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Long-run average cost minimization of a stochastic processing system

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ABSTRACT

A long-run average cost problem in stochastic control theory is addressed. This problem is related to the optimal control of a production-inventory system which is subjected to random fluctuation. The approach taken here is based on finding a smooth solution to the corresponding Hamilton-Jacobi-Bellman equation. This solution in turn is used to derive an optimal process for the above long-run average cost problem. Using the invariant distributions for positive recurrent diffusion processes, another derivation for the optimal long-run average cost is provided here.

1. Introduction

In this article, we would like to address a stochastic control problem associated with a stochastic system of production-inventory type. It produces a certain product which is typically of high demand in comparison with the supply rate. We model the *net-demand* by the state process and it is defined by the cumulative number of orders received in $[0, t]$ minus the cumulative number of products made in $[0, t]$. The available control is proportional to the idle time of the system. We introduce a holding cost which measures the deviation of the net-demand from the origin. Controller's objective is to keep the state in a low cost region and to minimize the long-run expected average cost.

Long-run average cost minimization problems are an important topic in stochastic control theory. Their applications appear in Queueing systems, math finance and stochastic networks. In [10], such a problem was addressed for a Brownian motion with a controlled drift. In math finance, target zones for exchange rates and identifying central bank interventions in the foreign exchange markets lead to such stochastic control problems [2, 4, 7, 9, 12, 13]. In buffer-length control problems of queueing systems under heavy traffic, the trade-off between the cost of rejected customers due to full buffer and the cost of abandoning customers due to long waiting times in the queue leads to similar long-run average cost minimization problems [11, 15, 16]. For example, these results [11, 16] have practical implications in finding optimal buffer lengths for waiting customers in telephone call centers. Long-run average cost minimization problems also known as *ergodic control* problems [16]. For general class of such *ergodic control* problem and associated techniques we refer to recent text book *Ergodic control of diffusion processes* [1].

Our work is based on [17] and we generalize their model. Our methods are substantially different from theirs and our approach can be applied to more general state-dependent supply and demand rates of a production-inventory system.

2. Mathematical Model

Consider a weak solution [14] to a stochastic differential equation

$$Z(t) = x + \int_0^t \theta(Z(s))ds + \int_0^t \sigma(Z(s))dW(s), \tag{2.1}$$

where $W(\cdot)$ is one dimensional standard Brownian motion on a probability space $(\omega, \mathcal{F}, P)$. The drift and diffusion coefficients are represented by the continuous functions $\theta(\cdot)$ and $\sigma(\cdot)$. They satisfy $M_0 < \theta(x) < \theta_0 < 0$ and $0 < \epsilon_0 \leq \sigma(x) \leq M$ for all $x \in \mathbb{R}$, respectively. Here M_0, θ_0, ϵ_0 and M are constants. Let $(\mathcal{F}_t)$ be the filtration of Brownian motion. Then $Z(\cdot)$ is adapted to the Brownian filtration $(\mathcal{F}_t)$ with initial starting position x. Since the drift coefficient $\theta(x)$ is uniformly negative, the process $Z(\cdot)$ has the tendency to drift towards negative infinity.

We introduce a controlled state process

$$Z(t) = x + \int_0^t \theta(Z(s))ds + \int_0^t \sigma(Z(s))dW(s) + Y(t), \tag{2.2}$$

where $Y(0) = 0$, $Y(\cdot)$ is non-decreasing continuous on $(0, \infty)$ and adapted to $(\mathcal{F}_t)$. $Y(\cdot)$ is considered to be the control process, since it resists the drifting of $Z(\cdot)$ towards negative infinity. We further assume that

$$\lim_{T \to \infty} \frac{E[Y(T)]}{T} = 0 \tag{2.3}$$

Hence, the control $Y(\cdot)$ grows much slower than t and it can be considered a "thin control".

Let $h(\cdot)$ represents a holding cost function which is continuous in $\mathbb{R}$, decreasing in $(-\infty, 0)$ and increasing in $(0, \infty)$. We do not make any convexity assumptions on $h(\cdot)$. The objective of the control is to keep the state process $Z(\cdot)$ in a low cost region so as to minimize the following cost functional:

$$\text{Minimize } J(x, Z, Y),$$

where

$$J(x, Z, Y) = \limsup_{T\to\infty} \frac{1}{T} E \int_0^T h(Z(t))dt. \tag{2.4}$$

$J(x, Z, Y)$ is referred as the long-run average cost with respect to the control process Y. Throughout we assume that $h(\cdot)$ need not be symmetric and $h(\cdot)$ satisfies the constraint $0 \le h(x) \le Ae^{Bx}$ where $0 < B < -\frac{2\theta_0}{\epsilon_0^2}$ and $A > 0$. Hence, all the functions h which are of polynomial growth as $|x| \to \infty$ are allowed.

When an $\mathcal{F}_t$-adapted non-decreasing process $Y(\cdot)$ satisfies the above assumptions and a corresponding state process $Z(\cdot)$ exists on $[0, \infty)$, we consider (x, Z, Y) to be an admissible control system. Let $\mathcal{A}(x)$ be the collection of all such admissible control systems (x, Z, Y).

We introduce the corresponding value function of the stochastic control problem by

$$V(x) = \inf_{(x,Z,Y)\in\mathcal{A}(x)} J(x, Z, Y). \tag{2.5}$$

Our objective is to find an optimal control process Y^* and the corresponding state process Z^* to achieve this value function $V(x)$.

To motivate, we consider the following generalized version of an example from [17].

Example: Consider a production-inventory system of a certain product. Let $Z(t)$ be the *net-demand* at time t. Thus, $Z(t)$ can be considered as the cumulative number of orders received in $[0, t]$ minus the cumulative number of products made in $[0, t]$. Observe that $Z(t) \ge 0$ represents the amount of demand waiting to be served and when $Z(t) < 0$, magnitude of $Z(t)$ represents the number of products waiting to be sold. Suppose the demand rate is a function of $Z(t)$ and given by $\lambda(Z(t)) > 0$ and the production rate is $\mu > 0$. Thus, in the absence of a control, Z satisfies $Z(t) = x + \int_0^t [\lambda(Z(u)) - \mu]du + \sigma W(t)$ where $\sigma W(t)$ models the Gaussian noise associated with the demand. We assume $\theta(z) = \lambda(z) - \mu \le \theta_0 < 0$. When $|Z(t)|$ is large and $Z(t) < 0$, there are large number of products in the storage waiting for orders. When $|Z(t)|$ is large and $Z(t) > 0$, then a large number of waiting orders exist but no products available. Since $\theta(Z) < 0$, the uncontrolled system has a tendency to drift towards negative infinity, which means typically demand is higher than the supply. Under these circumstances, there is no idling of the production. Now we introduce the idle time of the production system. Let $U(t)$ be the cumulative idle time during $[0, t]$. Then the number of produced items during $[0, t]$ becomes $\mu[t - U(t)]$ and we let $Y(t) = \mu U(t)$. Therefore, the controlled state process satisfied the equation (2.2).

