GENERATION OF GUIDED WAVES IN HOLLOW CYLINDERS BY WEDGE AND COMB TYPE TRANSDUCERS

John J. Ditri, Joseph L. Rose, Aleksander Pilarski
Engineering Science and Mechanics
114 Hallowell Building
Pennsylvania State University
State College, PA 16802

INTRODUCTION

It was shown by Denos C. Gazis in 1959[1] that in linearly elastic hollow circular cylinders there exists an infinite number of "normal modes", each of which has its own propagation characteristics such as phase and group velocity as well as their own displacement and stress distributions throughout the cross section of the cylinder. It was also shown that, even for a given mode, these characteristics changed with changing frequency. In general, when such a cylinder is loaded by external forces, all of the modes of the structure will be excited in varying strengths determined by the characteristics of the applied loading. From a nondestructive evaluation (NDE) point of view, however, there are some modes which, due to their unique characteristics, are more sensitive to the quantities being measured or the defects being sought. It would be advantageous, therefore, to modify the applied loading so as to excite with appreciable amplitude only those modes which are found to be sensitive to the quantity being measured. In order to do this however, the relationship between the applied loading and the amplitudes of the generated modes must be understood. In this paper, the general problem of determining the amplitudes with which each propagating mode is generated due to the application of specific types of separable, time harmonic loading is investigated. (The more general problem of non-separable loading can be found in a recent paper[2]). The general results for separable loading are then specialized to two types of transducers commonly used in NDE to determine how the parameters of these two types of sources affect the amplitudes of the generated modes.

BACKGROUND

Before beginning with the analysis, a few words must be said about the normal modes of an isotropic circular cylinder since these functions will be used as a basis with reference to which we expand the fields in the loaded cylinder. The notation which will hereafter be followed to identify the general form of the normal modes in the cylinder follows closely that used by Meeker and Meitzler[3] in describing the analogous modes in the solid rod. Recalling that there are actually a double infinity of modes (the modes are however, *countably* infinite) for the cylinder, we denote the fields of the normal modes by the two indices, μ and N so that the velocity field due to the "μ^{th}" mode of the "N^{th}" family can be written,

$$\vec{v}_{\mu}^{N}(r,\theta)e^{i(\omega t-\beta_{\mu}^{N}z)} = \sum_{\alpha=r,\theta,z} R_{\mu\alpha}^{N}(r)\Theta_{\alpha}^{N}(N\theta)e^{i(\omega t-\beta_{\mu}^{N}z)}\hat{e}_{\alpha} \tag{1}$$

where the functions $R_{\mu\alpha}^{N}(r)$ and $\Theta_{\alpha}^{N}(N\theta)$ denote the radial and angular field distributions of the"α^{th}" (physical) velocity component of the "μ^{th}" mode of the "N^{th}" family respectively. Similar expressions can be written for the other field variables as well.

The exact forms of the functions $R_{\mu\alpha}^{N}(r)$, and $\Theta_{\alpha}^{N}(N\theta)$ can be found in Ref. 1, where they are taken as real. The radial characteristic functions, $R(r)$, contain combinations of Bessel (and Modified Bessel) functions of the first and second kinds while the Angular characteristic functions, $\Theta_{\alpha}^{N}(N\theta)$, contain sines and cosines of argument $N\theta$ for the modes of "circumferential order" or "family", N.

In labelling the modes of circumferential order N by the index μ, no distinction has been made between the torsional, longitudinal and flexural modes. It must be understood however, that in general, any summations over the index μ implicitly include *all* of the types of modes of circumferential order N including the torsional, longitudinal (for N=0) and flexural. For instance, for $N = 0$, there are an infinite number of *Torsional* modes, say T_{μ}, $\mu \in \{0,1,2\ldots\}$, and an infinite number of *Longitudinal* modes, say L_{μ}, $\mu \in \{0,1,2\ldots\}$. For $N \neq 0$ we have the doubly infinite number of *Flexural* modes, $F(N,\mu)$, $N \in \{1,2,3\ldots\}$, $\mu \in \{0,1,2\ldots\}$.

