

3-15-2021

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We determine the inducibility of all tournaments with at most 4 vertices together with the extremal constructions. The 4-vertex tournament containing an oriented  $C_3$  and one source vertex has a particularly interesting extremal construction. It is an unbalanced blow-up of an edge, where the sink vertex is replaced by a quasi-random tournament and the source vertex is iteratively replaced by a copy of the construction itself.

### Disciplines

Discrete Mathematics and Combinatorics | Mathematics

### Comments

This preprint is made available through arXiv: <https://arxiv.org/abs/2103.07047>.

# Inducibility of 4-vertex tournaments

Dalton Burke\*    Bernard Lidický†    Florian Pfender‡    Michael Phillips§

March 15, 2021

## Abstract

We determine the inducibility of all tournaments with at most 4 vertices together with the extremal constructions. The 4-vertex tournament containing an oriented  $C_3$  and one source vertex has a particularly interesting extremal construction. It is an unbalanced blow-up of an edge, where the sink vertex is replaced by a quasi-random tournament and the source vertex is iteratively replaced by a copy of the construction itself.

## 1 Introduction

One of the central questions in extremal graph theory is to maximize the number of induced copies of a given graph  $H$  in a larger host graph on a fixed number of vertices. For a graph  $G$ , we denote the number of vertices by  $|H|$ . For this, let  $I(H, G)$  be the number of vertex subsets of  $G$  which induce a graph isomorphic to  $H$ , and let

$$I(H, n) = \max_{|G|=n} I(H, G).$$

For  $k = |H|$ , we normalize the answer and write  $i(H, n) \binom{n}{k} = I(H, n)$ . Clearly,  $0 \leq i(H, n) \leq 1$ , so we can think of  $i(H, n)$  as a subgraph density. An easy averaging argument shows that  $i(H, n)$  is monotone non-increasing and thus converging for  $n \rightarrow \infty$ . Pippenger and Golumbic [21] define the *inducibility of  $H$*  as

$$i(H) = \lim_{n \rightarrow \infty} i(H, n).$$

Determining inducibilities is notoriously difficult, and the answer is known only for very few explicit graphs  $H$ . A major breakthrough for the problem was the introduction of the flag algebra method by Razborov [22] in 2007, and since then the inducibility of a good number of small graphs has been determined with the help of this method [1, 2, 7]. Nevertheless, we do not even

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know  $i(P_4)$ , i.e. the inducibility of the path on four vertices, and we do not even have a conjecture for the answer.

On the other end of the spectrum, Fox, Huang, and Lee [9], and independently Yuster [24], have determined exact values for  $i(H, n)$  and thus  $i(H)$  for all  $n$  and almost all large enough graphs  $H$  by studying random graphs. There are numerous other results on inducibility [3, 6, 10–12, 16–19].

All of these questions can be studied for directed graphs as well. Falgas-Ravry and Vaughan [8] studied inducibility of small outstars. Huang [15] extended the result to all outstars. This was further generalized to other stars by Hu, Ma, Norin, and Wu [14]. Short paths with further restrictions were considered in [4] and orientation of a 4-cycle in [13]. In an REU in 2018, Burgher and Burke studied and conjectured extremal constructions for most oriented graphs (directed graphs without 2-cycles) of up to 4 vertices using the flag algebra method. In a similar and independent project, Bożyk, Grzesik and Kielak [23] established more bounds and found more constructions for oriented graphs.

In this paper, we look closer at the tournaments in this list, directed graphs with exactly one arc between any pair of vertices. The number of non-isomorphic tournaments on  $k$  vertices is slightly smaller than the number of graphs, and flag algebra computations tend to have similar power. The two projects mentioned in the previous paragraph both found inducibility bounds and closely matching constructions for all tournaments on up to 4 vertices, where the results are easy or trivial for all but three of these 8 small tournaments. These last three tournaments on 4 vertices have very interesting constructions, and in this paper we prove that these constructions are indeed optimal.

When edges are colored, Mubayi and Razborov [20] showed that for every tournament  $T$  on  $k \geq 4$  vertices whose edges are colored by  $\binom{k}{2}$  distinct colors, a structure on  $n \geq k$  vertices that maximizes the number of copies of  $T$  is a balanced iterated blow-up of  $T$ . This implies that  $i(T) = \frac{k!}{k^k - k}$  in this rainbow setting.

