0)}{dz} v_m h, \text{ if } z = z_k. \]
\[ A(z_0) + \frac{dA(z_0)}{dz} h \left( k - \frac{3}{2} \right) + \frac{dA(z_0)}{dz} v_m h + \frac{dA(z_0)}{dz} (z-z_k), \text{ if } z > z_k. \]
\[ A(z_0) + \frac{dA(z_0)}{dz} h(N-2) + \frac{dA(z_0)}{dz} v_m h, \text{ if } z = z_N. \]

Expressing \( \frac{dA(z_0)}{dz} \) via \( A(z_0) \) and \( A(z_N) \), taking \( k = (N+1)/2 \) and \( A(z_N) = -A(z_1) \) we write
\[
B = \begin{cases}
- \frac{2A(z_0)}{h(N - 2 + \frac{v_m}{v_o})} \frac{v_m}{v_o}, & \text{if } z_{k-1} < z < z_k, \\
- \frac{2A(z_0)}{h(N - 2 + \frac{v_m}{v_o})}, & \text{otherwise}
\end{cases}
\]

\[A(z_0) - \frac{2A(z_0)}{h(N - 2 + \frac{v_m}{v_o})} (z - z_0), \text{ if } z < z_{k-1}.
\]

\[A(z_0) - \frac{A(z_0)}{N - 2 + \frac{v_m}{v_o}} \left[ N - 2 + 2 \frac{v_m}{v_o} \frac{z - z_k - l}{h} \right], \text{ if } z_{k-1} < z < z_k.
\]

\[A(z_0) - \frac{A(z_0)}{N - 2 + \frac{v_m}{v_o}} \left[ N - 2 + 2 \frac{v_m}{v_o} + 2 \frac{z - z_k}{h} \right], \text{ if } z > z_k.
\]

as it is shown in Figure 17b for \(A(z_0) = 1, \frac{v_o}{v_m} = 20, N=2l, h=0.1.

If \(v\) is a piecewise linear function of \(z\),

\[
v = \begin{cases}
v_m + (v_o - v_m) \frac{z - z_{k-1}}{h}, & \text{if } z_{k-1} < z < z_k, \\
v_m + (v_o - v_m) \frac{z_{k+1} - z}{h}, & \text{if } z_k < z < z_{k+1}, \\
v_m, & \text{otherwise}
\end{cases}
\]

as it is shown in Figure 18a, with \(z_{j+1} - z_j = h\) for all \(j=0...N-1\). \(N\) - positive even integer.

then analytical solution for (16) is
A = \begin{cases} 
A(z_0) + \frac{dA(z_0)}{dz}(z - z_0), & \text{if } z < z_{k-1}, \\
A(z_0) + \frac{dA(z_0)}{dz} h(k - 1), & \text{if } z = z_{k-1}, \\
A(z_0) + \frac{dA(z_0)}{dz} h \left[ k - l + \frac{v_m}{v_o - v_m} \ln \left( 1 + \frac{v_o - v_m}{v_m} \frac{z - z_{k-1}}{h} \right) \right], & \text{if } z_{k-1} < z < z_k, \\
A(z_N) - \frac{dA(z_N)}{dz} (z_N - z), & \text{if } z > z_{k+1}, \\
A(z_N) - \frac{dA(z_N)}{dz} h(N - k - 1), & \text{if } z = z_{k+1} \\
A(z_N) - \frac{dA(z_N)}{dz} h \left[ N - k - 1 + \frac{v_m}{v_o - v_m} \ln \left( 1 + \frac{v_o - v_m}{v_m} \frac{z_{k+1} - z}{h} \right) \right], & \text{if } z_k < z < z_{k+1}. 
\end{cases}

Expressing $\frac{dA(z_0)}{dz}$ and $\frac{dA(z_N)}{dz}$ via $A(z_0)$ and $A(z_N)$, taking $k = N/2$. $A(z_N) = -A(z_0)$ and $
\frac{dA(z_0)}{dz} = \frac{dA(z_N)}{dz}$, we have $A(z_k) = 0$, and

Figure 18. (a) - $v$ as a piecewise linear function, (b) - corresponding MVP $A$ and induction $B$.
as it is shown in Figure 18b for \( A(z_0) = l, \frac{v_o}{v_m} = 20 \), \( N = 20 \), \( h = 0.1 \).

Let's assume that \( v \) is a piecewise polynomial of the form
\[
V = \begin{cases} 
V_m, & \text{if } z \leq z_k - 1, \\
V_m + (V_o - V_m) \left( \frac{z - z_k - 1}{h} \right)^n, & \text{if } z_k - 1 < z < z_k, \\
V_o, & \text{if } z = z_k, \\
V_m + (V_o - V_m) \left( \frac{z_k + 1 - z}{h} \right)^n, & \text{if } z_k < z < z_{k+1}, \\
V_m, & \text{if } z \geq z_{k+1}.
\end{cases}
\]

(24)

where \( h = z_k - z_{k-1} = z_{k+1} - z_k \) - grid stepsize, \( n \) - degree of polynomial, as it is shown in Figure 19a. Then, defining (see formula 2.142 in [28])

\[
\hat{s}(\zeta) = \begin{cases} 
-\frac{n-1}{2} \sum_{k=0}^{n} P_k \cos \frac{2k + 1}{n} - \pi + \frac{n}{2} \sum_{k=0}^{n/2 - 1} Q_k \sin \frac{2k + 1}{n} - \pi. & \text{for } n \text{ - even} \\
-\frac{1}{n} \ln(1 + \zeta) - \frac{n}{2} \sum_{k=0}^{n/2 - 1} P_k \cos \frac{2k + 1}{n} - \pi + \frac{n}{2} \sum_{k=0}^{n/2} Q_k \sin \frac{2k + 1}{n} - \pi. & \text{for } n \text{ - odd}
\end{cases}
\]

where

Figure 19. (a) - \( v \) as a piecewise polynomial, (b) - corresponding MVP A and induction B.
\[ P_k = \frac{1}{2} \ln \left( \xi^2 - 2\xi \cos \frac{2k + 1}{n} \pi + 1 \right), \]

\[ Q_k = \arctg \frac{x - \cos \frac{2k + 1}{n} \pi}{\sin \frac{2k + 1}{n} \pi}, \]

we write

\[
A = \begin{cases} 
A(z_0) + \frac{dA(z_0)}{dz}(z - z_0), & \text{if } z < z_{k-1}, \\
A(z_0) + \frac{dA(z_0)}{dz} h(k - 1), & \text{if } z = z_{k-1}, \\
A(z_0) + \frac{dA(z_0)}{dz} \left[ k - l + \left( \frac{v_m}{v_o - v_m} \right) \frac{l}{n} \left( \frac{v_o - v_m}{v_m} \right) \frac{l}{h} \right], & \text{if } z_{k-1} < z < z_k, \\
A(z_N) - \frac{dA(z_N)}{dz}(z_N - z), & \text{if } z > z_{k+1}, \\
A(z_N) - \frac{dA(z_N)}{dz} h(N - k - 1), & \text{if } z = z_{k+1}, \\
A(z_N) - \frac{dA(z_N)}{dz} \left[ N - k - l + \left( \frac{v_m}{v_o - v_m} \right) \frac{l}{n} \left( \frac{v_o - v_m}{v_m} \right) \frac{l}{h} \right], & \text{if } z_k < z < z_{k-1}.
\end{cases}
\]

Expressing \( \frac{dA(z_0)}{dz} \) and \( \frac{dA(z_N)}{dz} \) via \( A(z_0) \) and \( A(z_N) \), taking \( k = N/2 \). \( A(z_N) = -A(z_l) \)

and \( \frac{dA(z_0)}{dz} = \frac{dA(z_N)}{dz} \), we have \( A(z_k) = 0 \). and
\[
\frac{dA(z_0)}{dz} = -A(z_0) \left[ \frac{N}{2} - I + \left( \frac{v_m}{v_o - v_m} \right)^n \left( \frac{v_o - v_m}{v_m} \right)^{\frac{l}{n}} \right]
\]

so, that

\[
B = \begin{cases} 
\frac{-1}{N - I + \left( \frac{v_m}{v_o - v_m} \right)^n \left( \frac{v_o - v_m}{v_m} \right)^{\frac{l}{n}}} \cdot A(z_0) & z_{k-1} < z < z_k, \\
\frac{1}{N - I + \left( \frac{v_m}{v_o - v_m} \right)^n \left( \frac{v_o - v_m}{v_m} \right)^{\frac{l}{n}}} \cdot A(z_0) & z_k < z < z_{k+1}, \\
\frac{1}{h} \cdot \frac{A(z_0)}{h} & \text{otherwise}
\end{cases}
\]

(25)
\[
\begin{align*}
A(z) &= \begin{cases} 
\frac{A(z_0)(z-z_0)}{h \left[ \frac{N}{2} - 1 + \left( \frac{v_m}{v_o - v_m} \right)^{\frac{l}{n}} \left( \frac{v_o - v_m}{v_m} \right)^{\frac{l}{n}} \right]} & \text{if} \quad z < z_{k-1}, \\
\frac{N}{2} - 1 + \left( \frac{v_m}{v_o - v_m} \right)^{\frac{l}{n}} \left( \frac{v_o - v_m}{v_m} \right)^{\frac{l}{n}} \frac{z - z_{k-1}}{h} & \text{if} \quad z_{k-1} < z < z_k, \\
\frac{N}{2} - 1 + \left( \frac{v_m}{v_o - v_m} \right)^{\frac{l}{n}} \left( \frac{v_o - v_m}{v_m} \right)^{\frac{l}{n}} \frac{z_{k+1} - z}{h} & \text{if} \quad z_k < z < z_{k-1}, \\
\frac{N}{2} - 1 + \left( \frac{v_m}{v_o - v_m} \right)^{\frac{l}{n}} \left( \frac{v_o - v_m}{v_m} \right)^{\frac{l}{n}} \frac{z_N - z}{h} & \text{if} \quad z > z_{k+1}.
\end{cases}
\end{align*}
\]

(26)

as it is shown in Figure 19b for \(A(z_0) = 1, \frac{v_o}{v_m} = 20, N = 20, h = 0.1\).

If instead of \(v\) we plot \(\mu\) as a function of \(z\), as in Figure 20, we see, that the shape of the induction \(B\) is almost the same as that of \(\mu\). Lets try to use polynomial interpolation for \(\mu\), whereas \(v\) would be a corresponding reciprocal.
\[ \mu = \begin{cases} 
\mu_m, & \text{if } z \leq z_{k-1}, \\
\mu_m - (\mu_m - \mu_o) \left( \frac{z - z_{k-1}}{h} \right)^n, & \text{if } z_{k-1} < z < z_k, \\
\mu_o, & \text{if } z = z_k, \\
\mu_m - (\mu_m - \mu_o) \left( \frac{z_{k+1} - z}{h} \right)^n, & \text{if } z_k < z < z_{k+1}, \\
\mu_m, & \text{if } z \geq z_{k+1}, 
\end{cases} \]

(27)

\[ \nu = \begin{cases} 
\frac{l}{\mu_m}, & \text{if } z \leq z_{k-1}, \\
\frac{l}{\mu_m} - (\mu_m - \mu_o) \left( \frac{z - z_{k-1}}{h} \right)^n, & \text{if } z_{k-1} < z < z_k, \\
\frac{l}{\mu_o}, & \text{if } z = z_k, \\
\frac{l}{\mu_m} - (\mu_m - \mu_o) \left( \frac{z_{k+1} - z}{h} \right)^n, & \text{if } z_k < z < z_{k+1}, \\
\frac{l}{\mu_m}, & \text{if } z \geq z_{k+1}, 
\end{cases} \]

(28)

then we have

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig20.png}
\caption{\(\mu\), corresponding to \(\nu\) - piecewise linear function (a), and to \(\nu\) - piecewise polynomial (b).}
\end{figure}
\[
A(z_0) + \frac{dA(z_0)}{dz} (z - z_0), \quad \text{if} \quad z < z_{k-1}.
\]
\[
A(z_0) + \frac{dA(z_0)}{dz} h(k - 1), \quad \text{if} \quad z = z_{k-1}.
\]
\[
A(z_0) + \frac{dA(z_0)}{dz} \left[ z - z_0 - \frac{z - z_{k-1}}{n + 1} \left( \frac{z - z_{k-1}}{h} \right)^n \left( 1 - \frac{\mu_0}{\mu_m} \right) \right], \quad \text{if} \quad z_{k-1} < z < z_k.
\]
\[
A(z_N) - \frac{dA(z_N)}{dz} (z_N - z), \quad \text{if} \quad z > z_{k+1}.
\]
\[
A(z_N) - \frac{dA(z_N)}{dz} h(N - k - 1), \quad \text{if} \quad z = z_{k+1}
\]
\[
A(z_N) - \frac{dA(z_N)}{dz} \left[ z_N - z - \frac{z_{k+1} - z}{n + 1} \left( \frac{z_{k+1} - z}{h} \right)^n \left( 1 - \frac{\mu_0}{\mu_m} \right) \right], \quad \text{if} \quad z_k < z < z_{k+1}.
\]

Expressing \( \frac{dA(z_0)}{dz} \) and \( \frac{dA(z_N)}{dz} \) via \( A(z_0) \) and \( A(z_N) \), taking \( k = N/2 \).

\[
\mu_m = \frac{l}{\nu_m}, \quad \mu_o = \frac{l}{\nu_o}, \quad A(z_N) = -A(z_0), \quad \text{and} \quad \frac{dA(z_0)}{dz} = \frac{dA(z_N)}{dz}. \text{we have} \quad A(z_{k}) = 0. \text{and}
\]

\[
B = \left\{ \begin{array}{ll}
\frac{A(z_0)}{h} \left( 1 - \frac{z - z_{k-1}}{h} \right)^n \left( 1 - \frac{\nu_m}{\nu_o} \right), & \text{if} \quad z_{k-1} < z < z_k. \\
\frac{A(z_0)}{h} \left( 1 - \frac{z_{k+1} - z}{h} \right)^n \left( 1 - \frac{\nu_m}{\nu_o} \right), & \text{if} \quad z_k < z < z_{k+1}.
\end{array} \right.
\]

\[
\mu_m = \frac{l}{\nu_m}, \quad \mu_o = \frac{l}{\nu_o}, \quad A(z_N) = -A(z_0), \quad \text{and} \quad \frac{dA(z_0)}{dz} = \frac{dA(z_N)}{dz}. \text{we have} \quad A(z_{k}) = 0. \text{and}
\]

\[
B = \left\{ \begin{array}{ll}
\frac{A(z_0)}{h} \left( 1 - \frac{z - z_{k-1}}{h} \right)^n \left( 1 - \frac{\nu_m}{\nu_o} \right), & \text{if} \quad z_{k-1} < z < z_k. \\
\frac{A(z_0)}{h} \left( 1 - \frac{z_{k+1} - z}{h} \right)^n \left( 1 - \frac{\nu_m}{\nu_o} \right), & \text{if} \quad z_k < z < z_{k+1}.
\end{array} \right.
\]
\[ A(z_0) = \begin{cases} 
\frac{A(z_0)}{h \left( \frac{N}{2} - \frac{l}{n+1} \left( 1 - \frac{v_m}{v_o} \right) \right)} (z - z_0), & \text{if } z < z_{k-1}. \\
- \frac{z - z_0}{n+1} \left( \frac{z - z_{k-1}}{h} \right)^n \left( 1 - \frac{v_m}{v_o} \right), & \text{if } z_{k-1} < z < z_k. \\
\frac{z - z_0}{n+1} \left( \frac{z - z_{k+1} - z}{h} \right)^n \left( 1 - \frac{v_m}{v_o} \right), & \text{if } z_k < z < z_{k+1}. \\
- \frac{z - z_0}{n+1} \left( 1 - \frac{v_m}{v_o} \right) (z - z_{N}), & \text{if } z > z_{k+1}. 
\end{cases} \]

(30)

as it is shown in Figure 21a for \( A(z_0) = 1, \frac{v_o}{v_m} = 20, N=20, h=0.1 \).

For comparison, all four analytical solutions are combined in one Figure 22b.

Consider now a differential equation with a forcing term.

Figure 21. \( \mu \) as a piecewise polynomial (a), and the corresponding \( \nu \) (b).
Figure 22. (a) MVP A and induction B for piecewise polynomial $\mu$, and (b) a comparison of the four solutions.

\[
\frac{d}{dz} \left( \nu \frac{dA}{dz} \right) = J_s \tag{16a}
\]

(where $J_s$ is an external current source), which has a solution

\[
A = A(z_k) + \int_{z_i}^{z} \left[ \nu(z_k)B(z_k) + \int_{z_i}^{z} J_s ds \right] d\zeta.
\]

Also let $\nu$ be a piecewise constant.

\[
\nu = \nu_i, \quad \text{if} \quad z_i < z < z_{i+1}, \quad i = 0 \ldots N - 1.
\]

and

\[
J_s = \begin{cases} 
J, & \text{if} \quad z_s < z < z_{s+1} \\
0, & \text{otherwise}
\end{cases} \quad \text{for some} \quad 0 < s < N.
\]

then
\[
A = \begin{cases} 
A(z_0) + \frac{dA(z_0)v_0}{dz} & \sum_{i=0}^{k-1} \frac{z_{i+1} - z_i + z - z_k}{v_i v_k}, \quad \text{if } z_k < z < z_{k+1} \leq z_s \\
A(z_0) + \frac{dA(z_0)v_0}{dz} & \sum_{i=0}^{s-1} \frac{z_{i+1} - z_i + z - z_s}{v_i v_s} + \frac{J(z - z_s)^2}{2v_s}, \quad \text{if } z_s < z < z_{s+1} \\
A(z_0) + \frac{dA(z_0)v_0}{dz} & \sum_{i=0}^{k-1} \frac{z_{i+1} - z_i + z - z_k}{v_i v_k} + \frac{J(z_{s+1} - z_s)^2}{2v_s} + \\
+ J(z_{s+1} - z_s) & \sum_{i=s+2}^{k-1} \frac{z_{i+1} - z_i + z - z_k}{v_i v_k}, \quad \text{if } z_{s+1} \leq z_k < z < z_{k+1} \leq z_N 
\end{cases}
\]

\[
B = \begin{cases} 
\frac{dA(z_0)v_0}{dz} & v_k, \quad \text{if } z_k < z < z_{k+1} \leq z_s \\
\frac{dA(z_0)v_0}{dz} & v_s, \quad \frac{J(z - z_s)}{v_s}, \quad \text{if } z_s < z < z_{s+1} \\
\frac{dA(z_0)v_0}{dz} & v_k, \quad \frac{J(z_{s+1} - z_s)}{v_k}, \quad \text{if } z_{s+1} < z_k < z < z_{k+1} 
\end{cases}
\]

For zero Dirichlet boundary conditions \(A(z_0) = A(z_N) = 0\).

\[
\frac{dA(z_0)}{dz} v_0 = -J(z_{s+1} - z_s) \frac{\frac{z_{s+1} - z_s}{2v_s} + \sum_{i=s+2}^{N} \frac{z_{i+1} - z_i}{v_i}}{\sum_{i=0}^{N} \frac{z_{i+1} - z_i}{v_i}}.
\]

As it is shown in Figure 23b for \(N=5, s=2, J=1\), with \(v\) specified in Figure 23a.
Figure 23. Reluctivity (a) and analytical A and B (b) for the equation with current source.

