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Product Throttling

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Product Throttling

Abstract

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The resources used to accomplish a task can be blue vertices in zero forcing, Phasor Measurement Units (PMUs) in power domination, or cops in Cops and Robbers. The time is the number of rounds needed to complete the process (the propagation time or capture time).

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Product Throttling

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1 Introduction

Throttling addresses the question of minimizing the sum or the product of the resources used to accomplish a task and the time needed to complete that task for various graph searching processes. Graph parameters of interest include various types of zero forcing, power domination, and Cops and Robbers.

The resources used to accomplish a task can be blue vertices in zero forcing, Phasor Measurement Units (PMUs) in power domination, or cops in Cops and Robbers. The time is the number of rounds needed to complete the process (the propagation time or capture time).

We begin by defining the graph parameters for which we will discuss product throttling. Our focus is on connected graphs of order at least two (unless otherwise stated). Zero forcing is a coloring game on a graph, where the goal is to color all the vertices blue (starting with each vertex colored blue or white). White vertices are then colored blue by applying a color change rule; the type of zero forcing is determined by the color change rule. Standard zero forcing uses the *standard color change rule*:

If w is the unique white neighbor of a blue vertex v , then change the color of w to blue.

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Positive semidefinite (PSD) zero forcing uses the *PSD color change rule*:

Let B be the set of (currently) blue vertices and let W_1, \dots, W_k be the sets of vertices of the components of $G - B$. If $v \in B$, $w \in W_i$, and w is the only white neighbor of v in $G[W_i \cup B]$, then change the color of w to blue.

Note that it is possible that there is only one component of $G - B$, and in that case the effect of the PSD color change rule is the same as that of the standard color change rule.

A nonempty set $S \subseteq V(G)$ defines an initial set of blue vertices (with all vertices not in S colored white); this is called an *initial coloring of G* . Given an initial coloring S of G , the *final coloring of S* is the set of blue vertices obtained by applying the color change rule until no more changes are possible (other names for the final coloring include the *derived set* and the *closure* of S). A set S is a *standard zero forcing set* (respectively, *PSD zero forcing set*) of G if the final coloring of S is $V(G)$ using the standard (respectively, PSD color change rule). The *standard zero forcing number* (respectively, *PSD zero forcing number*), denoted by $Z(G)$ (respectively, $Z_+(G)$) is the minimum cardinality of a standard zero forcing set (respectively, a PSD zero forcing set). Hereafter, we will use the term *forcing set* to mean standard or PSD zero forcing set.

If v is used to change the color of w by a color change rule, we say v *forces* w and write $v \rightarrow w$. For a given forcing set S , we construct the final coloring, recording the forces. Depending on context, the symbol \mathcal{F} is used to denote the *set of forces* that produces the final coloring, or an ordered list of forces (in the order they were performed), called a *chronological list of forces*. For a given set S , there are often choices as to which vertex forces a particular vertex, so a set of forces, or a chronological list of forces, is usually not unique. However, the final coloring is unique for standard and PSD zero forcing [2, 27].

We can also approach (standard or PSD) zero forcing not as an individual sequence of forces but via *rounds*, where in each round we perform all possible forces that can be done independently of each other (rounds are also called *time steps* in the literature). Starting with $S \subseteq V(G)$, we define two sequences of sets, the set $S^{(i)}$ of vertices that turn blue in round i and the set $S^{[i]}$ of vertices that are blue after round i . Thus $S^{[0]} = S^{(0)} = S$ is the initial set of blue vertices. Assume $S^{(i)}$ and $S^{[i]}$ have been constructed and $S^{(i)} \neq \emptyset$. Then

$$S^{(i+1)} = \{w : w \text{ can be forced by some } v \text{ (given } S^{[i]} \text{ blue)}\} \quad \text{and} \quad S^{[i+1]} = S^{[i]} \cup S^{(i+1)}.$$

Let p denote the greatest integer such that $S^{(p)} \neq \emptyset$. Since $S^{(i)} = \emptyset$ implies $S^{(i+1)} = \emptyset$, S is a forcing set of G if and only if $S^{[p]} = V(G)$. When S is a forcing set, this p is called the *propagation time* of S in G , denoted $\text{pt}(G; S)$ or $\text{pt}_+(G; S)$ for standard and PSD zero forcing set, respectively; if S is not a forcing set, then $\text{pt}(G; S) = \infty$ or $\text{pt}_+(G; S) = \infty$. For $k \in \mathbb{Z}^+$, $\text{pt}(G, k) = \min_{|S|=k} \text{pt}(G; S)$ and $\text{pt}_+(G, k) = \min_{|S|=k} \text{pt}_+(G; S)$. The *standard propagation time of G* (respectively, *PSD propagation time of G*) is $\text{pt}(G) = \text{pt}(G, Z(G))$ (respectively, $\text{pt}_+(G) = \text{pt}_+(G, Z_+(G))$).

For each $v \in V(G)$, define the *round function* by $\text{rd}(v) = k$ for $v \in S^{(k)}$. A *propagating set of forces* is one in which $\text{rd}(u) < \text{rd}(v)$ implies u is forced before v in the associated chronological list of forces, and this is the only kind of forcing set we are concerned with. The round function will also be used for power domination and Cops and Robbers, but the meaning will be clear from the context or a subscript will be added to identify the parameter.

The name *zero forcing* comes from the fact that the process describes forcing zeros in the null vector of a symmetric matrix using only the pattern of off-diagonal nonzero entries of the matrix (a graph describes the nonzero off-diagonal pattern of a symmetric matrix). The zero forcing number was introduced in [2] as an upper bound for the maximum nullity, or equivalently, maximum multiplicity of an eigenvalue, among real symmetric matrices having this graph. Zero forcing was introduced independently in mathematical physics in the study of control of quantum systems [12], and later reintroduced as fast mixed graph searching [33]. Arguably its first appearance was as part of the power domination process, which we describe next.

Power domination models the observations that can be made by PMUs and was studied using graphs by Haynes et al. in [23]; Brueni and Heath [11] showed that a simplified version of the propagation rules is equivalent to the original version in [23], and we use their propagation rules. For a nonempty set S of vertices of G , $N[S]$ denotes the closed neighborhood of S . A set S is a *dominating set* of G if $N[S] = V(G)$, and the minimum cardinality of a dominating set is the *domination number* of G , denoted by $\gamma(G)$. Given $S \subseteq V(G)$, define the sequences of sets $P^{(i)}(S)$ and $P^{[i]}(S)$ by the following recursive rules:

$$(1) \ P^{[0]}(S) = P^{(0)}(S) = S, \ P^{[1]}(S) = N[S] \text{ and } P^{(1)}(S) = N[S] \setminus S.$$

$$(2) \text{ For } i \geq 1,$$

$$\begin{aligned} P^{(i+1)}(S) &= \{w \in V(G) \setminus P^{[i]}(S) : \exists u \in P^{[i]}(S), N_G(u) \setminus P^{[i]}(S) = \{w\}\}, \\ P^{[i+1]}(S) &= P^{[i]}(S) \cup P^{(i+1)}(S). \end{aligned}$$

Step (1) is called the domination step, because it results in $P^{[1]}(S) = N[S]$. Step (2) is called the zero forcing step, because $P^{(i+1)}(S) = N[S]^{(i)}$ for $i \geq 1$. For $v \in P^{(i)}(S)$, we say v is *observed in round i* or $\text{rd}(v) = i$ (if necessary to distinguish from zero forcing, we write $\text{rd}_{\text{pd}}(v)$). If every vertex is observed in some round, i.e., there is an i such that $P^{[i]}(S) = V(G)$, then S is a *power dominating set* of G ; S is a power dominating set of G if and only if $N[S]$ is a zero forcing set of G . The *power domination number* of G , denoted by $\gamma_P(G)$, is the minimum cardinality of a power dominating set. When S is a power dominating set, the least positive integer p with the property that $P^{[p]}(S) = V(G)$ is the *power propagation time* of S in G , denoted by $\text{pt}_{\text{pd}}(G; S)$; if S is not a power dominating set, then $\text{pt}_{\text{pd}}(G; S) = \infty$. Observe that $\text{pt}_{\text{pd}}(G; S) = \text{pt}(G; N[S]) + 1$. For $k \in \mathbb{Z}^+$, $\text{pt}_{\text{pd}}(G, k) = \min_{|S|=k} \text{pt}_{\text{pd}}(G; S)$ and the *power propagation time of G* is $\text{pt}_{\text{pd}}(G) = \text{pt}_{\text{pd}}(G, \gamma_P(G))$.

Cops and Robbers is a two-player game played on a graph. One player places and moves a collection of cops and the other places and moves a single robber. The goal for the cops is to capture the robber by having a cop occupy the same vertex the robber occupies. The goal of the robber is to avoid capture. After an initial placement of the cops on a multiset of vertices (meaning more than one cop can occupy a single vertex), followed by the placement of the robber, the game is played in a sequence of rounds during which the players take turns, both playing in a single round: The team of cops takes a turn by allowing each cop to move to an adjacent vertex or stay in place. Similarly, the robber takes a turn by moving to an adjacent vertex or staying in place. The cops win the game if after some finite number of rounds, a cop captures the robber. If the robber has a strategy to evade the cops indefinitely,

the robber wins. The *cop number* $c(G)$ of a graph G is the minimum number of cops required to capture the robber playing on G [1]. The *capture time*, denoted $\text{capt}(G)$, is the number of rounds it takes for $c(G)$ cops to capture the robber on the graph G (assuming all players follow optimal strategies) [8], and for any $k \geq c(G)$, the *k-capture time* of G , denoted by $\text{capt}_k(G)$, is the minimum number of rounds it takes for k cops to capture the robber on G (assuming that all players follow optimal strategies) [9]. If $k < c(G)$, then $\text{capt}_k(G) = \infty$.

Throttling originated with a question of Richard Brualdi to Michael Young in a talk about zero forcing and propagation time at the 2011 International Linear Algebra Society Conference in Braunschweig, Germany. This led Butler and Young to initiate the study of sum throttling for (standard) zero forcing in [13]. Sum throttling has been studied for numerous parameters including standard zero forcing, PSD zero forcing, and their minor monotone floors; power domination; Cops and Robbers (see [27, Chapter 10] for a survey). Here we define sum throttling for the four graph games we discuss, i.e., standard zero forcing, PSD zero forcing, power domination, and Cops and Robbers. The *standard throttling number* of S in G is $\text{th}(G; S) = |S| + \text{pt}(G; S)$, and the *standard k-throttling number* is $\text{th}(G, k) = k + \text{pt}(G, k)$. The *standard throttling number* of G is

$$\text{th}(G) = \min_{S \subseteq V(G)} \text{th}(G; S) = \min_{Z(G) \leq k \leq n} \text{th}(G, k).$$

The *PSD throttling number* of S in G is $\text{th}_+(G; S) = |S| + \text{pt}_+(G; S)$, and the *PSD k-throttling number* is $\text{th}_+(G, k) = k + \text{pt}_+(G, k)$. The *PSD throttling number* of G is

$$\text{th}_+(G) = \min_{S \subseteq V(G)} \text{th}_+(G; S) = \min_{Z_+(G) \leq k \leq n} \text{th}_+(G, k).$$

The *power domination throttling number* of S in G is $\text{th}_{\text{pd}}(G; S) = |S| + \text{pt}_{\text{pd}}(G; S)$, and the *power domination k-throttling number* is $\text{th}_{\text{pd}}(G, k) = k + \text{pt}_{\text{pd}}(G, k)$. The *power domination throttling number* of G is

$$\text{th}_{\text{pd}}(G) = \min_{S \subseteq V(G)} \text{th}_{\text{pd}}(G; S) = \min_{\gamma_P(G) \leq k \leq n} \text{th}_{\text{pd}}(G, k).$$

The *cop throttling number* of S in G is $\text{th}_c(G; S) = |S| + \text{capt}(G; S)$, and the *cop k-throttling number* is $\text{th}_c(G, k) = k + \text{capt}_k(G)$. The *cop throttling number* of G is

$$\text{th}_c(G) = \min_{S \subseteq V(G)} \text{th}_c(G; S) = \min_{c(G) \leq k \leq n} \text{th}_c(G, k).$$

Product throttling minimizes a product of the number of vertices and the propagation time. In order to make product throttling interesting, the case of a zero product (where each vertex has a cop/PMU/blue color and the propagation time is zero) must be excluded. This can be done by requiring that the cost of positioning cops/PMUs/blue vertices be considered (e.g., by adding one to the number of rounds before multiplying by the number of vertices used), and we describe this as *product throttling with initial cost*. Alternatively, a requirement that at least one round be performed must be added or the initial set consisting of all vertices must be excluded. The study of product throttling was initiated in [7] for Cops and Robbers. It was assumed there is a time cost to placing cops and the product throttling

number was defined as the the number of cops times one more than the propagation time. In contrast, PMUs remain in place but it is natural to assume that the domination step always occurs. In the study of product throttling for power domination in [4], at least one round was required, and the product throttling number was defined as the product of the number of PMUs and the power propagation time. Formal versions of these definitions are given in Section 2 for Cops and Robbers and Section 3 for power domination.

Requiring at least one round and excluding the zero round case are effectively the same for connected graphs of order at least two, but it is more convenient to exclude the zero round case by requiring that the number of vertices used is less than the order; this avoids having two different definitions of propagation time. We refer to this as *product throttling with no initial cost*. This is the approach taken in Section 4, where formal versions of the two definitions in universal notation are presented and discussed further, and in subsequent sections. Product throttling for the standard zero forcing number is not interesting for the first definition and not really a throttling question for the second definition (see Section 5). This fact (or at least the need to exclude the zero solution and that first definition results in product throttling number n for every graph of order n) may have delayed the introduction of product throttling. Using the universal perspective, we examine both definitions of product throttling for Cops and Robbers in Section 6, for power domination in Section 7, and for PSD zero forcing in Section 8. Section 9 compares product throttling for Cops and Robbers, power domination, and PSD zero forcing.

We need some additional notation. The path, cycle, and complete graph on n vertices are denoted by P_n , C_n , and K_n , respectively, and $K_{r,n-r}$ denotes a complete bipartite graph. For a graph G , $\alpha(G)$ denotes the independence number of G . For all the graph parameters discussed, the number of rounds is at least the maximum distance of any vertex to the initial set S , and this plays an important role in the analysis of throttling. Let S be a set of vertices of G . For $v \in V(G)$, the *distance* from v to S is $\text{dist}(S, v) = \min_{x \in S} \text{dist}(x, v)$. The *eccentricity* of S is defined by $\text{ecc}(S) = \max_{v \notin S} \text{dist}(S, v)$. The k -radius of G is $\text{rad}_k(G) = \min_{S \subseteq V(G), |S|=k} \text{ecc}(S)$.

2 Initial cost product throttling for Cops and Robbers

In this section, we follow the convention in [7] and do not assume G is connected or that its order n is at least two unless stated otherwise. The *product cop throttling number with initial cost* defined in [7] is

$$\text{th}_c^\times(G) = \min_{c(G) \leq k \leq n} \{k(1 + \text{capt}_k(G))\}$$

or equivalently, $\text{th}_c^\times(G) = \min_{S \subseteq V(G)} \text{th}_c^\times(G; S)$ where $\text{th}_c^\times(G; S) = |S|(1 + \text{capt}(G; S))$. This choice of definition for Cops and Robbers reflects the fact that there is a time cost to getting the cops in position. In this section we summarize results from [7].

Observation 2.1. [7] *There are two immediate upper bounds:*

(1) $\text{th}_c^\times(G) \leq c(G)(1 + \text{capt}(G))$.

$$(2) \text{ th}_c^\times(G) \leq 2\gamma(G).$$

Remark 2.2. [7] Let G be a graph of order n and suppose $S \subseteq V(G)$ with $|S| \geq c(G)$. Then

$$\text{th}_c^\times(G; S) = |S|(1 + \text{capt}(G; S)) = |S| + |S|\text{capt}(G; S) \geq |S| + \text{capt}(G; S) = \text{th}_c(G; S),$$

so $\text{th}_c^\times(G) \geq \text{th}_c(G)$. Equality occurs exactly when $|S| = 1$ or $\text{capt}(G; S) = 0$, so $\text{th}_c^\times(G) = \text{th}_c(G)$ if and only if $\text{th}_c(G) = \text{th}_c(G; S)$ with $|S| = 1$ or $|S| = n$, i.e., when the cop throttling number can be realized with a single cop or a cop on every vertex.