## 2 Results

Here we discuss tournaments on at most four vertices. For the tournaments  $T_1$  and  $T_2$  on one and two vertices, respectively, any tournament  $T$  has  $i(T_k, T) = 1$ . Similarly, for all transitive tournaments  $TT_k$  on  $k \geq 3$  vertices, the transitive tournament  $TT_n$  on  $n \geq k$  vertices is the unique tournament on  $n$  vertices with  $i(TT_k, T) = 1$ . On the other hand,  $i(TT_3, T)$  is minimized exactly if  $T$  has all out-degrees in  $\{\frac{n-2}{2}, \frac{n-1}{2}, \frac{n}{2}\}$ . This easily follows from counting  $TT_3$  by first choosing the source vertex, and then any two out-neighbors. As a consequence, we have for the only other tournament on three vertices  $C_3$ :

**Proposition 1.** *The number of induced copies of  $C_3$  is maximized if and only if every vertex of a tournament has out-degree in  $\{\frac{n-2}{2}, \frac{n-1}{2}, \frac{n}{2}\}$ .*

This implies  $i(C_3) = 1/4$  and leaves us with three 4-vertex tournaments to consider, see Figure 1: the tournaments we get from  $C_3$  by adding a source vertex ( $C_3^+$ ), a sink vertex ( $C_3^-$ ), and by adding a vertex of out-degree 1 or 2 ( $C_4$ ).

We will now define the class  $\mathcal{C}_n$  of *carousels* on  $n \geq 3$  vertices. A tournament  $T$  is in  $\mathcal{C}_n$  if its vertices can be labeled  $\{v_1, v_2, \dots, v_n\}$  such that  $v_i v_j \in E(T)$  if  $0 < j - i < \frac{n}{2}$  or if  $-n < j - i < -\frac{n}{2}$ .



Figure 1: The four 4-vertex tournaments.

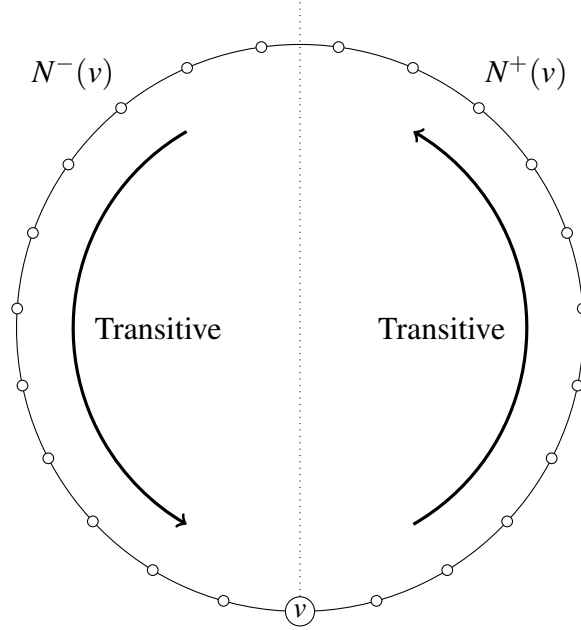


Figure 2: For odd  $n$ , the carousel  $C \in \mathcal{C}_n$  is unique and vertex transitive. In the even  $n$  case, an additional vertex  $v'$  would be placed opposite  $v$  on the circle with the arc containing  $v$  and  $v'$  chosen arbitrarily.

Said another way, a tournament  $T$  is in  $\mathcal{C}_n$  if for every  $x \in V(T)$ , the in- and out-neighborhoods induce transitive tournaments ( $T$  is *locally transitive*) and are as balanced as possible ( $T$  is *balanced* when  $|V(T)|$  is odd, or *nearly balanced* when  $|V(T)|$  is even). See Figure 2 for an illustration.

Observe that for odd  $n$  and for  $n = 4$ ,  $\mathcal{C}_n$  contains exactly one tournament, and we will call this unique carousel  $C_n$ . For even  $n \geq 6$ ,  $\mathcal{C}_n$  contains more than one tournament, depending on the directions of the arcs  $v_i v_{i+\frac{n}{2}}$ . For even  $n$ , we denote by  $C_n \in \mathcal{C}_n$  the unique tournament we get from deleting one vertex in  $C_{n+1}$ . Note that one can alternatively construct  $C_n$  from  $C_{n-1}$  by duplicating one vertex.