6.2. Numerical Solution Using Finite Element Method

For finite element methods, a weak formulation is used.

\[
\frac{dv}{dz} \frac{dA}{dz} + v \frac{d^2 A}{dz^2} = \frac{d}{dz}(v \frac{dA}{dz}).
\]

Multiplying by a test function \( \psi_j \), and integrating

\[
\int_{z_i}^{z_{i+1}} \psi_j \left( v \frac{dA}{dz} \right) dz = v \psi_j \left. \frac{dA}{dz} \right|_{z_i}^{z_{i+1}} - \int_{z_i}^{z_{i+1}} v \frac{dA}{dz} \frac{d\psi_j}{dz} dz = - \int_{z_i}^{z_{i+1}} v \frac{dA}{dz} \frac{d\psi_j}{dz} dz.
\]

With shape functions \( \phi_i \), describing \( v \), \( v = \sum_{i=0}^{N} v_i \phi_i \), shape functions \( \phi \) for \( A \),

\[
A = \sum_{i=0}^{N} A_i \phi_i
\]

(and test functions \( \psi_i = \phi_i \)).
From (16) we have

\[ I_j = 0, \quad j = 1 \ldots N - 1. \]  \hspace{1cm} (33a)

and imposing the boundary conditions (BC)

\[ A_0 = A_N = A(z_0). \]  \hspace{1cm} (33b)

finally write in matrix form

\[ SA = F. \]  \hspace{1cm} (34)

where

\[
A = \begin{bmatrix} A_0 \\ \vdots \\ A_N \end{bmatrix} \quad \text{- the unknown vector,} \quad F = \begin{bmatrix} A(z_0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{- forcing term (and BC),} \quad S \quad \text{- stiffness matrix, arising from the combining (33 a,b) into a system of equations. Solving (34) gives} \ A \ \text{in form of (31).} \ B \ \text{can be computed as}
\]

\[ B = \sum_{i=0}^{N} A_i \frac{d\phi_i}{dz}. \]  \hspace{1cm} (31a)
Taking various shape functions \( \varphi \) and \( \phi \), we can compare different approaches.

Let \( C_\varphi^N \) be a space, in which \( v \) is described, \( C_\phi^N \) - a space of the A approximation.

**6.2.1. Conventional approach.** \( C_\varphi^N \) - a space of piecewise constant functions. \( C_\phi^N \) - a space of piecewise linear functions. Variations in \( v \) are represented as abrupt changes (see Figure 12a).

\[
\varphi_i = \begin{cases} 
\frac{z_j - z}{h}, & \text{if } i = j - 1, z_{j-1} < z < z_j \\
\frac{z - z_{j-1}}{h}, & \text{if } i = j, z < z_j < z_{j+1} \\
1 - \frac{z - z_j}{h}, & \text{if } i = j, z_{j-1} < z < z_{j+1} \\
0, & \text{otherwise}
\end{cases}
\]

\[
\frac{d\varphi_i}{dz} = \begin{cases} 
-\frac{1}{h}, & \text{if } i = j - 1, z_{j-1} < z < z_j \\
\frac{1}{h}, & \text{if } i = j, z < z_j < z_{j+1} \\
\text{sign}(z_j - z), & \text{if } i = j, z_{j-1} < z < z_{j+1} \\
0, & \text{otherwise}
\end{cases}
\]

The integral (32) is now

\[
I_j = \frac{1}{h^2} \left[ \int_{z_{j-1}}^{z_j} (-A_{j-1} + A_j)v_j dz + \int_{z_j}^{z_{j+1}} (A_j - A_{j+1})v_{j+1} dz \right] = \frac{1}{h} \left[ -A_{j-1}v_j + A_j(v_j + v_{j+1}) - A_{j+1}v_{j+1} \right] \quad (32a)
\]

Now we solve system (34) with tridiagonal symmetric positive definite \( S \), and compare with (19) and (20), as shown in Figure 24a. Solutions coincide.
6.2.2. Linear (tight crack) approach. $C_{\phi}^N$ and $C_{\varphi}^N$ are both spaces of piecewise linear functions.

\[
\phi_i = \phi_i = \begin{cases} 
\frac{z_j - z}{h}, & \text{if } i = j - 1, z_{j-1} < z < z_j \\
\frac{z - z_{j-1}}{h}, & \text{if } i = j + 1, z_j < z < z_{j+1} \\
1 - \frac{z - z_j}{h}, & \text{if } i = j, z_{j-1} < z < z_{j+1} \\
0, & \text{otherwise}
\end{cases}
\]

Figure 24. FEM computed for constant $\nu$ (a), for linear $\nu$ (b), for polynomial $\nu$ (c), and for reciprocal polynomial $\nu$ (d).
\[
\frac{d\varphi_i}{dz} = \begin{cases} 
\frac{1}{h}, & \text{if } i = j, z_j - 1 < z < z_j \\
\frac{1}{h}, & \text{if } i = j + 1, z_j < z < z_{j+1} \\
\frac{\text{sign}(z_j - z)}{h}, & \text{if } i = j, z_{j-1} < z < z_{j+1} \\
0, & \text{otherwise}
\end{cases}
\]

The integral (32) is

\[
I_j = \frac{1}{h^3}\left[ \int_{z_{j-1}}^{z_j} (A_j - A_{j-1})[v_{j-1}(z_j - z) + v_j(z - z_{j-1})]dz + \int_{z_j}^{z_{j+1}} (A_j - A_{j+1})[v_j(z_{j+1} - z) + v_{j+1}(z - z_j)]dz \right] = \frac{l}{2h}[A_j - A_{j-1}(v_{j-1} + v_j) + A_j(v_{j-1} + 2v_j + v_{j+1}) - A_{j+1}(v_j + v_{j+1})] \tag{32b}
\]

Now we solve system (34), compare with (22) and (23), as shown in Figure 25a, and see the difference in Figure 25b.

If we use such a \( C_\varphi^N \), that in elements adjacent to the point \( z_k \), \( \varphi \) is different, i.e.

\[
\varphi_{k-1} = \begin{cases} 
\frac{z - z_k - 2}{h}, & \text{if } z_{k-2} < z < z_k - 1 \\
1 - \ln\left[ 1 + \frac{\mathcal{V}_o}{\mathcal{V}_m} \right] \frac{z - z_{k-1}}{h}, & \text{if } z_{k-1} < z < z_k \\
0, & \text{otherwise}
\end{cases}
\]
\[
\phi_k = \begin{cases} 
\frac{l}{\ln \frac{\nu_o}{v_m}} \ln \left[ 1 + \left( \frac{\nu_o - 1}{v_m - 1} \right) \frac{z - z_k - 1}{h} \right], & \text{if } z_k - 1 < z < z_k \\
\frac{l}{\ln \frac{\nu_o}{v_m}} \ln \left[ 1 + \left( \frac{\nu_o - 1}{v_m - 1} \right) \frac{z_{k+1} - z}{h} \right], & \text{if } z_k < z < z_{k+1} \\
0, & \text{otherwise}
\end{cases}
\]

\[
\phi_{k+1} = \begin{cases} 
\frac{z_{k+2} - z}{h}, & \text{if } z_{k+1} < z < z_{k+2} \\
\frac{l}{\ln \frac{\nu_o}{v_m}} \ln \left[ 1 + \left( \frac{\nu_o - 1}{v_m - 1} \right) \frac{z_{k+1} - z}{h} \right], & \text{if } z_k < z < z_{k+1} \\
0, & \text{otherwise}
\end{cases}
\]

Figure 25. FEM computed for linear \( v \) with improper \( \phi \) (a), and the error of this FEM solution (b).
\[
\frac{d\varphi_{k-1}}{dz} = \begin{cases} \frac{l}{h}, & \text{if } z_{k-2} < z < z_{k-1} \\ \frac{l}{(h\nu_m + z - z_{k-1}) \ln v_o / v_m}, & \text{if } z_{k-1} < z < z_k \\ 0, & \text{otherwise} \end{cases}
\]

\[
\frac{d\varphi_k}{dz} = \begin{cases} \frac{l}{h}, & \text{if } z_{k-1} < z < z_k \\ \frac{l}{(h\nu_m + z - z_k) \ln v_o / v_m}, & \text{if } z_k < z < z_{k+1} \\ 0, & \text{otherwise} \end{cases}
\]

\[
\frac{d\varphi_{k+1}}{dz} = \begin{cases} \frac{l}{h}, & \text{if } z_k < z < z_{k+2} \\ \frac{l}{(h\nu_m + z_{k+1} - z) \ln v_o / v_m}, & \text{if } z_{k+1} < z < z_{k+2} \\ 0, & \text{otherwise} \end{cases}
\]

For the three test functions \( \psi_{k-1}, \psi_k, \psi_{k+1} \) the integral (32) is

\[
I_{k-1} = \frac{v_m}{h}(A_{k-1} - A_{k-2}) + \frac{v_o - v_m}{h \ln v_o / v_m} (A_{k-1} - A_k)
\]

\[
I_k = \frac{v_o - v_m}{h \ln v_o / v_m} (-A_{k-1} + 2A_k - A_{k+1}) \quad (32b')
\]

\[
I_{k+1} = \frac{v_m}{h}(A_{k+1} - A_{k+2}) + \frac{v_o - v_m}{h \ln v_o / v_m} (A_{k+1} - A_k),
\]
for the other test functions \( \psi_j \), \( j = l...k-2,k+2...N-l \) it is as in (32a). Comparison of the FEM and analytical solutions is given in Figure 24b. They are identical.

6.2.3. Polynomial approach. \( C_{\phi}^N \) is a space of functions that are piecewise linear. except for two elements, adjacent to the point \( z_k \), where these functions are polynomials, as in (24). \( C_{\phi}^N \) is also a space of functions that are piecewise linear, except for two elements adjacent to point \( z_k \). Here functions \( \varphi \) are of the form

\[
\varphi_{k-1} = \begin{cases} \\
\frac{z-z_k-2}{h}, & \text{if } z_{k-2} < z < z_{k-1} \\
1 - \frac{1}{n} \left[ \left( \frac{\nu_o}{\nu_m} - 1 \right) \frac{z-z_{k-1}}{n} \right], & \text{if } z_{k-1} < z < z_k \\
0, & \text{otherwise} \\
\end{cases}
\]

\[
\varphi_k = \begin{cases} \\
\frac{1}{n} \left[ \left( \frac{\nu_o}{\nu_m} - 1 \right) \frac{z-z_{k-1}}{h} \right], & \text{if } z_{k-1} < z < z_k \\
\frac{1}{n} \left[ \left( \frac{\nu_o}{\nu_m} - 1 \right) \frac{z_{k+1} - z}{h} \right], & \text{if } z_k < z < z_{k+1} \\
0, & \text{otherwise} \\
\end{cases}
\]
\[\varphi_{k+1} = \left\{ \begin{array}{ll}
\frac{z_{k+2} - z}{h}, & \text{if } z_{k+1} < z < z_{k+2} \\
1 - \frac{l}{\left( \frac{v_o - l}{v_m - l} \right)^n} \left[ \left( \frac{v_o}{v_m - l} \right)^n \frac{l}{n} z_{k+1} - \frac{l}{n} z_k \right], & \text{if } z_k < z < z_{k+1}
\end{array} \right. \]

0. otherwise

\[\frac{d\varphi_{k-1}}{dz} = \left\{ \begin{array}{ll}
\frac{1}{h}, & \text{if } z_{k-2} < z < z_{k-1} \\
- \frac{l}{h} \left[ 1 + \left( \frac{v_o}{v_m - l} \right) \left( \frac{z - z_{k-1}}{h} \right)^n \right] \left[ \left( \frac{v_o}{v_m - l} \right)^n \frac{l}{n} \right] \left( \frac{v_m}{v_o - v_m} \right)^n, & \text{if } z_{k-1} < z < z_k
\end{array} \right. \]

0. otherwise

\[\frac{d\varphi_k}{dz} = \left\{ \begin{array}{ll}
\frac{1}{h} \left[ 1 + \left( \frac{v_o}{v_m - l} \right) \left( \frac{z_{k+1} - z}{h} \right)^n \right] \left[ \left( \frac{v_o}{v_m - l} \right)^n \frac{l}{n} \right] \left( \frac{v_m}{v_o - v_m} \right)^n, & \text{if } z_{k-1} < z < z_k
\end{array} \right. \]

0. otherwise
\[
\frac{d\varphi_{k+1}}{dz} = \begin{cases} 
-\frac{l}{h}, & \text{if } z_{k+1} < z < z_{k+2} \\
\frac{l}{h\left[1 + \left(\frac{v_o}{v_m} - 1\right) \frac{z_{k+1} - z}{h}\right]^n} \left(\frac{v_o - l}{v_m - l}\right)^n \left(\frac{v_m}{v_o - v_m}\right)^n, & \text{if } z_k < z < z_{k+1} \\
0, & \text{otherwise}
\end{cases}
\]

For the three test functions \(\psi_{k-1}, \psi_k, \psi_{k+1}\) the integral (32) is

\[
I_{k-1} = \frac{v_m}{h} (A_{k-1} - A_{k-2}) + \frac{v_m}{h\left[\left(\frac{v_o}{v_m} - 1\right) \frac{l}{n}\right]} \left(\frac{v_o - l}{v_m - l}\right)^n (A_{k-1} - A_k)
\]

(32c)

\[
I_k = \frac{v_m}{h\left[\left(\frac{v_o}{v_m} - 1\right) \frac{l}{n}\right]} \left(\frac{v_o - l}{v_m - l}\right)^n (-A_{k-1} + 2A_k - A_{k+1})
\]

\[
I_{k+1} = \frac{v_m}{h} (A_{k+1} - A_{k+2}) + \frac{v_m}{h\left[\left(\frac{v_o}{v_m} - 1\right) \frac{l}{n}\right]} \left(\frac{v_o - l}{v_m - l}\right)^n (A_{k+1} - A_k).
\]

for the other test functions \(\psi_j, \ j = 1 \ldots k-2, k+2 \ldots N-1\) it is as in (32a). Comparison of the FEM and analytical solutions is given in Figure 24c. They are identical. It is worth to mention, that by the same reason, as we had in 6.2.2, for space \(C^N_\varphi\) we can not use the space of piecewise linear functions (i.e. the same space as in 6.2.1).
6.2.4. Reciprocal polynomial approach. $C_{\phi}^N$ is a space of functions that are piecewise linear, except for two elements, adjacent to the point $z_k$, where these functions are reciprocal polynomials, as in (14). $C_{\phi}^N$ is also a space of functions that are piecewise linear, except for two elements adjacent to point $z_k$. Here functions $\varphi$ are of the form

$$
\varphi_{k-1} = \begin{cases} 
\frac{z - z_{k-2}}{h}, & \text{if } z_{k-2} < z < z_{k-1} \\
1 - \frac{(n+1) \frac{z - z_{k-1}}{h} - (1 - \frac{v_m}{v_o}) (\frac{z - z_{k-1}}{h})^{n+1}}{n + \frac{v_m}{v_o}}, & \text{if } z_{k-1} < z < z_k \\
0, & \text{otherwise}
\end{cases}
$$

$$
\varphi_k = \begin{cases} 
\frac{(n+1) \frac{z - z_{k-1}}{h} - (1 - \frac{v_m}{v_o}) (\frac{z - z_{k-1}}{h})^{n+1}}{n + \frac{v_m}{v_o}}, & \text{if } z_{k-1} < z < z_k \\
\frac{(n+1) \frac{z_{k+1} - z}{h} - (1 - \frac{v_m}{v_o}) (\frac{z_{k+1} - z}{h})^{n+1}}{n + \frac{v_m}{v_o}}, & \text{if } z_k < z < z_{k+1} \\
0, & \text{otherwise}
\end{cases}
$$
\[
\varphi_{k+1} = \begin{cases} 
\frac{z_{k+2} - z}{h}, & \text{if } z_{k+1} < z < z_{k+2} \\
(n+1) \frac{z_{k+1} - z}{h} - \left(1 - \frac{v_m}{v_o} \right) \left(\frac{z_{k+1} - z}{h}\right)^{n+1} / \left( n + \frac{v_m}{v_o} \right), & \text{if } z_k < z < z_{k+1} \\
0, & \text{otherwise}
\end{cases}
\]

\[
\frac{d\varphi_{k-1}}{dz} = \begin{cases} 
\frac{1}{h}, & \text{if } z_{k-2} < z < z_{k-1} \\
1 - \left(1 - \frac{v_m}{v_o} \right) \left(\frac{z - z_{k-1}}{h}\right)^n / h \left( n + \frac{v_m}{v_o} \right), & \text{if } z_{k-1} < z < z_k \\
0, & \text{otherwise}
\end{cases}
\]

\[
\frac{d\varphi_k}{dz} = \begin{cases} 
1 - \left(1 - \frac{v_m}{v_o} \right) \left(\frac{z_{k+1} - z}{h}\right)^n / h \left( n + \frac{v_m}{v_o} \right), & \text{if } z_k < z < z_{k+1} \\
0, & \text{otherwise}
\end{cases}
\]
\[ \frac{d\varphi_{k+l}}{dz} = \begin{cases} \frac{-1}{h} & \text{if } z_{k+l} < z < z_{k+2} \\ 1 - \left(1 - \frac{v_m}{v_o}\right) \left(\frac{z_{k+l} - z}{h}\right)^n & \text{if } z_k < z < z_{k+l} \\ 0 & \text{otherwise} \end{cases} \]

The integral (32) is

\[ I_{k-l} = \frac{v_m}{h} (A_{k-l} - A_{k-2}) + \frac{v_m(n+1)}{h \left( n + \frac{v_m}{v_o} \right)} (A_{k-l} - A_k) \]

\[ I_k = \frac{v_m(n+1)}{h \left( n + \frac{v_m}{v_o} \right)} (-A_{k-l} + 2A_k - A_{k+l}) \quad (32d) \]

\[ I_{k+l} = \frac{v_m}{h} (A_{k+l} - A_{k+2}) + \frac{v_m(n+1)}{h \left( n + \frac{v_m}{v_o} \right)} (A_{k+l} - A_k). \]

Now we solve system (34). and, using (31), compare with (29) and (30), as shown in Figure 24d. Results are also identical.