The product cop throttling number can be determined for low values by factoring the proposed value.

Proposition 2.3. [7] *Let G be a graph of order n .*

- (1) $\text{th}_c^\times(G) = 1$ if and only if $\text{th}_c(G) = 1$ if and only if $G = K_1$.
- (2) $\text{th}_c^\times(G) = 2$ if and only if $\text{th}_c(G) = 2$ if and only if either $G = 2K_1$ or $\gamma(G) = 1$.
- (3) $\text{th}_c^\times(G) = 3$ if and only if G satisfies at least one of the following conditions:
 - (a) $G = 3K_1$ or $G = K_1 \dot{\cup} K_2$.
 - (b) $\gamma(G) \geq 2$ and there exists $z \in V(G)$ such that
 - (i) for all $v \in V(G)$, $\text{dist}(z, v) \leq 2$, and
 - (ii) for all $w \in V(G) \setminus N[z]$, there is a vertex $u \in N[z]$ such that $N[w] \subset N[u]$.
- (4) $\text{th}_c^\times(G) = 4$ if and only if G satisfies at least one of the following conditions:
 - (a) $G = 4K_1$ or $G = 2K_1 \dot{\cup} K_2$.
 - (b) $\gamma(G) = 2$ and $\text{capt}_1(G) \geq 3$.
 - (c) $c(G) = 1$ and $\text{capt}(G) = 3$.

It is immediate that $\text{th}_c^\times(K_n) = 2$ and $\text{th}_c^\times(K_{1,n-1}) = 2$ for $n \geq 2$.

A graph G is a *chordal graph* if it has no induced cycle of length greater than 3. The next result is less elementary than the previous ones.

Theorem 2.4. [7] *Let H be a chordal graph. Then $\text{capt}_k(H) = \text{rad}_k(H)$. Furthermore,*

$$\text{th}_c^\times(H) = 1 + \text{rad}(H) = c(H) + \text{capt}(H).$$

From Theorem 2.4, $\text{th}_c^\times(P_n) = 1 + \lceil \frac{n-1}{2} \rceil$, and more generally, $\text{th}_c^\times(T) = 1 + \text{rad}(T)$ for any tree T .

Theorem 2.4 provides many examples of graphs G with $\text{th}_c^\times(G) = c(G)(1 + \text{capt}(G))$, thus achieving equality in the first upper bound in Observation 2.1. It can also be the case that $\text{th}_c^\times(G)$ is realized by small capture time and a larger number of cops, e.g., by using $\gamma(G)$ cops. One example of this is provided by a graph in the family $H(n)$ defined in [8];

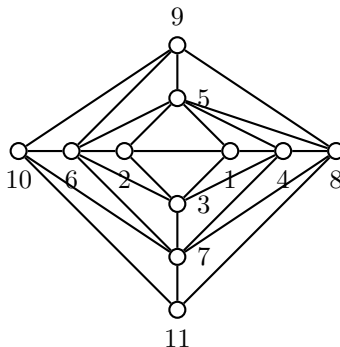


Figure 2.1: The graph $H(11)$.

it is shown there that $c(H(n)) = 1$ and $\text{capt}(H(n)) = n - 4$. It is observed in [7] that for $H(11)$ (see Figure 2.1), $\text{capt}(H(11)) = 7$, but vertices 5 and 7 dominate the graph, so $\text{th}_c^\times(H(11)) = 2(1 + 1) = 4$.

However, the next example provides a family of graphs G for which both $\text{th}_c^\times(G, c(G)) > \text{th}_c^\times(G)$ and $\text{th}_c^\times(G, \gamma(G)) > \text{th}_c^\times(G)$ for sufficiently large order. Fix a positive integer r , define $M'(r)$ to be the graph that is the union of C_4 and three disjoint copies of P_{r+1} where one of the end points of each of the paths is on a distinct vertex of C_4 , and define $M(r) = M'(r) \circ K_1$. The graph $M(3)$ is shown in Figure 2.2; the order of $M(r)$ is $6r + 8$.

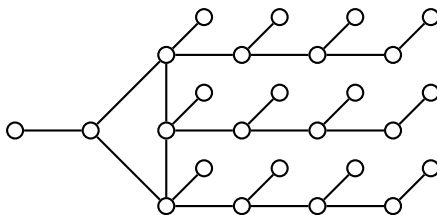


Figure 2.2: The graph $M(3)$.

Proposition 2.5. [7] *For $r \geq 7$, $\text{th}_c^\times(M(r)) < c(M(r))(1 + \text{capt}(M(r)))$ and $\text{th}_c^\times(M(r)) < 2\gamma(M(r))$.*

3 Product throttling for power domination with no initial cost

Let S be a power dominating set. In the papers that studied power propagation time and in this chapter, the power propagation time of S in G is defined to be the least nonnegative integer p such that $P^{[p]}(S) = V(G)$ and is denoted by $\text{pt}_{\text{pd}}(G; S)$. Thus $\text{pt}_{\text{pd}}(G; V(G)) = 0$. When product throttling for power domination was introduced in [4], the perspective was that the domination step always takes place, so the power propagation time of S is at least one even if $S = V(G)$. That is, the definition of power propagation time was modified to require p to be positive. Observe that $\text{pt}_{\text{pd}}(G; S) \geq 1$ for all $S \neq V(G)$. For power domination (and all other parameters discussed), it is immediate that $\text{pt}_{\text{pd}}(G, n - 1) = 1$ when G is a connected graph and has order $n \geq 2$. Thus the restriction that $\text{pt}_{\text{pd}}(G, k)$

be positive can be achieved by allowing the value zero for $\text{pt}_{\text{pd}}(G, k)$, excluding $k = |V(G)|$ from the definition of product power throttling, and requiring that G be connected and have order at least two.

For a connected graph G of order $n \geq 2$, the *product power throttling number with no initial cost*¹ is $\text{th}_{\text{pd}}^*(G) = \min_{1 \leq k < n} k \text{pt}_{\text{pd}}(G, k)$; equivalently, $\text{th}_{\text{pd}}^*(G) = \min_{S \subsetneq V(G)} \text{th}_{\text{pd}}^*(G; S)$ where $\text{th}_{\text{pd}}^*(G; S) = |S| \text{pt}_{\text{pd}}(G; S)$ for $S \neq V(G)$. Note that most results in [4] are for connected graphs but do not assume order at least two, but here we assume the order is at least two. Note that what we here denote by $\text{th}_{\text{pd}}^*(G)$ is denoted by $\text{th}_{\text{pd}}^\times(G)$ in [4]. It is immediate that

$$\text{th}_{\text{pd}}^*(G) = \min_{\gamma_P(G) \leq k \leq \gamma(G)} k \text{pt}_{\text{pd}}(G, k).$$

In this section, we summarize results from [4]. The next observation gives lower and upper bounds for $\text{th}_{\text{pd}}^*(G)$.

Observation 3.1. [4] *Let G be a connected graph of order at least two. Then*

- (1) $\text{th}_{\text{pd}}^*(G) \geq \gamma_P(G) \geq 1$.
- (2) $\text{th}_{\text{pd}}^*(G) \leq \gamma(G)$.
- (3) $\text{th}_{\text{pd}}^*(G) \leq \gamma_P(G) \text{pt}_{\text{pd}}(G)$.

The next result uses the maximum degree of a graph, denoted by $\Delta(G)$, to give a lower bound for $\text{th}_{\text{pd}}^*(G)$.

Proposition 3.2. [4] *Let G be a connected graph of order at least two. Then*

$$\text{th}_{\text{pd}}^*(G) \geq \left\lceil \frac{|V(G)|}{\Delta(G) + 1} \right\rceil.$$

Proposition 3.2 follows as a corollary of Theorem 2.1 in [18].

3.1 Graphs with $\text{th}_{\text{pd}}^*(G) = \gamma(G)$

The following results from [4] summarize conditions sufficient to ensure $\text{th}_{\text{pd}}^*(G) = \gamma(G)$ and give families of graphs for which $\text{th}_{\text{pd}}^*(G) = \gamma(G)$.

Observation 3.3. [4] *In any connected graph G of order $n \geq 2$ with $\gamma(G) = \left\lceil \frac{n}{\Delta(G)+1} \right\rceil$,*

$$\text{th}_{\text{pd}}^*(G) = \left\lceil \frac{n}{\Delta(G) + 1} \right\rceil.$$

Observation 3.3 follows as a result of Proposition 3.2. The next result gives two families of graphs for which the equality in Observation 3.3 holds.

¹Here ‘no initial cost’ refers to the act of monitoring, not the obvious initial cost of PMU placement.

Observation 3.4. [4] Let $n \geq 2$. Then $\text{th}_{\text{pd}}^*(P_n) = \lceil \frac{n}{3} \rceil$ because $\gamma(P_n) = \lceil \frac{n}{3} \rceil$ and $\Delta(P_n) = 2$. Similarly, $\text{th}_{\text{pd}}^*(C_n) = \lceil \frac{n}{3} \rceil$ since $\gamma(C_n) = \lceil \frac{n}{3} \rceil$ and $\Delta(C_n) = 2$, $\text{th}_{\text{pd}}^*(C_n) = \lceil \frac{n}{3} \rceil$.

Proposition 3.5. [4] Let G be a connected graph of order at least two. Then $\gamma_P(G) = \gamma(G)$ if and only if $\text{pt}_{\text{pd}}(G) = 1$. In this case $\text{th}_{\text{pd}}^*(G) = \gamma(G)$.

Corollary 3.6. Let $n \geq 2$.

(1) $\text{th}_{\text{pd}}^*(K_n) = 1 = \gamma(K_n)$.

(2) $\text{th}_{\text{pd}}^*(K_{1,n-1}) = 1 = \gamma(K_{1,n-1})$ and $\text{th}_{\text{pd}}^*(K_{r,n-r}) = 2 = \gamma(K_{r,n-r})$ for $2 \leq r \leq n-2$.

A graph G is a *unit interval graph* if there is a representation that assigns a closed unit length real interval $I(v)$ to each $v \in V(G)$ such that $I(x) \cap I(y) \neq \emptyset$ if and only if $xy \in E(G)$ for $x, y \in V(G)$. It was shown in [4] that the product power throttling number of a unit interval graph is its domination number.

Theorem 3.7. [4] If G is a connected unit interval graph, then $\text{th}_{\text{pd}}^*(G) = \gamma(G)$.

3.2 Cartesian products

The *Cartesian product* $G \square H$ of graphs G and H is the graph whose vertex set is $V(G \square H) = V(G) \times V(H)$ where two vertices (x_1, y_1) and (x_2, y_2) are adjacent in $G \square H$ if either $x_1 = x_2$ and $y_1 y_2 \in E(H)$ or $y_1 = y_2$ and $x_1 x_2 \in E(G)$. Bounds on the product power throttling number of a Cartesian product were presented in [4] and following results were shown.

Proposition 3.8. [4] For $1 \leq n \leq m$, $\text{th}_{\text{pd}}^*(K_n \square K_m) = \gamma(K_n \square K_m) = n$.

Proposition 3.9. [4] Let H be a connected graph of order n and let $G = H \square K_m$ with $m \geq \Delta(H)(n-1) + 1$. Then $\text{th}_{\text{pd}}^*(G) = n = \gamma(G)$. In particular, if $H = C_n$ or P_n and $m \geq 2n-1$, then $\text{th}_{\text{pd}}^*(G) = n$.

Theorem 3.10. [4] For all $n \geq 1$ and $m \geq 2$, $\text{th}_{\text{pd}}^*(J_n \square J_m) = \gamma(J_n \square J_m)$, where $J_r = P_r$ or $J_r = C_r$ for $r \geq 3$ and $J_r = P_r$ for $r = 1, 2$.

3.3 Extreme values

Proposition 3.11. [4] Let G be a connected graph of order at least two.

(1) $\text{th}_{\text{pd}}^*(G) = 1$ if and only if $\gamma(G) = 1$.

(2) $\text{th}_{\text{pd}}^*(G) = 2$ if and only if G satisfies at least one of the following conditions:

(a) $\gamma(G) = 2$.

(b) $\gamma_P(G) = 1$ and $\text{pt}_{\text{pd}}(G) = 2$.

A construction for creating any connected graph G with $\gamma_P(G) = 1$ and $\text{pt}_{\text{pd}}(G) = 2$ appears in [4].

Theorem 3.12. [24, Theorem 2.1] *Let G be a graph of order n having no isolated vertices. Then $\gamma(G) \leq \lfloor \frac{n}{2} \rfloor$.*

Theorem 3.12 and Observation 3.1 provide an upper bound that for any connected graph G of order $n \geq 2$: $\text{th}_{\text{pd}}^*(G) \leq \gamma(G) \leq \frac{n}{2}$.

For any graph H , define the *corona* of H with K_1 , denoted by $H \circ K_1$, to be the graph obtained from H by appending a leaf to each vertex of H . The next result will be used in Section 7.1 and was used in [4] to characterize graphs having $\text{th}_{\text{pd}}^*(G) = \frac{n}{2}$ (see Theorem 3.14).

Theorem 3.13. [4] *If H is a connected graph of order at least two and $G = H \circ K_1$, then $\text{th}_{\text{pd}}^*(G) = 2\gamma(H)$. Furthermore, any power dominating set for G that is a subset of $V(H)$ must be a dominating set for H .*

Theorem 3.14. [4] *Let G be a connected graph of order $n \geq 2$. Then $\text{th}_{\text{pd}}^*(G) = \frac{n}{2}$ if and only if $G = (H \circ K_1) \circ K_1$ for some connected graph H , $G = C_4 \circ K_1$, or $G = C_4$.*

4 Universal product throttling

Carlson introduced universal definitions for propagation time and sum throttling for a zero forcing parameter Y in [14]; here we use simplified versions. Recall that for $S \subseteq V(G)$, $\text{pt}_Y(G; S)$ is the least p such that $S^{[p]} = V(G)$, or infinity if S is not a Y -forcing set of G . For a positive integer k , $\text{pt}_Y(G, k) = \min_{|S|=k} \text{pt}_Y(G; S)$ and the sum throttling number for Y is

$$\text{th}_Y(G) = \min_{Y(G) \leq k \leq n} (k + \text{pt}_Y(G, k)).$$

In addition to (standard and PSD) zero forcing, these definitions apply immediately to power domination with $S^{[i]}$ defined to be $P^{[i]}(S)$. We define $\text{pt}_c(G, k) = \text{capt}_k(G)$ so that the universal notation can also be used for Cops and Robbers. Using the two definitions given for product cop throttling and product power throttling as models, we have two universal definitions for product throttling that apply to the four processes: Cops and Robbers, power domination, standard zero forcing, and PSD zero forcing,

$$\text{(with initial cost)} \quad \text{th}_Y^\times(G) = \min_{Y(G) \leq k \leq n} k(1 + \text{pt}_Y(G, k))$$

and

$$\text{(with no initial cost)} \quad \text{th}_Y^*(G) = \min_{Y(G) \leq k < n} k \text{pt}_Y(G, k),$$

where the definition of $\text{th}_Y^*(G)$ applies only to connected graphs of order at least two. Notice that the case $k = n$ is excluded for $\text{th}_Y^*(G)$. We also use the related notation $\text{th}_Y^\times(G, k) = k(1 + \text{pt}_Y(G, k))$ and $\text{th}_Y^*(G, k) = k \text{pt}_Y(G, k)$.