Based on certain properties of the angular characteristic functions, $\Theta_{\alpha}^{N}(N\theta)$, it can be shown[2,4] that the normal modes of the cylinder satisfy an orthogonality relationship of the form,

$$P_{nm}^{MN} = 0 \qquad \text{unless } M = N \quad \textit{and} \quad \beta_{m}^{N} = \beta_{n}^{N*} \tag{2}$$

where,

$$P_{nm}^{MN} \equiv -\frac{1}{4}\iint_{\mathcal{D}} (\vec{v}_{n}^{M*}\cdot \underline{\mathrm{T}}_{m}^{N} + \vec{v}_{m}^{N}\cdot \underline{\mathrm{T}}_{n}^{M*})\cdot \hat{e}_{z}\, d\sigma \tag{3}$$

In Eq. (3), $\mathcal{D}$ represents the cross section of the cylinder, $\underline{\mathrm{T}}_{\alpha}^{\beta}(r,\theta)$ represents the stress tensor of mode α of family β, $\hat{e}_z$ represents a unit vector in the axial direction of the cylinder (along the generator) and an asterisk denotes complex conjugation. For propagating modes, P_{nn}^{MM} represents the time average power carried down the cylinder by mode n of family M.

PROBLEM FORMULATION AND SOLUTION

Consider an infinite cylinder (Fig. 1) which is loaded on its inner surface ($r = a$) by time harmonic surface tractions of the general form,

$$T_{rr}(r=a) = \begin{cases} -p_1(\theta)p_2(z), & \text{if } |z| \leq L \\ 0, & \text{if } |z| > L \end{cases} \tag{4a}$$

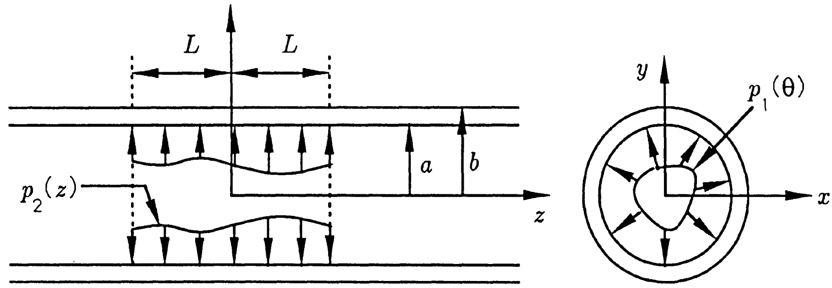

Fig. 1. Geometry of loaded cylinder.

$$T_{r\theta}(r = a) = T_{rz}(r = a) = 0 \qquad \forall\theta, z \tag{4b}$$

where $p_1(\theta)$ and $p_2(z)$ are arbitrary functions and the $e^{i\omega t}$ dependence has been omitted. The outer surface of the cylinder ($r = b$) is assumed to be traction free, $T_{rr}(b) = T_{rz}(b) = T_{r\theta}(b) = 0$, $\forall\theta, z$.

The shear traction is assumed to vanish over the entire length of the cylinder because the transducers to be modelled later are assumed to be coupled to the cylinder by a non-viscous liquid, thus not effectively transferring shear tractions. The load is seen to vanish outside the interval $|z| > L$ as shown in Fig. 1 and it is termed separable since it can be written as the product of individual functions of θ and z.

To solve the problem, the fields in the loaded waveguide are expanded in terms of the normal modes of the free waveguide multiplied by unknown, generally complex and "z" dependent expansion amplitudes, $A_\mu^N(z)$, *i.e.*,

$$\vec{v} = \sum_{N=0}^{\infty} \sum_{\mu} A_\mu^N(z) \vec{v}_\mu^N(r,\theta) \tag{5}$$

$$\underline{\mathrm{T}} = \sum_{N=0}^{\infty} \sum_{\mu} A_\mu^N(z) \underline{\mathrm{T}}_\mu^N(r,\theta) \tag{6}$$

These expansions are then used as the "1" solution in the complex reciprocity relation[5] and the "2" solutions are chosen as mode "n" of family "M" of the free structure, $\vec{v}_2 = \vec{v}_n^M(r,\theta)e^{-i\beta_n^M z}$, $\underline{\mathrm{T}}_2 = \underline{\mathrm{T}}_n^M(r,\theta)e^{-i\beta_n^M z}$.