6.2.5. Conventional method with forcing term. For the equation (16a), the right hand side of the equation (34) is now \( F_j = \int_{z_{j-l}}^{z_{j+l}} \psi_j J_z dz \). Figure 26 shows the course and fine mesh FEM approximations to the solution, shown in Figure 23b. Both approximations are error free.
Figure 26. Reluctivity specifications (a), and FEM approximation of $A$ and $B$, using the coarse (o) and fine (+) meshes (b).

### 6.3. Radial Direction

Consider now one more 1-D differential equation

$$
\nu \left( \frac{A}{r^2} + \frac{l}{r} \frac{dA}{dr} + \frac{d^2A}{dr^2} \right) + \frac{d\nu}{dr} \left( \frac{A}{r} + \frac{dA}{dr} \right) = \frac{d}{dr} \left[ \nu \left( \frac{A}{r} + \frac{dA}{dr} \right) \right] = J_s
$$

where $J_s$ is the density of a current source. Defining $B_z = \frac{A}{r} + \frac{dA}{dr} = \frac{l}{r} \frac{d}{dr} (rA)$ we solve (35) for $B$

$$
B_z(r_k) \nu_k + \int_{r_k}^{r} J_s dp = B_z
$$

and for $A$
\[ A = \frac{r_k A(r_k)}{r} + \frac{1}{r} \int_{r_k}^{r} \int_{r_k}^{\rho} \left( B_{z}(r_k) v_k + \int_{r_k}^{\rho} J_z ds \right) d\rho \]  

(36)

with \( r_k \) - some fixed point. Assuming piecewise constant \( v \),

\[ v = v_m, \quad \text{if} \quad r_m < r < r_{m+1}, \quad m = 0 \ldots M, \]

and

\[ J_z = \begin{cases} J, & \text{if} \quad r_s < r < r_{s+1} \\ 0, & \text{otherwise} \end{cases}, \quad \text{for some} \ s > 0. \]

get

\[ A = A(r_0) \frac{r_0}{r} + \frac{B_{z}(r_0)}{r} \frac{r^2 - r_0^2}{2}, \quad \text{if} \quad r_0 < r < r_1. \]

taking \( r_0 = 0 \), \( A(r_0) = 0 \).

\[ B_{z} = \begin{cases} B_{z}(0) \frac{v_0}{v_m}, & \text{if} \quad r_m < r < r_{m+1} \leq r_s \\ B_{z}(0) \frac{v_0}{v_s} + \frac{J}{v_s} (r - r_s), & \text{if} \quad r_s < r < r_{s+1} \\ B_{z}(0) \frac{v_0}{v_m} + \frac{J}{v_m} (r_{s+1} - r_s), & \text{if} \quad r_{s+1} \leq r_m < r < r_{m+1} \end{cases} \]
For zero Dirichlet boundary conditions, $A(r_M) = 0$.

$$
\frac{B_z(0)v_0}{2} = \begin{cases} 
  \frac{r}{v_{m-1}} + \frac{1}{r} \sum_{j=1}^{m-1} \left( \frac{l}{v_{j-1}} - \frac{l}{v_j} \right) r_j^2, & \text{if } r_{m-1} < r < r_m \leq r_s \\
  \frac{r_m}{v_{m-1}} + \frac{1}{r} \sum_{j=1}^{m-1} \left( \frac{l}{v_{j-1}} - \frac{l}{v_j} \right) r_j^2, & \text{if } r = r_m \leq r_s \\
  \frac{r_s}{v_{s}} + \frac{1}{r} \sum_{j=1}^{s} \left( \frac{l}{v_{j-1}} - \frac{l}{v_j} \right) r_j^2 + \frac{J}{v_s} \left( \frac{r_s^2}{3} - \frac{r_s r_{s+1}}{2} + \frac{r_s^3}{6} \right), & \text{if } r_s < r < r_{s+1} \\
  \frac{r_{s+1}}{v_{s+1}} + \frac{1}{r} \sum_{j=1}^{s} \left( \frac{l}{v_{j-1}} - \frac{l}{v_j} \right) r_j^2 + \frac{J}{v_{s+1}} \left( \frac{r_{s+1}^2}{3} - \frac{r_{s+1} r_{s+2}}{2} + \frac{r_{s+1}^3}{6} \right), & \text{if } r = r_{s+1} \\
  \frac{r_{s+1}}{v_{s+1}} + \frac{1}{r} \sum_{j=1}^{s} \left( \frac{l}{v_{j-1}} - \frac{l}{v_j} \right) r_j^2 + \frac{J}{v_{s+1}} \left( \frac{r_{s+1}^2}{3} - \frac{r_{s+1} r_{s+2}}{2} + \frac{r_{s+1}^3}{6} \right), & \text{if } r_{s+1} \leq r_{m-1} < r < r_m \\
  \frac{r_m}{v_{m-1}} + \frac{1}{r} \sum_{j=1}^{m-1} \left( \frac{l}{v_{j-1}} - \frac{l}{v_j} \right) r_j^2 + \frac{J}{v_m} \left( \frac{r_{s+1}^2}{3} - \frac{r_{s+1} r_{s+2}}{2} + \frac{r_{s+1}^3}{6} \right), & \text{if } r_{s+2} \leq r = r_m \\
  \frac{r}{v_{m-1}} + \frac{1}{r} \sum_{j=1}^{m-1} \left( \frac{l}{v_{j-1}} - \frac{l}{v_j} \right) r_j^2 + \frac{J}{v_{m-1}} \left( \frac{r_{s+1}^2}{3} - \frac{r_{s+1} r_{s+2}}{2} + \frac{r_{s+1}^3}{6} \right) \end{cases}
$$

Figure 27b (solid lines) shows the results for $J=1$, $s=3$, $M=7$, and reluctivity specifications, shown in Figure 27a.
For FEM solution a weak formulation is used,

\[
\int_{r_{i-1}}^{r_i} \psi_j \frac{d}{dr} \left[ \nu \left( \frac{A}{r} + \frac{dA}{dr} \right) \right] dr = \nu \psi_j \left( \frac{A}{r} + \frac{dA}{dr} \right)_{r_{i-1}}^{r_i} - \int_{r_{i-1}}^{r_i} \frac{d\psi_j}{dr} \nu \left( \frac{A}{r} + \frac{dA}{dr} \right) dr =
\]

\[
= \int_{r_{i-1}}^{r_i} \frac{d\psi_j}{dr} \nu \left( \frac{A}{r} + \frac{dA}{dr} \right) dr.
\]

\[
I_j = \int_{r_{i-1}}^{r_i} \frac{d\psi_j}{dr} \left[ \nu \left( \frac{A}{r} + \frac{dA}{dr} \right) \right] dr = \int_{r_{i-1}}^{r_i} \frac{d\phi_j}{dr} \nu \sum_{i=j-1}^{j+1} A_j \left( \frac{\phi_i}{r} + \frac{d\phi_i}{dr} \right) dr =
\]

\[
= \int_{r_{j-1}}^{r_j} \frac{d\phi_j}{dr} \nu_{j-1} \left[ A_{j-1} \left( \frac{\phi_{j-1}}{r} + \frac{d\phi_{j-1}}{dr} \right) + A_j \left( \frac{\phi_j}{r} + \frac{d\phi_j}{dr} \right) \right] dr +
\]

\[
+ \int_{r_j}^{r_{j+1}} \frac{d\phi_j}{dr} \nu_j \left[ A_j \left( \frac{\phi_j}{r} + \frac{d\phi_j}{dr} \right) + A_{j+1} \left( \frac{\phi_{j+1}}{r} + \frac{d\phi_{j+1}}{dr} \right) \right] dr
\]

Figure 27. (a) - reluctivity specifications, (b) - analytical (solid lines) and FEM A and B for the coarse (o) and fine (+) meshes.
For linear shape functions

\[ I_j = A_{j-1} \frac{v_{j-1}}{h_{j-1}^2} \left( r_j \ln \frac{r_j}{r_{j-1}} - 2r \frac{r_j}{r_{j-1}} \right) + A_j \frac{v_{j-1}}{h_{j-1}^2} \left( 2r \frac{r_j}{r_{j-1}} - r_{j-1} \ln \frac{r_j}{r_{j-1}} \right) + \]

\[ + \frac{v_j}{h_j^2} \left( 2r \frac{r_{j+1}}{r_j} - r_{j+1} \ln \frac{r_{j+1}}{r_j} \right) + A_{j+1} \frac{v_j}{h_j^2} \left( r_j \ln \frac{r_{j+1}}{r_j} - 2r \frac{r_{j+1}}{r_j} \right), \]

and

\[ \int_{r_{j-1}}^{r_{j+1}} \phi_j I_2 \, dr = \frac{J_{j-1} h_{j-1} + J_j h_j}{2}. \]

The results for the coarse and fine meshes are shown in Figure 27b. The errors are shown in Figures 28a and 28b.

It should be noticed that the matrix, corresponding to the resulting system is not symmetric. The symmetry may be obtained if the test functions are chosen as \( \psi_j = r \varphi_j \).

In this case

\[ I_j = \int_{r_{j-1}}^{r_j} \left( r \frac{d\varphi_j}{dr} + \varphi_j \right) v_{j-1} \left[ A_{j-1} \left( \frac{\varphi_j}{r} + \frac{d\varphi_j}{dr} \right) \right] + A_j \left( \frac{\varphi_j}{r} + \frac{d\varphi_j}{dr} \right) \, dr + \]

\[ + \int_{r}^{r_{j+1}} \left( r \frac{d\varphi_j}{dr} + \varphi_j \right) v_j \left[ A_j \left( \frac{\varphi_j}{r} + \frac{d\varphi_j}{dr} \right) + A_{j+1} \left( \frac{\varphi_{j+1}}{r} + \frac{d\varphi_{j+1}}{dr} \right) \right] \, dr. \]

For linear shape functions

\[ I_j = -A_{j-1} \frac{v_{j-1}}{h_{j-1}^2} r_j r_{j-1} \frac{r_j}{r_{j-1}} + A_j \frac{v_{j-1}}{h_{j-1}^2} \left( 2h_{j-1}^2 + r_{j-1}^2 \ln \frac{r_j}{r_{j-1}} \right) + \]
\[ + \frac{v_j}{h_j^2} \left( -2h_j^2 + r_{j+1}^2 \ln \frac{r_{j+1}}{r_j} \right) - A_{j+1} \frac{v_j}{h_j^2} r_j r_{j+1} \ln \frac{r_{j+1}}{r_j}. \]

and

\[ r_{r_{j+1}} \int r \phi_j J_s dr = \frac{J_{j-1} h_{j-1} \left( r_j - \frac{h_{j-1}}{3} \right) + J_j h_j \left( r_j + \frac{h_j}{3} \right)}{2}. \]

Figure 28. FEM error of \( A (o) \) and \( B (+) \) for the coarse (a, c) and fine (b, d) meshes for linear test functions (a, b) and test linear functions with factor \( r \) (c, d).
The errors for that test function are about the same as before, as can be seen from Figures 28c and 28d. However, the symmetry of the resulting matrix allows the utilization of more efficient algorithms and storage of only a half of the matrix.

In order to get the exact solution by FEM, we need to use the proper shape functions. Any function of the form

\[ f = \frac{a}{r} + br. \]  

(37)

can be uniquely determined by its two values \( F_0 = f(r_k) \) and \( F_I = f(r_k + h) \) at \( r = r_k \) and \( r = r_k + h \) respectively. Let's consider the two functions.

\[
F_{k,0} = \frac{r_k}{h(2r_k + h)} \left( \frac{(r_k + h)^2}{r} - r \right)
\]

and

\[
F_{k,1} = \frac{r_k + h}{h(2r_k + h)} \left( -\frac{r_k^2}{r} + r \right)
\]

of the form (37). Function \( F_0 \) passes the points \((r_k, 0), (r_k + h, 0)\). Function \( F_I \) passes the points \((r_k, 0), (r_k + h, 0)\). Hence, we have \( F = F_0 f_0 + F_I f_I \). Also,

\[
\frac{df_{k,0}}{dr} = -\frac{r_k}{h(2r_k + h)} \left( \frac{(r_k + h)^2}{r^2} + 1 \right), \quad \frac{df_{k,0}}{dr} = -2 \frac{r_k}{h(2r_k + h)}.
\]

\[
\frac{df_{k,1}}{dr} = \frac{r_k + h}{h(2r_k + h)} \left( \frac{r_k^2}{r^2} + 1 \right), \quad \frac{df_{k,1}}{dr} = 2 \frac{r_k + h}{h(2r_k + h)}.
\]
Functions $f_g$ and $f_l$ are shown in Figure 29a for $r_i = 5$ and $h=2$. They form the basis for the needed shape functions of the source free elements. The shape functions are

$$\varphi_i = \begin{cases} 
  f_{i,0}, & \text{if } i = j - 1, r_{j-1} < r < r_j \quad \text{or} \quad i = j, r_j < r < r_{j+1} \\
  f_{i-1,l}, & \text{if } i = j, r_{j-1} < r < r_j \quad \text{or} \quad i = j + 1, r_j < r < r_{j+1} \\
  0 & \text{otherwise}
\end{cases}$$

Similarly, a function of the form

$$g = \frac{a}{r} + br^2 + cr^4$$

(38)

can be uniquely determined by its values at three points, say $G_0 = g(r_k)$, $G_1 = g\left(r_k + \frac{h}{2}\right)$, and $G_2 = g(r_k + h)$. Let's consider the three functions of the form (38). function

$$g_{k,0} = \frac{a_0}{r} + b_0r^2 + c_0r$$

with

$$c_0 = -\frac{12r_k^2 + 18r_kh + 7h^2}{6r_k^2 + 6r_kh + h^2} \cdot \frac{r_k}{h^2},$$

$$b_0 = \frac{c_0 \left[ (r_k + h)^2 - r_k^2 \right] + r_k}{(r_k + h)^3 - r_k^3},$$

$$a_0 = -b_0 (r_k + h)^3 - c_0 (r_k + h)^2.$$
\[ b_I = -c_I \frac{(r_k + h)^2 - r_k^2}{(r_k + h)^3 - r_k^3}, \]

\[ a_I = -b_I(r_k + h)^3 - c_I(r_k + h)^2, \]

and function \( g_{k,2} = \frac{a_2}{r} + b_2 r^2 + c_2 r \) with

\[ b_2 = 8 \frac{r_k + h}{6 r_k^2 + 6 r_k h + h^2} \frac{r_k + h}{h^2}, \]

\[ c_2 = \frac{-b_2 [(r_k + h)^3 - r_k^3] + r_k + h}{(r_k + h)^2 - r_k^2}. \]

\[ a_2 = -b_2 r_k^3 - c_2 r_k^2. \]

Function \( g_0 \) passes the points \((r_k,0), (r_k + \frac{h}{2},0), (r_k + h,0)\). Function \( g_1 \) passes the points \((r_k,0), (r_k + \frac{h}{2},0), (r_k + h,0)\). Function \( g_2 \) passes the points \((r_k,0), (r_k + \frac{h}{2},0)\). Hence, we have \( g = G_0 g_0 + G_1 g_1 + G_2 g_2 \). Also,

\[ \frac{dg}{dr} = -\frac{a}{r^2} + 2br + c, \quad \frac{g}{r} + \frac{dg}{dr} = 3br + 2c. \]

Functions \( g_0, g_1 \) and \( g_2 \) are shown in Figure 29b for \( r_k = 5 \) and \( h = 2 \). They form the basis for the needed shape functions of the elements with current sources. In the elements with current sources we have two kinds of nodes: at the edges of elements (nodes with odd indices) and at the center of elements (nodes with even indices). The shape functions for odd \( j \) are
Figure 29. Shape functions for the source free elements, $r_k = 3, h=5$ (a), and shape functions for the elements with current sources, $r_k = 5, h=2$ (b).

\[
\varphi_i = \begin{cases} 
  g_{i,0} & \text{if } i = j - 2, r_{j-2} < r < r_j \quad \text{or} \quad i = j, r_j < r < r_{j+2} \\
  g_{i-2,2} & \text{if } i = j + 2, r_j < r < r_{j+2} \quad \text{or} \quad i = j, r_j < r < r_j \\
  g_{i-1,2} & \text{if } i = j - 1, r_{j-2} < r < r_j \quad \text{or} \quad i = j + 1, r_j < r < r_{j+2} \\
  0 & \text{otherwise}
\end{cases}
\]

whereas for even $j$ and $r_{j-1} < r < r_{j+1}$

\[
\varphi_i = \begin{cases} 
  g_{i,0} & \text{if } i = j - 1 \\
  g_{i-2,2} & \text{if } i = j + 1 \\
  g_{i-1,2} & \text{if } i = j \\
  0 & \text{otherwise}
\end{cases}
\]