**Theorem 2.** For  $n \geq 4$ , the tournaments maximizing  $I(C_4, T)$  are precisely the tournaments in  $\mathcal{C}_n$ . Consequently,  $i(C_4) = \frac{1}{2}$ , and for every  $n$ , we have

$$I(C_4, n) = \begin{cases} \frac{n(n^2-1)(n-3)}{48} & \text{if } n \text{ is odd,} \\ \frac{n(n^2-4)(n-3)}{48} & \text{if } n \text{ is even.} \end{cases}$$

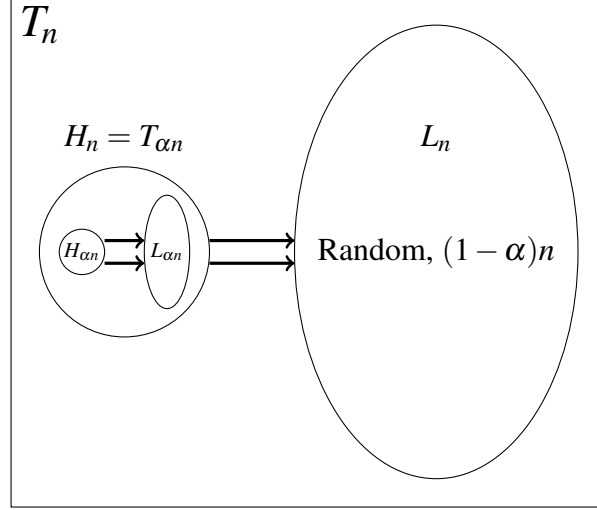


Figure 3: Construction maximizing the number of copies of  $C_3^+$ . For  $\alpha \in [0, 1]$  and  $n$  sufficiently large, the extremal construction  $T_n$  can be decomposed into subtournaments  $L_n$ , of size about  $(1 - \alpha)n$ , and  $H_n$ , of size about  $\alpha n$  with the properties shown above.

Note that the asymptotic statement that  $i(C_4) = \frac{1}{2}$  is also proved in [23], with a proof very similar to the one we provide in the next section. Flag algebra computations indicate that a similar statement is also true for  $C_5$ ,  $C_6$ ,  $C_7$  and  $C_8$ , and we conjecture it is true for all  $k$ . Observe that for  $k \geq 5$  and even  $n \geq k$ ,  $C_n$  contains more copies of  $C_k$  than the other members of  $\mathcal{C}_n$ , so our conjectured class of extremal tournaments is a bit smaller here.

**Conjecture 3.** For all  $k \geq 5$  and  $n \geq k$ , the unique  $n$ -vertex tournaments maximizing  $I(C_k, T)$  are the tournaments  $C_n$ .

The only tournaments on 4 vertices left are the two tournaments  $C_3^-$  and  $C_3^+$ . As one gets  $C_3^-$  from  $C_3^+$  by reversal of all arcs, the tournaments extremal for  $C_3^-$  are precisely the reversals of the tournaments extremal for  $C_3^+$ , so it suffices to only study  $C_3^+$ . Consider the following construction of a tournament  $T_n$  on  $n$  vertices. For some fixed  $\alpha \in (0, 1)$ , partition the vertices into two sets  $H_n$  (for high out-degree) and  $L_n$  (for low out-degree) of size  $\lceil \alpha n \rceil$  and  $\lfloor (1 - \alpha)n \rfloor$ , respectively. On the set  $L_n$ , direct the edges uniformly at random, i.e. insert a random tournament  $R_{\lfloor (1 - \alpha)n \rfloor}$ . All arcs between the sets are directed from  $H_n$  to  $L_n$ . On the set  $H_n$ , iterate the construction, i.e. insert the tournament  $T_{\lceil \alpha n \rceil}$  inductively. See Figure 3 for a sketch of the iterated construction.