Let $h(\cdot)$ be the holding cost function satisfying the above assumptions. The system manager's objective is to find out a $U^*(\cdot)$ which minimizes $\limsup_{T\to\infty} \frac{1}{T} E \int_0^T h(Z(t))dt$. We resolve this problem in the next two sections.

3. HJB Equation and Optimal Process

Consider the process Z defined in (2.2). Then the associated generator is given by

$$\mathcal{G}Q = \frac{\sigma^2(x)}{2}Q''(x) + \theta(x)Q'(x) \tag{3.1}$$

The formal Hamilton-Jacobi-Bellman (HJB) equation associated with this control problem is given by

$$\min\{\mathcal{G}Q(x) + h(x) - \lambda, Q'(x)\} = 0, \tag{3.2}$$

where $\mathcal{G}$ is the generator of a controlled state process and $\lambda > 0$ is a constant. The relationship between a smooth solution $Q(\cdot)$ to (3.2) and the stochastic control problem will be made clear in Lemma 2 below.

We intend to show that the constant $\lambda > 0$ associated with any smooth solution to HJB equation yields a lower bound for the value function [6]. Our approach to find a solution to (2.5) is two fold: First, we use a smooth solution to HJB equation to obtain a lower bound $\lambda^* > 0$ for the value function $V(x)$. Second, we construct a state process Z^* associated with a special admissible control Y^*, whose long-run average cost achieves this lower bound λ^*. Consequently, Y^* is an optimal control which minimizes the long-run average cost and the value function $V(x)$ is equal to λ^* which is independent of x.

Our next proposition guarantees the existence of a pair (Q^*, λ^*) which solves the HJB equation.

Proposition 1. *There exists a solution $Q^* \in C^2(\mathbb{R})$ and a corresponding constant $\lambda^* > 0$ so that Q^* satisfies the HJB equation (3.2). Moreover, ${Q^*}'$ is bounded.*

Proof. See the Appendix. □

Lemma 2 (Verification lemma)**.** *Let $Q^* \in C^2(\mathbb{R})$ and (Q^*, λ^*) be a solution to (3.2) as in Proposition 1, then $V(x) \geq \lambda^*$.*

Proof. We apply Itô's lemma to $Q^*(Z(T))$ to obtain

$$\begin{aligned}Q^*(Z(T)) =& Q^*(x) + \int_0^T {Q^*}'(Z(t))dZ(t) + \int_0^T \frac{1}{2}{Q^*}''(Z(t))[dZ(t)]^2 \\ =& Q^*(x) + \int_0^T \theta(Z(t)){Q^*}'(Z(t))dt + \int_0^T \sigma(Z(t)){Q^*}'(Z(t))dW(t) \\ &+ \int_0^T \frac{1}{2}\sigma^2(Z(t)){Q^*}''(Z(t))dt + \int_0^T {Q^*}'(Z(t))dY(t)\end{aligned}$$

Since we have $dZ(t) = \theta(Z(t))dt + \sigma(Z(t))dW(t) + dY(t)$, we obtain $[dZ(t)]^2 = \sigma^2(Z(t))dt$ above.

$$\begin{aligned}Q^*(Z(T)) =& Q^*(x) + \int_0^T [\frac{1}{2}\sigma^2(Z(t)){Q^*}''(Z(t)) + \theta(Z(t)){Q^*}'(Z(t))]dt \\ &+ \int_0^T \sigma(Z(t)){Q^*}'(Z(t))dW(t) + \int_0^T {Q^*}'(Z(t))dY(t)\end{aligned}$$

Since $\sigma(x)Q'(x)$ is bounded, the stochastic integral term $\int_0^t \sigma(Z(s)){Q^*}'(Z(s))dW(s)$ is a mean zero martingale. If we take expected value on both sides, the expectation of the stochastic integral term is zero. Then, we have the following

$$E[Q^*(Z(T))] = Q^*(x) + E\int_0^T [\frac{\sigma^2(Z(t))}{2}{Q^*}''(Z(t)) + \theta(Z(t)){Q^*}'(Z(t))]dt + E\int_0^T {Q^*}'(Z(t))dY(t)$$

Since the pair (Q^*, λ^*) satisfies the HJB equation (3.2) and the generator of the state process is given by $\mathcal{G}Q = \frac{\sigma^2(x)}{2}Q''(x) + \theta(x)Q'(x)$, we obtain that $\frac{\sigma^2(x)}{2}{Q^*}''(x) + \theta(x){Q^*}'(x) \geq \lambda^* - h(x)$ and ${Q^*}'(x) \geq 0$. This together with $dY(t) \geq 0$ yields

$$\begin{aligned}E[Q^*(Z(T))] + E\int_0^T h(Z(s))ds &\geq Q^*(x) + \lambda^* T \\ \frac{1}{T}E[Q^*(Z(T))] + \frac{1}{T}E\int_0^T h(Z(s))ds &\geq \frac{Q^*(x)}{T} + \lambda^*\end{aligned}$$

Taking lim sup on both sides as $T \to \infty$, we have

$$\begin{aligned}\lambda^* &\leq \limsup_{T\to\infty}\{\frac{1}{T}E[Q^*(Z(T))] + \frac{1}{T}E\int_0^T h(Z(t))dt\} \\ &\leq \limsup_{T\to\infty}\frac{1}{T}E[Q^*(Z(T))] + \limsup_{T\to\infty}\frac{1}{T}E\int_0^T h(Z(t))dt\end{aligned}$$

We claim that the first term is equal to 0. Without lose of generality, we consider any general $Q \in C^2(\mathbb{R})$ with $0 \le Q'(x) \le c$. Then, we have $0 \le Q(x) \le cx + d$. So we have $|Q(Z(t))| \le cZ(t) + d$. Now taking expectation and divided on both sides by T. We have

$$\frac{1}{T}E|Q(Z(T))| \le \frac{c}{T}E(Z(T)) + \frac{d}{T} \le \frac{c}{T}E(Z^+(T)) + \frac{d}{T}$$

Therefore, to show $\limsup_{T\to\infty} \frac{1}{T}E[Q(T))] = 0$, it is enough to show $\limsup_{T\to\infty} \frac{1}{T}E[Z^+(T))] = 0$. Consider $Z(t) = x + \theta(Z(t))t + \sigma(Z(t))W(t) + Y(t)$, we have

$$\begin{aligned}
Z^+(t) &= (x + \theta(Z(t))t + \sigma(Z(t))W(t) + Y(t)) \vee \{0\} \\
&\le (x + \sigma(Z(t))W(t) + Y(t)) \vee \{0\} \\
&\le [x + \sigma(Z(t))W(t) + Y(t)]^+ \\
&\le [x + \sigma(Z(t))W(t)]^+ + Y(t) \\
&\le (x + MW(t))^+ + Y(t),
\end{aligned}$$

where $M > 0$ is a constant. Because we have $(f+g)^+ \le f^+ + g^+$ and $Y(t) \ge 0$. Taking expectation and divided by T on both sides, we have $\frac{1}{T}E[Z^+(T)] \le \frac{1}{T}E[(x + MW(T))^+] + \frac{1}{T}E[Y(T)]$. Since $E[W(T)^2] = T$, the first term approaches 0 and the second term also approaches to 0 by the constraint (2.3) on $Y(t)$. In particular, this implies that $\limsup_{T\to\infty} \frac{1}{T}E[Q^*(Z(t))] = 0$.