In this section, we record some simple consequences of these definitions, including possible low values of product throttling numbers. But first we make some comments based on prior results for $\text{th}_c^\times(G)$ and $\text{th}_{\text{pd}}^*(G)$. The nature of the application may motivate the

choice of definition (e.g., cops need to move to their positions whereas PMUs are fixed and immediately available). It seems that $\text{th}_Y^\times(G)$ favors a small number of cops/PMUs/blue vertices, whereas $\text{th}_Y^*(G)$ seems to favor a small propagation/capture time. For example, $\text{th}_c^\times(H) = \text{th}_c^\times(H, 1)$ for H chordal and other graphs for which $\text{th}_c^\times(H) = c(H) \text{capt}(H)$, whereas there are many graphs with $\text{th}_{\text{pd}}^*(G) = \gamma(G)$. Note that $\gamma(G)$ is the smallest number of vertices that power dominate G in one round. Just as we have defined $\text{pt}_Y(G, k)$ to be the minimum possible propagation time using k vertices, it is useful to record the minimum number of vertices that can be used to achieve propagation time p more generally: Define $k_Y(G, p) = \min\{k : \text{pt}_Y(G, k) = p\}$. With this definition, we expect $\text{th}_Y^*(G) = k_Y(G, 1)$ to be common, although it may be written as $\gamma(G)$, because $k_{\gamma_P}(G, 1) = k_c(G, 1) = \gamma(G)$.

For the future, it would be desirable to consider the effect of initial cost values other than one in the case of product throttling with initial cost, i.e. to consider product throttling with with variable initial cost

$$\text{th}_Y^\times(G, \omega) = \min_{Y(G) \leq k \leq n} k(\omega + \text{pt}_Y(G, k)).$$

The results listed below assume that for all graphs G of order n , we have $1 \leq Y(G) \leq n$, $\text{pt}_Y(G, n) = 0$, and $S' \subseteq S$ implies $\text{pt}_Y(G; S) \leq \text{pt}_Y(G; S')$; furthermore, $Y(G) \leq n - 1$ and $\text{pt}_Y(G, n - 1) = 1$ when G is connected and $n \geq 2$. These conditions are satisfied by Cops and Robbers, power domination, and both standard and PSD zero forcing.

4.1 General observations about $\text{th}_Y^\times(G)$ and $\text{th}_Y^*(G)$

We begin with observations that involve both definitions.

Observation 4.1. *For any connected graph G of order at least two, $\text{th}_Y^\times(G, k) = k + \text{th}_Y^*(G, k)$ and thus $\text{th}_Y^*(G) < \text{th}_Y^\times(G)$.*

Remark 4.2. Let Y be one of (standard or PSD) zero forcing, power domination, or Cops and Robbers. Since $\text{pt}_Y(G, k) \geq \text{rad}_k(G)$,

$$\text{th}_Y^\times(G) \geq \min_{Y(G) \leq k \leq n} k(1 + \text{rad}_k(G)) \text{ and } \text{th}_Y^*(G) \geq \min_{Y(G) \leq k < n} k \text{rad}_k(G)$$

(G must be connected and of order at least two for $\text{th}_Y^*(G)$ to be defined).

Next we consider $\text{th}_Y^\times(G)$.

Observation 4.3. *For any graph G of order n , $Y(G) \leq \text{th}_Y^\times(G) \leq n$ since $\text{th}_Y^\times(G, k) = \infty$ for $k < Y(G)$ and $\text{th}_Y^\times(G, n) = n$. If G is connected and $n \geq 2$, then $Y(G) + 1 \leq \text{th}_Y^\times(G)$, because $Y(G) \leq n - 1$.*

Observation 4.4. *For every graph G :*

- (1) $\text{th}_Y^\times(G) \leq Y(G)(1 + \text{pt}_Y(G))$.
- (2) $\text{th}_Y^\times(G) \leq 2k_Y(G, 1)$.

Remark 4.5. For any graph G of order n , $Y(G) \geq \frac{n}{2}$ implies $\text{th}_Y^\times(G) = n$: If $Y(G) \leq |S| < n$, then $\text{pt}_Y(G; S) \geq 1$ so $1 + \text{pt}_Y(G; S) \geq 2$ and $|S|(1 + \text{pt}_Y(G; S)) \geq n$.

The previous remark is not very useful for connected graphs when $Y(G) \leq \gamma(G)$, including Cops and Robbers and power domination, since $\gamma(G) \leq \frac{n}{2}$ for a connected graph of order $n \geq 2$ [24, Theorem 2.1]. However, it can be useful for other parameters (and for all parameters when disconnected graphs are considered).

Remark 2.2, which relates sum and product cop throttling, is valid more generally for $\text{th}_Y^\times(G)$.

Remark 4.6. Let G be a graph of order n . Since $k \geq 1$,

$$\text{th}_Y^\times(G, k) = k(1 + \text{pt}_Y(G, k)) \geq k + \text{pt}_Y(G, k) = \text{th}_Y(G, k),$$

so $\text{th}_Y^\times(G) \geq \text{th}_Y(G)$. Since we have equality exactly when $k = 1$ or $\text{pt}_Y(G, k) = 0$, $\text{th}_Y^\times(G) = \text{th}_Y(G)$ if and only if $\text{th}_Y(G) = \text{th}_Y(G, 1)$ or $\text{th}_Y(G) = \text{th}_Y(G, n)$.

Finally we consider $\text{th}_Y^*(G)$.

Observation 4.7. For any connected graph G of order $n \geq 2$, $Y(G) \leq \text{th}_Y^*(G) \leq k_Y(G, 1) \leq n - 1$. Furthermore, $\text{th}_Y^*(G) \leq Y(G) \text{pt}_Y(G)$ and $\text{th}_Y^*(G) = Y(G)$ if and only if $\text{pt}_Y(G) = 1$.

The next remark could be stated more generally, but we present it in a form we find useful in characterizing graphs with high $\text{th}_Y^*(G)$.

Remark 4.8. Let Y be one of (standard or PSD) zero forcing, power domination, or Cops and Robbers. Suppose G' is a connected graph of order $n' \geq 2$ that is an induced subgraph of a connected graph G of order n such that $\text{th}_Y^*(G') = k_Y(G, 1)$. Then there is a set $S' \subset V(G')$ such that $\text{th}_Y^*(G') = \text{th}_Y^*(G'; S')$ and $\text{pt}_Y(G'; S') = 1$. Define $S = S' \cup (V(G) \setminus V(G'))$. Since $\text{pt}_Y(G; S) \leq \text{pt}_Y(G'; S') = 1$,

$$\text{th}_Y^*(G) \leq n - n' + \text{th}_Y^*(G').$$

For $\text{th}_Y^*(G)$ there is a relationship with sum throttling.

Proposition 4.9. For any connected graph of order $n \geq 2$, $\text{th}_Y^*(G) \geq \text{th}_Y(G) - 1$. If $\text{th}_Y(G) = \text{th}_Y(G, 1) = 1 + \text{pt}_Y(G, 1)$ or $\text{th}_Y(G) = \text{th}_Y^*(G, k_Y(G, 1)) = k_Y(G, 1) + 1$, then $\text{th}_Y^*(G) = \text{th}_Y(G) - 1$.

Proof. Let $t = \text{th}_Y(G)$. Then $\text{pt}_Y(G, k) \geq t - k$ for all $k = 1, \dots, t$. It is straightforward to verify that $k(t - k) \geq t - 1$ for all $k = 1, \dots, t - 1$. Thus $k \text{pt}_Y(G, k) \geq \text{th}_Y(G) - 1$ for all $k = 1, \dots, t - 1$. Furthermore, $k \text{pt}_Y(G, k) \geq t > \text{th}_Y(G) - 1$ for $k = t, \dots, n - 1$. Thus $\text{th}_Y^*(G) \geq \text{th}_Y(G) - 1$ since $\text{th}_Y^*(G) = k \text{pt}_Y(G, k)$ for some k with $1 \leq k \leq n - 1$.

If $\text{th}_Y(G) = \text{th}_Y(G, 1)$, then $\text{th}_Y^*(G) \leq \text{th}_Y^*(G, 1) = \text{th}_Y(G) - 1$. The argument for $\text{th}_Y(G) = \text{th}_Y^*(G, k_Y(G, 1))$ is similar. \square

4.2 Low values of the product throttling number

For low values of $\text{th}_Y^\times(G)$ or $\text{th}_Y^*(G)$, results can usually be described using graphs with low values of $Y(G)$ and $\text{pt}_Y(G)$. Recall that all graphs are connected and of order at least two.

Remark 4.10. Let G be a connected graph of order $n \geq 2$. Setting $\text{th}_Y^\times(G) = t$ or $\text{th}_Y^*(G) = t$ and factoring t yields the following results for small t .

- (a) No graph of order two or more has $\text{th}_Y^\times(G) = 1$.
- (b) $\text{th}_Y^*(G) = 1$ if and only if $Y(G) = 1$ and $\text{pt}_Y(G) = 1$.
- (c) $\text{th}_Y^\times(G) = 2$ if and only if $Y(G) = 1$ and $\text{pt}_Y(G) = 1$.
- (d) $\text{th}_Y^*(G) = 2$ if and only if G satisfies exactly one of the following conditions:
 - (i) $Y(G) \leq 2$, $\text{pt}_Y(G, 2) = 1$, and $\text{pt}_Y(G, 1) > 2$.
 - (ii) $Y(G) = 1$ and $\text{pt}_Y(G, 1) = 2$.
- (e) $\text{th}_Y^\times(G) = 3$ if and only if G satisfies exactly one of the following conditions:
 - (i) $n = 3$ and $Y(G) > 1$.
 - (ii) $Y(G) = 1$ and $\text{pt}_Y(G, 1) = 2$.
- (f) $\text{th}_Y^*(G) = 3$ if and only if G satisfies exactly one of the following conditions:
 - (i) $Y(G) \leq 3$, $\text{pt}_Y(G, 3) = 1$, $\text{pt}_Y(G, 2) > 1$, and $\text{pt}_Y(G, 1) > 3$.
 - (ii) $Y(G) = 1$, $\text{pt}_Y(G, 1) = 3$, and $\text{pt}_Y(G, 2) > 1$.
- (g) $\text{th}_Y^\times(G) = 4$ if and only if G satisfies exactly one of the following conditions:
 - (i) $n = 4$, $Y(G) > 1$, and $\text{pt}_Y(G, 2) > 1$.
 - (ii) $Y(G) \leq 2$, $\text{pt}_Y(G, 2) = 1$, and $\text{pt}_Y(G, 1) > 3$.
 - (iii) $Y(G) = 1$ and $\text{pt}_Y(G, 1) = 3$.

For (c), note that $\text{th}_Y^\times(G) = \text{th}_Y^\times(G, 2)$ implies $\text{pt}_Y(G, 2) = 0$ and thus $G = K_2$, which is covered by $Y(G) = 1$ and $\text{pt}_Y(G) = 1$. For (e)(ii) and (g)(ii), note that $\text{pt}_Y(G, 1) \geq 2$ implies $n \geq 3$, so $\text{pt}_Y(G, 2) \geq 1$ and $\text{th}_Y^\times(G, 2) \geq 4$.

Observation 4.11. *Let p be a prime number.*

- (1) *If $\text{th}_Y^\times(G) = p$, then $Y(G) = 1$ and $\text{pt}_Y(G) = p - 1$, or $|V(G)| = p$.*
- (2) *If $\text{th}_Y^*(G) = p$, then $Y(G) = 1$ and $\text{pt}_Y(G) = p$, or $\text{pt}_Y(G, p) = 1$.*

5 Product throttling for standard zero forcing

For any set $S \subseteq V(G)$, at most $|S|$ forces can be performed in each round, so $\frac{n-|S|}{|S|} \leq \text{pt}(G; S)$. This implies for $1 \leq k \leq n$,

$$\frac{n-k}{k} \leq \text{pt}(G, k). \quad (5.1)$$

This fundamental bound is the reason that the initial cost version of product throttling for standard zero forcing, defined by $\text{th}^\times(G) := \min_{S \subseteq V(G)} |S|(1 + \text{pt}(G; S))$, is not interesting.

Remark 5.1. For any graph G of order n , $\min_{S \subseteq V(G)} |S|(1 + \text{pt}(G; S)) = n$, which is achieved by coloring all vertices blue, because

$$k(1 + \text{pt}(G, k)) \geq k \left(1 + \frac{n-k}{k} \right) = n$$

by (5.1).

5.1 Characterization of $\text{th}^*(G)$

Next we consider the version of product throttling for standard zero forcing that has no initial cost and show that $\text{th}^*(G) := \min_{Z(G) \leq k < n} k \text{pt}(G, k)$ is the least k such that $\text{pt}(G, k) = 1$. First we need to define some terms and prove a lemma. Given a standard zero forcing set S and a propagating set of forces \mathcal{F} , a *forcing chain* is a maximal sequence of vertices (v_1, v_2, \dots, v_s) such that for $i = 1, \dots, s-1$, $v_i \rightarrow v_{i+1}$. The set of vertices of a forcing chain necessarily induces a path. A *reversal* of S is the set of last vertices of the zero forcing chains of a propagating set of forces \mathcal{F} , i.e., the vertices that do not perform forces. Note that a set S often has more than one set of forces, but a given propagating set of forces \mathcal{F} has a unique set of forcing chains that thus defines one reversal of S . Any reversal of S can be denoted by $\text{rev}(S)$. Note that the cardinality of $\text{rev}(S)$ is the same as the cardinality of S . It is well-known that if S is a zero forcing set of G , then any reversal of S is also a zero forcing set of G [26].

Lemma 5.2. *Let G be a graph, S a zero forcing set of G , and let $t = \text{pt}(G; S)$. Then for any reversal of S and $i = 0, \dots, t$,*

$$S^{(t-i)} \subseteq (\text{rev}(S))^{[i]}.$$

Proof. Let \mathcal{F} be a propagating set of forces that produces $\text{rev}(S)$. The result is established by induction on i . For $i = 0$, $S^{(t-0)} = S^{(t)} \subseteq \text{rev}(S) = (\text{rev}(S))^{[0]}$. Assume that $S^{(t-j)} \subseteq (\text{rev}(S))^{[j]}$ for $0 \leq j < i$. Let $w \in S^{(t-i)}$. If w does not perform a force in \mathcal{F} , then $w \in \text{rev}(S) \subseteq (\text{rev}(S))^{[i]}$. So assume $w \rightarrow u$ in round $t-j > t-i$, so $j < i$. By the induction hypothesis, $u \in (\text{rev}(S))^{[j]} \subseteq (\text{rev}(S))^{[i-1]}$. Now consider a neighbor $v \neq w$ of u . Since v does not force w and $w \rightarrow u$ in round $t-j$, either v does not force or $v \rightarrow x$ in round $t-j'$ with $j' < j$. In either case, $v \in (\text{rev}(S))^{[i-1]}$. So if $w \notin (\text{rev}(S))^{[i-1]}$, then u can force w in the i th round of forcing starting with $\text{rev}(S)$. Thus $w \in (\text{rev}(S))^{[i]}$. \square

As in Section 4.1, define $k(G, p) = \min\{|S| : \text{pt}(G; S) = p\}$.

Theorem 5.3. *For any graph G , $\text{th}^*(G)$ is the least k such that $\text{pt}(G, k) = 1$, i.e., $\text{th}^*(G) = k(G, 1)$. Necessarily $k(G, 1) \geq \frac{n}{2}$.*

Proof. Let $k \geq Z(G)$, let $t = \text{pt}(G, k)$ and let $S \subsetneq V(G)$ be such that $\text{pt}(G; S) = t$. If $\text{pt}(G, k) = 1$, there is nothing to prove for this k , so suppose $t \geq 2$. Define $\hat{S} = S \cup \text{rev}(S)$ for some reversal of S . Then for $i = 1, \dots, t$,

$$S^{[i]} \cup \bigcup_{j=0}^{i-1} S^{(t-j)} \subseteq S^{[i]} \cup (\text{rev}(S))^{[i]} \subseteq \hat{S}^{[i]}$$

by Lemma 5.2. In particular (since $t \geq 2$),

$$V(G) = \bigcup_{j=0}^{t-1} S^{(j)} \subseteq \hat{S}^{[\lceil \frac{t-1}{2} \rceil]}.$$

Thus $\text{pt}(G, 2k) \leq \lceil \frac{t-1}{2} \rceil \leq \frac{t}{2}$ and $(2k) \text{pt}(G, 2k) \leq k \text{pt}(G, k)$. Apply this repeatedly as needed to show that $\min_{Z(G) \leq k < n} k \text{pt}(G, k) = \min\{k : \text{pt}(G, k) = 1\} = k(G, 1)$.