In this construction, all copies of  $C_3^+$  lie completely in  $H_n$ , completely in  $L_n$ , or have exactly one vertex in  $H_n$  and three vertices forming a  $C_3$  in  $L_n$ . Notice that  $i(C_3, R_{\lfloor (1 - \alpha)n \rfloor}) = 1/4 + o(1)$  and  $i(C_3^+, R_{\lfloor (1 - \alpha)n \rfloor}) = 1/8 + o(1)$ . As  $i(C_3^+, T_{\lceil \alpha n \rceil}) = i(C_3^+, T_n) + o(1)$ , we have

$$\begin{aligned} i(C_3^+, T_n) &= \alpha^4 i(C_3^+, T_n) + 4\alpha(1 - \alpha)^3 i(C_3, R_{\lfloor (1 - \alpha)n \rfloor}) + (1 - \alpha)^4 i(C_3^+, R_{\lfloor (1 - \alpha)n \rfloor}) + o(1) \\ &= \alpha^4 i(C_3^+, T_n) + 4\alpha(1 - \alpha)^3 \frac{1}{4} + (1 - \alpha)^4 \frac{1}{8} + o(1), \end{aligned}$$

so

$$i(C_3^+, T_n) = \frac{\alpha(1 - \alpha)^3 + \frac{1}{8}(1 - \alpha)^4}{1 - \alpha^4} + o(1).$$

Maximizing this quantity gives us  $\alpha = \frac{1}{5}(2\sqrt[3]{9} - 2 - \sqrt[3]{3}) \approx 0.1435836$ , and

$$i(C_3^+, T_n) = \frac{1}{8} \left( 8 - 9\sqrt[3]{3} + 3\sqrt[3]{9} \right) + o(1) \approx 0.1575006670 + o(1).$$

We show that all large extremal tournaments for  $C_3^+$  essentially look this way, and that the limit object is unique.

**Theorem 4.** *Let  $(T_n)_{n=1}^\infty$  be a sequence of tournaments on  $n$  vertices with  $I(C_3^+, T_n) = I(C_3^+, n)$ . Let  $\alpha = \frac{1}{5}(2\sqrt[3]{9} - 2 - \sqrt[3]{3})$ . For sufficiently large  $n$ , the vertex set of  $T_n$  can be partitioned into sets  $L_n$  and  $H_n$  so that  $|H_n| = \alpha n + o(1)$ , all arcs between these sets are from  $H_n$  to  $L_n$ , the sequence of tournaments  $(T_n[L_n])_{n=1}^\infty$  is quasi-random, and  $I(C_3^+, T_n[H_n]) = I(C_3^+, |H_n|)$ . Hence*

$$i(C_3^+) = \frac{1}{8} \left( 8 - 9\sqrt[3]{3} + 3\sqrt[3]{9} \right) \approx 0.1575006670.$$

This construction is also mentioned in [23], together with an almost matching upper bound obtained by the flag algebra method. While  $C_3^+$  may not be the most interesting tournament to consider, we find this extremal construction combining quasi-random parts with iterated blow-ups fascinating.

### 3 Proof of Theorem 2

*Proof of Theorem 2.* We begin by observing the following identity for all tournaments  $T$  on at least 4 vertices:

$$i(C_3, T) = \frac{1}{2}i(C_4, T) + \frac{1}{4}i(C_3^+, T) + \frac{1}{4}i(C_3^-, T). \quad (1)$$

This follows from the fact that the probability to find a  $C_3$  when picking three vertices at random is equal to the probability to first find  $C_4$ ,  $C_3^+$ , or  $C_3^-$  when picking four vertices, times the appropriate probability that removing one of these vertices leaves a  $C_3$ .

Multiplying both sides by  $\binom{n}{4}$ , we can express this relationship in terms of a direct count of induced  $C_4$  for any tournament  $T$ :

$$I(C_3, T) \cdot \frac{n-3}{4} = \frac{1}{2}I(C_4, T) + \frac{1}{4}(I(C_3^+, T) + I(C_3^-, T)),$$

implying that

$$I(C_4, T) = I(C_3, T) \cdot \frac{n-3}{2} - \frac{1}{2}(I(C_3^+, T) + I(C_3^-, T)).$$

Let  $T \in \mathcal{C}_n$ . Then every vertex in  $T$  has out-degree in  $\{\frac{n-2}{2}, \frac{n-1}{2}, \frac{n}{2}\}$ , so by Proposition 1,  $I(C_3, T)$  is maximized. On the other hand, the out-neighborhoods and in-neighborhoods of all vertices in  $T$  induce transitive tournaments, so  $I(C_3^+, T) = I(C_3^-, T) = 0$ . This shows that  $T$  maximizes  $I(C_4, T)$ .