Therefore,

$$\lambda^* \le \limsup_{T\to\infty} \frac{1}{T}E\int_0^T h(Z(t))dt = J(x, Z, Y)$$

Then, if we take infimum on both sides over the class of admissible controls $\mathcal{A}(x)$, we conclude that $V(x) \ge \lambda^*$. □

Now we know that the lower bound of the value function $V(x)$ is given by λ^*, i.e. $\lambda^* \le V(x)$. Next, we show λ^* is an achievable lower bound by a special choice of control and derive an optimal strategy.

Theorem 3. *Let (Q^*, λ^*) be a solution to the HJB equation (3.2) and $r_{\lambda^*} < 0$ is defined by $h(r_{\lambda^*}) = \lambda^*$ as given in the proof of Proposition 1. Consider the state process Z^* defined* *by*

$$Z^*(t) = x + \int_0^t \theta(Z^*(s))ds + \int_0^t \sigma(Z^*(s))dW(s) + L^*(t) \tag{3.3}$$

where $L^*(t) = \lim_{\epsilon\to\infty} \frac{1}{2\epsilon}\int_0^t I_{[r_{\lambda^*}, r_{\lambda^*}+\epsilon)}(Z^*(s))ds$ *is the local-time process of* Z^* *at* $x = r_{\lambda^*}$*. Then,* (x, Z^*, L^*) *is an admissible control system and* $\lim_{T\to\infty} \frac{1}{T} E\int_0^T h(Z^*(t))dt = \lambda^*$*. Consequently, the control process* L^* *is optimal.*

Moreover, λ^* *is given by*

$$\lambda^* = \frac{\int_{r^*}^\infty h(x)\frac{2}{\sigma^2(x)} e^{\int_0^x \frac{2\theta(u)}{\sigma^2(u)}du} dx}{\int_{r^*}^\infty \frac{2}{\sigma^2(z)} e^{\int_0^z \frac{2\theta(u)}{\sigma^2(u)}du} dz}.$$

Proof. Consider the reflected state process (3.3) with a special control L^* on the state space $[r_{\lambda^*}, \infty)$ (see [8] Pages 17-34). We know this reflected diffusion process reflects at r^* which is a simplified notation for r_{λ^*} and the control is given by the local-time process L^*. The local-time process L^* is a continuous increasing function which increases only at times t where $Z^*(t) = r^*$ (see [5] Pages 141-156). Similar to the proof of Lemma 2, instead of an inequality, we obtain the following equation:

$$\begin{aligned} E[Q(Z^*(T))] &= Q(x) + \lambda^* T - E\int_0^T h(Z^*(t))dt + E\int_0^T Q'(Z^*(t))dL(t) \\ &= Q(x) + \lambda^* T - E\int_0^T h(Z^*(t))dt + E\int_0^T Q'(r^*)dL(t) \\ &= Q(x) + \lambda^* T - E\int_0^T h(Z^*(t))dt. \end{aligned}$$

Dividing both sides by T and taking $\limsup$, we conclude that $\lim_{T\to\infty} \frac{1}{T} E\int_0^T h(Z^*(t))dt = \lambda^*$. Therefore, We proved that the lower bound λ^* is achievable, i.e. $\lambda^* = J(x, Z^*, L)$. Consequently, (x, Z^*, L^*) is an admissible control system as needed.

In the proof of Proposition 1, we obtain $F_{\lambda^*}(r_{\lambda^*}) = 0$, which further implies

$$\lambda^* = \frac{\int_{r^*}^\infty h(x)\frac{2}{\sigma^2(x)} e^{\int_0^x \frac{2\theta(u)}{\sigma^2(u)}du} dx}{\int_{r^*}^\infty \frac{2}{\sigma^2(z)} e^{\int_0^z \frac{2\theta(u)}{\sigma^2(u)}du} dz}.$$

□

Consequently, the value function is actually a constant independent of x, i.e. $V(x) \equiv \lambda^*$. Notice that all the statements above happened in the case of starting position $Z(0) = x \geq r_{\lambda^*}$. In the case of $x < r_{\lambda^*}$, it has no effect on our statements, if there is an initial jump to r_{λ^*} immediately without any cost. Hence, the minimum long-run average cost of our state process is given by $V(x) \equiv \lambda^*$ for all x and the corresponding r^* (or denoted by r_{λ^*}) is the optimal reflective position.

Remark. *Consider the constraint (2.3) on $Y(t)$. What happens in the case $\liminf_{T\to\infty} \frac{E[Y(T)]}{T} > \delta$ for any $\delta > 0$? Here we observe that the cost $J(x, Z, Y) = +\infty$ in such a situation, where the holding cost is a convex function. By the definition of $\liminf$, we obtain $E[Y(t)] > \frac{\delta}{2}t$ for $t > T$ large enough. Under an additional assumption that $h(\cdot)$ is a convex function, it is valid to apply Jensen's inequality. We obtain the following inequality:*

$$\begin{aligned}\frac{1}{T}E\int_0^T h(Z(t))dt &\geq Eh(\frac{1}{T}\int_0^T Z(t)dt) \\ &\geq h(\frac{1}{T}E\int_0^T Z(t)dt).\end{aligned}$$

Then, we consider $\frac{1}{T}E\int_0^T Z(t)dt$. Substitute the state process into the integrand

$$\begin{aligned}\frac{1}{T}E\int_0^T Z(t)dt &= \frac{1}{T}E\int_0^T \big(x + \int_0^t \theta(Z(s))ds + \int_0^t \sigma(Z(s))dW(s) + Y(t)\big)dt \\ &= x + \frac{1}{T}E\int_0^T\int_0^t \theta(Z(s))dsdt + \frac{1}{T}E\int_0^T\int_0^t \sigma(Z(s))dsdt + \frac{1}{T}E\int_0^T Y(t)dt \\ &> x + \frac{1}{T}E\int_0^T M_0 t dt + \frac{1}{T}E\int_0^T \epsilon_0 W(t)dt + \frac{1}{T}E\int_0^T Y(t)dt \\ &> x + \frac{M_0}{2}T + \frac{1}{T}E\int_0^T Y(t)dt \\ &> x + \frac{M_0}{2}T + \frac{\delta}{4}T\end{aligned}$$

Since T large enough, we conclude that the average cost function approaches infinity. In order to make the cost structure with finite cost as $T \to \infty$, we impose the constraint $\lim_{T\to\infty} \frac{E[Y(T)]}{T} = 0$. When $\limsup_{T\to\infty} \frac{E[Y(T)]}{T} \geq \delta > 0$ and $\liminf_{T\to\infty} \frac{E[Y(T)]}{T} = 0$, the solution to the control problem is not known.