If $\text{pt}(G, k) = 1$, then $1 \geq \frac{n-k}{k}$, so $k \geq \frac{n}{2}$. \square

Remark 5.4. For a connected graph G of order $n \geq 2$, $k(G, 1) \geq \gamma(G)$ since to achieve $\text{pt}(G; S) = 1$, every vertex must be in S or adjacent to a vertex in S . However, $k(G, 1)$ can be much larger than the domination number. For example, $k(K_n, 1) = n - 1$ whereas $\gamma(K_n) = 1$.

While the question of determining $k(G, 1) = \text{th}^*(G)$ is interesting, it seems more like a question about a form of domination or zero forcing rather than a throttling question (since there is no balancing of resources and time). However, in the next two sections we offer characterizations of extreme values of $\text{th}^*(G)$.

5.2 Low values of $\text{th}^*(G)$

Observation 5.5. *By Theorem 5.3, $\text{th}^*(G) = \frac{n}{2}$ if and only if G has a zero forcing set S with $\text{pt}(G; S) = 1$ and $|S| = \frac{n}{2}$. Analogously, in the case n is odd, $\text{th}^*(G) = \frac{n+1}{2}$ is equivalent to the existence of a zero forcing set S such that $\text{pt}(G; S) = 1$ and $|S| = \frac{n+1}{2}$.*

Next we use Observation 5.5 to show that paths have the minimum possible product throttling number.

Proposition 5.6. *For $n \geq 2$, $\text{th}^*(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$.*

Proof. Assume $V(P_n) = \{v_1, \dots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$. Define $T_n \subset V(P_n)$ by

$$T_n = \begin{cases} \{v_i \in V(P_n) : i \equiv 2 \pmod{4} \text{ or } i \equiv 3 \pmod{4}\} & \text{if } n \not\equiv 1 \pmod{4} \\ \{v_i \in V(P_n) : i \equiv 2 \pmod{4} \text{ or } i \equiv 3 \pmod{4}\} \cup \{v_n\} & \text{if } n \equiv 1 \pmod{4} \end{cases}$$

When n is even,

$$N(v_i) \setminus T_n = \begin{cases} v_{i-1} & \text{if } i \equiv 2 \pmod{4} \\ v_{i+1} & \text{if } i \equiv 3 \pmod{4} \end{cases}$$

for every $v_i \in T_n$. As a result, each vertex in T_n forces in a single round. Now suppose n is odd. The same is true for all vertices except v_n , which has no neighbor outside T_n if $n \equiv 3 \pmod{4}$ and which shares a neighbor with v_{n-2} if $n \equiv 1 \pmod{4}$. Thus propagation time is again one and $\text{th}^*(P_n) = |T_n| = \frac{n+1}{2}$. \square

The next result is a direct consequence of Theorem 5.3 and Proposition 5.6.

Corollary 5.7. *For any graph connected graph G of order n , $\text{th}^*(G) \geq \left\lceil \frac{n}{2} \right\rceil$ and this bound is tight.*

A characterization of all graphs attaining the lower bound in Corollary 5.7 will be presented after introducing some terminology and obtaining a few preliminary results. We start by recalling the definition of matched-sum graphs, as introduced by Georges and Mauro in [19]: Let G_1 and G_2 be graphs and let M be a matching between $V(G_1)$ and $V(G_2)$. Then the M -matched-sum of G_1 and G_2 (or simply, M -sum of G_1 and G_2), denoted by $G_1 M^+ G_2$, is the graph with $V(G_1 M^+ G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 M^+ G_2) = E(G_1) \cup E(G_2) \cup M$. A graph of the form $G_1 M^+ G_2$ is also called a *matched-sum graph*.

Example 5.8. A path of even order is an example of a matched-sum graph. We show this by identifying graphs G_1 , G_2 and M such that $G_1 M^+ G_2$ is isomorphic to P_n , for any even integer n . Let G_1 be the subgraph of P_n induced by the corresponding zero forcing set T_n , as defined in Proposition 5.6. Let G_2 be the subgraph of P_n induced by $V(P_n) \setminus T_n$, and let M be the matching defined by the edges used in the propagation where each vertex of G_1 forces its only neighbor in G_2 . It is straightforward to verify that $G_1 M^+ G_2 = P_n$. See Figure 5.1.

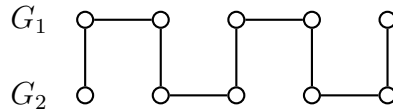


Figure 5.1: P_{10} drawn as a matched-sum graph.

Observation 5.9. *For any two connected graphs G_1 and G_2 , a graph $G_1 M^+ G_2$ is connected for any matching M . However, it is not necessary for G_1 and G_2 to be connected, to obtain a connected graph matched-sum graph $G_1 M^+ G_2$ (cf. Example 5.8).*

In our work we assume a matched-sum graph $G_1M^+G_2$ is connected but we do not require G_1 and G_2 to be connected. Readers interested in details on the connectivity of matched-sum graphs are referred to [6].

Observation 5.10. *Any matched-sum graph $G_1M^+G_2$ necessarily has even order.*

As shown in Proposition 5.6, it is possible for a graph G of odd order n to satisfy $\text{th}^*(G) = \left\lceil \frac{n}{2} \right\rceil$. As a result, matched-sum graphs are not sufficient to characterize all graphs G attaining the lower for $\text{th}^*(G)$ in Corollary 5.7, but they capture essential structural properties necessary for a graph to have a zero forcing set with propagation time 1 of minimum size.

We note that matched-sum graphs have also been studied under other names, such as (G_1, G_2) -permutation graphs [5] or matching graphs [26]. In particular, matched-graphs were used by Hogben et al. [26] to study graphs having a minimum zero forcing set of propagation time 1. We recall a result from [26] with a particular connection to the work in this section.

Proposition 5.11. [26] *Let G be a graph of order n . Then any two of the following conditions imply the third.*

- (1) $n = 2Z(G)$.
- (2) $\text{pt}(G) = 1$.
- (3) G is a matched-sum graph.

Let G be a graph of even order n . Observe that Conditions 1 and 2 in Proposition 5.11 directly imply $\text{th}^*(G) = \frac{n}{2} = Z(G)$ and, by Proposition 5.11, G is a matched-sum graph. As shown in Example 5.8, the condition $\frac{n}{2} = Z(G)$ is not necessary to obtain $\text{th}^*(G) = \frac{n}{2}$. In our next result, we prove that the condition $\text{th}^*(G) = \frac{n}{2}$ is both necessary and sufficient for G to be a matched-sum graph.

Theorem 5.12. *A connected graph G of even order n satisfies $\text{th}^*(G) = \frac{n}{2}$ if and only if G is a connected matched-sum graph.*

Proof. Suppose G is a matched-sum graph. Let H and H' be a pair of vertex-disjoint graphs, and let M be a matching between $V(H)$ and $V(H')$ such that $G = HM^+H'$. It is immediate that $V(H)$ is a zero forcing set of G with order $\frac{n}{2}$ such that $\text{pt}(G, V(H)) = 1$, which implies $\text{th}^*(G) = \frac{n}{2}$ by Observation 5.5.

Let G be a connected graph of even order $n = 2r$ such that $\text{th}^*(G) = r$. By Observation 5.5, there exists $S \subset V(G)$ zero forcing set of G with $|S| = r$ and $\text{pt}(G; S) = 1$. Let H and H' be the subgraphs of G induced by S and $V(G) \setminus S$, respectively. Let M be the set of edges uv where $u \rightarrow v$; we show $G = HM^+H'$. It is immediate that H and H' are vertex-disjoint graphs of order r satisfying $V(G) = V(HM^+H')$. By definition of matched-sum graph, $E(HM^+H') = E(H) \cup M \cup E(H')$, and the selection of M , H and H' guarantees $E(HM^+H') \subseteq E(G)$. To conclude the proof, it is sufficient to show $xy \in E(G)$ with $x \in S$

and $y \notin S$ implies $xy \in M$. Suppose to the contrary that $xy \notin M$. Then there exists $w \notin S$ such that $xw \in M$. But then x has two neighbors in $V(G) \setminus S$, so $\text{pt}(G; S) > 1$, which is a contradiction. \square

The characterization of all graphs G of odd order n such that $\text{th}^*(G) = \lceil \frac{n}{2} \rceil$ is given in terms of the a graph operation we recall next. If G is a graph of order $n \geq 2$ and v is a vertex of G , then the graph $G - v$ is defined by $V(G - v) = V(G) \setminus \{v\}$ and $E(G - v) = E(G) \setminus \{uv : u \in N_G(v)\}$. That is, $G - v$, the graph obtained by removing the vertex v in all edges incident with v in G .

Theorem 5.13. *A connected graph G of order n satisfies $\text{th}^*(G) = \lceil \frac{n}{2} \rceil$ if and only if there exists $v \in V(G)$ such that $G - v$ is a matched-sum graph.*

Proof. Suppose $v \in V(G)$ and $G - v$ is a matched-sum graph. Then by Theorem 5.12, there is a set $S \subset V(G - v)$ such that $|S| = \frac{n-1}{2}$ and $\text{pt}(G - v; S) = 1$. Define $S' = S \cup \{v\}$. Then $\frac{n+1}{2} = |S'| = \text{th}^*(G; S') \geq \text{th}^*(G)$, and $\text{th}^*(G) = \lceil \frac{n}{2} \rceil$ by Theorem 5.3.

Let G be a graph of odd order $n = 2r + 1$ such that $\text{th}^*(G) = r + 1$. By Observation 5.5, there exists $S \subset V(G)$ with $|S| = p + 1$ such that S is a zero forcing set of G and $\text{pt}(G; S) = 1$. Let \mathcal{F} be a propagating set of forces and let $v \in S$ be the one vertex that does not perform a force. Then $S' = S \setminus \{v\}$ is zero forcing set for $G - v$ and $\text{pt}(G - v; S') = 1$. Thus $\text{th}^*(G - v) = r = \frac{|V(G-v)|}{2}$, so $G - v$ is a matched-sum graph by Theorem 5.12. \square

Matched-sum graphs include several interesting graph families, and for these graphs the product zero forcing throttling number is obtained by applying Theorem 5.12. For example, the d -dimensional hypercube $Q_d = K_2 \square \cdots \square K_2$ (with d copies of K_2) is a matched-sum graph for any $d \geq 2$, so Theorem 5.12 yields $\text{th}^*(Q_d) = 2^{d-1}$. The generalized Petersen graph $G(r, s)$ is also a matched-sum graph, so $\text{th}^*(G(r, s)) = r$.

5.3 High values of $\text{th}^*(G)$

The maximum value of $\text{th}^*(G)$ over connected graphs G of order $n \geq 2$ is $n - 1$ (see Observation 4.7) and this is realized by K_n . We can use Carlson and Kritschgau's characterization of graphs having $\text{th}(G) = n$ in [16] to characterize graphs having $\text{th}^*(G) = n - 1$.

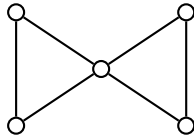


Figure 5.2: The bowtie graph.

Theorem 5.14. [16] *For a connected graph G of order n , $\text{th}(G) = n$ if and only if G does not have a P_4, C_4 , or bowtie graph as an induced subgraph.*

Corollary 5.15. *For a connected graph G of order $n \geq 2$, $\text{th}^*(G) = n - 1$ if and only if G does not have a P_4, C_4 , or bowtie graph as an induced subgraph.*

Proof. Since $\text{th}^*(G) \leq n - 1$, it suffices to establish that $\text{th}^*(G) \leq n - 2$ if and only if G has a P_4, C_4 , or bowtie graph as an induced subgraph.

Let G' be one of P_4, C_4 , or the bowtie graph and let n' be the order of G' . Observe that $\text{pt}(G', n' - 2) = 1$, so $\text{th}^*(G') \leq n' - 2$. Suppose that a connected graph G of order n contains G' as an induced subgraph. Then by Remark 4.8, $\text{th}^*(G) \leq n - n' + (n' - 2) = n - 2$.

Now suppose G is a connected graph of order $n \geq 2$ and $\text{th}^*(G) \leq n - 2$. Then $\text{th}(G) \leq \text{th}^*(G) + 1 = n - 1$ by Proposition 4.9, so G has a P_4, C_4 , or bowtie graph as an induced subgraph by Theorem 5.14. \square

6 Product throttling for Cops and Robbers revisited

In Section 2, we summarized known results on the product cop throttling number with initial cost, $\text{th}_c^\times(G)$. In this section, we study product cop throttling number with no initial cost. Let G be a connected graph of order at least two. For a set $S \subseteq V(G)$ with $c(G) \leq |S| \leq \gamma(G)$, $\text{th}_c^*(G; S) = |S| \text{capt}(G; S)$. Define $\text{th}_c^*(G, k) = k \text{capt}_k(G)$. The *no initial cost product cop throttling number* of G is

$$\text{th}_c^*(G) = \min_{c(G) \leq k \leq \gamma(G)} k \text{capt}_k(G) = \min_{c(G) \leq k \leq \gamma(G)} \text{th}_c^*(G, k).$$

While initial cost product throttling seems more realistic if considering actual police officers, no initial cost product throttling is useful in other searching applications. This parameter has been studied as *work* $w_k = k \text{capt}_k(G)$ and *speedup* between using $j > i$ cops, defined as w_i/w_j [28, 29, 30]. The product cop power throttling number with no initial cost extends this idea by considering the number of cops that yields the largest possible speed-up. We give bounds for $\text{th}_c^*(G)$ and determine this number exactly for certain families of graphs, including paths, cycles, complete graphs, complete bipartite graphs, full t -ary trees and unit interval graphs. We also establish a few additional results for $\text{th}_c^\times(G)$.

6.1 General observations

Observation 6.1. *Let G be a connected graph of order at least two. There are several immediate upper bounds for $\text{th}_c^*(G)$:*

- (1) $\text{th}_c^*(G) \leq \gamma(G)$.
- (2) $\text{th}_c^*(G) \leq c(G) \text{capt}(G)$.
- (3) $\text{th}_c^*(G) \leq \text{th}_c^\times(G) - c(G)$.

There are several immediate lower bounds for $\text{th}_c^(G)$:*

- (4) $\text{th}_c^*(G) \geq c(G)$.
- (5) $\text{th}_c^*(G) \geq \text{th}_c(G) - 1$.

$$(6) \text{ th}_c^*(G) \geq \min_{c(G) \leq k \leq \gamma(G)} k \text{ rad}_k(G).$$

Furthermore, $\text{th}_c^*(G) = c(G)$ if and only $\text{capt}(G) = 1$.

Remark 6.2. The following values for K_n and $K_{r,n-r}$ follow immediately from Observation 6.1. Capture time agrees with power propagation time for paths and cycles, so the values of these graphs follow from Observation 3.4.

$$(1) \text{ th}_c^*(K_n) = 1 = \gamma(K_n).$$

$$(2) \text{ For } 2 \leq r \leq n-2, \text{ th}_c^*(K_{r,n-r}) = 2 = \gamma(K_{r,n-r}) \text{ and } \text{th}_c^*(K_{1,n-1}) = 1 = \gamma(K_{1,n-1}).$$

$$(3) \text{ th}_c^*(P_n) = \lceil \frac{n}{3} \rceil = \gamma(P_n).$$

$$(4) \text{ th}_c^*(C_n) = \lceil \frac{n}{3} \rceil = \gamma(C_n) \text{ for } n \geq 4.$$

The next result is immediate from Remark 4.10 and Proposition 2.3.

Remark 6.3. Let G be a connected graph of order at least two.

$$(1) \text{ th}_c^*(G) = 1 \text{ if and only if } \gamma(G) = 1.$$

$$(2) \text{ th}_c^*(G) = 2 \text{ if and only if } G \text{ satisfies at least one of the following conditions:}$$

$$(a) \gamma(G) = 2.$$

$$(b) \gamma(G) \geq 3 \text{ and there exists } z \in V(G) \text{ such that}$$

$$(i) \text{ for all } v \in V(G), \text{ dist}(z, v) \leq 2, \text{ and}$$

$$(ii) \text{ for all } w \in V(G) \setminus N[z], \text{ there is a vertex } u \in N[z] \text{ such that } N[w] \subset N[u].$$

$$(3) \text{ th}_c^*(G) = 3 \text{ if and only if } G \text{ satisfies at least one of the following conditions:}$$

$$(a) \gamma(G) = 3 \text{ and } \text{th}_c^*(G) \neq 2.$$

$$(b) c(G) = 1, \text{ capt}(G, 1) = 3, \text{ and } \text{capt}(G, 2) \geq 2.$$

Note that whether or not $\text{capt}(G, 2) \geq 2$ can be determined by a polynomial time algorithm (see [21] or [10, Algorithm 2]).