It remains to show that no other tournament shares this property. For this, let  $T$  be any  $\{C_3^+, C_3^-\}$ -free, (near) regular tournament, and let  $v_1 \in V(T)$  with  $d^+(v_1) = k \in \{\frac{n-2}{2}, \frac{n-1}{2}, \frac{n}{2}\}$ . As  $T$  is  $C_3^+$ -free, the out-neighborhood of  $v_1$  is  $C_3$ -free and therefore transitive, and we may relabel the out-neighbors in this induced order as  $\{v_2, v_3, \dots, v_{k+1}\}$ . Similarly, the in-neighborhood is transitive, and we may relabel it in the induced order as  $\{v_{k+2}, \dots, v_n\}$ .

Now suppose, for the sake of contradiction, that  $T \notin \mathcal{C}_n$ , and thus there exists an arc  $v_i v_j$  with  $0 < i - j < \frac{n}{2}$  or if  $-n < i - j < -\frac{n}{2}$ . Let us first assume that  $0 < i - j < \frac{n}{2}$ . As  $\{v_2, v_3, \dots, v_{k+1}\}$  and









*Proof.* Let  $x \in L_n, y \in H_n$ , so  $x$  has normalized out degree in  $[0.415, 0.441]$  and  $y$  has normalized out-degree at least 0.8849. We color  $V(T_n) - \{x, y\}$  with the following scheme, in which the top color represents the relation to  $x$ , and the bottom color represents the relation to  $y$  (see also Figure 4):

- Assign color black-black  $\bullet$  to  $N^+(x) \cap N^+(y)$ ,
- Assign color black-white  $\ominus$  to  $N^+(x) \cap N^-(y)$ ,
- Assign color white-black  $\omin�$  to  $N^-(x) \cap N^+(y)$ ,
- Assign color white-white  $\circ$  to  $N^-(x) \cap N^-(y)$ .

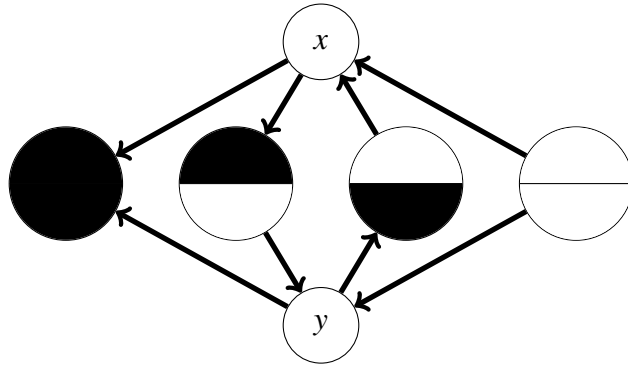


Figure 4: Four-coloring scheme for  $T_n - \{x, y\}$

In order to model the out-degree assumptions, we will use the following constraints:

$$0.416 \leq \bullet + \ominus \leq 0.44057 \quad \text{and} \quad 0.8849 \leq \bullet + \omin�.$$

As in the proof of Claim 4.3, any programs involving this color scheme can include a constraint to ensure that  $x$  is in the right number of  $C_3^+$  with vertices in  $V(T_n) \setminus y$ , and that  $y$  is in the right number of  $C_3^+$  with vertices in  $V(T_n) \setminus x$ .

The purpose of this set up is to show that  $x \rightarrow y$  results in fewer  $C_3^+$  than  $y \rightarrow x$ , so we need to determine how to construct  $C_3^+$  which include both of these vertices. For this, we look again at Figure 4. If  $x \rightarrow y$ , we create a  $C_3^+$  with each arc  $\omin� \rightarrow \omin�$  and with each arc  $\bullet \rightarrow \omin�$ . On the other hand, if  $y \rightarrow x$ , we create a  $C_3^+$  with each arc  $\omin� \rightarrow \bullet$  and with each arc  $\bullet \rightarrow \omin�$ .

Similarly as above, we can now pose the following program bounding the difference between  $C_3^+$  containing  $x \rightarrow y$  and containing  $y \rightarrow x$ . Note that there are up to 96 different  $C_3^+$  in  $T_n - \{x, y\}$  with 4 colors. Also, when counting the  $C_3^+$  in  $T_n - y$  containing  $x$ , we have to account for the colors induced by the arcs with  $y$ .