Remark. *In our discussion, we only consider a general continuous control processes. However, a control may include an initial jump so that the state process reach the reflected point*

r at time $t = 0$. The diffusion jumps to some particular position b and the diffusion starts again at b. Because of the negative drift part, the process will keep dropping from time to time. And continue the same jumping procedure. In this case, we may need to keep an eye on the generalized Ito's lemma for semi-martingale.

4. Invariant distribution for $Z(t)$

Here we intend to characterize the long-run average cost of reflected diffusion processes by using their invariant distribution. Since the drift term is strictly negative, such an invariant distribution exists. We intend to apply some propositions and theorems in [3] to find the invariant distribution if exists.

Let a stochastic state process be defined on $S = (a, b)$ where a and b could be infinity. For fixed $x_0 \in S$, the scale function and speed function are defined by

$$\begin{aligned} s(x) \equiv s(x_0; x) &:= \int_{x_0}^{x} e^{-\int_{x_0}^{z} \frac{2\theta(u)}{\sigma^2(u)} du} dz \\ m(x) \equiv m(x_0; x) &:= \int_{x_0}^{x} \frac{2}{\sigma^2(z)} e^{\int_{x_0}^{z} \frac{2\theta(u)}{\sigma^2(u)} du} dz \end{aligned} \tag{4.1}$$

for $x \in S$, respectively. Let a stochastic state process $Z^*(\cdot)$ be given as in (3.3) on $S = [r^*, b)$ where r^* is reflecting boundary and b could be infinity. In [3], Proposition 10.3 shows that the diffusion is positive recurrent if and only if $s(b) = \infty$ and $m(b) < \infty$.

Proposition 4. *The reflected diffusion $Z^*(t)$ defined in (3.3) is positive recurrent.*

Proof. It suffices to show $s(b) = \infty$ and $m(b) < \infty$ as b approaches infinity. Without loss of generality, let $x_0 = 0$. We have

$$s(b) = \int_0^b e^{-\int_{x_0}^{z} \frac{2\theta(u)}{\sigma^2(u)} du} dz > \int_0^b e^{-\frac{2M_0}{M^2} z} dz.$$

Then, let $b \to \infty$, $\int_0^b e^{-\frac{2\theta_0}{\sigma_0^2} z} dz \to \infty$ since $-\frac{2M_0}{M^2} > 0$. Therefore, $s(b) = \infty$ as needed. Similarly, we consider speed function

$$m(b) = \int_0^b \frac{2}{\sigma^2(z)} e^{\int_{x_0}^{z} \frac{2\theta(u)}{\sigma^2(u)} du} dz < \int_0^b \frac{2}{\epsilon_0^2} e^{\frac{2\theta_0}{\epsilon_0^2} z} dz.$$

Since $\frac{2\theta_0}{\epsilon_0^2} < 0$, the integral is calculable and finite as b approaches infinity. Therefore, $m(b) < \infty$ as needed. By the Proposition 10.3 in [3], we know the reflected diffusion $Z^*(\cdot)$ is positive recurrent. □

In [3], Theorem 12.2 gives us the existence and uniqueness of the invariant distribution of $Z^*(\cdot)$ and also an approach to compute the invariant distribution.

Theorem 12.2. Suppose the diffusion is positive recurrent on $S = (a, b)$, we have the following:

a) There exists a unique invariant distribution $\pi(dx)$.

b) For every real-valued function f such that $\int_S |f(x)|\pi(dx) < \infty$, the strong law of large numbers holds, i.e. with probability 1,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(Z(s))ds = \int_S f(x)\pi(dx),$$

no matter what the initial distribution may be.

c) If the end points a, b of S are inaccessible or reflecting, then the invariant probability distribution has a density $\pi(x)$, which is the unique normalized integrable solution of $\mathcal{G}^*\pi(x) = 0$, i.e.

$$\frac{1}{2}\frac{d^2}{dx^2}(\sigma^2(x)\pi(x)) - \theta(x)\pi(x) = 0.$$

Indeed, the invariant measure is the normalized speed measure,

$$\pi(x) = \frac{m'(x)}{m(b) - m(a)}.$$

Since we have proved that the reflected diffusion $Z^*(\cdot)$ is positive recurrent, we are able to compute the invariant distribution by the third conclusion above. We obtain

$$\pi(x) = \frac{m'(x)}{m(b) - m(a)} = \frac{\frac{2}{\sigma^2(x)} e^{\int_0^x \frac{2\theta(u)}{\sigma^2(u)}du}}{\int_0^\infty \frac{2}{\sigma^2(z)} e^{\int_0^z \frac{2\theta(u)}{\sigma^2(u)}du} dz - \int_0^{r^*} \frac{2}{\sigma^2(z)} e^{\int_0^z \frac{2\theta(u)}{\sigma^2(u)}du} dz}$$
$$= \frac{\frac{2}{\sigma^2(x)} e^{\int_0^x \frac{2\theta(u)}{\sigma^2(u)}du}}{\int_{r^*}^\infty \frac{2}{\sigma^2(z)} e^{\int_0^z \frac{2\theta(u)}{\sigma^2(u)}du} dz}.$$

By the second conclusion, in our case the strong law of large numbers implies

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t h(Z(s))ds &= \int_S h(x)\pi(dx)\\ &= \frac{\int_{r^*}^{\infty} h(x)\frac{2}{\sigma^2(x)}e^{\int_0^x \frac{2\theta(u)}{\sigma^2(u)}du}dx}{\int_{r^*}^{\infty}\frac{2}{\sigma^2(z)}e^{\int_0^z \frac{2\theta(u)}{\sigma^2(u)}du}dz},\end{aligned}$$

which is equal to λ^* in Theorem 3. We also observe that the left-hand side of the equation actually represents the long-run average cost functional (2.4). Consequently, the optimal control process we got in our method agrees with the invariant distribution method.