Let G_1 and G_2 be graphs such that $G_1 \cap G_2 = K_m$ for some m , and $G_1, G_2 \neq K_m$. Then $G_1 \cup G_2$ is the *clique sum* of G_1 and G_2 . Next we state bounds for the (sum) cop throttling number for clique sums and establish analogous bounds for the product cop throttling numbers of clique sums.

Theorem 6.4. [10] *Let G be a clique sum of G_1 and G_2 . Let k_1 and k_2 be numbers such that $\text{th}_c(G_i) = \text{th}_c(G_i, k_i)$ for $i \in \{1, 2\}$. Then*

$$\max\{\text{th}_c(G_1, k_1), \text{th}_c(G_2, k_2)\} \leq \text{th}_c(G) \leq k_1 + k_2 + \max\{\text{capt}_{k_1}(G_1), \text{capt}_{k_2}(G_2)\}.$$

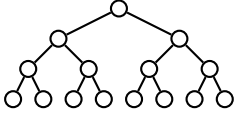
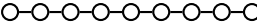
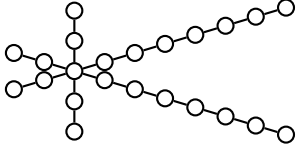
G	$\text{rad}(G)$	$\gamma(G)$	$\text{th}_c^*(G)$
	3	5	$3 = \text{rad}(G) < \gamma(G)$
	4	3	$3 = \gamma(G) < \text{rad}(G)$
	7	9	$6 < \min\{\text{rad}(G), \gamma(G)\}$

Table 6.1: Examples illustrating relationships among $\text{rad}(G)$, $\gamma(G)$, and $\text{th}_c^*(G)$.

Proposition 6.5. *Let G be a connected non-trivial clique sum of G_1 and G_2 . Let k_1, k_2, ℓ_1, ℓ_2 be such that for $i \in \{1, 2\}$, $\text{th}_c^*(G_i) = k_i \cdot \text{capt}_{k_i}(G_i)$ and $\text{th}_c^\times(G_i) = \ell_i \cdot (\text{capt}_{\ell_i}(G_i) + 1)$. Then*

$$\begin{aligned} \max\{\text{th}_c^*(G_1), \text{th}_c^*(G_2)\} &\leq \text{th}_c^*(G) \leq (k_1 + k_2) \max\{\text{capt}_{k_1}(G_1), \text{capt}_{k_2}(G_2)\}, \text{ and} \\ \max\{\text{th}_c^\times(G_1), \text{th}_c^\times(G_2)\} &\leq \text{th}_c^\times(G) \leq (\ell_1 + \ell_2)(\max\{\text{capt}_{\ell_1}(G_1), \text{capt}_{\ell_2}(G_2)\} + 1). \end{aligned}$$

Proof. Let $k = k_1 + k_2$. As in [10], note that G_1 and G_2 are retracts of G , so $\text{capt}_k(G) \leq \max\{\text{capt}_{k_1}(G_1), \text{capt}_{k_2}(G_2)\}$ (see [9]). Therefore

$$\text{th}_c^*(G) \leq (k_1 + k_2) \cdot \max\{\text{capt}_{k_1}(G_1), \text{capt}_{k_2}(G_2)\}.$$

For the lower bound, as in [10], note that for $i \in \{1, 2\}$, if a robber's movement within G is restricted to $V(G_i)$, then for k_i cops, there is no benefit to the cops starting outside $V(G_i)$. The k_i cops then catch the robber in time $\text{capt}_{k_i} G_i$.

The proof for $\text{th}_c^\times(G)$ is similar. □

6.2 Chordal graphs

Recall that a graph G is a *chordal graph* if it has no induced cycle of length greater than 3. The next result is immediate from Theorem 2.4 and Observation 6.1.

Remark 6.6. Let H be a connected chordal graph of order $n \geq 2$. Then

$$\text{th}_c^*(H) = \min_{1 \leq k \leq \gamma(H)} k \text{rad}_k(H) \leq \min\{\text{rad}(H), \gamma(H)\}.$$

Each of the upper bounds $\text{rad}(G)$ and $\gamma(G)$ is tight, but the inequality in Remark 6.6 cannot be changed to an equality. In Table 6.1, we provide examples of graphs such that $\text{th}_c^*(G) = \text{rad}(G)$ and $\text{th}_c^*(G) = \gamma(G)$ as well as an example of a graph where $\text{th}_c^*(G) < \min\{\text{rad}(G), \gamma(G)\}$.

We can find $\text{th}_c^*(G)$ exactly for certain families of chordal graphs, including split graphs, full t -ary trees, and unit interval graphs. A *split graph* is a graph whose vertices can be partitioned into a clique and an independent set.

Remark 6.7. Let G be a connected split graph of order two or more. If $\gamma(G) = 1$, then $\text{th}_c^*(G) = 1$ by Remark 6.3. If $\gamma(G) > 1$, then $\text{th}_c^*(G) = 2$ by Remark 6.6 since $\text{rad}(G) = 2$.

A full r -ary tree of height h , denoted by $T_{r,h}$, is a rooted tree in which each node has r children unless it is at distance h from the root, and the distance between a vertex and the root is at most h ; $T_{3,2}$ is shown in Figure 6.1.

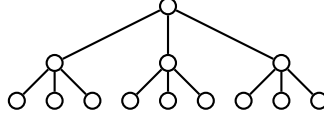


Figure 6.1: The full ternary tree of height two $T_{3,2}$.

Proposition 6.8. Let $T_{r,h}$ be the full r -ary tree of height h with $h, r \geq 2$. Then $\text{th}_c^*(T_{r,h}) = h$.

Proof. Consider a set $S \subseteq V(T_{r,h})$ of cardinality $k \geq 2$ such that $\text{capt}(T_{r,h}; S) = \text{capt}(T_{r,h}, k)$. If $k \text{capt}(T_{r,h}; S) \leq h$, then each of the leaves must be within distance $\frac{h}{k}$ of a vertex in S . This requires at least $r^{h-\frac{h}{k}} \geq t^{\frac{h}{2}}$ vertices; it is most efficiently done by taking every vertex at depth $h - \lfloor \frac{h}{k} \rfloor$, so

$$k \text{capt}(T_{r,h}; S) \geq 2 \cdot 2^{\frac{h}{2}} > h. \quad \square$$

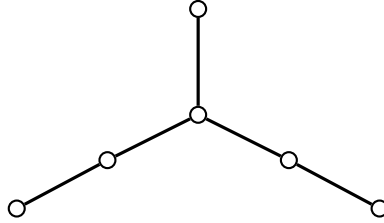


Figure 6.2: An interval graph T with $\text{th}_c^*(T) < \gamma(T)$.

As shown in [4], the graph T in Figure 6.2 is an interval graph (and chordal). As with power domination, $\text{th}_c^*(T) = 2 < 3 = \gamma(T)$. We show that $\text{th}_{\text{pd}}^*(G) = \gamma(G)$ for a unit interval graph G ; some of the ideas come from the proof that $\text{th}_{\text{pd}}^*(G) = \gamma(G)$ in [4], but there are substantial differences. Instead of partitioning the vertices by the round in which they are observed, we partition the vertices of G by their distance from $S \subset V(G)$: Define $S^{(k)} = \{v : \text{dist}(S, v) = k\}$. Then $V(G) = S \dot{\cup} S^{(1)} \dot{\cup} \dots \dot{\cup} S^{(\text{ecc}(S))}$. For a unit interval graph G , fix a unit representation of G with induced order $<$. For any vertex v , define $L(v)$ to be the least vertex in $N[v]$, $L^1(v) = L(v)$ and $L^{k+1}(v) = L(L^k(v))$. Define $R(v)$ to be the greatest vertex in $N[v]$, and define $R^k(v)$ analogously.

Lemma 6.9. Let G be a connected unit interval graph of order at least two with a fixed unit representation and induced order, and let $S = \{x\} \subset V(G)$. For $k = 1, \dots, \text{ecc}(S)$, $\{L^k(x), R^k(x)\}$ dominates $S^{(k)} \cup S^{(k+1)}$, and $L^1(x)$ dominates x .

Proof. It is immediate that x is dominated by $L^1(x)$. Let $v \in S^{(i)}$ with $i \geq 1$, so there is a path $(x = v_0, v_1, \dots, v_i = v)$ and $v_j \in S^{(j)}$ for $j = 0, \dots, i$. Suppose $v \leq x$, so $v = v_i < \dots < v_1 < x = v_0$. From the definition of $L^k(x)$, $(x = v_0, L^1(x), \dots, L^{i-1}(x), v_i = v)$ is also a path from x to v . By setting $i = k + 1$, we see that $L^k(x)$ dominates v . Now suppose $i = k$. Then v is a neighbor of $L^{k-1}(x)$ so $L^k(x) \leq v < L^{k-1}(x)$. Since neighborhoods are consecutive and $L^{k-1}(x) \in N[L^k(x)]$, $v \in N[L^k(x)]$. Thus $L^k(x)$ dominates all $v \in S^{(k)} \cup S^{(k+1)}$ such that $v < x$. The case of $v > x$ is handled by $R^k(x)$. \square

Theorem 6.10. *If G is a unit interval graph, then $\text{th}_c^*(G) = \gamma(G)$.*

Proof. Let G be a connected unit interval graph of order at least two with a fixed unit representation and induced order. It suffices to show $\gamma(G) \leq \text{th}_c^x(G, k)$ for $1 \leq k < \gamma(G)$. Let $t = \text{rad}_k(G)$, and choose $S \subset V(G)$ such that $|S| = k$ and $\text{ecc}(S) = t$; note that $t \geq 2$. Define $T^k(S) = \cup_{x \in S} \{L^k(x), R^k(x)\}$. We consider two cases, t is even and t is odd.

Assume first that t is even. Then $\hat{S} = T^1(S) \cup T^3(S) \cup \dots \cup T^{t-1}(S)$ dominates $V(G) = S \cup S^{(1)} \cup \dots \cup S^{(t)}$ by Lemma 6.9. Then $\gamma(G) \leq |\hat{S}| \leq |S|2^{\frac{t}{2}} = kt = \text{th}_c^x(G, k)$.

Now assume that t is odd. Let $\hat{S} = S \cup S^{(2)} \cup S^{(4)} \cup \dots \cup S^{(t-1)}$. Then $\hat{S} = S \cup T^2(S) \cup T^4(S) \cup \dots \cup T^{t-1}(S)$ dominates $V(G) = S \cup S^{(1)} \cup \dots \cup S^{(t)}$ by Lemma 6.9 and since the vertices in $S^{(1)}$ are dominated by S by definition. Then $\gamma(G) \leq |\hat{S}| \leq |S|(1 + 2^{\frac{t-1}{2}}) = kt = \text{th}_c^x(G, k)$. \square

7 Product throttling for power domination revisited

In this section we determine $\text{th}_{\text{pd}}^*(G)$, the product power throttling number with no initial cost, for additional families of graphs G and explore the definition of product throttling with initial cost, $\text{th}_{\text{pd}}^x(G)$.

7.1 Determination of $\text{th}_{\text{pd}}^*(G)$ for additional families of graphs

The values of $\text{th}_{\text{pd}}^*(G)$ for some families of graphs were established in [4] and several families of graphs for which $\text{th}_{\text{pd}}^*(G) = \gamma(G)$ were presented (see Section 3). In this section, we establish $\text{th}_{\text{pd}}^*(G)$ for some additional families of graphs. We also construct infinite families of graphs where $\text{th}_{\text{pd}}^*(G) \neq \gamma(G)$. For these families, $\text{th}_{\text{pd}}^*(G) < \min\{\gamma(G), \gamma_P(G) \text{pt}_{\text{pd}}(G)\}$.

We need the following definitions. For $j \geq 2$ and $d \geq 4$, construct the j, d -generalized necklace $N_{j,d}$ by connecting j copies of $K_d - e$ arranged cyclically to create a $d - 1$ regular graph; $N_{3,5}$ is shown in Figure 7.1.

The results in the next remark follow from results stated in Section 3.

Remark 7.1. Let $n \geq 2$.

- (1) $\text{th}_{\text{pd}}^*(Q_d) = \gamma(Q_d) = 2^{2^\ell - \ell - 1}$ for $d = 2^\ell - 1$ using Corollary 3.3 because the order of Q_d is 2^d , $\Delta(Q_d) = d$, and for $d = 2^\ell - 1$, $\gamma(Q_d) = 2^{2^\ell - \ell - 1}$ by [22].
- (2) $\text{th}_{\text{pd}}^*(P_r \circ K_1) = 2 \lceil \frac{r}{3} \rceil$ by Theorem 3.13 and $\gamma(P_n) = \lceil \frac{n}{3} \rceil$.

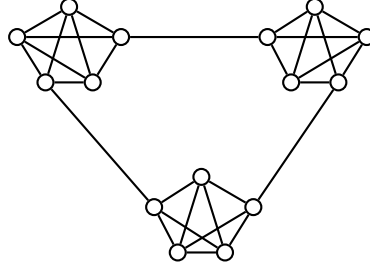


Figure 7.1: The 3,5-generalized necklace $N_{3,5}$.

(3) $\text{th}_{\text{pd}}^*(C_r \circ K_1) = 2 \lceil \frac{r}{3} \rceil$ by Theorem 3.13 and $\gamma(C_n) = \lceil \frac{n}{3} \rceil$.

(4) $\text{th}_{\text{pd}}^*(N_{j,d}) = j$ by Corollary 3.3 because the order of $N_{j,d}$ is jd , $\Delta(N_{j,d}) = d - 1$, and $\gamma(N_{j,d}) = \lceil \frac{jd}{(d-1)+1} \rceil = j$.

We now construct a family $G(n, s, m)$ of 2-connected graphs for which $\text{th}_{\text{pd}}^*(G(n, s, m))$ is less than both $\gamma(G(n, s, m))$ and $\gamma_P(G(n, s, m)) \cdot \text{pt}_{\text{pd}}(G(n, s, m))$. These examples lead to a family of r -connected graphs with the same properties, for any integer $r \geq 2$.

Let K_n be the complete graph on vertices $\{u_1, u_2, \dots, u_n\}$. Replace each edge $u_i u_j$ of K_n with $s \geq 1$ disjoint paths of length $m \geq 1$ between u_i and u_j , for $1 \leq i < j \leq n$. Call the resulting graph $G(n, s, m)$; Figure 7.2 shows $G(3, 3, 4)$.

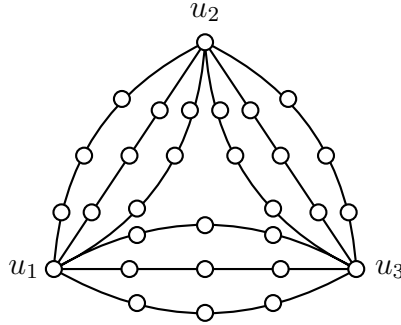


Figure 7.2: $G(3, 3, 4)$ is formed by replacing each edge of K_3 by 3 paths of length 4.

Proposition 7.2. *Let $n \geq 2$, $s \geq 3$, and $m \geq 4$. Then*

(1) $\gamma(G(n, s, m)) \geq s \frac{n(n-1)}{2} \lceil \frac{m-3}{3} \rceil$.

(2) $\gamma_P(G(n, s, m)) = n - 1$, $\text{pt}_{\text{pd}}(G(n, s, m)) = m$, and $\gamma_P(G(n, s, m)) \cdot \text{pt}_{\text{pd}}(G(n, s, m)) = (n - 1)m$.

(3) $\text{th}_{\text{pd}}^*(G(n, s, m)) \leq n \lceil \frac{m-1}{2} \rceil$.

For $n, m \geq 5$, $\gamma(G(n, s, m)) \geq \gamma_P(G(n, s, m)) \cdot \text{pt}_{\text{pd}}(G(n, s, m)) > \text{th}_{\text{pd}}^*(G(n, s, m))$.