Taking the corresponding integrals, we have for the source free elements

\[
I_j = \int_{r_{j-1}}^{r_{j+1}} \varphi_j \frac{d}{dr} \left[ v \left( \frac{A}{r} + \frac{dA}{dr} \right) \right] dr =
\]
\[ \int_{r_{j-1}}^{r_j} \frac{d\varphi_j}{dr} v_{j-1} \sum_{i=j-1}^{j} A_i \left( \frac{\varphi_i}{r} + \frac{d\varphi_i}{dr} \right) dr + \int_{r_j}^{r_{j+1}} \frac{d\varphi_j}{dr} v_{j} \sum_{i=j}^{j+1} A_i \left( \frac{\varphi_i}{r} + \frac{d\varphi_i}{dr} \right) dr = \]

\[ \int_{r_{j-1}}^{r_j} \frac{df_{j-1,1}}{dr} v_{j-1} \sum_{i=0}^{l} A_{j-i} \left( \frac{f_{j-i,1-i}}{r} + \frac{df_{j-i,1-i}}{dr} \right) dr + \]

\[ \int_{r_j}^{r_{j+1}} \frac{df_{j,0}}{dr} v_{j} \sum_{i=0}^{l} A_{j+i} \left( \frac{f_{j+i,i}}{r} + \frac{df_{j+i,i}}{dr} \right) dr = \]

\[ \int_{r_{j-1}}^{r_j} \frac{r_j v_{j-1}}{h_{j-1}(r_j + r_{j-1})} \left( \frac{r_{j-1}}{r^2} + 1 \right) \left[ A_j \frac{2r_j}{h_{j-1}(r_j + r_{j-1})} - A_{j-1} \frac{2r_{j-1}}{h_{j-1}(r_j + r_{j-1})} \right] dr + \]

\[ \int_{r_j}^{r_{j+1}} \frac{r_j v_{j}}{h_{j}(r_j + r_{j+1})} \left( \frac{r_{j+1}}{r^2} + 1 \right) \left[ A_j \frac{2r_j}{h_{j}(r_j + r_{j+1})} - A_{j+1} \frac{2r_{j+1}}{h_{j}(r_j + r_{j+1})} \right] dr = \]

\[ -\frac{2A_{j-1} r_j v_{j-1}}{h_{j-1} r_j (r_j + r_{j-1})} + A_j \left( \frac{2v_{j-1}}{h_{j-1}(r_j + r_{j-1})} + \frac{2v_j}{h_{j}(r_j + r_{j+1})} \right) - \frac{2A_{j+1} r_{j+1} v_{j}}{h_{j+1} r_j (r_j + r_{j+1})}. \]

and

\[ \int_{r_j}^{r_{j+1}} \varphi_j dr = \frac{r_j^2 r_{j+1}}{h_j(r_j + r_{j+1})} \ln \frac{r_{j+1}}{r_j} - \frac{r_j}{2}. \]

\[ \int_{r_j}^{r_{j+1}} \varphi_{j+1} dr = -\frac{r_j^2 r_{j+1}}{h_j(r_j + r_{j+1})} \ln \frac{r_{j+1}}{r_j} + \frac{r_{j+1}}{2}. \]

The error for the case, when shape functions \( f \) are used only (use \( f \) instead of \( g \) in elements with current sources, so that we neglect constant and quadratic terms), is shown in Figure
30a. We see an improvement compared to the error shown in Figure 28a. As could be anticipated, the biggest error occurs at the element with a source.

Taking the integral over an element with current source, we have

\[
\int_{r_k}^{r_{k+1}} \frac{d\phi_j}{dr} v_k A_i \left( \frac{\phi_i}{r} + \frac{d\phi_i}{dr} \right) dr = \int_{r_k}^{r_{k+1}} \left( -\frac{a_j}{r^2} + 2b_j r + c_j \right) v_k A_i \left( 3b_i r + 2c_i \right) dr =
\]

\[
v_k A_i \left[ \frac{2a_j c_i}{r} - 3a_j b_i \ln r + 2b_j b_i r^3 + \frac{4b_j c_i + 3b_i c_j}{2} r^2 + 2c_j c_i r \right]_{r_k}^{r_{k+1}}.
\]

Also for elements with current sources

\[
\int_{r_k}^{r_{k+1}} \phi_j dr = \int_{r_k}^{r_{k+1}} \left( \frac{a_j}{r} + b_j r^2 + c_j r \right) dr = a_j \ln \frac{r_{k+1}}{r_k} + b_j \frac{r_{k+1}^3 - r_k^3}{3} + c_j \frac{r_{k+1}^2 - r_k^2}{2}.
\]

The FEM solution with these properly chosen shape functions is exact. The error due to the round-off errors of the computer is shown in Figure 30b.

However, the exact solution can be obtained without increasing the number of degrees of freedom in elements with current sources. We notice from the analytical solution, that for the element \(s \) with current source \(J\) and three nodal representation of \(A\),

\[
A = \left( A_s a_0 + A_{s+1} a_1 + A_{s+2} a_2 \right) r + \left( A_s b_0 + A_{s+1} b_1 + A_{s+2} b_2 \right) r^2 +
\]

\[
\left( A_s c_0 + A_{s+1} c_1 + A_{s+2} c_2 \right) r.
\]
Figure 30. FEM errors of $A$ (o) and $B$ (+) for the shape functions with quadratic term neglected (a) and for the proper shape functions (b).

$A_{s+l}$ can be found from the identity $A_s b_0 + A_{s+l} b_l + A_{s+2} b_2 = \frac{J}{3v_s}$ and, hence, $A$ can be represented by two nodes.

$$A = \left[ A_s \left( a_0 - a_l \frac{b_0}{b_l} \right) + A_{s+2} \left( a_2 - a_l \frac{b_2}{b_l} \right) \right] \frac{l}{r} +$$

$$+ \left[ A_s \left( c_0 - c_l \frac{b_0}{b_l} \right) + A_{s+2} \left( c_2 - c_l \frac{b_2}{b_l} \right) \right] r + \frac{J}{3v_s} \left( \frac{a_l}{b_l} \frac{l}{r} + r^2 + \frac{c_l}{b_l} r \right).$$

Therefore, instead of three equations with three unknowns,

$$\sum_{i=0}^{2} A_{s+i}^* S_{s+i,s+j}^* = F_j^* \quad j=0, 1, 2,$$

two equations can be left.
\[ \sum_{i=0}^{l} A_{s+i} S_{s+i,s+j} = F_j, \quad j=0, l. \]

where

\[ A_s = A_{s+1}^*, \quad A_{s+1} = A_{s+2}. \]

\[ S_{s,s} = S_{s,s}^* - S_{s,s+1}^* \frac{b_0}{b_1}, \quad S_{s+1,s} = S_{s+1,s+2}^* + S_{s+2,s+1}^* \frac{b_0}{b_1}. \]

and the unknown \( A_{s+1}^* \) is eliminated. The FEM solution in this case is exact, and the error due to the round-off errors of the computer is similar to that shown in Figure 30b.

### 6.4. 2-D Axisymmetric Case

The PDE under consideration is

\[ \nabla \times \frac{l}{\mu} (\nabla \times \vec{A}) = \vec{J}_s, \]

where \( \vec{J}_s \) is the density of an external current source. For a source-free element with constant material properties, \( \vec{J}_s = 0 \) and \( \nu = \text{const.} \). Let's call elements of this type regular. For the corresponding governing equation

\[ \left( -\frac{A}{r^2} + \frac{l}{r} \frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial r^2} + \frac{\partial^2 A}{\partial \zeta^2} \right) = 0 \]
the variables can be separated: \( A = RZ \).

\[
-\frac{1}{r^2} + \frac{1}{R} \frac{\partial R}{\partial r} + \frac{1}{R} \frac{\partial^2 R}{\partial r^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = 0
\]

and we get a system

\[
\begin{align*}
\frac{\partial^2 Z}{\partial z^2} &= \pm \lambda^2 Z \\
r^2 \frac{\partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} + R\left( \pm \lambda^2 r^2 - 1 \right)
\end{align*}
\]

(39)

For \( \lambda \neq 0 \), denote \( x = \lambda r \), so that \( R(r) = R\left(\frac{x}{\lambda}\right) \). and

\[
x^2 \frac{\partial^2 R}{\partial x^2} + x \frac{\partial R}{\partial x} + R\left( \pm x^2 - 1 \right) = 0
\]

For positive sign at \( \lambda^2 \) in (39) the solution is

\[
\begin{align*}
Z &= Z_0 e^{-\lambda z} + Z_1 e^{\lambda z} \\
R &= R_0 J_1(r\lambda) + R_1 Y_1(r\lambda)
\end{align*}
\]

where \( J_1 \) and \( Y_1 \) are first order Bessel functions of the first and the second kind respectively. \( Z_0, Z_1, R_0, R_1 \) - constants.

For negative sign at \( \lambda^2 \) in (39) the solution is

\[
\begin{align*}
Z &= Z_0 \sin \lambda z + Z_1 \cos \lambda z \\
R &= R_0 I_1(r\lambda) + R_1 K_1(r\lambda)
\end{align*}
\]

where \( I_1 \) and \( K_1 \) are first order modified Bessel functions of the first and the second kind respectively. For reference, first order Bessel functions are shown in Figure 31.

For \( \lambda = 0 \) we already found the solution, namely
\[
\begin{align*}
Z &= Z_0 + Z_l z \\
R &= R_0 r + R_l \frac{l}{r}
\end{align*}
\]

General solution for a regular element with arbitrary boundary conditions is

\[
A = (Z_0 + Z_l z) \left( R_0 r + \frac{R_l}{r} \right) + \sum_{\lambda > 0} \left[ (Z_{0\lambda} \sin \lambda z + Z_{1\lambda} \cos \lambda z) \left( R_{0\lambda} I_1(\lambda r) + R_{l\lambda} K_1(\lambda r) \right) \right] + \\
&+ \sum_{\mu > 0} \left[ (Z_{0\mu} e^{-\mu z} + Z_{1\mu} e^{\mu z}) \left( R_{0\mu} J_1(\mu r) + R_{l\mu} Y_1(\mu r) \right) \right]
\]

(40)

In certain cases the expression for \( A \) simplifies:

- if \(|A| \neq \infty\) at \( r=0 \), then \( R_l = 0 \), \( R_{l\lambda} = 0 \), \( R_{l\mu} = 0 \).

- if \( \frac{\partial A}{\partial z} = 0 \) at \( z = z_0 \) and at \( z = -z_0 \), then \( Z_l = 0 \), \( Z_{0\mu} = 0 \), \( Z_{1\mu} = 0 \).

with either \( Z_{0\lambda} = 0 \) and \( \lambda = \frac{\pi}{z_0} n \), \( n = 1, 2, 3, \ldots \), or \( Z_{1\lambda} = 0 \) and \( \lambda = \frac{\pi}{z_0} \left( n - \frac{1}{2} \right) \).

- if \( A = 0 \) at \( r = r_0 \), then

\[
\begin{align*}
R_l &= -R_0 r_0^2 \\
R_{l\lambda} &= -R_0 \lambda \frac{I_1(r_0 \lambda)}{K_1(r_0 \lambda)} \\
R_{l\mu} &= -R_0 \mu \frac{J_1(r_0 \mu)}{Y_1(r_0 \mu)}
\end{align*}
\]

For example, if \(|A| \neq \infty\) at \( r = 0 \), \( \frac{\partial A(z_0)}{\partial z} = 0 \) and \( \frac{\partial A(-z_0)}{\partial z} = 0 \), then

\[
A = Z_0 R_0 r + \sum_{k=1}^{\infty} \left[ Z_{0k} \sin \left( \frac{\pi r}{z_0} \left( k - \frac{1}{2} \right) \right) R_{0k} I_1 \left( \frac{\pi r}{z_0} \left( k - \frac{1}{2} \right) \right) \right] + \\
+ Z_{1k} \cos \left( \frac{\pi r}{z_0} k \right) R_{1k} I_1 \left( \frac{\pi r}{z_0} k \right)
\]

whereas if \( A = 0 \) at \( r = r_0 \) and \( \frac{\partial A(z_0)}{\partial z} = 0 \), \( \frac{\partial A(-z_0)}{\partial z} = 0 \), then
Figure 31. First order Bessel functions of the first kind (a) and of the second kind (b), and modified Bessel functions of the first kind (c) and of the second kind (d).

\[ A = Z_0 R_0 \left( r - \frac{r_0^2}{r} \right) + \]

\[ + \sum_{k=1}^{8} \left[ Z_{0k} \sin \left( \frac{\pi r}{z_0} \left( k - \frac{1}{2} \right) \right) R_{0k} \left( I_1 \left[ \frac{\pi r}{z_0} \left( k - \frac{1}{2} \right) \right] - \frac{I_1 \left[ \frac{\pi r_0}{z_0} \left( k - \frac{1}{2} \right) \right]}{K_1 \left[ \frac{\pi r_0}{z_0} \left( k - \frac{1}{2} \right) \right]} \right) \right] \]
For an element with current source and constant material properties, \( v=\text{const} \) and \( J_s \neq 0 \).

For the corresponding governing equation
\[
\left( -\frac{A}{r^2} + \frac{l}{r} \frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial r^2} + \frac{\partial^2 A}{\partial z^2} \right) = J_s
\]
the variables can be separated if we represent the solution as \( A = R^*Z^* + J_s \frac{r^2}{3} \).

General solution for an element with a current source and arbitrary boundary conditions is
\[
A = \frac{J_s}{3} r^2 + (Z_0 + Z_I z) \left( R_0 r + \frac{R_I}{r} \right) + \sum_{\lambda>0} \left[ (Z_{0\lambda} \sin \lambda z + Z_{I\lambda} \cos \lambda z) \left( R_{0\lambda} I_I(\lambda r) + R_{I\lambda} K_I(\lambda r) \right) + \sum_{\mu>0} \left[ (Z_{0\mu} e^{-\mu z} + Z_{I\mu} e^{\mu z}) \left( R_{0\mu} J_I(\mu r) + R_{I\mu} Y_I(\mu r) \right) \right]
\]

(41)

For FEM a weak formulation is used
\[
\int_{\Omega_j} \left( \nabla \times \frac{l}{\mu} (\nabla \times A) \right) \psi_{ij} d\omega = \int_{\Omega_j} \left( \nabla \times \frac{\psi_{ij}}{\mu} (\nabla \times A) \right) d\omega + \int_{\Omega_j} \left( \frac{l}{\mu} \nabla \times A \right) \times \nabla \psi_{ij} d\omega = \\
= \oint_{L_j} \frac{\psi_{ij}}{\mu} (\nabla \times A) d\ell + \int_{\Omega_j} \left( \frac{l}{\mu} \nabla \times A \right) \times \nabla \psi_{ij} d\omega = \int_{\Omega_j} \left( \frac{l}{\mu} \nabla \times A \right) \times \nabla \psi_{ij} d\omega
\]

Assuming
\[ \nabla \psi_{ij} = \hat{r} \frac{\partial \psi_{ij}}{\partial r} + \hat{z} \frac{\partial \psi_{ij}}{\partial z}, \quad \overline{J}_s = \hat{f} J_s \]

\[ \hat{f} I_{ij} = \oint_{\Omega_y} \left( \frac{1}{\mu} \nabla \times \tilde{A} \right) \times \nabla \psi_{ij} d\omega = \hat{f} \oint_{\Omega_y} \left[ \frac{\partial A}{\partial z} \frac{\partial \psi_{ij}}{\partial z} + \left( \frac{A}{r} + \frac{\partial A}{\partial r} \right) \frac{\partial \psi_{ij}}{\partial r} \right] drdz = \]

\[ = \hat{f} \int_{\Omega_y} \frac{1}{\mu} \sum_{k=i-1}^{k=i+1} \sum_{l=j-1}^{l=j+1} A_{kl} \left[ \frac{\partial \phi_{ij}}{\partial r} \left( \frac{\phi_{kl}}{r} + \frac{\partial \phi_{kl}}{\partial r} \right) + \frac{\partial \phi_{ij}}{\partial z} \frac{\partial \phi_{kl}}{\partial z} \right] drdz \quad (42) \]

and the system of linear equations is

\[ I_{ij} = \oint_{\Omega_y} J_s \psi_{ij} d\omega \]

For axisymmetric 2-D numerical simulations, we use a dual approach, in which the chosen shape functions have properties different in different directions, and \( \varphi_{ij}(r,z) = \varphi_{ri}(r)\varphi_{zj}(z) \).