APPENDIX A. Proof of Proposition 1

Our aim in this section is to provide a proof for Proposition 1. Consider the reflected diffusion process $Z(t) = x + \int_0^t \theta(Z(s))ds + \int_0^t \sigma(Z(s))dW(s) + Y(t)$. If we apply Itô's lemma to $Q(Z(t))$ and taking expectation to cancel out the mean zero martingale term, we end up with the generator term (3.1) and the control term, i.e.

$$E[Q(Z(T))] = Q(x) + E\int_0^T \mathcal{G}Q(t)dt + E\int_0^T Q'(Z(t))dY(t). \tag{A.1}$$

If we have $\mathcal{G}Q \geq \lambda - h(x)$ for some $\lambda \geq 0$ and $Q'(x) \geq 0$ and some other assumptions, we can find the lower bound of average cost function, more specifically, a lower bound for the value function. Therefore, the optimal problem can be reduced to find $Q(x)$ and $\lambda \geq 0$ such that

a) $Q'(x)$ is bounded and $Q'(x) \geq 0$;

b) $Q \in C^2(\mathbf{R})$ and $\mathcal{G}Q + h(x) \geq \lambda$, where the generator $\mathcal{G}$ is given by in (3.1).

Obviously, we can consider the reduced problem as proving the existence of a solution to the HJB equation and vice versa.

Proposition 1. *Given $\lambda > 0$, the solution to the ODE*

$$\begin{cases} \mathcal{G}Q + h(x) = \lambda \\ Q'(x) \geq 0 \end{cases} \tag{A.2}$$

is given by

$$Q'(x) = e^{-\int_0^x \frac{2\theta(u)}{\sigma^2(u)}du}\Big[-\int_x^\infty \frac{2}{\sigma^2(t)}(\lambda - h(t))e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}dt\Big], \tag{A.3}$$

where the generator $\mathcal{G}$ is given in (3.1).

Proof. Notice that above ODE is essentially a first order ODE in terms of $Q'(x)$. Simplifying this the equation gives

$$Q''(x) + \frac{2\theta(x)}{\sigma^2(x)}Q'(x) = \frac{2}{\sigma^2(x)}(\lambda - h(x)).$$

Multiply both sides by integrating factor and take integral from 0 to x. Keep in mind that in this step, we take integral from 0 to x. Then, we obtain

$$\begin{aligned}
&\int_0^x d(e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}Q'(t)) = \int_0^x \frac{2}{\sigma^2(t)}(\lambda - h(t))e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}dt \\
\Rightarrow\ & e^{\int_0^x \frac{2\theta(u)}{\sigma^2(u)}du}Q'(x) = Q'(0) + \int_0^x \frac{2}{\sigma^2(t)}(\lambda - h(t))e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}dt \\
\Rightarrow\ & Q'(x) = e^{-\int_0^x \frac{2\theta(u)}{\sigma^2(u)}du}[Q'(0) + \int_0^x \frac{2}{\sigma^2(t)}(\lambda - h(t))e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}dt].
\end{aligned}$$

We require Q' to be bounded. When x approaches infinity, we know the integrand of the exponential term is bounded and therefore the exponential term will approach infinity. In order to cancel out the increasing part of exponential term, we need $Q'(0)$ given by

$$Q'(0) = -\int_0^\infty \frac{2}{\sigma^2(t)}(\lambda - h(t))e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}dt.$$

Consequently, we solved the first order ODE for $Q'(x)$, given by (A.3).

We denote $F(x) = -\int_x^\infty \frac{2}{\sigma^2(t)}(\lambda - h(t))e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}dt$. Then, the solution (A.3) to the ODE problem (A.2) can be written as

$$Q'(x) = e^{-\int_0^x \frac{2\theta(u)}{\sigma^2(u)}du}F(x). \tag{A.4}$$

□

We intend to find a $\lambda^* > 0$ and corresponding Q so that $Q^{'}(r_{\lambda^*}) = 0$. For this we gather few facts. Taking derivative with respect to x, we have $F'(x) = \frac{2}{\sigma^2(x)}(\lambda - h(x))e^{\int_0^x \frac{2\theta(u)}{\sigma^2(u)}}$. In addition, we know $\lim_{x\to\infty} F(x) = 0$, since the integrand is dominated by an integrable function. To see this, we need to apply the assumptions stated in section 2: the holding cost function $0 \le h(x) \le Ae^{Bx}$ with $0 < B < -\frac{2\theta_0}{\epsilon_0}$, drift coefficient $M_0 < \theta(x) < \theta_0 < 0$ and diffusion coefficient $0 < \epsilon_0 \le \sigma(x) \le M$. Therefore, it is valid to apply Lebesgue dominated convergence theorem.

In the next three lemmas, in order to characterize the graphs of Q', we could observe that it is a good idea to compare λ and $h(x)$ for any x over different intervals. If we consider some x such that $h(x) = \lambda$, we only have two roots based on the assumption of holding cost function $h(x)$. It suffices to state that h has only one negative root which is denoted by r_λ. The corresponding positive root is denoted by $\eta(r_\lambda)$. If h is symmetric, such as $h(x) = x^2$, $\eta(r_\lambda)$ is equal to $-r_\lambda$. Consequently, we have $\lambda = h(r_\lambda) = h(\eta(r_\lambda))$. Consider the collection of (r_λ, λ) given by $\{(r_\lambda, \lambda) : \lambda > 0,\ r_\lambda < 0 \textit{ and } h(r_\lambda) = \lambda\}$. To justify the sign of $F'(x)$, it is enough to consider the sign of $\lambda - h(x)$ for which we analyze case by case.

a) If $r_\lambda < x < \eta(r_\lambda)$, we have $h(x) < \lambda$. Then, $F'(x) > 0$ which implies $F(x)$ is strictly increasing in this region.

b) If $x > \eta(r_\lambda)$, we have $h(x) > \lambda$. Then, $F'(x) < 0$ which implies $F(x)$ is strictly increasing in this region.

c) If $x < r_\lambda$, we have $h(x) > \lambda$. Then, $F'(x) < 0$ which implies $F(x)$ is strictly decreasing in this region.