Objective: maximize

$$\left( \begin{array}{c} \omin� \\ \uparrow \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \uparrow \\ \bullet \end{array} \right) - \left( \begin{array}{c} \bullet \\ \uparrow \\ \omin� \end{array} + \begin{array}{c} \bullet \\ \uparrow \\ \bullet \end{array} \right).$$





with out-degree  $\frac{n}{2} + o(n)$ . Now observe that

$$\begin{aligned}
\frac{1}{8} \binom{n}{4} + o(n^4) &= I(C_3^+, G_n) \\
&= \sum_{v \in V(G_n)} I(C_3, G_n[N^+(v)]) \\
&= \sum_{v \in V(G_n)} i(C_3, G_n[N^+(v)]) \binom{n/2}{3} + o(n^3), \text{ by regularity} \\
&\leq \sum_{v \in V(G_n)} \frac{1}{4} \binom{n/2}{3} + o(n^3) \\
&= \frac{1}{8} \binom{n}{4} + o(n^4).
\end{aligned}$$

This implies that  $i(C_3, G_n[N^+(v)]) = \frac{1}{4} + o(1)$  for all but at most  $o(n)$  vertices  $v \in V(G_n)$ . This equality also implies that  $i(TT_4, G_n) = \frac{3}{8} + o(1)$ . Now

$$\begin{aligned}
\frac{1}{4} + o(1) &= i(C_3, G_n) &= \frac{1}{2}i(C_4, G_n) + \frac{1}{4}i(C_3^+, G_n) + \frac{1}{4}i(C_3^-, G_n), \\
\frac{1}{4} + o(1) &= \frac{1}{3}i(TT_3, G_n) &= \frac{1}{6}i(C_4, G_n) + \frac{1}{4}i(C_3^+, G_n) + \frac{1}{4}i(C_3^-, G_n) + \frac{1}{3}i(TT_4, G_n),
\end{aligned}$$

so

$$o(1) = i(C_3, G_n) - \frac{1}{3}i(TT_3, G_n) = \frac{1}{3}i(C_4, G_n) - \frac{1}{8} + o(1),$$

and thus  $i(C_4, G_n) = \frac{3}{8} + o(1)$ . This last statement is property  $P_2$  in [5] which is equivalent to  $(G_n)$  being quasi-random.  $\square$

**Claim 4.8.** *In any tournament  $T$  on  $n$  vertices,  $i(C_3^+, T) \leq \frac{1}{8} + \frac{2}{3}(\frac{1}{4} - i(C_3, T)) + o(1)$ .*

*Proof.* Using flag algebras, we show

$$3i(C_3^+, T) + 2i(C_3, T) \leq \frac{7}{8} + o(1).$$

The claim follows by rearranging the inequality. Certificates can be found at <http://lidicky.name/pub/tournaments>.  $\square$

**Claim 4.9.** *The sequence  $(T_n[L_n])$  is quasi-random.*

*Proof.* Let  $L = \frac{1}{n}|L_n|$ . By Claim 4.5,  $i(C_3^+, T_n[H_n]) = i(C_3^+, T_n) + o(1)$ . Looking at the density of the  $C_3^+$  which are not completely contained in  $H_n$ , we have, ignoring  $o(1)$  terms,

$$\begin{aligned}
L^4 \frac{1}{8} + 4(1-L)L^3 \frac{1}{4} &\leq (1 - (1-L)^4)i(C_3^+, T_n) \\
&= L^4 \cdot i(C_3^+, T_n[L_n]) + 4(1-L)L^3 \cdot i(C_3, T_n[L_n]) \\
&\leq L^4 \cdot \left(\frac{1}{8} + \frac{2}{3}(\frac{1}{4} - i(C_3, T_n[L_n]))\right) + 4(1-L)L^3 \cdot i(C_3, T_n[L_n]) \\
&= \frac{7}{24}L^4 + L^3 i(C_3, T_n[L_n])(4 - \frac{14}{3}L) \\
&\leq \frac{7}{24}L^4 + L^3 \frac{1}{4}(4 - \frac{14}{3}L) = L^4 \frac{1}{8} + 4(1-L)L^3 \frac{1}{4}.
\end{aligned}$$