Let's first use the piecewise linear \( C^N_{\Phi} \) and piecewise constant \( C^N_{\Theta} \) approach (as in section 6.2.1) for \( r \)-direction. We also use the same shape functions in \( z \)-direction, except for the elements, adjacent to the crack \( (z=z_c) \), where they are reciprocal polynomials (the same as we had in section 6.2.4). Then for the defect-free elements

\[ I_{ij} = \frac{V_{i,lj-l}}{h_{zj-l}^2 h_{rj-l}^2} \int_{r}^{\bar{r}} \int_{z}^{\bar{z}} \left\{ A_{l-lj-l} \left[ \left( z - z_{j-l} \right) \left( z_j - \bar{z} \right) \left( \frac{r_i}{r} - 2 \right) - (r - r_{i-l})(r_i - r) \right] + \right. \]

\[ + A_{l-lj} \left[ \left( z - z_{j-l} \right)^2 \left( \frac{r_i}{r} - 2 \right) + (r - r_{i-l})(r_i - r) \right] + \]

\[ + A_{ij-l} \left[ \left( z - z_{j-l} \right) \left( z_j - \bar{z} \right) \left( 2 - \frac{r_{i-l}}{r} \right) - (r - r_{i-l})^2 \right] + \]

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\[ \left. + \right. \]
\[ + A_{ij}\left[ (z-z_{j-1})^2 \left( 2 - \frac{r_{i-1}}{r} \right) + (r-r_{i-1})^2 \right] \] drdz +

\[ \frac{v_{i-l,j}}{h_{y}^2 h_{r}^2} \int_{r_i}^{z_i} \int_{r_{j-1}}^{z_{j-1}} \left\{ A_{i-l,j} \left[ (z_{j-1} - z)^2 \left( \frac{r_i}{r} - 2 \right) + (r - r_{i-1})(r_i - r) \right] + \right. \]

\[ + A_{i-l,j+1} \left[ (z_{j+1} - z)(z - z_j) \left( \frac{r_i}{r} - 2 \right) - (r - r_{i-1})(r_i - r) \right] + \]

\[ + A_{ij} \left[ (z_{j+1} - z)(z - z_j) \left( 2 - \frac{r_{i-1}}{r} \right) - (r - r_{i-1})^2 \right] \] drdz +

\[ + A_{ij+1} \left[ (z_{j+1} - z)(z - z_j) \left( 2 - \frac{r_{i-1}}{r} \right) - (r - r_{i-1})^2 \right] \] drdz +

\[ \frac{v_{ij-l}}{h_{y}^2 h_{r}^2} \int_{r_i}^{z_i} \int_{r_{j-1}}^{z_{j-1}} \left\{ A_{ij-l} \left[ (z_{j-1} - z)(z - z_j) \left( 2 - \frac{r_{i-1}}{r} \right) - (r_{i-1} - r)^2 \right] + \right. \]

\[ + A_{ij} \left[ (z - z_{j-1})^2 \left( 2 - \frac{r_{i-1}}{r} \right) + (r_{i-1} - r)^2 \right] + \]

\[ + A_{i+1,j-1} \left[ (z - z_{j-1})(z_j - z) \left( \frac{r_i}{r} - 2 \right) - (r - r_i)(r_{i-1} - r) \right] + \]

\[ + A_{i+1,j} \left[ (z - z_{j-1})^2 \left( \frac{r_i}{r} - 2 \right) + (r - r_i)(r_{i-1} - r) \right] \] drdz +

\[ \frac{v_{i-l,j}}{h_{y}^2 h_{r}^2} \int_{r_i}^{z_i} \int_{r_{j-1}}^{z_{j-1}} \left\{ A_{ij} \left[ (z_{j+1} - z)^2 \left( 2 - \frac{r_{i+1}}{r} \right) + (r_{i+1} - r)^2 \right] + \right. \]

\[ + A_{ij+1} \left[ (z_{j+1} - z)(z - z_j) \left( 2 - \frac{r_{i+1}}{r} \right) - (r_{i+1} - r)^2 \right] + \]
Performing the integration

\[ \int_{r_i}^{r} \left( \frac{r_i}{r} - 2 \right) dr = r_i \ln \frac{r_i}{r_{i-1}} - 2h_{r_{i-1}}. \]

\[ \int_{r_i}^{r} (r - r_{i-1})(r - r) dr = \frac{h_{r_{i-1}}^3}{6}. \]

\[ \int_{r_i}^{r} (r - r_{i-1})^2 dr = \frac{h_{r_{i-1}}^3}{3}. \]

and the like. We get the equations for the nodes in the regular elements in the form

\[ A_{i-1} \frac{v_{i-1,j-1}}{6h_{j-1}h_{r_{i-1}}} \left[ h_{j-1}^2 \left( r_i \ln \frac{r_i}{r_{i-1}} - 2h_{r_{i-1}} \right) - h_{r_{i-1}}^3 \right] + \]

\[ + A_{i-1} \frac{v_{i-1,j}}{6h_{r_{i-1}}^2} \frac{v_{i,j-1}}{h_{j-1}^2} \left[ 2h_{j-1}^2 \left( r_i \ln \frac{r_i}{r_{i-1}} - 2h_{r_{i-1}} \right) + h_{r_{i-1}}^3 \right] + \]

\[ + \frac{v_{i-1,j}}{h_{j-1}} \left[ h_{j-1}^2 \left( r_i \ln \frac{r_i}{r_{i-1}} - 2h_{r_{i-1}} \right) + h_{r_{i-1}}^3 \right] + \]

\[ + A_{i-1} \frac{v_{i-1,j+1}}{6h_{j+1}h_{r_{i-1}}} \left[ h_{j+1}^2 \left( r_i \ln \frac{r_i}{r_{i-1}} - 2h_{r_{i-1}} \right) - h_{r_{i-1}}^3 \right] + \]

\[ + A_{i-1} \frac{v_{i,j}}{6h_{r_{i-1}}^2} \frac{v_{i,j+1}}{h_{j+1}^2} \left[ h_{j+1}^2 \left( 2h_{r_{i-1}} - r_{i-1} \ln \frac{r_i}{r_{i-1}} \right) - 2h_{r_{i-1}}^3 \right] + \]

\[ + A_{i-1} \frac{v_{i,j+1}}{6h_{j+1}h_{r_{i-1}}} \left[ 2h_{j+1}^2 \left( 2h_{r_{i-1}} - r_{i-1} \ln \frac{r_i}{r_{i-1}} \right) + 2h_{r_{i-1}}^3 \right]. \]
\[ + \frac{V_{ij-1}}{h_{ri}^2} \left[ h_{ij-1}^2 \left( 2h_{ri} - r_{i+1} \ln \frac{r_i}{r_{i+1}} - 2h_{ri}^3 \right) \right] + \]

\[ + \frac{A_{ij}}{3} \left[ \frac{V_{i-1,j-1}h_{ij-1} + V_{i-1,j}h_{ij}}{h_{ri-1}^2} \left( 2h_{ri-1} - r_{i-1} \ln \frac{r_i}{r_{i-1}} \right) + h_{ri-1} \left( \frac{V_{i-1,j-1}}{h_{ij-1}} + \frac{V_{i-1,j}}{h_{ij}} \right) \right] + \]

\[ + \frac{V_{ij-1}h_{ij-1} + V_{ij}h_{ij}}{h_{ri}^2} \left( 2h_{ri} - r_{i+1} \ln \frac{r_i}{r_{i+1}} \right) + h_{ri} \left( \frac{V_{ij-1}}{h_{ij-1}} + \frac{V_{ij}}{h_{ij}} \right) \] +

\[ + \frac{A_{ij+1}}{6h_{ji}} \left[ \frac{V_{i-1,j}h_{ij-1}}{h_{ri}^2} \left( 2h_{ri-1} - r_{i-1} \ln \frac{r_i}{r_{i-1}} - 2h_{ri}^3 \right) \right] + \]

\[ + \frac{V_{ij}}{h_{ri}^2} \left( 2h_{ri} - r_{i+1} \ln \frac{r_i}{r_{i+1}} - 2h_{ri}^3 \right) \] +

\[ + \frac{A_{i+1,j-1}}{6h_{ji}} \left[ \frac{V_{ij-1}h_{ij-1}}{h_{ri}^2} \left( h_{ij-1}^2 \left( r_i \ln \frac{r_i}{r_i} - 2h_{ri} \right) - h_{ri}^3 \right) \right] + \]

\[ + \frac{A_{i+1,j}}{6h_{ri}} \left[ h_{ij-1}^2 \left( 2h_{ji} - r_{i+1} \ln \frac{r_i}{r_i} - 2h_{ri} \right) + h_{ri}^3 \right] + \frac{V_{ij}}{h_{ri}} \left[ 2h_{ij}^2 \left( r_i \ln \frac{r_i}{r_i} - 2h_{ri} \right) + h_{ri}^3 \right] \] +

\[ + \frac{A_{i+1,j+1}}{6h_{ji}h_{ri}^2} \left[ h_{ij}^2 \left( r_i \ln \frac{r_i}{r_i} - 2h_{ri} \right) - h_{ri}^3 \right] = 0. \]

It should be noticed that if a factor \( r \) stands at \( drdz \), the variables of integration, then the resulting stiffness matrix becomes symmetric, like we had in the 1-D case in the radial direction. Alternatively, we can use the shape functions of the form (37) in the radial direction, which give for the regular elements

\[ \frac{V_{i-1,j-1}}{h_{ji}^2} \int_{r_{i-1}, z_{i-1}} r_{i-1, z_{i-1}} \left[ A_{i-1,j-1} \left( (z - z_{j-1})(z_j - z) \left( -\frac{a_{i-1,j}}{r^2} + b_{i-1,j} \right) 2b_{i-1,0} - \right. \right. \]

\[ \left. \left. \left. + \frac{V_{i-1,j}h_{ij}}{h_{ri}^2} \left( 2h_{ri} - r_{i+1} \ln \frac{r_i}{r_{i+1}} - 2h_{ri}^3 \right) \right) + h_{ri} \left( \frac{V_{i-1,j}}{h_{ij}} \right) \right] + \]

\[ + \frac{A_{i+1,j-1}}{6h_{ji}} \left[ V_{i-1,j}h_{ij-1} \left( 2h_{ji} - r_{i+1} \ln \frac{r_i}{r_{i+1}} - 2h_{ri} \right) + h_{ri} \left( V_{i-1,j} + V_{i-1,j} \right) \right] + \]

\[ + \frac{V_{ij}}{h_{ri}^2} \left( 2h_{ri} - r_{i+1} \ln \frac{r_i}{r_{i+1}} - 2h_{ri}^3 \right) \] +

\[ + \frac{A_{i+1,j}}{6h_{ri}} \left[ h_{ij-1} \left( 2h_{ji} - r_{i+1} \ln \frac{r_i}{r_{i+1}} - 2h_{ri} \right) + h_{ri} \left( V_{ij} \right) \right] + \frac{V_{ij}}{h_{ri}} \left[ 2h_{ij}^2 \left( r_i \ln \frac{r_i}{r_i} - 2h_{ri} \right) + h_{ri}^3 \right] ] +

\[ + \frac{A_{i+1,j+1}}{6h_{ji}h_{ri}^2} \left[ h_{ij} \left( r_i \ln \frac{r_i}{r_i} - 2h_{ri} \right) - h_{ri}^3 \right] = 0. \]
\[\begin{align*}
&- \left( \frac{a_{i-1,0}}{r} + b_{i-1,0} r \right) \left( \frac{a_{i-1,1}}{r} + b_{i-1,1} r \right) + \\
&+ A_{i-1,j} \left( \left( z - z_{j-1} \right)^2 \left( - \frac{a_{i-1,1}}{r^2} + b_{i-1,1} \right) 2b_{i-1,0} + \left( \frac{a_{i-1,0}}{r} + b_{i-1,0} r \right) \left( \frac{a_{i-1,1}}{r} + b_{i-1,1} r \right) \right) + \\
&+ A_{j-1} \left( \left( z - z_{j-1} \right) \left( z_j - z \right) \left( - \frac{a_{i-1,1}}{r^2} + b_{i-1,1} \right) 2b_{i-1,1} - \left( \frac{a_{i-1,1}}{r} + b_{i-1,1} r \right)^2 \right) + \\
&+ A_{j,i} \left( \left( z - z_{j-1} \right)^2 \left( - \frac{a_{i-1,1}}{r^2} + b_{i-1,1} \right) 2b_{i-1,1} + \left( \frac{a_{i-1,1}}{r} + b_{i-1,1} r \right)^2 \right) \right] drdz + \\
&+ \frac{\nu_{i-1,j}}{h_{j-1}} \int_{r_{i,j-1}}^{r_{i,j}} \int_{z_{j-1}}^{z_j} \left[ A_{i-1,j} \left( \left( z_{j+1} - z \right)^2 \left( - \frac{a_{i-1,1}}{r^2} + b_{i-1,1} \right) 2b_{i-1,0} + \\
&+ \left( \frac{a_{i-1,0}}{r} + b_{i-1,0} r \right) \left( \frac{a_{i-1,1}}{r} + b_{i-1,1} r \right) \right) \right] drdz + \\
&+ A_{i-1,j+1} \left( \left( z_{j+1} - z \right) \left( z - z_j \right) \left( - \frac{a_{i-1,1}}{r^2} + b_{i-1,1} \right) 2b_{i-1,0} - \left( \frac{a_{i-1,0}}{r} + b_{i-1,0} r \right) \left( \frac{a_{i-1,1}}{r} + b_{i-1,1} r \right) \right) + \\
&+ A_{j} \left( \left( z_{j+1} - z \right)^2 \left( - \frac{a_{i-1,1}}{r^2} + b_{i-1,1} \right) 2b_{i-1,1} + \left( \frac{a_{i-1,1}}{r} + b_{i-1,1} r \right)^2 \right) + \\
&+ A_{j+1} \left( \left( z_{j+1} - z \right) \left( z - z_j \right) \left( - \frac{a_{i-1,1}}{r^2} + b_{i-1,1} \right) 2b_{i-1,1} - \left( \frac{a_{i-1,0}}{r} + b_{i-1,0} r \right)^2 \right) \right] drdz + \\
&+ \frac{\nu_{j,i}}{h_{j-1}^2} \int_{r_{j,i}}^{r_{j,i+1}} \int_{z_{i-1}}^{z_i} \left[ A_{j-1} \left( \left( z - z_{j-1} \right) \left( z_j - z \right) \left( - \frac{a_{i,0}}{r^2} + b_{i,0} \right) 2b_{i,0} - \left( \frac{a_{i,0}}{r} + b_{i,0} r \right)^2 \right) + \\
&+ A_{j} \left( \left( z - z_{j-1} \right)^2 \left( - \frac{a_{i,1}}{r^2} + b_{i,1} \right) 2b_{i,1} + \left( \frac{a_{i,1}}{r} + b_{i,1} r \right)^2 \right) \right] drdz + \\
&+ \frac{\nu_{j,i}}{h_{j-1}^2} \int_{r_{j,i}}^{r_{j,i+1}} \int_{z_{i-1}}^{z_i} \left[ A_{j+1} \left( \left( z - z_{j-1} \right) \left( z_j - z \right) \left( - \frac{a_{i,1}}{r^2} + b_{i,1} \right) 2b_{i,1} - \left( \frac{a_{i,1}}{r} + b_{i,1} r \right)^2 \right) \right] drdz + \\
&+ \frac{\nu_{j,i}}{h_{j-1}^2} \int_{r_{j,i}}^{r_{j,i+1}} \int_{z_{i-1}}^{z_i} \left[ A_{j} \left( \left( z - z_{j-1} \right)^2 \left( - \frac{a_{i,1}}{r^2} + b_{i,1} \right) 2b_{i,1} + \left( \frac{a_{i,1}}{r} + b_{i,1} r \right)^2 \right) \right] drdz.
\end{align*}\]
+ \Delta_{ij} \left( z - z_j \right)^2 \left( -\frac{a_{i,0}}{r^2} + b_{i,0} \right) 2b_{i,0} + \left( \frac{a_{i,0}}{r} + b_{i,0}r \right)^2 + \\
+ A_{i+1,j-1} \left( z - z_{j-1} \right) \left( z_j - z \right) \left( -\frac{a_{i,0}}{r^2} + b_{i,0} \right) 2b_{i,1} - \left( \frac{a_{i,0}}{r} + b_{i,0}r \right) \left( \frac{a_{i,1}}{r} + b_{i,1}r \right) + \\
+ A_{i+1,j} \left( z - z_{j-1} \right)^2 \left( -\frac{a_{i,0}}{r^2} + b_{i,0} \right) 2b_{i,1} + \left( \frac{a_{i,0}}{r} + b_{i,0}r \right) \left( \frac{a_{i,1}}{r} + b_{i,1}r \right) \right] dr dz + \\
+ \frac{v_{ij}}{h_{ji}} \int \int_{r_i}^{r_j} \left[ A_{ij} \left( z_{j+1} - z \right)^2 \left( -\frac{a_{i,0}}{r^2} + b_{i,0} \right) 2b_{i,0} + \left( \frac{a_{i,0}}{r} + b_{i,0}r \right)^2 \right] dr dz + \\
+ A_{ij+1} \left( z_{j+1} - z \right) \left( z - z_j \right) \left( -\frac{a_{i,0}}{r^2} + b_{i,0} \right) 2b_{i,0} - \left( \frac{a_{i,0}}{r} + b_{i,0}r \right)^2 + \\
+ A_{i+1,j} \left( z_{j+1} - z \right)^2 \left( -\frac{a_{i,0}}{r^2} + b_{i,0} \right) 2b_{i,1} + \left( \frac{a_{i,0}}{r} + b_{i,0}r \right) \left( \frac{a_{i,1}}{r} + b_{i,1}r \right) + \\
+ A_{i+1,j+1} \left( z_{j+1} - z \right) \left( z - z_j \right) \left( -\frac{a_{i,0}}{r^2} + b_{i,0} \right) 2b_{i,1} - \left( \frac{a_{i,0}}{r} + b_{i,0}r \right) \left( \frac{a_{i,1}}{r} + b_{i,1}r \right) \right] dr dz.

where

\[ a_{i,0} = \frac{r_i r_{i+1}^2}{h_{ii}(r_i + r_{i+1})}, \quad b_{i,0} = -\frac{r_i}{h_{ii}(r_i + r_{i+1})}, \]

\[ a_{i,1} = -\frac{r_i^2 r_{i+1}}{h_{ii}(r_i + r_{i+1})}, \quad b_{i,1} = \frac{r_{i+1}}{h_{ii}(r_i + r_{i+1})}. \]

Performing the integration

\[ \int_{r_i}^{r_{j+1}} \left( -\frac{a_{i,0}}{r^2} + b_{i,0} \right) dr = \left( b_{i,0} - \frac{a_{i,0}}{r_i r_{j+1}} \right) h_{ii} = -1, \]

\[ \int_{r_{i-1}}^{r_i} \left( -\frac{a_{i-1,1}}{r^2} + b_{i-1,1} \right) dr = 1. \]
\[
\begin{align*}
\int_{r_i}^{r_{i+1}} \left( \frac{a_{i,0}}{r} + b_{i,0} r \right) \left( \frac{a_{i,l}}{r} + b_{i,l} r \right) \, dr &= \left( \frac{a_{i,0} a_{i,l}}{r_i r_{i+1}} + a_{i,0} b_{i,l} + a_{i,l} b_{i,0} \right) h_r + b_{i,0} b_{i,l} \frac{r_{i+1}^3 - r_i^3}{3} = \\
&= \frac{2r_i r_{i+1} h_{ri}}{3(r_i + r_{i+1})^2},
\end{align*}
\]

\[
\begin{align*}
\int_{r_{i-1}}^{r_i} \left( \frac{a_{i-1,l}}{r} + b_{i-1,l} r \right) \, dr &= \left( \frac{a_{i-1,l}^2}{r_i r_{i-1}} + 2a_{i-1,l} b_{i-1,l} \right) h_{ri-1} + b_{i-1,l}^2 \frac{r_i^3 - r_{i-1}^3}{3} = \\
&= \frac{r_i (3r_{i-1} + r_i) h_{ri-1}}{3(r_{i-1} + r_i)^2}.
\end{align*}
\]

\[
\begin{align*}
\int_{r}^{r_{i+1}} \left( \frac{a_{i,0}}{r} + b_{i,0} r \right) \, dr &= \left( \frac{a_{i,0}^2}{r_i r_{i+1}} + 2a_{i,0} b_{i,0} \right) h_{ri} + b_{i,0}^2 \frac{r_{i+1}^3 - r_i^3}{3} = \\
&= \frac{r_i (r_i + 3r_{i+1}) h_{ri}}{3(r_i + r_{i+1})^2}.
\end{align*}
\]

we get the equations for the nodes in the regular elements in the form

\[
\begin{align*}
A_{i-1,j-1} \frac{v_{i-1,j-1} r_{i-1}}{3h_{j-1}(r_{i-1} + r_i)} \left( - \frac{h_{j-1}^2}{h_{ri-1}} - \frac{2r_i h_{ri-1}}{r_{i-1} + r_i} \right) + \\
+ A_{i-1,j} \frac{v_{i-1,j} r_{i-1}}{6h_{ri-1}^2} \left[ 2h_{j-1}^2 \left( r_i \ln \frac{r_i}{r_{i-1}} - 2h_{ri-1} \right) + h_{ri-1}^2 \right] + \\
+ \frac{v_{i-1,j}}{h_{j}} \left[ 2h_{j}^2 \left( r_i \ln \frac{r_i}{r_{i-1}} - 2h_{ri-1} \right) + h_{ri-1}^2 \right] + \\
+ A_{i-1,j+1} \frac{v_{i-1,j+1} r_{i-1}}{6h_{j} h_{ri-1}^2} \left[ h_{j}^2 \left( r_i \ln \frac{r_i}{r_{i-1}} - 2h_{ri-1} \right) - h_{ri-1}^2 \right] +
\end{align*}
\]
For an element with a tight crack at \( z = z_k \), consider a reciprocal polynomial model.