Proof. (1): Since the domination number of a path is the ceiling of its number of vertices divided by 3 and $m + 1 - 4$ vertices are not accessible from the original vertices of the K_n , any dominating set of $G(n, s, m)$ will have to contain at least $\lceil \frac{m-3}{3} \rceil$ vertices from each of the s paths between u_i and u_j for $1 \leq i < j \leq n$.

(2): Note that $\{u_1, u_2, \dots, u_{n-1}\}$ is a power dominating set of $G(n, s, m)$, because each path between vertices u_i and u_j will have a degree 2 neighbor of a blue endpoint turn blue in the first round. To show that $\gamma_P(G(n, s, m)) \geq n - 1$, if S is a minimum power dominating set that contains neither u_i nor u_j , then S must contain at least $s - 1 \geq 2$ vertices on the paths between them, and replacing these vertices by u_i results in a power dominating set S' with $|S'| < |S|$, contradicting the minimality of S . Thus the only minimum power dominating sets are $\{u_1, u_2, \dots, u_n\} \setminus \{u_i\}$ for $i = 1, \dots, n$. To evaluate $\text{pt}_{\text{pd}}(G(n, s, m); \{u_1, \dots, u_{n-1}\})$, note that the last vertex to be observed will be u_n in round m .

(3): Note that $\text{pt}_{\text{pd}}(G(n, s, m); \{u_1, \dots, u_n\}) = \lceil \frac{m-1}{2} \rceil$, so $\text{th}_{\text{pd}}^*(G(n, s, m); \{u_1, \dots, u_n\}) = n \lceil \frac{m-1}{2} \rceil$.

Let $n, m \geq 5$. Then $nm \geq 2m + 3n$, so $n(n-1)(m-3) \geq 2(n-1)m$. Since $s \geq 3$, $s \frac{n(n-1)(m-3)}{2 \cdot 3} \geq (n-1)m$. Also $nm > n + 2m$ implies $(n-1)m > \frac{n(m+1)}{2} = n(\frac{m-1}{2} + 1) > n \lceil \frac{m-1}{2} \rceil$. \square

We now construct a family of $(r+2)$ -connected graphs for any positive integer $r \geq 1$ for which $\text{th}_{\text{pd}}^*(G) < \min\{\gamma(G), \gamma_P(G) \text{pt}_{\text{pd}}(G)\}$.

Theorem 7.3. *Let $r \geq 1$ and let $s \geq 2^{r+1} + 1$. Define $H_r = G(3, s, 4) \square Q_r$. Then $\text{th}_{\text{pd}}^*(H_r) < \gamma(H_r)$, $\text{th}_{\text{pd}}^*(H_r) < \gamma_P(H_r) \text{pt}_{\text{pd}}(H_r)$, and H_r is $(r+2)$ -connected.*

Proof. Clearly H_r contains 2^r copies of $G(3, s, 4)$. By Proposition 7.2(1), $\gamma(G(3, s, 4)) \geq 3s \geq 3(2^{r+1} + 1)$. Let S be a power dominating set of H_r .

Assume first that $|S| \leq 2^{r+1} - 1$. Then there is a copy of $G(3, s, 4)$ in which at most one of the vertices from the original K_3 are chosen. Hence there are at least two vertices in the original K_3 of that copy, say u_1 and u_2 , that are not in S . For at least $s - 1$ of the paths between u_1 and u_2 , there must be a vertex in S that is adjacent to a vertex on the path. No vertex in a different copy of $G(3, s, 4)$ is adjacent to two of these paths, so there are at least $s - 1 = 2^{r+1}$ vertices needed, contradicting our assumption that $|S| \leq 2^{r+1} - 1$. Thus any power dominating set of H_r must contain at least 2^{r+1} vertices.

Now suppose S is a set that has the same two vertices in each copy of $G(3, s, 4)$, so $|S| = 2 \cdot 2^r$. Since power domination will occur simultaneously in each copy, S is a power dominating set of H_r and $\text{pt}_{\text{pd}}(H_r; S) = \text{pt}_{\text{pd}}(G(3, s, 4); S) = \text{pt}_{\text{pd}}(G(3, s, 4))$. Thus $\gamma_P(H_r) = 2^{r+1}$. Since any minimum power dominating set must have this form, $\text{pt}_{\text{pd}}(H_r) = \text{pt}_{\text{pd}}(G(3, s, 4))$. By Proposition 7.2(2), $\text{pt}_{\text{pd}}(G(3, s, 4)) = 4$ and therefore $\gamma_P(H_r) \text{pt}_{\text{pd}}(H_r) = 2^{r+1} \cdot 4 = 2^{r+3}$.

Finally, suppose S is the set of three original vertices in each copy of $G(3, s, 4)$. Then $|S| = 3 \cdot 2^r$. As in the proof of Proposition 7.2(3), the number of rounds needed to complete the power domination simultaneously in each copy is 2. Thus,

$$\text{th}_{\text{pd}}^*(H_r) \leq 3 \cdot 2^r \cdot 2 = 3 \cdot 2^{r+1} < 3(2^{r+1} + 1) < 4 \cdot 2^{r+1} = 2^{r+3}.$$

If G is s -connected and H is r -connected, then $G \square H$ is $(s+r)$ -connected [31]. Thus Q_r is r -connected and $G(3, s, 4) \square Q_r$ is $(2+r)$ -connected. \square

7.2 The initial cost definition $\text{th}_{\text{pd}}^{\times}(G)$

In this section we summarize some basic results about $\text{th}_{\text{pd}}^{\times}(G)$, and we prove that $\text{th}_{\text{pd}}^{\times}(G) < |V(G)|$ when G is a connected graph of order at least three. In contrast, we present examples where $\text{th}_{\text{pd}}^{\times}(G) = \frac{6}{7}|V(G)|$. The graphs $G = H \circ K_1$ have high domination number, but we show that $\text{th}_{\text{pd}}^{\times}(H \circ K_1) \leq \frac{3}{4}|V(G)|$ if H is connected and nontrivial. We compare the results found in [4] about $\text{th}_{\text{pd}}^*(P_n \square P_m)$ to new upper bounds on $\text{th}_{\text{pd}}^{\times}(P_n \square P_m)$ that show that the best ways to power dominate are different in each case.

For a graph G of order n , recall that $\text{th}_{\text{pd}}^{\times}(G, k) = k(1 + \text{pt}_{\text{pd}}(G, k))$ and

$$\text{th}_{\text{pd}}^{\times}(G) = \min_{\gamma_P(G) \leq k \leq n} k(1 + \text{pt}_{\text{pd}}(G, k)) = \min_{\gamma_P(G) \leq k \leq n} \text{th}_{\text{pd}}^{\times}(G, k).$$

The next result follows from Observations 4.3 and 4.4.

Observation 7.4. *For every graph G of order n :*

- (1) $\text{th}_{\text{pd}}^{\times}(G) \leq \gamma_P(G)(1 + \text{pt}_{\text{pd}}(G))$.
- (2) $\text{th}_{\text{pd}}^{\times}(G) \leq 2\gamma(G)$.
- (3) $\gamma_P(G) \leq \text{th}_{\text{pd}}^{\times}(G) \leq n$ and $\gamma_P(G) + 1 \leq \text{th}_{\text{pd}}^{\times}(G)$ if G is connected and $n \geq 2$.
- (4) If $\gamma_P(G) \geq \frac{n}{2}$, then $\text{th}_{\text{pd}}^{\times}(G) = n$.

The next result follows from Remark 4.10 in the case G is connected, and the analysis of the disconnected case is straightforward.

Remark 7.5.

- (1) $\text{th}_{\text{pd}}^{\times}(G) = 1$ if and only if $G = K_1$.
- (2) $\text{th}_{\text{pd}}^{\times}(G) = 2$ if and only if $\gamma(G) = 1$ or $G = 2K_1$.
- (3) $\text{th}_{\text{pd}}^{\times}(G) = 3$ if and only if $\text{pt}_{\text{pd}}(G, 1) = 2$ or $G = 3K_1$ or $G = K_2 \dot{\cup} K_1$.

A description of a construction for a connected graph G with $\text{pt}_{\text{pd}}(G, 1) = 2$, which is equivalent to $\gamma_P(G) = 1$ and $\text{pt}_{\text{pd}}(G) = 2$, appears in [4]. In particular, $\text{th}_{\text{pd}}^{\times}(C_4) = 3$.

Remark 7.6. Power domination on trees behaves like Cops and Robbers, and power domination on C_n behaves like power domination on P_n . The remaining parts of the next result follow from Observation 7.4 and Remark 7.5.

- (1) If T is a tree, then $\text{th}_{\text{pd}}^{\times}(T) = 1 + \text{rad } T$.
- (2) $\text{th}_{\text{pd}}^{\times}(P_n) = 1 + \text{rad}(P_n) = 1 + \lceil \frac{n-1}{2} \rceil$.
- (3) $\text{th}_{\text{pd}}^{\times}(C_n) = 1 + \text{rad}(P_n) = 1 + \lceil \frac{n-1}{2} \rceil$.
- (4) $\text{th}_{\text{pd}}^{\times}(K_n) = 2$.

$$(5) \text{ th}_{\text{pd}}^{\times}(K_{1,n-1}) = 2.$$

$$(6) \text{ For } n \geq 4, \text{ th}_{\text{pd}}^{\times}(K_{2,n-2}) = 3.$$

$$(7) \text{ For } p, q \geq 3, \text{ th}_{\text{pd}}^{\times}(K_{p,q}) = 4.$$

In Theorem 7.8 we show that $\text{th}_{\text{pd}}^{\times}(G) < |V(G)|$ for a connected graph of order at least three, using the next result.

Theorem 7.7. [24, Theorem 2.2] *A connected graph G of order $n \geq 2$ has $\gamma(G) = \frac{n}{2}$ if and only if $G = H \circ K_1$ for some connected graph H or $G = C_4$.*

Theorem 7.8. *Let G be a connected graph of order $n \geq 3$. Then $\text{th}_{\text{pd}}^{\times}(G) < n$. Furthermore, if $G = H \circ K_1$ for a connected graph H of order at least two, then $\text{th}_{\text{pd}}^{\times}(G) = 3\gamma(H)$ and $\text{th}_{\text{pd}}^{\times}(G) = \frac{3n}{4}$.*

Proof. By Theorem 3.12, $\gamma(G) \leq \frac{n}{2}$. If $\gamma(G) < \frac{n}{2}$, then by Observation 7.4(2), $\text{th}_{\text{pd}}^{\times}(G) \leq 2\gamma(G) < n$. So assume $\gamma(G) = \frac{n}{2}$. By Theorem 7.7, either $G = C_4$ or G consists of a connected graph H with a leaf attached to each vertex. If $G = C_4$, then $\text{th}_{\text{pd}}^{\times}(C_4) = 3 < n$ by Remark 7.5(3).

So suppose $G = H \circ K_1$ for a connected graph H . Since $n \geq 3$, H has at least 2 vertices, so by Theorem 3.12, $\gamma(H) \leq \frac{|V(H)|}{2}$, and this proves that $3\gamma(H) \leq \frac{3n}{4}$. Let S be a dominating set of H with $|S| = \gamma(H)$. Then S is a power dominating set of G with propagation time two. Thus $\text{th}_{\text{pd}}^{\times}(G) \leq 3\gamma(H) \leq \frac{3n}{4} < n$.

To show that $\text{th}_{\text{pd}}^{\times}(G) = 3\gamma(H)$ for $G = H \circ K_1$, we choose $S \subset V(G)$ such that $\text{th}_{\text{pd}}^{\times}(G; S) = \text{th}_{\text{pd}}^{\times}(G)$ and show that $\text{th}_{\text{pd}}^{\times}(G; S) \geq 3\gamma(H)$. Note that S is not a dominating set of G since $\gamma(G) = \frac{n}{2}$ and $\text{th}_{\text{pd}}^{\times}(G) < n$. Without loss of generality, we may assume that $S \subseteq V(H)$ since we can always replace a leaf of G by its neighbor in H . The set S must be a dominating set of H in order to be a power dominating set of G , so $|S| \geq \gamma(H)$ and the propagation time for S is two. Therefore $\text{th}_{\text{pd}}^{\times}(G; S) = 3|S| \geq 3\gamma(H)$. \square

The next result is immediate from Theorem 7.8.

Corollary 7.9. *For $m \geq 2$, $\text{th}_{\text{pd}}^{\times}(P_m \circ K_1) = 3 \lceil \frac{m}{3} \rceil$.*

Next we present a family of connected graphs that satisfy $\text{th}_{\text{pd}}^{\times}(G) = \frac{6|V(G)|}{7}$.

Example 7.10. Let G_1 consist of a P_4 and a P_3 with an edge connecting them between a vertex of degree 2 on each, as shown in Figure 7.3. The vertices of degree 3 are colored green in the figure. Then $\gamma(G_1) = 3$, hence $\text{th}_{\text{pd}}^{\times}(G_1) \leq 3 \cdot 2 = 6$. However, any power dominating set of G_1 that is not a dominating set must contain a vertex from each path and have propagation time at least 2. Thus $\text{th}_{\text{pd}}^{\times}(G_1) \geq 2 \cdot 3 = 6$. Since $|V(G_1)| = 7$, $\text{th}_{\text{pd}}^{\times}(G_1) = \frac{6|V(G_1)|}{7}$.

Now let G consist of r disjoint copies of G_1 , say G_1, G_2, \dots, G_r , with any subset of edges between the green vertices in each copy of G_1 . Any dominating set of G must contain at least 3 vertices from each G_i , $1 \leq i \leq r$, in order to dominate the leaves. Any power dominating set of G must contain at least two vertices from each G_i , $1 \leq i \leq r$. Hence $\text{th}_{\text{pd}}^{\times}(G) = \frac{6|V(G)|}{7}$.

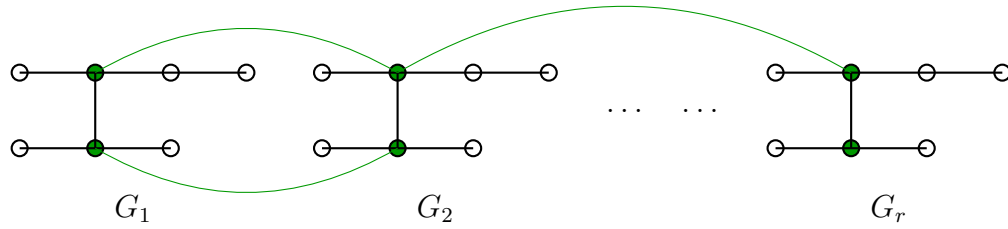


Figure 7.3: A graph G with $|V(G)| = 7r$ and $\text{th}_{\text{pd}}^{\times}(G) = \frac{6|V(G)|}{7}$. Any subset of edges between the green vertices may be included.

Let G be j copies of P_3 and i copies of P_4 (all disjoint, so the order of G is $3j + 4i$). We show $\frac{\text{th}_{\text{pd}}^{\times}(G)}{3j+4i} \geq \frac{6}{7}$ if and only if $j = i$. Note that $\gamma(G) = j+2i$, $\gamma_P(G) = j+i$, and $\text{pt}_{\text{pd}}(G) = 2$. Then $\text{th}_{\text{pd}}^{\times}(G) = \min\{2(j+2i), 3(j+i)\}$ because $\text{pt}_{\text{pd}}(G, k) = 2$ for $\gamma_P(G) \leq k < \gamma(G)$. Thus $i = j$ implies $\text{th}_{\text{pd}}^{\times}(G) = \frac{6}{7}$. By choosing a dominating set, $\frac{\text{th}_{\text{pd}}^{\times}(G)}{3j+4i} \leq \frac{2\gamma(G)}{3j+4i} = \frac{2(j+2i)}{3j+4i} = \frac{2j+4i}{3j+4i}$. If $i < j$, then $14j + 28i < 18j + 24i$, which implies $\frac{2j+4i}{3j+4i} < \frac{6}{7}$. By choosing a minimum power dominating set, $\frac{\text{th}_{\text{pd}}^{\times}(G)}{3j+4i} \leq \frac{3(j+i)}{3j+4i} = \frac{3j+3i}{3j+4i}$. If $j < i$, then $21j + 21i < 18j + 24i$, which implies $\frac{3j+3i}{3j+4i} < \frac{6}{7}$. Hence $i \neq j$ implies $\frac{\text{th}_{\text{pd}}^{\times}(G)}{3j+4i} < \frac{6}{7}$.