The first inequality is true as the left side is the value we would expect if we replace  $T_n[L_n]$  by a random tournament. The second inequality follows from Claim 4.8. For the last inequality, note that  $0 < L < \frac{6}{7}$ , and so the left side is maximized if and only if  $C_3$  is maximized at  $\frac{1}{4}$ . As the first and the last term in this chain of inequalities are equal, we have equality throughout. Thus  $i(C_3, T_n[L_n]) = \frac{1}{4}$  and  $i(C_3^+, T_n[L_n]) = \frac{1}{8}$ , proving the claim using Claim 4.7.  $\square$

**Claim 4.10.** *The normalized size of  $L_n$  is  $L = \frac{1}{5}(7 + \sqrt[3]{3} - 2\sqrt[3]{9}) + o(1)$ , and our conjectured structure is the limit object for the inducibility of  $C_3^+$ .*

*Proof.* We know that  $L < 6/7$ , that  $T_n[L_n]$  is quasi-random, that all arcs between  $H_n$  and  $L_n$  point towards the vertex in  $L_n$ , and that  $i(C_3^+, T_n[H_n]) = i(C_3^+, T_n) + o(1)$ . Thus,

$$i(C_3^+, T_n) = L^4 \cdot \frac{1}{8} + \binom{4}{1} L^3 (1-L) \cdot \frac{1}{4} + (1-L)^4 (i(C_3, T_n) + o(1)).$$

This is maximized when  $i(C_3^+, T_n) = \frac{1}{8} (8 - 9\sqrt[3]{3} + \sqrt[3]{3^5}) + o(1)$  and  $L = 1 - \alpha + o(1)$ .  $\square$

We have thus shown that every extremal tournament matches our construction, completing the proof of this theorem.  $\square$

## 5 Discussion

In this section, we discuss some of the peculiarities of this problem and its solutions, including the novel strategies introduced in this paper. First and foremost, we know of no other inducibility problem for which all extremal constructions include a quasi-random component as in the case of  $C_3^+$  and  $C_3^-$  and ask the following question:

**Problem 1.** *For what classes of graphs (undirected or directed) do the extremal constructions for the corresponding inducibility problem involve non-trivial quasi-random components?*

For  $C_3^+$ , the extremal construction was conjectured by noting that our tournament can be decomposed into a source vertex and a  $C_3$ ; described another way, we begin with an arc and blow up the head into a  $C_3$ . Essentially, we ask the following: for a digraph  $G = (V, E)$  with cut  $C = (S, T)$  and cut-set of size  $|S| \cdot |T|$ , for what structures  $G[S]$  and  $G[T]$  does the resulting inducibility problem have as extremal solutions constructions for which  $\alpha \cdot 100\%$  of the vertices induce a “typical random graph structure” for some  $\alpha \in (0, 1)$ ? Natural candidates for consideration would include  $G[T] \cong C_3$  and  $G[S]$  isomorphic to any 2-vertex digraph or 3-vertex tournament.

Historically, flag algebra techniques have been leveraged to determine bounds on global graph densities. The models developed in Claims 4.3 and 4.4, however, resulted in bounds on localized information. In the case of Claim 4.3, we were able to determine something very powerful regarding the distribution of out-degrees in extremal constructions, namely that all vertices have normalized out-degrees in a very specific set. In the case of Claim 4.4, we were able to determine the direction of an arc between any pair of vertices which satisfy basic constraints related to their out-degrees.

Finally, we want to make an observation about Conjecture 3. Let  $k \geq 5$  be odd, and let  $n > k$ . Let  $X \subset V(C_n)$  be a set of  $k$  vertices such that  $C_n[X] \cong C_k$ . Observe that for every vertex  $v \in$

$V(C_n) \setminus X$ , we have  $C_n[X \cup \{x\}] \cong C_{k+1}$ . If we now express  $i(C_k, T)$  in a tournament  $T$  in terms of densities of  $(k+1)$ -vertex graphs similarly to (1), we can easily conclude that Conjecture 3 is true for  $k+1$  if it is true for  $k$ , so it suffices to prove it for all odd  $k$ . Standard flag algebra computations show that  $C_n$  is  $o(n^2)$  arc flips away from every extremal tournament for  $C_5$  and  $C_7$  (and thus for  $C_6$  and  $C_8$  by this observation), but we have not seriously tried to show the full conjecture for these cases.

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