Using solutions "1" and "2" in the complex reciprocity relation and performing several manipulations, including the use of the orthogonality relation, Eq. (2), and assuming that the "2" solution corresponds to a propagating mode (real wavenumber β_n^M), it can be shown that the expansion amplitudes of the forward propagating modes must satisfy the first order ordinary linear differential equation[2],

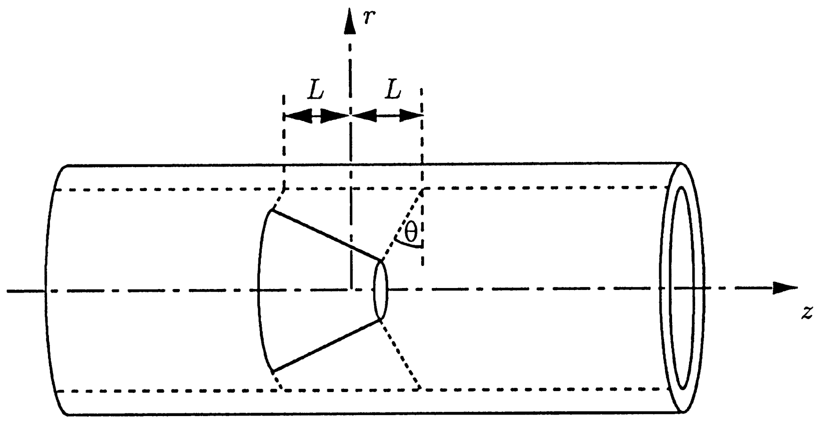

Fig. 2. Typical wedge type transducer

$$4P_{nn}^{MM}\left(\frac{d}{dz}+i\beta_n^M\right)A_n^M(z)=\oint_{\partial_1\mathcal{D}}\vec{v}_n^{M*}\cdot(\underline{\mathrm{T}}\cdot\hat{n}_1)\,ds \tag{7}$$

where $\partial\mathcal{D}_1$ represents a curve around the inside surface of the cylinder at an arbitrary "z" location. Using the boundary condition that the forward propagating modes are not present to the left of the generation region, or equivalently, that $A_n^M(z)=0$ for $z\leq -L$, along with the fact that the $\underline{\mathrm{T}}\cdot\hat{n}_1$ term in the integral on the right is the applied traction given by Eq. (4), the general solution of Eq. (9) can be written,

$$A_{+n}^M(z)=-a\frac{R_{nr}^{M*}(a)e^{-i\beta_n^M z}}{4P_{nn}^{MM}}\int_{-\infty}^{\infty}p_2(z)e^{i\beta_n^M z}\,dz\int_{\theta_o}^{\theta_o+2\pi}p_1(\theta)\Theta_r^M(\theta)\,d\theta \qquad z\geq L \tag{8}$$

where θ_o is any convenient reference angle.

WEDGE TYPE TRANSDUCER

A typical "wedge type" transducer is shown in Fig. 2. The conically shaped piezoelectric element produces a cylindrically spreading, axially symmetric beam which impinges onto the inner surface of the cylinder in the region $|z|\leq L$ at a given angle θ_i. The piezoelectric element can either be contained in a solid housing (Plexiglass for instance), or surrounded by a liquid. In the case of a solid housing, the housing itself should be coupled to the cylinder via a thin couplant layer. In either case, it is assumed that this type of loading produces traction on the inner surface of the cylinder for which the functions $p_1(\theta)$ and $p_2(z)$ of Eq. (4) can be written,

$$p_2(z)=\begin{cases}|\Phi|e^{-ik_s\sin(\theta_i)z}, & \text{if } |z|\leq L\\ 0, & \text{if } |z|>L\end{cases}$$
$$p_1(\theta)=P\ \ (Constant) \tag{9}$$

where $|\Phi|$ represents a "reflection" factor similar to that used in Ref. 6 to account for the fact that the true traction at the cylinder surface is not only due to the incident wave.