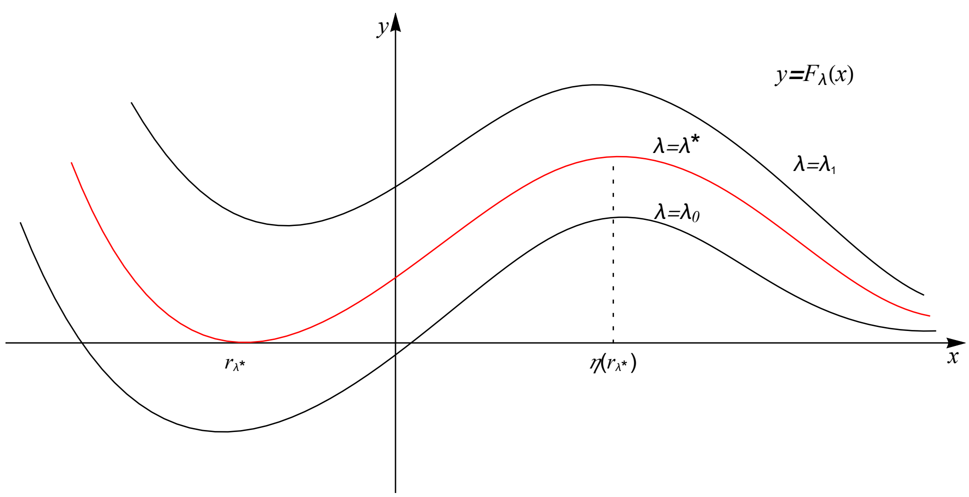

Figure A.1 Graphs of $y = F_\lambda(x)$ for different values of $\lambda > 0$

Hence, we have the graphs of $F(x)$ (see Figure A.1). Since we have $\lim_{x\to\infty} F(x) = 0$ and F is strictly decreasing in $(\eta(r_\lambda), \infty)$, we have $F(\eta(r_\lambda))$ must be positive. From now on, we denote the $F(x)$ associated with $\lambda > 0$ by $F_\lambda(x)$. In Figure A.1, we can see that all three forms are possible function F_λ associated with different $\lambda > 0$.

Since we look for the solution which also need to satisfy $Q'(x) \geq 0$ and Q' is represented in (A.4), it is sufficient and necessary to consider $F(x) \geq 0$. We consider the following lemma.

Lemma 2. $\lim\limits_{\lambda\to\infty} F_\lambda(r_\lambda) = -\infty$.

Proof. From the definition of $F(x)$, we have $F_\lambda(r_\lambda) = -\int_{r_\lambda}^{\infty} \frac{2}{\sigma^2(t)}(\lambda - h(t))e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}dt$. We can split the integral into two parts

$$\begin{aligned}
F_\lambda(r_\lambda) &= \int_{r_\lambda}^{\infty} \frac{2}{\sigma^2(t)}(h(t)-\lambda)e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}dt \\
&= \int_{r_\lambda}^{\eta(r_\lambda)} \frac{2}{\sigma^2(t)}(h(t)-\lambda)e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}dt + \int_{\eta(r_\lambda)}^{\infty} \frac{2}{\sigma^2(t)}(h(t)-\lambda)e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}dt \\
&:= I_1(\lambda) + I_2(\lambda)
\end{aligned}$$

Note that $I_2(\lambda)$ approaches 0 as λ approaches positive infinity, since r_λ approaches negative infinity as λ approaches positive infinity. If r_λ approaches negative infinity, we have $\eta(r_\lambda)$ approaches positive infinity simultaneously. Note that the integrand of $I_1(\lambda) \leq 0$, since $h(t) - \lambda$ is always negative over $(r_\lambda, \eta(r_\lambda))$. Consider the following:

$$\begin{aligned}
I_1 &= \int_{r_\lambda}^{\eta(r_\lambda)} \frac{2}{\sigma^2(t)}(h(t)-\lambda)e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}dt \\
&> \int_{r_\lambda}^{0} \frac{2}{\sigma^2(t)}(h(t)-\lambda)e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}dt \\
&> \int_{-1}^{0} \frac{2}{\sigma^2(t)}(h(t)-\lambda)e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}dt \\
&> \int_{-1}^{0} \frac{2}{M^2}e^{\frac{2M_0}{M^2}t}(h(t)-\lambda)dt.
\end{aligned}$$

The above inequalities hold since the integration over $(-1, 0)$ must be smaller than the integration over $(r_\lambda, 0)$ as r_λ approaches negative infinity. Meanwhile, we have λ approaches positive infinity. Therefore, I_1 approaches negative infinity as needed. □

Lemma 3. $\lim\limits_{\lambda\to 0} F_\lambda(r_\lambda)$ *exists and strictly positive.*

Proof. We split the integral into two parts,

$$\begin{aligned}F_\lambda(r_\lambda) &= -\int_{r_\lambda}^{\infty} \frac{2}{\sigma^2(t)}(\lambda - h(t))e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du} dt = \int_{r_\lambda}^{\infty} \frac{2}{\sigma^2(t)}(h(t) - \lambda)e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du} dt \\ &= \int_{r_\lambda}^{\eta(r_\lambda)} \frac{2}{\sigma^2(t)}(h(t) - \lambda)e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du} dt + \int_{\eta(r_\lambda)}^{\infty} \frac{2}{\sigma^2(t)}(h(t) - \lambda)e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du} dt \\ &= J_1(\lambda) + J_2(\lambda)\end{aligned}$$

If λ approaches zero, we have r approaches 0 from the left, meanwhile, $\eta(r)$ approaches 0 from the right. Therefore, the first integral term converges to 0. we have for each $\epsilon > 0$, there exists δ such that $|J_1| < \epsilon$ whenever $|x| > \delta$. Consider the second term J_2

$$\begin{aligned}J_2(\lambda) &\geq \int_{\eta(r_\lambda)}^{\infty} \frac{2}{\sigma^2(t)}(h(t) - \lambda)e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du} dt \\ &\geq \int_{1}^{\infty} \frac{2}{\sigma^2(t)}(h(t) - \lambda)e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du} dt = L\end{aligned}$$

Note that the above inequality holds since the integration over $(1, \infty)$ is less than the integration over $(\eta(r_\lambda), \infty)$. As long as we can show L is finite, we can pick $\epsilon = \frac{L}{1000}$. Therefore, $-\frac{L}{1000} < J_1 < \frac{L}{1000}$. Then we have

$$F_\lambda(r_\lambda) = J_1(\lambda) + J_2(\lambda) > L - \frac{L}{1000} > 0,$$

which means $\lim_{\lambda \to 0} F_\lambda(r_\lambda)$ is positive.

It remains to show that L is finite. Consider the integrand, we know $h(t) - \lambda$ is positive over $(\eta(r_\lambda), \infty)$. Hence, the integrand is positive over $(\eta(r_\lambda), \infty)$. Then,

$$\begin{aligned}L &= \int_{1}^{\infty} \frac{2}{\sigma^2(t)}(h(t) - \lambda)e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du} dt \\ &\leq \int_{0}^{\infty} \frac{2}{\sigma^2(t)}(h(t) - \lambda)e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du} dt \\ &< \frac{2}{{\epsilon_0}^2}\int_{0}^{\infty} (h(t) - \lambda)e^{\frac{2\theta_0}{{\epsilon_0}^2}t} dt \\ &= \frac{2}{{\epsilon_0}^2}\int_{0}^{\infty} h(t)e^{\frac{2\theta_0}{{\epsilon_0}^2}t} dt - \frac{2\lambda}{{\epsilon_0}^2}\int_{0}^{\infty} e^{\frac{2\theta_0}{{\epsilon_0}^2}t} dt = J_3 + J_4\end{aligned}$$