Denote \( \zeta = \frac{z-z_{k-1}}{h_{z_{k-1}}} \) and \( \nu = l - \frac{v_m}{v_o} \). Then for the element \( (k-l,i-l) \) we have
\[
\varphi_{z,k} \varphi_{z,k-1} = \frac{n + \frac{v_m}{v_0} - (n + 1)\zeta + \left(1 - \frac{v_m}{v_0}\right)\zeta^{n+1}}{(n + \frac{v_m}{v_0})^2 \left[1 - \left(1 - \frac{v_m}{v_0}\right)\zeta^n\right]}
\]

\[
= \frac{v_m \zeta^{n+2}}{(n + l - v)^2} - \frac{(2n + 1)\zeta^2 v_m}{(n + l - v)^2} + \frac{v_m \zeta}{n + l - v} - \frac{v_m^2 \zeta^2}{(n + l - v)^2 \left[1 - v\zeta^n\right]} + \frac{v_m n \zeta}{(n + l - v) \left[1 - v\zeta^n\right]},
\]

\[
\int \frac{\varphi_{z,k} \varphi_{z,k-1}}{\mu} \, dz = h_{z,k-1} v_m \left\{ \frac{\nu}{(n + l - v)^2 (n + 3)} - \frac{2n + 1}{3(n + l - v)^2} + \frac{I}{2(n + l - v)} - \left(\frac{n}{n + l - v}\right)^2 \frac{1}{\nu^{3/n}} \hat{s}_2\left(\nu^{1/n}\right) + \frac{l}{n + l - v} - \frac{l}{\nu^{2/n}} \hat{\zeta}_1\left(\nu^{1/n}\right) \right\},
\]

\[
\varphi_{z,k}^2 = \frac{n + \frac{v_m}{v_0} - (n + 1)\zeta - v\zeta^{n+1}}{(n + \frac{v_m}{v_0})^2 \left[1 - v\zeta^n\right]}
\]

\[
= -\frac{v_m \zeta^{n+2}}{(n + l - v)^2} + \frac{(2n + 1)v_m \zeta^2}{(n + l - v)^2} + \frac{n^2 \zeta^2}{(n + l - v)^2 \left[1 - v\zeta^n\right]}.
\]

\[
\int \frac{\varphi_{z,k}^2}{\mu} \, dz = h_{z,k-1} v_m \left\{ \frac{\nu}{(n + l - v)^2 (n + 3)} - \frac{2n + 1}{3(n + l - v)^2} + \right.
\]

\[
+ \left(\frac{n}{n + l - v}\right)^2 \frac{1}{\nu^{3/n}} \hat{s}_2\left(\nu^{1/n}\right) \right\},
\]

\[
\frac{l}{\mu} \left( \frac{\partial \varphi_{z,k}}{\partial z} \right)^2 = -\frac{l}{\mu} \frac{\partial \varphi_{z,k}}{\partial z} \frac{\partial \varphi_{z,k-1}}{\partial z} = \frac{v_m}{h_{z,k-1} \left(n + l - v\right)^2}.
\]
\[
\int_{z_{n-1}}^{z_{n}} \frac{1}{\mu} \left( \frac{\partial \varphi_{z_{k}}}{\partial z} \right)^2 dz = \frac{v_m}{h_{z_{k}-1}(n + l)(n + l - v)}, \tag{45}
\]

\[
\varphi_{z_{k-1}}^2 - \mu = -\frac{v_m v_{\zeta}^2}{(n + l - v)^2} + \frac{v_m v_{\zeta}^2}{(n + l - v)^2} - \frac{2\zeta}{(n + l - v)} + \frac{v_m v_{\zeta}^2}{l - v_{\zeta}^n} \left( \frac{n}{n + l - v} \right)^2 - \frac{2\zeta}{l - v_{\zeta}^n} \left( \frac{n}{n + l - v} \right) + \frac{v_m}{l - v_{\zeta}^n},
\]

\[
\int_{z_{n-1}}^{z_{n}} \frac{\varphi_{z_{k-1}}^2}{\mu} dz = h_{z_{k}-1} v_m \left\{ \frac{v}{(n + 3)(n + l - v)^2} + \frac{2n + l}{3(n + l - v)^2} - \frac{l}{n + l - v} + \left( \frac{n}{n + l - v} \right)^2 \frac{l}{v^{3/n}} \hat{\alpha}_2 \left( u^{l/n} \right) - \frac{n}{n + l - v} \frac{2}{v^{2/n}} \hat{\alpha}_1 \left( u^{l/n} \right) + \frac{l}{u^{l/n}} \hat{\alpha}_1 \left( u^{l/n} \right) \right\}, \tag{46}
\]

where

\[
\hat{\alpha}_1(x) = \int_{0}^{l} \frac{d\zeta}{x - \zeta^n}.
\]

\[
\hat{\alpha}_1(x) = \int_{0}^{l} \frac{\zeta}{x - \zeta^n} d\zeta.
\]

\[
\hat{\alpha}_2(x) = \int_{0}^{l} \frac{\zeta^2}{x - \zeta^n} d\zeta.
\]

Substitution of these integrals into (42) gives the solution.

The magnetic flux density was computed for a magnetic flux leakage (MFL) tool. For
the conventional approach, where the reluctivity is piecewise constant, the distribution of the
magnetic field is as in Figure 32a, and the field around the defect is shown in Figure 32b
(because of axisymmetry only one half of the tool is shown). This approach does not allow
thin cracks unless a sufficiently dense discretization is used. In contrast, the mesh can be uniform in the $z$-direction for the tight crack model. The corresponding pictures are presented in Figure 33. Accordingly, the computed signals from the crack are different for the two methods. Figure 34 and Figure 35 show the radial and axial components of the magnetic flux density at a sensor's position. Figure 34a and Figure 34b correspond to the conventional approach, Figure 35a and Figure 35b correspond to the new model.

Figure 32. Magnetic flux lines for a rectangular 50-% deep 19 mm long (grid size) defect (a) and the field around the defect (b).
Figure 33. Magnetic flux lines for the tight crack (a) and the field around the crack (b).

Figure 34. Radial (a) and axial (b) components of the magnetic induction $B$ in the pipe. The tool passes a 50% deep, 19 mm long (grid size) defect.
a) b)

Figure 35. Signals from a 50% deep tight crack: $Br$ and $B_z$.

### 6.5. 2-D in Polar Coordinates

Consider the case when $\vec{A} = \hat{z} A$, $\vec{J}_s = \hat{z} J_s$, $\frac{\partial A}{\partial z} = 0$, $\frac{\partial \sigma}{\partial z} = \frac{\partial \nu}{\partial z} = 0$.

$$\nabla \times \vec{A} = \vec{r} \left( \frac{\partial A}{\partial \phi} \right) + \phi \left( \frac{\partial A}{\partial r} \right),$$

and we assume $\frac{\partial \sigma}{\partial r} = \frac{\partial \nu}{\partial r} = 0$ within each element. Let's first also take (for simplicity) $\sigma_z = 0$, which is also a reasonable assumption for insulated laminations. Then for elements without current source we have

$$\nabla \times (\nu(\nabla \times \vec{A})) = 0.$$

and since

$$\nabla \times (\nu(\nabla \times \vec{A})) = \nu(\nabla \times \nabla \times \vec{A}) + \nabla \nu \times (\nabla \times \vec{A}).$$
\[ \nabla \times \nabla \times \vec{A} = -\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( \frac{\partial A}{\partial r} \right) + \frac{1}{r} \frac{\partial^2 A}{\partial \phi^2} \right]. \]

\[ \nabla \nabla = \rho \frac{\partial v}{\partial r} + \phi \frac{\partial v}{\partial \phi} = \phi \frac{\partial v}{\partial \phi}, \quad (\nabla \nabla) \times (\nabla \times \vec{A}) = -\frac{\partial}{\partial r} \left( \frac{\partial v}{\partial \phi} \right). \]

we finally have

\[ \nabla \left[ r \frac{\partial}{\partial r} \left( \frac{\partial A}{\partial r} \right) + \frac{\partial^2 A}{\partial \phi^2} \right] + \frac{\partial v}{\partial \phi} \frac{\partial A}{\partial \phi} = 0 \]

In regular elements \( \frac{\partial v}{\partial \phi} = 0 \) where, hence,

\[ r \frac{\partial}{\partial r} \left( \frac{\partial A}{\partial r} \right) + \frac{\partial^2 A}{\partial \phi^2} = 0. \quad \text{or} \quad r^2 \frac{\partial^2 A}{\partial r^2} + r \frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial \phi^2} = 0. \]

Using the separation of variables: \( A = R\Phi \).

\[ r^2 \frac{\partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} + \frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial \phi^2} = 0. \]

we obtain the system

\[ \begin{align*}
\frac{r^2 \partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} &\pm \lambda^2 R = 0, \\
\frac{\partial^2 \Phi}{\partial \phi^2} &= \pm \lambda^2 \Phi.
\end{align*} \]

For positive signs at \( \lambda^2 \)

\[ R = R_0 \cos(\lambda \ln r) + R_1 \sin(\lambda \ln r), \]
\[ \Phi = \Phi_0 e^{\lambda \phi} + \Phi_1 e^{-\lambda \phi}. \]

for negative signs at \( \lambda^2 \)
\[
\begin{align*}
R &= R_0 r^\lambda + R_1 r^{-\lambda} \\
\Phi &= \Phi_0 \cos \lambda \phi + \Phi_1 \sin \lambda \phi.
\end{align*}
\]

and for \( \lambda = 0 \)
\[
\begin{align*}
R &= R_0 \ln r + R_1 \\
\Phi &= \Phi_0 \phi + \Phi_1.
\end{align*}
\]

so that the general solution is
\[
A = (R_0 \ln r + R_1)(\Phi_0 \phi + \Phi_1) + \sum_{\lambda > 0} \left( R_{0\lambda} r^\lambda + R_{1\lambda} r^{-\lambda} \right) \left( \Phi_{0\lambda} \cos \lambda \phi + \Phi_{1\lambda} \sin \lambda \phi \right) + \sum_{\mu > 0} \left( R_{0\mu} \mu \ln r + R_{1\mu} \mu \ln r \right) \left( \Phi_{0\mu} e^{\lambda \phi} + \Phi_{1\mu} e^{-\lambda \phi} \right) \tag{48}.
\]

To choose a shape function we limit ourselves to the null approximation, i.e. the shape functions are of the form
\[
\varphi = (R_0 \ln r + R_1)(\Phi_0 \phi + \Phi_1) \tag{49},
\]
that corresponds to \( \lambda = 0 \) solution. In other words, their general form is a products of two general solutions to 1-D differential equations. namely
\[
\frac{r}{r} \frac{\partial^2 R}{\partial r^2} + \frac{\partial R}{\partial r} = 0 \quad \text{and} \quad \frac{\partial^2 \Phi}{\partial \phi^2} = 0 \tag{50}.
\]

In elements adjacent to the cracks \( \frac{\partial v}{\partial \phi} \neq 0 \), and the separation of variables gives
\[
\sqrt{r^2 \frac{\partial^2 R}{\partial r^2} + \frac{r}{R} \frac{\partial R}{\partial r} + \frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial \phi^2}} + \frac{1}{\Phi} \frac{\partial v}{\partial \phi} \frac{\partial \Phi}{\partial \phi} = 0.
\]

Limiting ourselves to the null approximation, we consider the following two differential equations,
Solution for (51b) and shape functions for various crack models, i.e. for various distributions of material properties around the crack as functions of $\phi$, are the same as that given in section 6.2 for functions of $z$.

The weak formulation implies

\[
\int_{S_{\text{int}}} (\nabla \times \mathbf{A}) \times \nabla \psi_{ij} \, ds + j \omega \int_{S_{\text{ext}}} \sigma \psi_{ij} \, ds = \int_{S_{\text{int}}} J_s \psi_{ij} \, ds. \quad \text{and we have}
\]

\[
\nabla \times \mathbf{A} = \hat{\mathbf{r}} \left( \frac{\partial A}{\partial \phi} \right) + \hat{\phi} \left( -\frac{\partial A}{\partial r} \right), \quad \nabla \times \psi_{ij} = \hat{\mathbf{r}} \frac{\partial \psi_{ij}}{\partial r} + \hat{\phi} \frac{\partial \psi_{ij}}{\partial \phi},
\]

so that

\[
(\nabla \times \mathbf{A}) \times \nabla \psi_{ij} = 2 \left( \frac{\partial A}{\partial r} \frac{\partial \psi_{ij}}{\partial r} + \frac{l}{r^2} \frac{\partial A}{\partial \phi} \frac{\partial \psi_{ij}}{\partial \phi} \right), \quad \text{resulting for constant \(\nu\) and \(\sigma\) in}
\]

\[
\int_{S_{\text{int}}} \left[ \nu \left( \frac{\partial A}{\partial r} \frac{\partial \psi_{ij}}{\partial r} + \frac{l}{r^2} \frac{\partial A}{\partial \phi} \frac{\partial \psi_{ij}}{\partial \phi} \right) \right] \, ds = \int_{S_{\text{int}}} J_s \psi_{ij} \, ds.
\]

If linear shape functions are used for elements without cracks, the integrations give

\[
\begin{align*}
\int_{r_{i-1}}^{r_{i+1}} \left( \frac{r - r_{i-1}}{h_{i-1}} \right) \, dr &= \frac{h_{i-1}}{12} (r_{i-1} + r_i), \\
\int_{r_{i-1}}^{r_{i+1}} \left( \frac{r - r_{i-1}}{h_{i-1}} \right)^2 \, dr &= \frac{h_{i-1}^2}{12} (r_{i-1} + 3r_i).
\end{align*}
\]

\[
\begin{align*}
\int_{r_{i-1}}^{r_{i+1}} \left( \frac{r - r_{i-1}}{h_i} \right) \, dr &= \frac{h_i}{12} (r_{i} + r_{i+1}), \\
\int_{r_{i-1}}^{r_{i+1}} \left( \frac{r - r_{i-1}}{h_i} \right)^2 \, dr &= \frac{h_i^2}{12} (r_{i} + 3r_i).
\end{align*}
\]

\[
\begin{align*}
\int_{r_{i-1}}^{r_{i+1}} \left( \frac{r - r_{i-1}}{h_{i-1}} \right) \, dr &= \frac{l}{h_{i-1}^2} \left( r_{i-1} h_{i-1} + \frac{h_{i-1}^2}{2} - r_{i-1} r_i \ln \frac{r_i}{r_{i-1}} \right), \\
\int_{r_{i-1}}^{r_{i+1}} \left( \frac{r - r_{i-1}}{h_{i-1}} \right)^2 \, dr &= \frac{l}{h_{i-1}^2} \left( r_{i-1} h_{i-1}^2 + \frac{h_{i-1}^3}{3} - r_{i-1} r_i \ln \frac{r_i}{r_{i-1}} \right).
\end{align*}
\]

\[
r \frac{\partial^2 R}{\partial r^2} + \frac{\partial R}{\partial r} = 0 \quad \text{and} \quad \frac{\partial}{\partial \phi} (\nu \Phi) = 0 \quad \text{(51)}.
\]
Magnetic flux lines for 3-phase excitation in a pipe without defects are shown in Figures 36a and 36b.

For an element with an axial tight crack at \( \phi = \phi_k \), a reciprocal polynomial model is considered, so that we have integrals similar to (43)-(46), where \( z \) is replaced by \( \phi \). The magnetic flux lines for a 50\% deep OD axial crack in a pipe around a 3-phase rotor generating the rotating magnetic field are shown in Figures 36c and 36d. Lets introduce, for
Figure 36. Magnetic flux lines around 3-phase rotor in a pipe without a defect (a) and zoom into the pipe wall (b), magnetic flux lines around 3-phase rotor in a pipe with a 50% deep OD axial tight crack (c) and zoom into the pipe wall (d).
Figure 36 (continued). Magnetic flux lines corresponding to differential magnetic field around rotor (e) and the tight crack (f).