Substituting the functions p_1 and p_2 as given in Eq. (9) into the general solution, Eq. (8) gives for the expansion amplitudes,

$$A_{+n}^{M}(z) = -a\pi P|\Phi|\delta_{M0}\frac{R_{nr}^{M*}(a)e^{-i\beta_n^M z}}{P_{nn}^{MM}}\frac{\sin\left(\frac{(\beta_n^M - k_s\sin\theta_i)D}{2\cos\theta_i}\right)}{(\beta_n^M - k_s\sin\theta_i)} \qquad z \geq L \qquad (10)$$

where δ_{M0} represents the Kronecker delta.

According to most simplified approaches to this type of problem, it is assumed that, given the properties of the housing material (or liquid) and the angle of incidence of the wedge source, that only modes with a wavenumber satisfying Snell's law, $\beta_n^M = k_s \sin\theta_i$, can be generated. As can be seen from the third factor in Eq. (10), however, this is *not* true when using a finite source. For the finite source, there is a spectrum of wavenumbers (or phase velocities) which can be generated, the width of which depends on the ratio of the loading length to the wavelength at the Snell's law phase velocity[4]. In other words, although the third term in Eq. (10) is a maximum when Snell's law is satisfied, it does not identically vanish when Snell's law is not satisfied. It can be shown[4] that as the length of the transducer approaches infinity, the third term approaches, in a distributional sense, the Dirac delta function $\delta(V_{\text{ph}} - V_{\text{ph}}^{o})$ where V_{ph}^{o} represents the phase velocity calculated using Snell's law. In this case (plane wave incidence) Snell's law is strictly applicable. The size of the transducer is, however, limited by the requirement that it fit inside the cylinder, and this can cause a very large phase velocity bandwidth, especially at low inspection frequencies[4,7].

Due to the presence of the δ_{M0} term in Eq. (10), only the $M = 0$ or axially symmetric modes are generated by this type of source which is itself axially symmetric. Finally, the presence of the $R_{nr}^{M*}(a)/P_{nn}^{MM}$ term, which is simply the power normalized radial component of particle velocity at the inner surface of the cylinder, shows that for this type of internal pressure loading, the excitability of a given mode is proportional to its radial component of particle velocity (or displacement) at the surface in contact with the source.

COMB TYPE TRANSDUCER

A typical "comb type" transducer is shown in Fig. 3. In this case, which is analogous to that treated in Ref. 6 for flat layers, there are discreet "fingers" which are in contact with the inner surface of the cylinder. This type of source can be achieved by several methods, including the use of a radially expanding and contracting piezoelectric element or an interdigital type transducer. In either case, it is assumed that the loading functions appearing in Eq. (4) can be written for this case as,

$$p_2(z) = \begin{cases} 1, & \text{if } z \in \mathcal{G} \\ 0, & \text{if } z \notin \mathcal{G} \end{cases}$$
$$p_1(\theta) = P \quad (Constant) \qquad (11)$$

where

$$\mathcal{G} \equiv \bigcup_{\eta=0}^{N/2-1} g_\eta \quad ; \qquad g_\eta \equiv \{z : (2\eta+1)b + 2\eta a \leq |z| \leq (2\eta+1)b + 2a(\eta+1)\} \qquad (12)$$

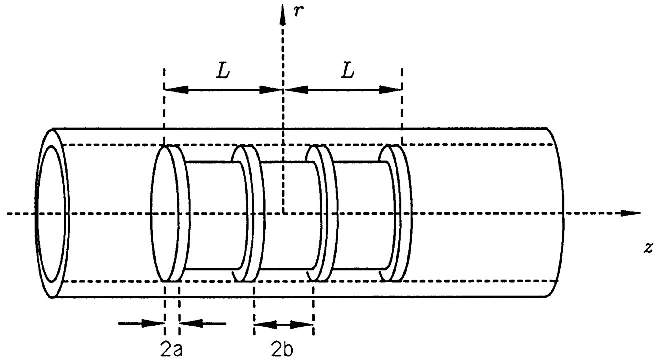

Fig. 3. Typical comb type transducer with N = 4 "fingers".

and N represents the number of "fingers" in the comb source and the constants "a" and "b" are defined in Fig. 3.