If G is connected and not K_1 or K_2 , then Theorem 7.8 shows that the ratio $\frac{\text{th}_{\text{pd}}^{\times}(G)}{|V(G)|}$ is less than 1, and Example 7.10 shows that it can be as large as $\frac{6}{7}$.

Question 7.11. *Is $\frac{6}{7}$ the largest possible value of $\frac{\text{th}_{\text{pd}}^{\times}(G)}{|V(G)|}$ that is achieved for connected graphs of arbitrarily large order?*

Next we examine grid graphs, which are natural to consider in PMU placement problems. It is interesting to compare the value of $\text{th}_{\text{pd}}^{\times}(P_m \circ K_1)$ in Corollary 7.9 with the value of $\text{th}_{\text{pd}}^{\times}(P_m \square P_2)$ in Theorem 7.14.

Proposition 7.12. *For $n, m \geq 2$, $\text{th}_{\text{pd}}^{\times}(P_n \square P_m) \leq \min\{\lceil \frac{m}{3} \rceil (n+1), \lceil \frac{n}{3} \rceil (m+1)\}$.*

Proof. Arrange $P_n \square P_m$ with n rows and m columns. By symmetry, we need only show that $\text{th}_{\text{pd}}^{\times}(P_n \square P_m) \leq \lceil \frac{m}{3} \rceil (n+1)$. Let S be a minimum dominating set of the top row of $P_n \square P_m$, so $|S| = \gamma(P_m)$. After the first round, each vertex in the top row is observed and has at most one unobserved neighbor in the second row. Thus zero forcing can proceed row by row, so $\text{pt}_{\text{pd}}(P_n \square P_m; S) \leq n$ and $\text{th}_{\text{pd}}^{\times}(P_n \square P_m) \leq \lceil \frac{m}{3} \rceil (1+n)$. \square

In [4], it was shown that $\text{th}_{\text{pd}}^*(P_n \square P_m) = \gamma(P_n \square P_m)$ for all m, n (see Theorem 3.10). As for all graphs, $\text{th}_{\text{pd}}^{\times}(P_n \square P_m) \leq 2\gamma(P_n \square P_m)$. For example, $\gamma(P_2 \square P_m) = \lfloor \frac{m+2}{2} \rfloor$ and $\gamma(P_3 \square P_m) = \lfloor \frac{3m+4}{4} \rfloor$ (see [3]), and therefore $\text{th}_{\text{pd}}^{\times}(P_2 \square P_m) \leq 2\gamma(P_2 \square P_m) = 2 \lfloor \frac{m+2}{2} \rfloor \leq m+2$ and $\text{th}_{\text{pd}}^{\times}(P_3 \square P_m) \leq 2\gamma(P_3 \square P_m) = 2 \lfloor \frac{3m+4}{4} \rfloor$. However, these bounds are not tight, as shown in the next remark.

Remark 7.13. In [20], it is shown that $\gamma(P_n \square P_m) = \left\lfloor \frac{(m+2)(n+2)}{5} - 4 \right\rfloor$ for $m, n \geq 16$. Since $\left\lceil \frac{m}{3} \right\rceil (n+1) < 2 \left(\left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor - 4 \right)$, $\text{th}_{\text{pd}}^\times(P_n \square P_m) < 2\gamma(P_n \square P_m)$ for $m, n \geq 16$.

To establish the exact value of $\text{th}_{\text{pd}}^\times(P_2 \square P_m)$ in Theorem 7.14, we need some definitions. We use the notation $u \rightarrow v$ or u forces v to mean u observes v (this may involve a choice among several vertices that can observe v). For a power dominating set $S \subseteq V(G)$, create a *propagating power domination set of forces* \mathcal{F} of S as follows: Initially, $\mathcal{F} = \emptyset$. For each $w \in N[S] \setminus S$, choose $x \in S \cap N(w)$ and add $x \rightarrow w$ to \mathcal{F} . Then choose a propagating set of forces for $N[S]$ (using the standard color change rule for zero forcing) and add that to \mathcal{F} . Suppose S is a power dominating set of G and \mathcal{F} is a propagating power domination set of forces of S . For a vertex $x \in S$, define S_x to be the set of all vertices w such that there is a sequence of forces $x = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k = w$ in \mathcal{F} ; the empty sequence of forces is permitted, i.e., $x \in S_x$.

Theorem 7.14. For $m \geq 2$, $\text{th}_{\text{pd}}^\times(P_2 \square P_m) = m$ if $m \equiv 0 \pmod{3}$ and $\text{th}_{\text{pd}}^\times(P_2 \square P_m) = m + 1$ if $m \not\equiv 0 \pmod{3}$.

Proof. Let $G = P_2 \square P_m$. By Proposition 7.12, $\text{th}_{\text{pd}}^\times(G) \leq \left\lceil \frac{2}{3} \right\rceil (m+1) = m+1$. If m is divisible by 3, then $\text{th}_{\text{pd}}^\times(G) \leq 3 \left\lceil \frac{m}{3} \right\rceil = 3 \cdot \frac{m}{3} = m$. This proves the upper bound. Next we prove the lower bound.

For any k such that $\text{pt}_{\text{pd}}(G, k) = 1$, $k \geq \gamma(G) = \left\lfloor \frac{m+2}{2} \right\rfloor$ by [3], so $\text{th}_{\text{pd}}^\times(G, k) \geq 2 \left\lfloor \frac{m+2}{2} \right\rfloor \geq m+1$. Thus we need consider only sets S such that $\text{pt}_{\text{pd}}(G; S) \geq 2$.

Arrange G with 2 rows and m columns. Choose S such that $\text{th}_{\text{pd}}^\times(G) = \text{th}_{\text{pd}}^\times(G; S) = |S|(1+p)$ where $p = \text{pt}_{\text{pd}}(G; S)$. Create a propagating power domination set of forces for S by choosing the forcing vertex in the same row whenever there is a choice (row-forcing is preferred). For $x \in S$, we show that $|S_x| \leq 2(1+p)$ and $|S_x| < 2(1+p)$ except under the additional conditions that $|P^{(1)}(S) \cap S_x| = 3$, $|P^{(2)}(S) \cap S_x| = 2$, and $p = 2$.

By the rules of power domination, $|P^{(i)}(S) \cap S_x| \leq |P^{(1)}(S) \cap S_x|$ for $i \geq 2$. Thus $|P^{(1)}(S) \cap S_x| \leq 2$ implies $|S_x| \leq 1 + 2p < 2(1+p)$. So assume $|P^{(1)}(S) \cap S_x| = 3$ and x is in the top row. Denote the east, south, and west neighbors of x by x_E, x_S and x_W , and name additional vertices similarly, according to their direction from x . Since row-forcing is preferred, $x_{SE}, x_{SW} \notin S$.

Let $r \geq 2$ be the first round in which any of x_W, x_S, x_E performs a force. We analyze the situation based on which force(s) occur in round r . To obtain a contradiction, suppose that $x_W \rightarrow x_{WW}$ in round r , which requires $x_{SW} \in P^{[r-1]}(S)$ and x_{SW} is not forced by x_S . If $x_{SW} \in P^{[1]}(S)$, then $x_{WSW} \in S$ and so $x_{WW} \in P^{(1)}(S)$, which is a contradiction. Otherwise, $x_{SW} \in P^{(i)}(S)$ with $1 < i < r$ and $x_{WSW} \rightarrow x_{SW}$ in round i requires that x_{WW} must already be observed before round i , which is a contradiction. Therefore, $x_W \rightarrow x_{WW}$ cannot happen in round r . Similarly, $x_E \rightarrow x_{EE}$ cannot happen in round r .

Now suppose that $x_S \rightarrow x_{SW}$ in round r . This requires $x_{SE} \in P^{[r-1]}(S)$, which in turn requires $x_{ESE}, x_{EE} \in P^{[r-1]}(S)$. So x_E can never force. Then $|P^{(r)}(S) \cap S_x| = 1$ since x_W cannot force in round r , $|P^{(i)}(S) \cap S_x| \leq 2$ for $i \geq r+1$, and $|P^{(i)}(S) \cap S_x| = 0$ for $1 < i < r$.

Thus

$$|S_x| \leq \sum_{i=0}^p |P^{(i)}(S) \cap S_x| \leq 1 + 3 + 0 + \cdots + 0 + 1 + 2(p-r) < 2(1+p).$$

The case $x_S \rightarrow x_{SE}$ in round r is similar.

It remains to consider the case when x_S does not force in round r , and in this case, $|P^{(r)}(S) \cap S_x| \leq 2$. By definition of r , one of x_W, x_E must force in round r , and we have shown that it cannot force along a row. Without loss of generality, let $x_W \rightarrow x_{SW}$ in round r . Necessarily, $x_{WW} \in P^{[r-1]}(S)$ or x_W and x_{SW} are the leftmost vertices in G . We show that $x_{WSW} \in P^{[r]}(S)$ if $x_{WW} \in P^{[r-1]}(S)$, and so in either case x_{SW} never performs a force. If $x_{WW} \in S$, then $x_{WSW} \in P^{[1]}(S) \subset P^{[r]}(S)$. If $x_{WWW} \rightarrow x_{WW}$ in round $r-1$, then $x_{WW} \rightarrow x_{WSW}$ in round r . Note that x_{WSW} cannot force x_{WW} in round $r-1$ because its neighbor x_{SW} is also unobserved until round r . If $x_E \rightarrow x_{SE}$ in round r , a similar proof shows that x_{SE} can never force. Thus $|P^{(i)}(S) \cap S_x| \leq 2$ for $i \geq r+1$, and

$$|S_x| \leq \sum_{i=0}^p |P^{(i)}(S) \cap S_x| \leq 1 + 3 + 0 + \cdots + 0 + 2 + 2(p-r) \leq 2(1+p).$$

If $r \geq 3$, then $|P^{(2)}(S) \cap S_x| = 0$, and $|S_x| < 2(1+p)$. If $r = 2$ and $x_W \not\rightarrow x_{SW}$ or $x_E \not\rightarrow x_{SE}$, then x_W or x_E cannot force in round 2, and $|S_x| \leq 1 + 3 + 1 + 2(p-2) < 2(1+p)$. If $r = 2$ and $x_W \rightarrow x_{SW}$ and $x_E \rightarrow x_{SE}$ in round 2, then as described above, none of x_{SW}, x_{SE} , or x_S can ever force, so $|P^{(i)}(S) \cap S_x| = 0$ for $i \neq 0, 1, 2$ and $|S_x| < 2(1+p)$ if $p > 2$. Thus we have shown that $|S_x| \leq 2(1+p)$ in all cases, and $|S_x| < 2(1+p)$ unless $|P^{(1)}(S) \cap S_x| = 3$, $|P^{(2)}(S) \cap S_x| = 2$, and $p = 2$.

Since $|S_x| \leq 2(1+p)$ in all cases, $2m = \sum_{x \in S} |S_x| \leq |S|2(1+p) = 2 \text{th}_{\text{pd}}^\times(G)$ and $\text{th}_{\text{pd}}^\times(G) \geq m$ for all G . Furthermore $\text{th}_{\text{pd}}^\times(G) > m$ unless $|P^{(1)}(S) \cap S_x| = 3$, $|P^{(2)}(S) \cap S_x| = 2$, and $p = 2$ for all $x \in S$. In this case $|S_x| = 6 = 2(1+p)$ for all $x \in S$ and hence $2m = \sum_{x \in S} |S_x| = 6|S|$ and $m = 3|S|$, and m is divisible by 3. \square

8 Product throttling for PSD zero forcing

The next result implies that product throttling for Z_+ is nontrivial for $\text{th}_+^\times(G)$ and $\text{th}_+^*(G)$, based on results from Cops and Robbers. As in Section 4, define $k_+(G, p) = \min\{|S| : \text{pt}_+(G; S) = p\}$.

Theorem 8.1. [10] *Let $S \subseteq V(G)$ be a PSD zero forcing set. Then $|S| \geq c(G)$, $\text{capt}(G; S) \leq \text{pt}_+(G; S)$, and $c(G) \leq Z_+(G)$. If T is a tree and $S \subseteq V(T)$, then $\text{capt}(T; S) = \text{pt}_+(T; S) = \text{ecc}(S)$ for $S \subseteq V(T)$. Thus $\text{capt}_k(T) = \text{pt}_+(T, k) = \text{rad}_k(T)$.*

Note that $c(G)$ and $Z_+(G)$ can be substantially different, resulting in very different product throttling numbers. For example, $Z_+(K_n) = n-1$ whereas $c(K_n) = 1$.

8.1 Initial cost definition $\text{th}_+^\times(G)$

Let G be a graph of order n . Define $\text{th}_+^\times(G, k) = k(1 + \text{pt}_+(G, k))$ and

$$\text{th}_+^\times(G) = \min_{Z_+(G) \leq k \leq n} \text{th}_{\text{pd}}^\times(G, k) = \min_{Z_+(G) \leq k \leq n} k(1 + \text{pt}_+(G, k)).$$

The results in the next remark follow immediately from the universal forms of these results in Section 4.1.

Remark 8.2. Let G be a graph of order n .

- (1) $Y(G) \leq \text{th}_+^\times(G) \leq n$. If G is connected and $n \geq 2$, then $Y(G) + 1 \leq \text{th}_+^\times(G)$.
- (2) $\text{th}_Y^\times(G) \leq Y(G)(1 + \text{pt}_Y(G))$.
- (3) $\text{th}_Y^\times(G) \leq 2k_Y(G, 1)$.
- (4) $\text{th}_+^\times(G) \geq \min_{Z_+(G) \leq k \leq n} k(1 + \text{rad}_k(G))$.
- (5) If $Z_+(G) \geq \frac{n}{2}$, then $\text{th}_+^\times(G) = n$. Examples include $\text{th}_+^\times(K_n) = n$ and $\text{th}_+^\times(Q_d) = 2^d$.

The next result follows from Theorems 2.4 and 8.1 since a tree is chordal.

Corollary 8.3. *Let T be a tree. Then $\text{th}_+^\times(T, k) = \text{th}_c^\times(T, k)$ and $\text{th}_+^\times(T) = \text{th}_c^\times(T) = 1 + \text{rad } T$. In particular, $\text{th}_+^\times(P_n) = 1 + \lceil \frac{n-1}{2} \rceil$.*

For any graph G , $\text{th}_c^\times(G, k) \leq \text{th}_+^\times(G, k)$ for $c(G) \leq k \leq n$.

The next result follows from Remark 4.10 in the case G is connected, and the analysis of the disconnected case is straightforward (for $n = 4$ see [25]).

Remark 8.4. Let G be a graph.

- (1) $\text{th}_+^\times(G) = 1$ if and only if $G = K_1$.
- (2) $\text{th}_+^\times(G) = 2$ if and only if $Z_+(G) = 1$ and $\text{pt}_+(G) = 1$ (i.e., $G = K_{1,n-1}$) or $G = 2K_1$.
- (3) $\text{th}_+^\times(G) = 3$ if and only if G satisfies exactly one of the following conditions:
 - (a) $G = K_3$, $G = K_2 \dot{\cup} K_1$, or $G = 3K_1$.
 - (b) $Z_+(G) = 1$ and $\text{pt}_+(G, 1) = 2$ (i.e., G is a tree and $\text{rad } G = 2$).
- (4) $\text{th}_+^\times(G) = 4$ if and only if G satisfies at least one of the following conditions:
 - (a) $G = K_4$, $G = K_3 \dot{\cup} K_1$, $G = K_2 \dot{\cup} 2K_1$, or $G = 4K_1$.
 - (b) $\text{pt}_+(G, 2) = 1$ and $\text{pt}_+(G, 1) > 3$.
 - (c) $Z_+(G) = 1$ and $\text{pt}_+(G, 1) = 3$.