Consider J_4 first, we can directly calculate the integral since $\theta_0 < 0$. It turns out to be $-\frac{{\epsilon_0}^2}{2\theta_0}$. For J_3, we have

$$\begin{aligned}J_3 &= \frac{2}{{\epsilon_0}^2}\int_{0}^{\infty} h(t)e^{\frac{2\theta_0}{{\epsilon_0}^2}t} dt \\ &\leq \frac{2A}{{\epsilon_0}^2}\int_{0}^{\infty} e^{(\frac{2\theta_0}{{\epsilon_0}^2} + B)t} dt\end{aligned}$$

Since we have $0 < B < -\frac{2\theta_0}{{\epsilon_0}^2}$ in our assumptions, we have $B + \frac{2\theta_0}{{\epsilon_0}^2} < 0$. Then, we have $I_3 \leq -\frac{2}{{\epsilon_0}^2}\frac{A}{B+\frac{2\theta_0}{{\epsilon_0}^2}}$. Therefore, combine those together

$$\begin{aligned} L &< J_3 + J_4 \\ &\leq \frac{2}{{\epsilon_0}^2}[-\frac{A}{B+\frac{2\theta_0}{{\epsilon_0}^2}} + \frac{\lambda{\epsilon_0}^2}{2\theta_0}] = \frac{2}{{\epsilon_0}^2}\frac{aA + \lambda(B-a)}{a(a-B)} \end{aligned}$$

where $a = -\frac{2\theta_0}{{\epsilon_0}^2}$.

Observe that even though L is bounded above by some constant with respect to λ, L still approaches to a constant as λ goes to 0 as anticipated. □

From above discussion, we know that some large enough $\lambda > 0$ implies $F_\lambda(r_\lambda) < 0$ and some small $\lambda > 0$ implies $F_\lambda(r_\lambda) > 0$. Let $\lambda_1 > \lambda_2 > 0$, we have $r_{\lambda_1} < r_{\lambda_2} < 0$ since the holding cost function is decreasing over $(-\infty, 0)$. Meanwhile, we could derive the following monotonicity property.

Lemma 4 (Monotonicity property)**.** *Consider function $F_\lambda(x)$. Then for any x, we have $F_{\lambda_1}(x) < F_{\lambda_2}(x)$ whenever $\lambda_1 > \lambda_2 > 0$.*

Proof. Since $\lambda_1 > \lambda_2 > 0$, we have $h(x) - \lambda_1 < h(x) - \lambda_2$ for all $x \in \mathbb{R}$. By definition of $F_\lambda(x) = -\int_x^\infty \frac{2}{\sigma^2(t)}(\lambda - h(t))e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}dt$, we obtain the monotonocity property given by $F_{\lambda_1} < F_{\lambda_2}$ whenever $\lambda_1 > \lambda_2 > 0$. □

Monotonicity property implies that the graphs of F_λ do not intersect with each other as λ varies.

Define $\lambda^* = \sup\{\lambda > 0 : F_\lambda(r_\lambda) > 0\}$ where r_λ is the local minimum of $F_\lambda(x)$. Then, there exists a corresponding $r_{\lambda^*} < 0$ denoted by r^* such that $h(r^*) = \lambda^*$. Notice that $r_\lambda < 0$ is continuous as a function of λ, since the inverse function theorem of $h(r_\lambda) = \lambda$ by the definition we made before. It is trivial to show $F'_{\lambda^*}(r^*) = 0$. As long as we can show $F_{\lambda^*}(r^*) = 0$, we are able to derive the exact formula for λ^* and the corresponding F_{λ^*} and so does Q^*.

Lemma 5. *$F_\lambda(r_\lambda) = \int_{r_\lambda}^{\infty} \frac{2}{\sigma^2(t)} e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}(h(t)-\lambda)dt$ is continuous function with respect to r_λ. In particular, with respect to λ.*

Proof. The proof of this consists of two parts. Note that the pair (r_λ, λ) are related by the function $h(x)$. By the definition and the inverse function theorem, we know it is injective. So, r_λ is continuous in λ trivially. Now we are left to show $F_\lambda(r_\lambda)$ is continuous wit respect to λ. The r_λ can be considered as the composition of two functions.

Let $\lambda_n \to \lambda$ as $n \to \infty$. In this case, the function is given by $F_{\lambda_n}(r_{\lambda_n}) = \int_{r_{\lambda_n}}^{\infty} \frac{2}{\sigma^2(t)} e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du}(h(t)-\lambda_n)dt$. To interchange the integration and limit, it is enough to should LDCT is valid in this case. Consider the integrand together with the indicator function $\mathcal{X}_{[r_{\lambda_n},\infty)}$

$$\begin{aligned} |\frac{2}{\sigma^2(t)} e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)}du} h(t)\mathcal{X}_{[r_{\lambda_n},\infty)}| &\leq \frac{2}{{\epsilon_0}^2} e^{\frac{2\theta_0}{{\epsilon_0}^2}t}|h(t)| \\ &\leq \frac{2A}{{\epsilon_0}^2} e^{(\frac{2\theta_0}{{\epsilon_0}^2}+B)t} \end{aligned}$$

In addition, $\frac{2A}{{\epsilon_0}^2} e^{(\frac{2\theta_0}{{\epsilon_0}^2}+B)t}$ is integrable on $[M,\infty)$ for some fixed $M < 0$. Similarly, the λ_n term can be verified directly by computation. Therefore, $F_{\lambda_n}(r_{\lambda_n}) \to F_\lambda(r_\lambda)$ as $n \to \infty$ which implies continuity. $\square$

We saw in Lemma 2 and Lemma 3 that different choices of λ gives us either positive or negative value of $F_\lambda(r_\lambda)$. In addition, we have shown that $F_\lambda(r_\lambda)$ is continuous with respect to λ in Lemma 5. The intermediate value theorem implies there exists λ' and the corresponding $r_{\lambda'}$ such that $F_{\lambda'}(r_{\lambda'}) = 0$. Note that since they never intersect for different values of λ, we have the uniqueness for such a λ'. Continuity of $F_\lambda(r_\lambda)$ with respect to r_λ, in particular, with respect to λ implies $\lambda' \in \{\lambda > 0 : F_\lambda(r_\lambda) \geq 0\}$. Since $\lambda' \leq \sup\{\lambda > 0 : F_\lambda(r_\lambda) \geq 0\} = \sup\{\lambda > 0 : F_\lambda(r_\lambda) > 0\} = \lambda^*$ and together with monotonicity property, we have $F_{\lambda'} \geq F_{\lambda^*}$. Therefore, we have $F_{\lambda^*} \leq 0$. Note that we have $F_{\lambda^*} \geq 0$. Then, $F_{\lambda^*}(r_{\lambda^*}) = 0$. That is to say $r' \equiv r^*$.