Substituting these functions into the general solution, Eq. (8), results in the expression,

$$A_{+n}^{M}(z) = -2a\pi P\delta_{M0}\frac{R_{nr}^{M*}(a)e^{-i\beta_n^M z}}{4P_{nn}^{MM}}\sum_{\eta=0}^{N/2-1}\int_{z\in g_\eta} e^{i\beta_n^M z}\,dz \qquad z \geq L \tag{13}$$

which, using the fact that,

$$\int_{z\in g_\eta}\phi(z)\,dz = \int_{-[(2\eta+1)b+2a(\eta+1)]}^{-[(2\eta+1)b+2\eta a]}\phi(z)\,dz + \int_{(2\eta+1)b+2\eta a}^{(2\eta+1)b+2a(\eta+1)}\phi(z)\,dz \tag{14}$$

can be written in closed form as,

$$A_{+n}^{M}(z) = -a\pi P\delta_{M0}\frac{R_{nr}^{M*}(a)e^{-i\beta_n^M z}}{P_{nn}^{MM}}\times$$
$$\frac{\sin(\beta_n^M a)}{a}\left\{\frac{\cos[\beta_n^M(L-a)] - \cos\left[\frac{(N+1)\beta_n^M(L-a)}{(N-1)}\right]}{1-\cos\left[\frac{2\beta_n^M(L-a)}{(N-1)}\right]}\right\} \tag{15}$$

It can be seen that the first two terms (upper row) in the expression for the expansion amplitudes, Eq. (15), except for a multiplicative factor, are identical to those of the wedge source. This is due to the fact that these terms are functions only of the modes being excited and not the source (actually, the δ_{M0} term which appears for both comes about from the fact that both sources are axially symmetric and therefore *is* a property of the source). The third terms in each expression carry information about the source used to excite the waveguide. The expression for the comb type transducer is seen to depend on the number of fingers, N, as well as the size of each finger, $2a$, and the spacing between fingers, $2b$. The function in curly brackets goes through a number of maxima and minima, and it can be shown that local maxima (and/or minima) occur whenever the the wavenumber of a given mode satisfies the relation,

$$\beta_n^M = \frac{m\pi(N-1)}{(L-a)} \qquad m \in \{0,1,2,\ldots\} \tag{16}$$

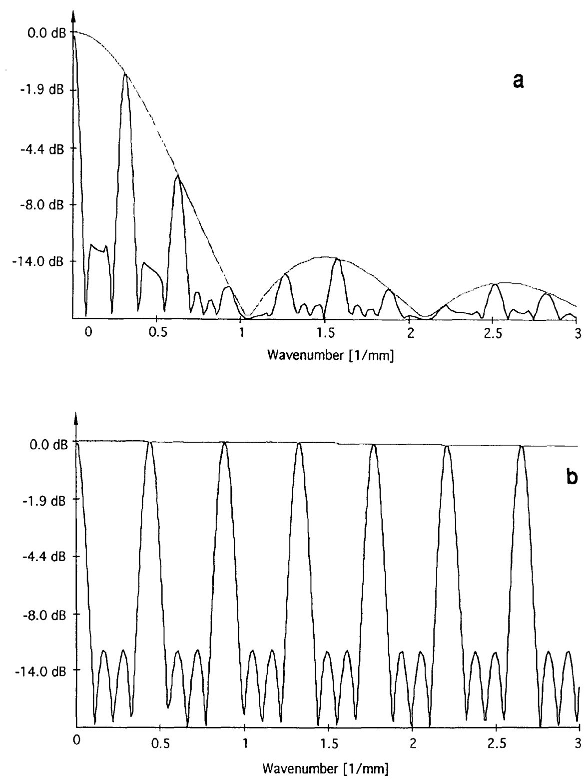

Fig. 4. A plot of the third factor of Eq. (15) as a function of wavenumber showing the efficiency with which the comb source can generate modes with large wavenumbers (small phase velocities). (a) $2a$ = 6mm, (b) $2a$ = .2 mm