8.2 No initial cost definition $\text{th}_+^*(G)$

Let G be a connected graph of order $n \geq 2$. Define $\text{th}_+^*(G, k) = k \text{pt}_+(G, k)$ and

$$\text{th}_+^*(G) = \min_{Z_+(G) \leq k < n} k \text{pt}_+(G, k) = \min_{Z_+(G) \leq k < n} \text{th}_+^*(G, k).$$

The results in the next remark follow immediately from the universal forms of these results in Section 4.1.

Remark 8.5. Let G be a connected graph of order $n \geq 2$.

- (1) $Z_+(G) \leq \text{th}_+^*(G) \leq k_+(G, 1) \leq n - 1$.
- (2) $\text{th}_+^*(G) \leq Z_+(G) \text{pt}_+(G)$ and $\text{th}_+^*(G) = Z_+(G)$ if and only if $\text{pt}_+(G) = 1$.
- (3) If G' is a connected graph of order $n' \geq 2$ that is an induced subgraph of G , and $\text{th}_+^*(G') = k_+(g, 1)$, then
$$\text{th}_Y^*(G) \leq n - n' + \text{th}_Y^*(G').$$
- (4) $\text{th}_Y^*(G) \geq \text{th}_Y(G) - 1$.
- (5) If $\text{th}_Y(G) = \text{th}_Y(G, 1) = 1 + \text{pt}_Y(G, 1)$ or $\text{th}_Y(G) = \text{th}_Y^*(G, k_Y(G, 1)) = k_Y(G, 1) + 1$, then $\text{th}_Y^*(G) = \text{th}_Y(G) - 1$.
- (6) $\text{th}_+^*(G) \geq \min_{Z_+(G) \leq k < n} k \text{rad}_k(G)$

The next result follows from Theorem 8.1.

Corollary 8.6. *Let T be a tree. Then $\text{th}_+^*(T, k) = \text{th}_c^*(T, k)$ and*

$$\text{th}_+^*(T) = \text{th}_c^*(T) = \min_{Z_+(G) \leq k < n} k \text{rad}_k(G).$$

For any graph G , $\text{th}_c^*(G, k) \leq \text{th}_+^*(G, k)$ for $c(G) \leq k < n$.

The next remark lists values of $\text{th}_+^*(G)$ for various families of graphs G .

Remark 8.7. PSD propagation time agrees with capture time for paths and cycles, so the values for these graphs follow from Remark 6.2. For the remaining graphs G , it is well known that $\text{pt}_+(G) = 1$ [32].

- (1) For $n \geq 2$, $\text{th}_+^*(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.
- (2) For $n \geq 4$, $\text{th}_+^*(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$.
- (3) For $n \geq 2$, $\text{th}_+^*(K_n) = Z_+(K_n) = n - 1$.
- (4) For $1 \leq p \leq q$, $\text{th}_+^*(K_{p,q}) = Z_+(K_{p,q}) = p$.
- (5) For $d \geq 1$, $\text{th}_+^*(Q_d) = Z_+(Q_d) = 2^{d-1}$ where Q_d is the hypercube.

Let $\alpha(G)$ denote the independence number of G . The next result is analogous to Proposition 2.9 in [15].

Proposition 8.8. $\text{th}_+^*(G) \leq n - \alpha(G)$.

Proof. Let S' be a maximum independent set of vertices (so $|S'| = \alpha(G)$) and let $S = V(G) \setminus S'$. Then $\text{pt}_+(G; S) = 1$ and $\text{th}_+^*(G) \leq n - \alpha(G)$. \square

Extreme values of $\text{th}_+^*(G)$

Remark 8.9. Let G be a connected graph of order $n \geq 2$. From Remark 4.10 we have:

- (1) $\text{th}_+^*(G) = 1$ if and only if $Z_+(G) = 1$ and $\text{pt}_+(G) = 1$, i.e., $G = K_{1,n-1}$.
- (2) $\text{th}_+^*(G) = 2$ if and only if G satisfies one of the following conditions:
 - (a) $Z_+(G) \leq 2$, $\text{pt}_+(G, 2) = 1$, and $\text{pt}_+(G, 1) > 2$.
 - (b) $Z_+(G) = 1$ and $\text{pt}_+(G, 1) = 2$ (i.e., G is a tree and $\text{rad}(G) = 2$).
- (3) $\text{th}_+^*(G) = 3$ if and only if G satisfies one of the following conditions:
 - (a) $Z_+(G) \leq 3$, $\text{pt}_+(G, 3) = 1$, $\text{pt}_+(G, 2) \geq 2$, and $\text{pt}_+(G, 1) > 3$.
 - (b) $Z_+(G) = 1$, $\text{pt}_+(G, 1) = 3$, and $\text{pt}_+(G, 2) \geq 2$ (so G is a tree, $\text{rad}(G) = 3$, and no two vertices dominate G).

Remark 8.10. Since we require G to be connected and of order at least two, $\text{th}_+^*(G) \leq n-1$. Furthermore, $\text{th}_+^*(G) = n-1$ if and only if $G = K_n$ because $\text{th}_+^*(G) = n-1$ implies $\alpha(G) = 1$ by Proposition 8.8, so G is complete (and $\text{th}_+^*(K_n) = n-1$).

Define the set \mathcal{H} to be the set of all connected graphs G of order at least two such that $\alpha(G) = 2$ and G does not have an induced 5-cycle or house subgraph (see Figure 8.1).

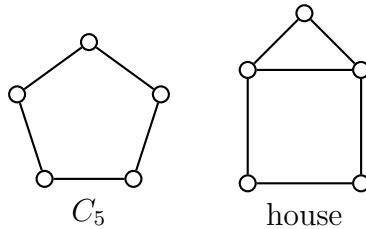


Figure 8.1: Graphs forbidden as induced subgraphs of $G \in \mathcal{H}$.

Remark 8.11. Suppose $G \in \mathcal{H}$ and let $S \subset V(G)$. Since $\alpha(G) = 2$, $G - S$ is connected or consists of two connected components. If $G - S$ has two connected components, then each is a clique.

It is straightforward to verify the statements in the next lemma.

Lemma 8.12. *Let $n \geq 3$.*

- (1) *For $k = 1, \dots, n-2$, $k(n-k-1) \geq n-2$.*
- (2) *For $k = 2, \dots, n-2$, $k \frac{n-k}{2} \geq n-2$.*

Lemma 8.13. [15] *Let $G \in \mathcal{H}$ and let $S \subset V(G)$ be such that $G - S$ is connected and $|S| \leq n-3$. Then $\text{pt}_+(G; S) = n - |S| - 1$.*

Lemma 8.14. *Let $G \in \mathcal{H}$ and let $S \subset V(G)$ be such that $G - S$ consists of two nonempty connected components, $G[W_1]$ and $G[W_2]$. Then $|S^{(j)} \cap W_i| \leq 1$ for $i = 1, 2$ and $j = 1, \dots, \text{pt}_+(G; S)$.*

Proof. We show that $|S^{(j)} \cap W_i| \geq 2$ and $\alpha(G) = 2$ implies G has an induced house graph or 5-cycle; the construction is illustrated in Figure 8.2. Suppose that $w, w' \in S^{(j)} \cap W_1$ and $w \neq w'$. Then there exist $x, x' \in S^{(j-1)}$ such that $x \rightarrow w$ and $x' \rightarrow w'$ in round j . This means that $xw, x'w' \in E(G)$ and $xw', x'w \notin E(G)$. Since there are no edges between W_1 and W_2 and $G[W_1]$ is a clique, $x, x' \in S$ and $w, w' \in E(G)$. Let $u \in W_2$. Then $wu, w'u \notin E(G)$. Since $\alpha(G) = 2$, $wu, wx' \notin E(G)$ implies $ux' \in E(G)$ and $w'u, w'x \notin E(G)$ implies $ux \in E(G)$. Thus $G[\{u, x, w, w', x'\}]$ is a 5-cycle or house graph depending on whether $xx' \notin E(G)$ or $xx' \in E(G)$. \square

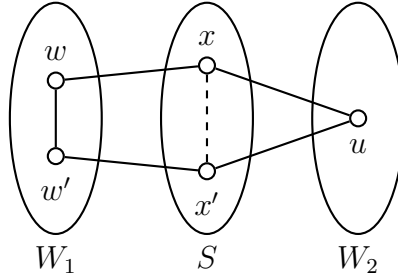


Figure 8.2: Diagram for Lemma 8.14. The dotted edge may or may not be present.

Theorem 8.15. *Let G be a connected graph of order $n \geq 2$. Then $\text{th}_+^*(G) = n - 2$ if and only if $G \in \mathcal{H}$.*

Proof. Let H denote the house graph. Observe that $\text{th}_+^*(C_5) = 2$, $\text{pt}_+(C_5, 2) = 1$, $\text{th}_+^*(H) = 2$, and $\text{pt}_+(H, 2) = 1$. If G has an induced C_5 or H , then $\text{th}_+^*(G) \leq n - 5 + 2 = n - 3$ by Remark 4.8 applied to PSD throttling. If $\alpha(G) \geq 3$, then $\text{th}_+^*(G) \leq n - 3$ by Proposition 8.8, and $\alpha(G) = 1$ implies $G = K_n$ and $\text{th}_+^*(G) = n - 1$. Thus $G \notin \mathcal{H}$ implies $\text{th}_+^*(G) \neq n - 2$.

Now assume $G \in \mathcal{H}$ and let $S \subset V(G)$ such that $\text{th}_+^*(G) = \text{th}_+^*(G; S) = \text{th}_+^*(G, k)$ for $k = |S|$. Since G is connected and $\alpha(G) = 2$, $n \geq 3$, $\text{th}_+^*(G) \leq n - 2$, and it suffices to show that $\text{th}_+^*(G, k) \geq n - 2$. Observe first that $\text{th}_+^*(G, k) \geq n - 2$ for $k = n - 2, n - 1$, so assume $k \leq n - 3$. If $G - S$ is connected, then $\text{pt}_+(G; S) = n - k - 1$ by Lemma 8.13, so $\text{th}_+^*(G, k) \geq k(n - k - 1) \geq n - 2$ by Lemma 8.12. Suppose $G - S$ is not connected. Then $G - S$ has only two connected components. By Lemma 8.14, $|S^{(j)} \cap W_i| \leq 1$ for $i = 1, 2$ and $j = 1, \dots, \text{pt}_+(G; S)$, so $|S^{(j)}| \leq 2$ for $j = 1, \dots, \text{pt}_+(G; S)$. Thus $\text{pt}_+(G; S) \geq \frac{n-k}{2}$, and $\text{th}_+^*(G; S) = k \text{pt}_+(G; S) \geq k \frac{n-k}{2} \geq n - 2$ for $k \geq 2$ by Lemma 8.12. If $k = 1$, then G is a tree, $\alpha(G) = 2$ implies $G = P_3$ or $G = P_4$, and thus $\text{th}_+^*(G) = n - 2$. \square

9 Comparisons of product throttling numbers

Recall that $\text{capt}(G; S) \leq \text{pt}_+(G; S) \leq \text{pt}(G; S)$ for any $S \subseteq V(G)$ (see Theorem 8.1 for the first inequality), and thus

$$\text{th}_c^\times(G) \leq \text{th}_+^\times(G) \leq \text{th}^\times(G) \text{ and } \text{th}_c^*(G) \leq \text{th}_+^*(G) \leq \text{th}^*(G).$$

From the definitions, $\text{pt}_{\text{pd}}(G; S) \leq \text{pt}(G; S)$ for any $S \subseteq V(G)$, so

$$\text{th}_{\text{pd}}^\times(G) \leq \text{th}^\times(G) \text{ and } \text{th}_{\text{pd}}^*(G) \leq \text{th}^*(G).$$

We establish the noncomparability of the following pairs of parameters: $\text{th}_c^\times(G)$ and $\text{th}_{\text{pd}}^\times(G)$, $\text{th}_c^*(G)$ and $\text{th}_{\text{pd}}^*(G)$, $\text{th}_+^\times(G)$ and $\text{th}_{\text{pd}}^\times(G)$, $\text{th}_+^*(G)$ and $\text{th}_{\text{pd}}^*(G)$.

Example 9.1. Let $r, h \geq 2$. Recall that $T_{r,h}$ denotes a full r -ary tree of height h (see Proposition 6.8). Since $Z_+(T_{r,2}) = c(T_{r,2}) = 1$ and $\text{capt}(T_{r,2}) = \text{pt}_+(T_{r,2}) = 2$, $\text{th}_+^\times(T_{r,2}) = \text{th}_c^\times(T_{r,2}) = 3$ and $\text{th}_+^*(T_{r,2}) = \text{th}_c^*(T_{r,2}) = 2$. Since $\gamma_P(T_{r,2}) = \gamma(T_{r,2}) = r$, $\text{th}_{\text{pd}}^\times(T_{r,2}) = 2r > 3$ and $\text{th}_{\text{pd}}^*(T_{r,2}) = r > 2$ for $r \geq 3$.

Example 9.2. The *generalized wheel* $GW(k, r)$ with $k \geq 4$ and $r \geq 1$ is the graph of order $kr + 1$ constructed from $C_k \square P_r$ by adding a new vertex adjacent to every vertex in one end copy of C_k ; Figure 9.1 shows $GW(6, 2)$. Let $k \geq 5$, so $\gamma(GW(k, 2)) \geq 3$. It is immediate that $\text{th}_{\text{pd}}^\times(GW(k, 2)) = 3$ and $\text{th}_{\text{pd}}^*(GW(k, 2)) = 2$, whereas $c(GW(k, 2)) = 2$ and $\text{capt}(GW(k, 2)) \geq 2$, so $\text{th}_+^\times(GW(k, 2)) \geq \text{th}_c^\times(GW(k, 2)) \geq 6$ and $\text{th}_+^*(GW(k, 2)) \geq \text{th}_c^*(GW(k, 2)) \geq 3$.

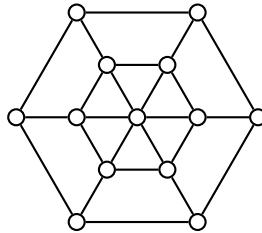


Figure 9.1: The generalized wheel $GW(6, 2)$.

From Theorem 5.3, Theorem 3.12, and Observation 3.1, it follows that for any graph G ,

$$\text{th}^*(G) \geq \frac{n}{2} \geq \gamma(G) \geq \text{th}_{\text{pd}}^*(G).$$

Moreover, there exist graphs G such that $\text{th}^*(G) = \frac{n}{2} = \gamma(G) = \text{th}_{\text{pd}}^*(G)$. For example, a straightforward verification shows $\text{th}^*(C_4) = 2$, $\gamma(G) = 2$ and $\text{th}_{\text{pd}}^*(G) = 2$. We now combine results from previous sections to obtain the following characterization of graphs G such that $\text{th}^*(G) = \frac{n}{2} = \gamma(G) = \text{th}_{\text{pd}}^*(G)$.

Proposition 9.3. *Let G be a connected graph of order n such that $\text{th}_{\text{pd}}^*(G) = \frac{n}{2}$. Then $\text{th}^*(G) = \frac{n}{2} = \gamma(G)$.*

Proof. By Theorem 3.14, $\text{th}_{\text{pd}}^*(G) = \frac{n}{2}$ if and only if $G = (H \circ K_1) \circ K_1$ for some connected graph H , $G = C_4 \circ K_1$, or $G = C_4$. To conclude the proof we show that G is a matched-sum graph, and apply Theorem 5.12 to conclude $\text{th}^*(G) = \frac{n}{2}$. If $G = C_4$, then $G = H_1 M^+ H_2$ where $H_1 = K_2$, $H_2 = K_2$ and M is any matching between $V(H_1)$ and $V(H_2)$. Now suppose $G = H_1 \circ K_1$ for some connected graph H_1 . Let $V(H_2) = V(G) \setminus V(H_1)$. For each $v \in V(H_2)$, v has a unique neighbor u_v in G ; note that $u_v \in V(H_1)$. Let $M = \{vu_v : v \in V(H_2)\}$. Then G is a matched-sum graph, specifically, $G = H_1 M^+ H_2$. \square

Corollary 9.4. *A connected graph G satisfies $\text{th}^*(G) = \frac{n}{2} = \gamma(G) = \text{th}_{\text{pd}}^*(G)$ if and only if $G = (H \circ K_1) \circ K_1$ for some connected graph H , $G = C_4 \circ K_1$, or $G = C_4$.*

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