Alternative proof: Since we need to show $F_{\lambda^*}(r_{\lambda^*}) = 0$, we can prove this by contradiction. Notice that $F_{\lambda^*}(r_{\lambda^*}) \geq 0$ by definition. Suppose we have $F_{\lambda^*}(r_{\lambda^*}) > 0$. Assume

$g(\lambda) = F_\lambda(r_\lambda)$ which means we consider function $F_\lambda(r_\lambda)$ as a function of λ. From previous lemma 5, we know $g(\lambda)$ is continuous in λ. If $g(\lambda^*) > 0$, there exists a $\delta > 0$ such that $g(\lambda) > 0$ for $\lambda^* - \delta < \lambda < \lambda^* - \delta$ which implies $\lambda \in \{\lambda > 0 : F_\lambda(r_\lambda) > 0\}$. When we have $\lambda^* \leq \lambda < \lambda^* + \delta$ and $\lambda \in \{\lambda > 0 : F_\lambda(r_\lambda) > 0\}$ at the same time, we derive a contradiction since λ^* is the least upper bound of the set $\{\lambda > 0 : F_\lambda(r_\lambda) > 0\}$. Therefore, $F_{\lambda^*}(r_{\lambda^*}) > 0$ cannot be true. Hence, we have $F_{\lambda^*}(r_{\lambda^*}) = 0$. Consequently, from (A.4) we obtain $Q'(r_{\lambda^*}) = 0$ as needed.

It is easy to derive the exact formula for λ^* from $F_{\lambda^*}(r_{\lambda^*}) = 0$. And the corresponding r_{λ^*} is given by $\lambda^* = h(r_{\lambda^*})$.

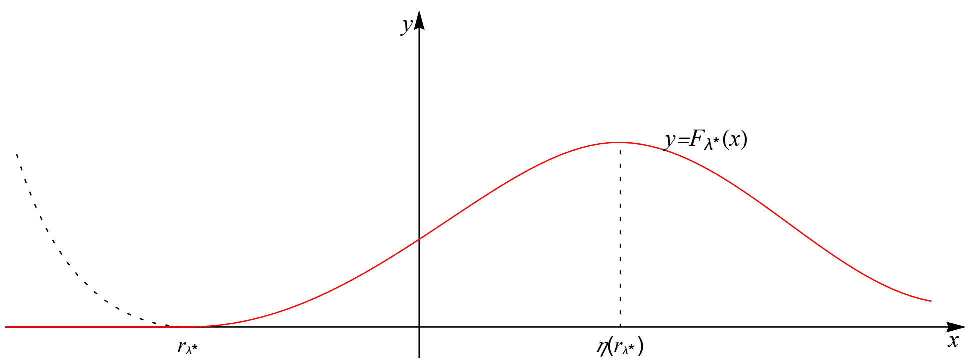

Figure A.2 Graph of $y = F_{\lambda^*}(x)$ for $\lambda^* > 0$

Proof of Proposition 1. See Figure A.2, the red line represents F_{λ^*} and we define ${Q^*}' : \mathbb{R} \to \mathbb{R}$ by

$$
{Q^*}'(x) = \begin{cases} 0, & if\ x \leq r_{\lambda^*} \\ e^{-\int_0^x \frac{2\theta(u)}{\sigma^2(u)} du} F_{\lambda^*}(x), & if\ x > r_{\lambda^*}, \end{cases}
$$

where $F_{\lambda^*}(x) = -\int_x^\infty \frac{2}{\sigma^2(t)}(\lambda^* - h(t)) e^{\int_0^t \frac{2\theta(u)}{\sigma^2(u)} du} dt$. Let $Q^*(x) = \int_{r_{\lambda^*}}^x {Q^*}'(u) du$. Obviously, we have $Q^* \in C^2(\mathbb{R})$ by the construction above. Now we show ${Q^*}'$ is bounded. Based on the definition of ${Q^*}'$, it is trivial to show the boundedness for $x \leq r_{\lambda^*}$. Consider $x > r_{\lambda^*}$, we have $\lim_{x\to\infty} F_{\lambda^*}(x) = 0$ and $\lim_{x\to\infty} e^{\int_0^x \frac{2\theta(u)}{\sigma^2(u)} du} = 0$. To compute the limit of ${Q^*}'(x)$ as x

approaches infinity, we need to apply L'Hôspital's rule as the following:

$$\begin{aligned}\lim_{x\to\infty} Q^{*'}(x) &= \lim_{x\to\infty} \frac{F_{\lambda^*}(x)}{e^{\int_0^x \frac{2\theta(u)}{\sigma^2(u)}du}} = \lim_{x\to\infty} \frac{F'_{\lambda^*}(x)}{\frac{2\theta(x)}{\sigma^2(x)} e^{\int_0^x \frac{2\theta(u)}{\sigma^2(u)}du}} \\ &= \lim_{x\to\infty} \frac{\frac{2}{\sigma^2(x)}(\lambda^* - h(x))}{\frac{2\theta(x)}{\sigma^2(x)}} = \lim_{x\to\infty} \frac{\lambda^* - h(x)}{\theta(x)} \\ &\le \lim_{x\to\infty} \frac{\lambda^*}{\theta(x)} < \frac{\lambda^*}{M_0} < \infty.\end{aligned}$$

We also have the continuity of $Q^{*'}(\cdot)$ with respect to x. Therefore, we obtain the boundedness property for $Q^{*'}$ as needed.

We are left to show Q^* defined above satisfies the HJB equation (3.2). It is trivial to show that the equation (3.2) is satisfied on (r_{λ^*}, ∞) by construction stated above. Consider the HJB equation (3.2) at $x = r_{\lambda^*}$. As shown above, we obtain $Q^{*'}(r_{\lambda^*}) = 0$. In addition, in this case we obtain

$$\mathcal{G}Q + h(x) - \lambda^* = \frac{1}{2}\sigma^2(r_{\lambda^*})Q^{*''}(r_{\lambda^*}) + \theta(x)Q^{*'}(r_{\lambda^*}) + h(r_{\lambda^*}) - \lambda^* = 0,$$

which implies the HJB equation holds. Consider the interval $(-\infty, r_{\lambda^*})$ on which we have $Q^{*'}(x) = 0$. In this case, we are able to simplify the HJB equation. We have

$$\min\{\frac{1}{2}\sigma^2(x)Q^{*''}(x) + \theta(x)Q^{*'}(x) + h(x) - \lambda^*,\ Q^{*'}(x)\} = \min\{h(x) - \lambda^*,\ 0\} = 0,$$

since $h(x) > \lambda^*$ for any $x \in (-\infty, r_{\lambda^*})$. Consequently, $Q^{*'}(\cdot)$ defined above satisfies the HJB equation (3.2) as anticipated. □

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