which, in terms of wavelengths, can be written,

$$\lambda_n^M = \frac{2(a+b)}{m} \qquad m \in \{0, 1, 2, \ldots\} \tag{17}$$

Recognizing the quantity $2(a + b)$ as the spacing between the centers of the fingers, Eq. (17) shows that the comb source will generate all modes which, at the given frequency of excitation, have wavelengths which are equal the the spacing between centers of the fingers divided by some integer. This same conclusion was obtained by Viktorov[6] for the special case of $a = b$ and a comb source on flat layers. One major advantage of the comb source, as pointed out by Viktorov, is the fact that there is essentially no lower limit to phase velocity which can be excited. This is only true however, if the size of the fingers, "$2a$" is small. An example is shown in Fig. 4 in which the third term (second row) of Eq. (15) is plotted (in dB scale) versus wavenumber. Both parts "a" and "b" correspond to comb sources with four fingers, operating at 3.5 MHz and with b = 7.0 mm. In Fig. 4a, the dimension of the fingers was $2a$ = 6mm whereas for Fig. 4b, $2a$ = .2 mm. As can be seen from part "a" of the figure, although there are still resonant peaks at large wavenumbers (hence small phase velocities) the decrease in amplitude is around 14 dB. For the smaller finger size of Fig. 4b, the higher wavenumbers (and hence lower phase velocities) are excited with virtually no loss in efficiency. For this particular case, phase velocities down to 1.0 mm/μsec can be generated with roughly an 8 dB efficiency loss (as compared to the maximum at infinite phase velocity or zero wavenumber).

CONCLUSIONS

The excitation of hollow cylinder waveguides by wedge and comb type transducers has been studied in detail. It has been shown that, due to the finite size of the loading region for either type source, there will be a phase velocity spectrum which is excited by the source as opposed to a single "Snell's" law phase velocity. For the wedge type transducer, the range of phase velocities which can be effectively generated by the source is proportional to the ratio of the wavelength at the Snell's law phase velocity (and the given frequency) to the size of the loading region. For small sources and/or large wavelengths, this range can be over 100 percent of the Snell's law phase velocity. This is effect is particularly important when inspecting small diameter tubes since the size of the conical element is limited by the cylinder walls (for a non-zero incident angle). The comb type source will generate most efficiently those modes which have a wavelength satisfying Eq. (17). The ability to excite modes with small phase velocities, which is one of the major advantages of the comb type source over the wedge type, is critically dependent on the size of the fingers used in the comb. The smaller the fingers, the wider the range of wavenumbers, and hence phase velocities, which can be excited by the comb type transducer.

ACKNOWLEDGEMENTS

Thanks are given to the Electric Power Research Institute and Florida Power and Light company for their support of this work with special thanks to Frank Carr of FPL for technical discussions.

REFERENCES

1 Gazis, D.C., *J. Acoust. Soc. Am.* (31), 5, 568-573, (1959).

2 Ditri, J.J. and Rose, J.L., *J. Appl. Phys.* 72, (7), 2589-2597, (1992).

3 Meeker, T.H., Meitzler, A.H., *Physical Acoustics* Vol 1, Part A, Ed. W.P. Mason, pp. 111-167, (1964).

4 Ditri, J.J., Ph.D. Dissertation, Drexel University, Philadelphia PA, (1992).

5 Auld, B.A., *Acoustic Fields and Waves in Solids Vol II*, 2^{nd} Ed., Kreiger Publishing Co., FL, pp. 151-155, (1990).

6 Viktorov, I.A., *Rayleigh and Lamb Waves: Physical Theory and Applications*, Plenum Press, pp. 16-17, (1967).

7 Ditri, J.J., Rose, J.L., submitted to *J. Appl. Mech.*, (1992).