In convenience, a notion of differential magnetic field as the difference between the field without cracks and the field in presence of a crack. The magnetic flux lines of such a differential magnetic field around the tight crack are shown in Figure 36e and 36f. Also a circumferential ED scan of the magnetic flux density (circumferential component $B_c$) was computed for these cases. The scan results for the differential magnetic field, corresponding to a 50% deep tight crack is shown in Figure 37a, whereas the dependence of the peak value vs. crack depth is shown in Figure 37b. It is clearly seen from Figure 36f, that cracks indeed behave like dipoles [28].
Figure 37. Circunferential scan of the magnetic flux density for the differential magnetic field, corresponding to 50% deep tight crack (a), and dependence of the peak value of the scan vs. crack depth (b).
CHAPTER 7. EXPERIMENTAL VALIDATION

To calibrate and assess the correctness of the FEM simulations, experiments were done with the rotor, generating the rotating magnetic field in the pipe. The simplest experimental setup consists of rotor, excited with a 3-phase power supply at frequency 40 Hz. A coil with approximately 10,000 turns with dimensions $2 \times 1 \times 0.5 \text{ cm}$ was used as a sensor for the measurement of the magnetic flux density components in axial, radial, and circumferential directions.

Circumferential scans were made at a range of distances $z$ (along the main axis) from the rotor's edge at a constant radius $r$. The magnitude and a phase of the signal for $z=6 \text{ cm}$ and $z=19 \text{ cm}$ at $r=6.5 \text{ cm}$ are presented in Figure 38. It is seen, that the magnitude is relatively stable (within the accuracy of the setup), whereas the phase varies linearly with the angle along the circumference. Slight variation of the magnitude could be explained by deviation of the scan surface from the ideal cylindrical form, imperfect current phase balance in power supply, asymmetry of the windings and the rotor core laminations, and other uncontrolled conditions of the experiment. Also axial scans of all the magnetic flux density components were taken. The corresponding magnitude and phase of the signal are shown in Figure 39. For comparison, the FEM computed values of the magnetic flux density are presented in Figure 40. The form of the scan signals are similar to experimental, so that by appropriate choice of material properties the numerical simulations can be close to reality. The dependence of the three components $B_c$, $B_r$ and $B_z$ of the magnetic flux density on the axial and the circumferential position of the sensor (for constant radial position $r=6.5 \text{ cm}$), obtained by the FEM simulations, are shown in Figures 41-43.
Figure 38. Magnitude $U$ (a and c) and phase $F$ (b and d) of the axial, radial, and circumferential components of the measured signal from the circumferentially scanned coil at a distance $z=6 \text{ cm}$ (a and b) and $z=19 \text{ cm}$ (c and d) from the rotor's edge (the rotor is in air).
Figure 39. Magnitude $U$ (a) and phase $F$ (b) of axial, radial, and circumferential components of the measured signal from the axially scanned coil (for the rotor in air).

Figure 40. Components of the magnetic flux density for the rotor in air, obtained by FEM simulations.
Figure 41. Circumferential component $B_c$ of the magnetic flux density around the rotor in air obtained by FEM simulations: real part (a); imaginary part (b); absolute value (c); phase (d); decay of the magnitude in logarithmic scale, 3-D view (e) and side view (f).
Figure 42. Radial component $B_r$ of the magnetic flux density around the rotor in air, obtained by FEM simulations: real part (a); imaginary part (b); absolute value (c); phase (d); decay of the magnitude in logarithmic scale, 3-D view (e) and side view (f).
Figure 43. Axial component $B_z$ of the magnetic flux density around the rotor in air, obtained by FEM simulations: real part (a); imaginary part (b); absolute value (c); phase (d); decay of the magnitude in logarithmic scale, 3-D view (e) and side view (f).
CHAPTER 8. CONCLUSIONS

8.1. Summary of Accomplishments

The main contribution of this work is introduction and development of the tight crack models and incorporation of these models into the finite element electromagnetic NDT codes. The cracks are described in a variety of rational polynomial representations of the line pipe material electromagnetic properties $\mu$ and $\sigma$ around these cracks. This approach avoids dense mesh discretization of the region around the crack, thus improving the efficiency of the FEM calculations.

The FEM formulation and the choice of the shape functions are done through the analytical analysis of the general solutions of the corresponding partial differential equations in Cartesian and cylindrical coordinates, with careful consideration of periodic boundary conditions, as well as conditions of planar and axial symmetry. The chosen shape functions not only allow introduction of the tight cracks, but also considerably improve accuracy of computation.

Also a procedure, allowing considerable reduction in the required computer memory size and allowing utilization of high performance multiprocessor clusters with parallel code execution, was developed and used for solving banded matrix equations.

8.2. Future Work

It is well understood that an accurate choice of the tight crack model can be done only upon the results of experimental study. Initial design of the experimental work has already
been started, and appropriate equipment and specimens are being prepared. The broad research activity of the Material Characterization Research Group (MCRG) of the Department of Electrical and Computer Engineering at Iowa State University includes an intensive study of possible means of the SCC detection.

A study of crack colonies is an important topic, where the models could be tested. The numerical generation (including random) of the SCC colonies may be done for a variety of colony shapes and crack orientations. The ultimate goal of the numerical simulations as well as experimental work remains the use of new knowledge in the test instrumentation.
Since the FEM simulations are done for long cylindrical objects such as pipes, the number of nodes in the axial direction is much (up to 15 times) greater than in radial or circumferential directions. This leads to relatively narrow (compared to the matrix dimensions) band of the resulting matrix. The direct methods, such as the Gaussian elimination method, can be used for solving the system. Some modifications to the Gaussian elimination method, allowing reduction of the required computer memory size, such as the frontal solution approach, have been suggested in the literature [30]. However, in the case of a structured rectangular mesh it can save no more than about 3% of the memory. A radical solution to the problem is chosen in this work. The band is divided into parts. The size of the parts depends on the memory capacity of a computer (or a processor in a cluster of parallel machines). Figure 44 shows a banded matrix divided into 3 (overlapping) parts AD, CF, and EH, with shaded regions corresponding to non-zero entries.

The idea is that we can start forward or backward elimination from any point of the matrix. Let's do forward elimination from point B down to point D, and then backward elimination from point E to point C. This results in filling out the matrix with non-zero entries in regions B"BUD" and VWXE, and making elements above the line EC and below the line BD to be zero, as shown in Figure 45a. These two (forward and backward) elimination procedures lead to expression of the unknowns, standing in segment C'D', of the system, as a linear combination of the unknowns from segments AB' and E'F'. Similarly, we do the eliminations:
Figure 44. A bandlimited matrix divided into 3 overlapping parts AD, CF and EH. Shaded regions correspond to non-zero entries.

Figure 45. Process of forward and backward elimination (a), the result of discarding the equations, not needed for computations of representative nodes (b), and the condensed matrix (c).
• from C to A,
• from D to F and from G to E,
• from F to H.

Let's call the unknowns from the segments AB', CD', EF' and GH' of the system as representative nodes. We notice that the expressions for these representative nodes do not contain any more the unknowns from segments BC', DE' and FG'. The equations from B' to C', from D' to E' and from F' to G' are not needed for computation of the representative nodes, and hence, can be temporarily discarded from the system, as shown in Figure 45b. This results in a condensed square matrix shown in Figure 45c, which in turn, can be solved by regular Gaussian elimination method.

Once the representative nodes are found, the rest of the unknowns are computed by solving smaller, independent (i.e. now decoupled) matrices, built from sectors laying between the points B and C, D and E, F and G. It is seen, that if only one processor is available, than the size of the computer memory should be large enough to store two matrices: the (bandlimited) matrix sector between B and C, and the condensed matrix. However, this size may be significantly smaller than the size of the original matrix, especially for finite element meshes with one dominating axis, such as cylindrical mesh for a long pipe.

It is also seen, that
• each pair of eliminations (forward and backward) for obtaining the condensed matrix can be done independently, and hence, by different (parallel) processors (if available).
• each decoupled matrix, made of a sector of the original matrix, can also be solved separately (and in parallel) by different processors.
A version of the code (that can be implemented on any type of computer having .5 G memory space), allowing reduction of the required computer memory size through the construction of a condensed matrix, is presented below.

```
* implicit double precision (a-h,o-z)
PARAMETER(pi=4.0*datan(1.0d0),npols=2)
PARAMETER(mf=6+1,mr=46,mz=204,mf=1,mr1=mr+1,mz1=mz+1)
PARAMETER(nels=mf*mr*mz,numat=102,NNPE=8,mfr=mf*mr1)
PARAMETER(nodes=(mf1*mr1)*mz1.ibs=mr1*(mf1+2)+3)
PARAMETER((mf1*mr1)*2)
PARAMETER(lnn=lengths((mf1-1)*(mr-1)+1),lnn=17,lnn1=lnn+1)
PARAMETER(iunk=lnn*mz1,nbw=ibs*2+1,ibs=ibs+1)
PARAMETER(nn=(iunk-1)/n3,nn=2*nn.len.nn3=nn-1)
PARAMETER(nn=nn*lnn1,len.nn=nn+nn1)
PARAMETER(sipi3=dsin(pi/3.),sm=-sipi3)

* SK,Q,c0,c1,dncur,PP,QQ,AA
dimension
* XORD(nodes),YORD(nodes),ZORD(nodes),NBC(nodes),
* NP(nels,8),MAT(nels,8),RX(numat,3),SIG(numat,3),
* SF(4,8,8),dncur(numat,8,3),SK(nbw,nlen),Q(nlen),
* XORD1(inodes),YORD1(inodes),AA(len.ibs1),
* PP(len,nngl),QQ(iunk)
open(2,file='meshout9.dat',status='old')
open(7,file='e9sa',status='unknown')

FREQ=40.0

OMEGA=2.*pi*FREQ
do i=1,nodes
   READ(2,22)XORD(i),YORD(i),ZORD(i),NBC(i)
enddo
doi=1,nels
   READ(2,24)(NP(i,j),j=1,8),MAT(i)
enddo

22 FORMAT(3d28.15,17)
```
24  FORMAT(9I7)
do k=1,3
   do j=1,8
      do i=1,numat
         READ(2.*) dncur(i,j,k)
      enddo
   enddo
do i=1,inodes
   READ(2.*) XORD1(i),YORD1(i)
enddo
c0=(.5,sm)
c1=(.5,ipi3)
ir1=0
geomf=2.*pi/(3.*npols)
cosg=dcos(geomf)
sing=dsin(geomf)
do l=1,numat
   do J=1,3
      SIG(LJ)=0.0
      RX(LJ)=7.9577e5
   enddo
enddo
c aluminum shield
do i=1,3
   SIG(2.I)=3.5e7
enddo
c rotor
   RX(4.1)=RX(1.1)/200.0
   RX(4.2)=RX(1.1)/200.0
   RX(4.3)=RX(1.1)/20.0
   SIG(4,1)=1.0e6
   SIG(4,2)=1.0e6
C******************************************************************************C
   CALL SHAFAAC(SF,NNPE)
   print*,'Stiffness matrix'
   lenp=len+1
   lenm=len-1
   mfmr=mf*mr
   icou=0
   ipos=1
   ibslen1=ibs1-len
   ibslen=ibslen1-1
   nulst=1
   nulend=nnlen
   nbwlen=nbw-len
   nnz1=nnz+1
do i=1,iunk
    QQ(i)=0
enddo
do i=1,nngl
    do j=1,len
        PP(j,i)=0
    enddo
enddo
do ii=1,lnn
    do j=nulst,nnlen
        Q(j)=0
        do k=1,nbw
            SK(k,j)=0
        enddo
    enddo
    nbeg=(ii-1)*nn
    if(ii.eq.lnn) nnzl=nnz
    do jzz=1,nnz
        jz=jzz+(ii-1)*nnz
        do jr=1,mr-1
            do jf=1,mf
                jxfr=jf+mf*(jr-1)
                jx=jxfr+mfmr*(jz-1)
                call poisson(jx,jxm,MAT,RX,SIG,NP,cosg,sing,OMEGA.
                SF,XORD,YORD,ZORD,nbc.dncur,c0,SK.Q.i.
                XORD1,YORD1,nulst,nulend,iir1,0)
                if(iir1.eq.1) goto 7001
            enddo
        enddo
    enddo
    nulst=len+1
    lenii=len*(ii-1)
    if(ii.ne.lnn) then
        call greg6(SK.Q.nnlen.ibs,AA,len)
        do j=1,len
            jn=j+lenii
            QQ(j+nbeg)=Q(j+len)
            jibslen1=1+j
            do k=1,len
                PP(k,jn)=AA(k,jibslen1)
            enddo
        enddo
    endif
enddo
enddo

the last chunk
call gauss(SK.Q.nnlen.nnlen.ibs)
nmin=unk-nnlen
do j=1.nnlen
   jn=nmin+j
   QQ(jn)=Q(j)
endo
c intermediate points
  jb2=(lnn-1)*nn
do j=1.lnn-1
     jb1=jb2-nn
     lenj=len*(lnn-j-1)
do k=1.len
        jbk1=jb1+k
        lenjk=lenj+k
do l=1.len
           jbl2=jb2+l
           QQ(jbk1)=QQ(jbk1)-PP(lenjk)*QQ(jbl2)
endo
do
  jb2=jb1
endo
c everything in between
  jbs=0
  nulst=ien+l
  nulend=nn
do j=1.Inn-1
     jbl=jbs+nn
     jb=jbs+len
  c recalling the middle part
     do i=1.nn3
        Q(i)=0
        do i1=1.nbw
           SK(i1,i)=0
        enddo
endo
endo
do jzz=1.nnz
   jz=jzz+(j-1)*nnz
   do jr=1.mr-1
      do jf=1.mf
         jxf=jf+mf*(jr-1)
         jx=jxf+mfmr*(jz-1)
         jxm=jxf+mfmr*(jzz-1)
call poisson(jx,jxm.MAT.RX.SIG.NP.cosg.sing.OMEGA,
* SF.XORD.YORD.ZORD.nbc.dncur.c1.SK.Q.ii.
* XORD1.YORD1.nulst.nulend.ir1.len)
endo
endo
c forming the r.h.s. from intermediate points
  do k=1,ibslen1
    n3lk=nn3+l-k
    ibsk=ibs+k
    do l=1,len
      Q(k)=Q(k)-QQ(l+jbs)*SK(l+ibslen1,k)
      Q(n3lk)=Q(n3lk)-QQ(l+jbl)*SK(ibs+k,n3lk)
    enddo
  enddo
  do k=ibslen1+1,len
    kl=k-ibslen
    k1=kl-1
    n3lk=nn3+l-k
    do l=1,k1,len
      Q(k)=Q(k)-QQ(l+jbs)*SK(l,k1)
      Q(n3lk)=Q(n3lk)-QQ(l+jbl)*SK(nbslen+l,n3lk)
    enddo
  enddo
  c final pieces
  call gauss(SK,Q,nnlen,nn3,ibs)
  do k=1,nn3
    QQ(jb+k)=Q(k)
  enddo
  jbs=jbl
  enddo
  do l=1,iunk/3
    M1=3*l-2
    M2=3*l-1
    M3=3*l
    WRITE(7,*)real(QQ(M1))
    WRITE(7,*)imag(QQ(M1))
    WRITE(7,*)real(QQ(M2))
    WRITE(7,*)imag(QQ(M2))
    WRITE(7,*)real(QQ(M3))
    WRITE(7,*)imag(QQ(M3))
  enddo
  goto 188
C CALCULATION FOR CURRENT PROBE POSITION COMPLETED
7001 print *,"Error: negative jacobian ".ir1.detj
  do intr=1,NNPE
    print *,XORD(NP(ir1,intr)), YORD(NP(ir1,intr)),ZORD(NP(ir1,intr)),intr
  enddo
188 continue
STOP 88
END
C****************************************************************** C
SUBROUTINE SHAFAC(SF,NNPE)
  implicit double precision (a-h,o-z)
  DIMENSION SF(4,8,8),QPT(8,3),Q3(2)
C GENERATION OF QUADRATURE POINTS AND WEIGHTS
  P=1.0/(SQRT(3.0))
  Q3(1)=(1.0+P)/2.0
  Q3(2)=(1.0-P)/2.0
C INITIALIZATION
  do I=1,8
    do J=1,2
      QPT(I,J)=0.0
    enddo
  do K=1,NNPE
    do L=1,4
      SF(L,K,I)=0.0
    enddo
  enddo
  do J=1,2
    do K=1,2
      do I=1,4
        N=N+1
        QPT(N,1)=Q3(I)
        QPT(N,2)=Q3(J)
        QPT(N,3)=Q3(K)
      enddo
    enddo
  enddo
C FORMING WT AND QPT ARRAYS
  N=0
  do I=1,2
    do J=1,2
      do K=1,2
        N=N+1
        QPT(N,1)=Q3(I)
        QPT(N,2)=Q3(J)
        QPT(N,3)=Q3(K)
      enddo
    enddo
  enddo
C SHAPE FUNCTIONS AND THEIR DERIVATIVES
  do J=1,8
    do I=1,NNPE
      U=QPT(J,1)
      V=QPT(J,2)
      W=QPT(J,3)
      U1=1-U
      V1=1-V
      W1=1-W
      SF(1,I,J)=SFN(U,V,W,U1,V1,W1,1)
      SF(2,I,J)=SFNU(U,V,W,U1,V1,W1,1)
      SF(3,I,J)=SFNV(U,V,W,U1,V1,W1,1)
      SF(4,I,J)=SFNW(U,V,W,U1,V1,W1,1)
    enddo
  enddo
  return
END
FUNCTION SFN(U,V,W,U1,V1,W1,I)
implicit double precision (a-h,o-z)
GOTO(1,2,3,4,5,6,7,8),I
1 SFN=U1*V1*W1
RETURN
2 SFN=U*V1*W1
RETURN
3 SFN=U*V*W1
RETURN
4 SFN=U1*V*W1
RETURN
5 SFN=U1*V1*W
RETURN
6 SFN=U*V1*W
RETURN
7 SFN=U*V*W
RETURN
8 SFN=U1*V*W
RETURN
END

FUNCTION SFNU(U,V,W,U1,V1,W1,I)
implicit double precision (a-h,o-z)
GOTO(1,2,3,4,5,6,7,8),I
1 SFNU=-V1*W1
RETURN
2 SFNU=V1*W1
RETURN
3 SFNU=V*W1
RETURN
4 SFNU=-V*W1
RETURN
5 SFNU=-V1*W
RETURN
6 SFNU=V1*W
RETURN
7 SFNU=V*W
RETURN
8 SFNU=-V*W
RETURN
END

FUNCTION SFNV(U,V,W,U1,V1,W1,I)
implicit double precision (a-h,o-z)
GOTO(1,2,3,4,5,6,7,8),I
1 SFNV=-U1*W1
RETURN
2 SFNV=U*W1
RETURN
3 SFNV=U*W1
RETURN
4 SFNV=U1*W1
RETURN
5 SFNV=U1*W
RETURN
6 SFNV=U*W
RETURN
7 SFNV=U*W
RETURN
8 SFNV=U1*W
RETURN
END
C********************************************************************
FUNCTION SFNW(U,V,W,U1,V1,W1,I)
implicit double precision (a-h.o-z)
GOTO(1,2,3,4,5,6,7,8).I
1 SFNW=U1*V1
RETURN
2 SFNW=U*V1
RETURN
3 SFNW=U*V
RETURN
4 SFNW=U1*V
RETURN
5 SFNW=U1*V1
RETURN
6 SFNW=U*V1
RETURN
7 SFNW=U*V
RETURN
8 SFNW=U1*V
RETURN
END
C******************************************************************************C
subroutine greg6(SK,Q,nnlen,ibs,AA,len)
implicit double precision (a-h.o-z)
double complex SK,Q,AA,AB,c
dimension SK(ibs*2+1,nnlen),Q(nnlen),AA(len,ibs+1)
dimension AB(len)
lenp=len+1
lenm=len-1
nn=nnlen-len
nbw=ibs*2+1
ibs1=ibs+1
ibs1 = ibs1 + 1
iters1 = nnlen - ibs
ibsm = ibs - len
print *, 'flood elimination'
do i = 1, iters1
   c = SK(ibs1, i)
do j = ibs1 + 1, nbw
   SK(j, i) = SK(j, i) / c
endo
SK(ibs1, i) = 1
Q(i) = Q(i) / c
do k = 1, ibs
   ik = i + k
   ibs1k = ibs1 - k
   c = SK(ibs1k, ik)
c = SK(i + k, ibs1 - k) = right beneath the diagonal element
do j = 1, ibs
   ibs1kj = ibs1k + j
   SK(ibs1kj, ik) = SK(ibs1kj, ik) - SK(ibs1 + j, i) * c
endo
SK(ibs1k, ik) = 0
Q(ik) = Q(ik) - Q(i) * c
endo
do i = 1, ibsm
   isk = i + iters1
   c = SK(ibs1, isk)
do j = ibs1 + 1, nbw - i
   SK(j, isk) = SK(j, isk) / c
endo
SK(ibs1, isk) = 1
Q(isk) = Q(isk) / c
do k = 1, ibs - i
   lk = isk + k
   ibs1k = ibs1 - k
   c = SK(ibs1k, ik)
c = SK(i + k, ibs1 - k) = right beneath the diagonal element
do j = 1, ibs - i
   ibs1kj = ibs1k + j
   SK(ibs1kj, ik) = SK(ibs1kj, ik) - SK(ibs1 + j, isk) * c
endo
SK(ibs1k, ik) = 0
Q(ik) = Q(ik) - Q(isk) * c
endo
do i = 1, ibslen
   nni = nn + 1 - i
   Ibsm = ibs1 + 1 - i
120

ibsp=ibs+i
do k=1.len
   AA(k.ibsm)=SK(ibsp+k.nni)
enddo
endif

doi=ibs1-ibslen
do k=1.len
   AA(k,i)=0
enddo

doi=ibslen+1.len
   nni=nn+1-i
   ibsm=ibs1+1-i
   ibsp=ibs+i
   il=il-ibslen-1
   do k=1.len-il
      AA(k.ibsm)=SK(ibsp+k.nni)
   enddo

doi=nn-ibs
   nni=nn+1-i
   do j=1.len
      AB(j)=AA(j.ibs1)
   enddo

   do j=1.ibs
      nnj=nn-j
      c=SK(ibsl+j.nnj)
      ibs1j=ibs1-j
      ibs1lj=ibs1j+1
      do k=1.len
         AA(k.ibs1lj)=AA(k.ibs1j)-c*AB(k)
      enddo
   enddo

   Q(nnij)=Q(nnij)-c*Q(nni)
enddo

doi=nn-ibs+1.nn-l
   l=nn-i
   il2=il+2
   nni=nn+1-i
   do j=1.len
      AB(j)=AA(j,il2)
   enddo

doi=1.i1
   nnj=nn-j
   c=SK(ibsl+j.nnj)
   il2j=il2-j
   do k=1.len
      AA(k.il2j)=AA(k.il2j)-c*AB(k)
   enddo
enddo
Q(nnij)=Q(nnij)-c*Q(nni)
enddo
enddo
doi=1,len
Q(len+i)=Q(i)
Q(i)=Q(nn+i)
ibs1i=ibs1-i
inn=i+nn
doj=1.ibs+i
jibs1i=j+ibs1i
SK(jibs1i.i)=SK(jibs1i.inn)
endo
dendo
return
dendroutinesubroutine gauss(A,B,maxlen,numeq,isb)
imPLICIT double precision (a-h),(o-z)
double complex A,B,c
dimension A(2*isb+1,maxlen),B(maxlen)
isbl=isb+1
ibw=isb*2+1
print *,’forward elimination’
do i=1,numeq
num=numeq-i
if(num.gt.isb) num=isb
c=A(isbl.i)
doj=isbl.isbl+num
A(j,i)=A(j,i)/c
endo
B(i)=B(i)/c
do k=1.num
ik=i+k
if(nik.gt.isb) nik=isb
isbk=isb1-k
c=A(isblk,ik)
doj=1.num
isbkj=isbk+j
A(isbkj,ik)=A(isbkj,ik)-A(isbk1+j,i)*c
endo
B(ik)=B(ik)-B(i)*c
endo
print *,’backward substitution’
do i=1,numeq-1
num=i
nq=numeq-i
if(i.gt.isb) num=isb

    do j=1,num
        B(nq)=B(nq)-B(nq+j)*A(isb+j,nq)
    enddo

enddo

return
end

subroutine poisson(jx, jxm, MAT, RX, SIG, NP, cosg, sing, OMEGA,
* SF, XORD, YORD, ZORD, nbc, dncur, c0, c1, SK, Q, ii.
* XORD1, YORD1, null, nullend, ir1, nshift)
!

implicit double precision (a-h),(o-z)
!
PARAMETER(pi=4.0*datan(1.0d0), npols=2)
PARAMETER(mf=6+1, mr=46, mz=204, mfl=mf+1, mr1=mr+1, mz1=mz+1)
PARAMETER(nels=mf*mr*mz, numat=102, NNPE=8, mfr=mfl*mr+1)
PARAMETER(nodes=(mfl*mr+1)*mz1.ibs=(mr1*(mf-1)+2)*3)
PARAMETER(inodes=(mfl*mr+1)*2)
PARAMETER(len=((mf-1)*(mr-1)+1)*3, lnn=17, ln1=lnn+1)
PARAMETER(iunk=len*mz1.nnw=ibs*2+1, ibs=1)
PARAMETER(nn=(iunk-len)lnn.n3=2*nn-len, nn3=nn-len)
PARAMETER(nng=lnn(nn-1), nnz=mz/nnlen=nn+len)
PARAMETER(sipi3=dsin(pi/3.), sm=-sipi3)
!
double  complex
!
    dimension * XORD(nodes), YORD(nodes), ZORD(nodes), nbc(nodes),
* NP(nels, 8), MAT(nels), RX(numat, 3), SIG(numat, 3),
* DNDX(8), DNDY(8), COOR(3, 8), SF(4, 8), RII(24, 24),
* RJAC(3, 3), RJACI(3, 3), QE(24), RRR(24, 24), DNDZ(8),
* dncur(numat, 8, 3), SK(nbw, nnlen), Q(nnlen),
* XORD1(inodes), YORD1(inodes)
!
MJ=MAT(JX)
RXI=RX(MJ, 1)
RYI=RX(MJ, 2)
RZI=RX(MJ, 3)
SX=SIG(MJ, 1)
SY=SIG(MJ, 2)
SZ=SIG(MJ, 3)
do K=1,24
    QE(K)=0.0
    do L=1,24
        RRR(K, L)=0.0
        RII(K, L)=0.0
    enddo
enddo
!
do il=1,NNPE
    COOR(1, il)=XORD(NP(JX, il))
    COOR(2, il)=YORD(NP(JX, il))
C CALCULATE THE JACOBIAN AND ITS INVERSE FOR CURRENT ELEMET

DO N=1,NNPE
  RJAC(1,1)=0.0
  RJAC(1,2)=0.0
  RJAC(1,3)=0.0
  RJAC(2,1)=0.0
  RJAC(2,2)=0.0
  RJAC(2,3)=0.0
  RJAC(3,1)=0.0
  RJAC(3,2)=0.0
  RJAC(3,3)=0.0
  DO K=1,NNPE
    RJAC(1,1)=RJAC(1,1)+SF(2,N)*COOR(I,K)
    RJAC(1,2)=RJAC(1,2)+SF(3,N)*COOR(I,K)
    RJAC(1,3)=RJAC(1,3)+SF(4,N)*COOR(I,K)
    RJAC(2,1)=RJAC(2,1)+SF(2,N)*COOR(2,K)
    RJAC(2,2)=RJAC(2,2)+SF(3,N)*COOR(2,K)
    RJAC(2,3)=RJAC(2,3)+SF(4,N)*COOR(2,K)
    RJAC(3,1)=RJAC(3,1)+SF(2,N)*COOR(3,K)
    RJAC(3,2)=RJAC(3,2)+SF(3,N)*COOR(3,K)
    RJAC(3,3)=RJAC(3,3)+SF(4,N)*COOR(3,K)
  END DO
  DETJ=0.0
  DETJ=RJAC(1,1)*RJAC(2,2)*RJAC(3,3)+RJAC(2,1)*RJAC(3,2)*RJAC(1,3)+
       RJAC(3,1)*RJAC(1,2)*RJAC(2,3)
  IF(DETJ.LE.0.0) THEN
    IRL=1
    RETURN
  ENDIF
  DO JI1=1,3
    I1=MOD(JI1,3)+1
    I2=MOD(JI1,3)+2
    IF(I2.EQ.4) I2=1
    DO JI2=1,3
      J1=MOD(JI2,3)+1
      J2=MOD(JI2,3)+2
      IF(J2.EQ.4) J2=1
      RJAC(JI2,JI1)=(RJAC(1,JI)*RJAC(I2,1)-RJAC(I2,JI)*RJAC(1,1))/DETJ
    END DO
  END DO
END DO

C CALCULATE DERIVATIVES OF SHAPE FUNCTIONS
C IN GLOBAL COORDINATES

DO K=1,NNPE
DNDX(K) = RJACI(1,1) * SF(2, K, N) + RJACI(2,1)

DNDY(K) = RJACI(1,2) * SF(2, K, N) + RJACI(2,2)

DNDZ(K) = RJACI(1,3) * SF(2, K, N) + RJACI(2,3)

C FORM THE ELEMENTAL MATRICES FOR ELEMENT

J (QE()) AND S()  
do K = 1, NNPE
  M = 3 * K
  IM = M - 1
  JM = M - 2
QE(JM) = QE(JM) + SF(1, K, N) * dncur(mj, k, 1) * DETJ
QE(IM) = QE(IM) + SF(1, K, N) * dncur(mj, k, 2) * DETJ
QE(M) = QE(M) + SF(1, K, N) * dncur(mj, k, 3) * DETJ
do L = 1, NNPE
    NN1 = 3 * L
    IN = NN1 - 1
    JN = NN1 - 2
RR(JM, JN) = RII(JM, JN) + SF(1, K, N)
RR(IM, IN) = RII(IM, IN) + SF(1, K, N)
RR(M, IN) = RII(M, IN) + SF(1, K, N)
RR(M, NN1) = RII(M, NN1) + SF(1, K, N)
RRR(M, NN1) = RRR(M, NN1) + (DNDY(K) - RYI * DNDY(L) + DNDX(K) * RXI * DNDX(L)) * DETJ
RRR(M, JN) = RRR(M, JN) + DNDX(K) * (-RZI) * DNDZ(L) * DETJ
RRR(M, IN) = RRR(M, IN) + DNDY(K) * (-RZI) * DNDZ(L) * DETJ
RRR(JM, NN1) = RRR(JM, NN1) + DNDZ(K) * (-RXI) * DNDX(L) * DETJ
RRR(JM, JN) = RRR(JM, JN) + DNDZ(K) * (-RXI) * DNDX(L) * DETJ
RRR(JM, IN) = RRR(JM, IN) + DNDZ(K) * (-RXI) * DNDX(L) * DETJ
endo
if(nbc(npjk1).ne.3) then
  if(nbc(npjk1).eq.2.and.ii.eq.1) then
    ibplane=1
    npfar=npjk1+mfr
    dzz=ZORD(npfar)-ZORD(npjk1)
    dxx=XORD1(npfar)-XORD1(npjk1)
    dyy=YORD1(npfar)-YORD1(npjk1)
    dxz=dxx/dzz/2.
    dyz=dyy/dzz/2.
    c on the plane of symmetry
    else
    ibplane=0
    c not on the plane of symmetry
  endif
  c # of layers
  npjk2=npjk1/mfr
  c exclude boundaries of the previous layers
  npjk1=npjk1-npkj2*(mf-1+2*mr)
  c exclude previous layers and subtract 1
  npjk2=npjk1-npkj2*((mf-1)*(mr-1)+1)-1
  ns1=1
  if(npkj2.ne.0) then
    c not on the main axis
    ibound=0
    c exclude axial point
    npjk2=npjk2-1
    c # of previous rows in the current layer
    npjk3=npjk2/mf1
    c exclude first-and-last nodes of the previous rows
    c and the first node of the current row
    npjk1=npjk1-npkj3*2-1
    c if the first or last node in current row
    npkif=npjk2-npkj3*mf1
    if(npkif.eq.mf.or.npkif.eq.0) then
      c do not want a hat function on the edge
      ns1=0
    endif
    else
    c on the axis of symmetry
    if((jmod(jxm,mf).eq.1.and.(K.eq.1.or.K.eq.5)).or.
      =
      (jmod(jxm,mf).eq.0.and.(K.eq.2.or.K.eq.6))) then
      ns1=0
    else
      ibound=1
    endif
  endif
  K1=3*npjk1-3
  do ki=1,3
K0=K1+ki
if(K0.gt.nulend.or.K0.lt.nulst) then
   ns1=0
else
   K0=K0-nshift
endif
K3ki=3*K-3+ki
if(ns1.ne.0) then
   ns1=1
   if(ibound.eq.0.or.ki.ne.3) then
      Q(K0)=Q(K0)+QE(3*K-3+ki)
   else
      Q(K0)=(0,0,0,0)
      SK(ibsl,K0)=(1,0,0,0)
      ns1=2
   endif
   if(ki.ne.3.and.ibplane.eq.1) then
      Q(K0)=(0,0,0,0)
      SK(ibsl,K0)=1.0
      if(ki.eq.1) then
         SK(ibsl+2,K0)=-dxz
      elseif(ki.eq.2) then
         SK(ibsl+1,K0)=-dyz
      endif
      ns1=2
   endif
endif
do L=1,NNPE
   npjll=np(jxm,l)
   if(nbc(npjll).ne.3) then
      npjll=npjll/mfr
      npjll=npjll-npjll2*(mf-1+2*mr)
      npjll2=npjll-npjll2*((mf-1)*(mr-1)+1)-1
      ns2=1
      if(npjll2.ne.0) then
         npjl2=npjll2-1
         npjll3=npjll3/mfl
         npjl1=npjll1-npjll3*2
         npjlif=npjll2-npjll3*mf1
         if(npjlif.eq.mf) then
            npjl1=npjl1-mf
            ns2=0
         elseif(npjlif.eq.0) then
            npjl1=npjl1+mf-2
            ns2=2
         else
            npjl1=npjl1-1
         endif
      else
endif
L1=3*nplj1-3-nshift

do li=1,3
    L0=L1+li-KO+ibsl
    L3li=3*L-3+li

    if(ns1.eq.1) then
        if(L0.gt.2*ibsl+l) return
        if(ns2.eq.1) then
            SK(L0,K0)=SK(L0,K0)+RRR(K3kiX3li)+
            * (0,1)*RII(K3kiL3li)
        elseif(ns2.eq.0) then
            if(li.eq.3) then
                elseif(li.eq.1) then
                    elseif(li.eq.1) then
                        endif
                    endif
                elseif(li.eq.1) then
                    endif
                endif
            else
                elseif(ns2.eq.2) then
                    elseif(li.eq.3) then
                        elseif(li.eq.1) then
                            elseif(li.eq.1) then
                                endif
                            endif
                        elseif(li.eq.1) then
                            endif
                        endif
                    elseif(li.eq.1) then
                        endif
                    endif
                else
                    elseif(ns2.eq.2) then
                        elseif(li.eq.3) then
                            elseif(li.eq.1) then
                                elseif(li.eq.1) then
                                    endif
                                endif
                            elseif(li.eq.1) then
                                endif
                            endif
                        elseif(li.eq.1) then
                            endif
                        endif
                    else
                        endif
                    endif
                endif
            endif
        endif
    endif
enddo
c do li=1,3
    endif
    c if(nbc(npjl1).ne.3) then
        enddo
    c do L=1,NNPE
        enddo
    c do ki=1,3
        endif
    c if(nbc(npjk1).ne.3) then
        enddo
        return
        end
end
REFERENCES


[10] V.E. Shcherbinin and N.N. Zatsepin, “Calculation of the magnetostatic field of surface defects. II. Experimental verification of the principal theoretical relationships.”


