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Sensitivity boundary integral equations with applications in engineering mechanics

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Sensitivity boundary integral equations with applications in engineering mechanics

by

Daming Zhang

A dissertation submitted to the graduate faculty in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Engineering Mechanics
Major Professor: Frank J. Rizzo

Iowa State University
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CHAPTER 1. INTRODUCTION

With the advent of the high speed computer age, what is now the boundary integral equation (BIE) method was first suggested as a modern numerical analysis technique by Jaswon for potential problems [1]. After Rizzo used it to solve elastostatic problems in the 1960's [2], the boundary integral equation method gave rise to the boundary element method (BEM), and it became a real engineering tool. Some mature concepts in the well-known finite element method (FEM) were also introduced into BEM [3-4]. Through 30 year's development, the BIE/BEM method has found more and more engineering applications in engineering mechanics. These applications include, but are not limited to, elastodynamics, fracture mechanics, contact problems, fluid flow, design sensitivity analysis and optimization, and inverse problems such as flaw detecting [5-14].

While the BIE method is a viable alternative to FEM for most engineering mechanics problems, some applications in the field such as acoustics, wave propagation, fracture mechanics, design sensitivity analysis and optimization are especially suited for BIE/BEM treatment. This research is about design sensitivity analysis and its application to fracture mechanics.

1.1. Sensitivity Analysis and Sensitivity BIE

In ordinary engineering structural analysis, the response of structures, such as displacements, stresses, natural frequencies, etc., under certain boundary conditions including
constraints and loads, are obtained. These responses can be used by engineers and designers to estimate the mechanical performance of an engineering design. When a specific design needs to be improved, a powerful tool is needed to give directions for the modifications. This tool is called design sensitivity analysis. The automated procedure of performing this kind of repeated ordinary structural analysis, design sensitivity analysis, and redesign to obtain an optimum design is called structural shape optimization. The design sensitivity analysis, in addition to other useful applications, is by itself, a key step in structural shape optimization.

Generally speaking, design sensitivity analysis is a kind of numerical analysis technique performed to study the sensitivity of a specific design of the structure, or the gradient of the responses of the structure, to the possible changes of the design parameters. Those structural responses, also known as design constraints, could be the displacements, stresses, or forces of the structure in static analysis. The responses could also be displacements, velocities, accelerations, stresses, or forces of the structure in frequency response analysis, etc. Those design parameters, also known as design variables, could be in one of two main categories, sizing/property design variables or shape design variables.

In sizing/property design sensitivity analysis, the design sensitivity analysis results provide the derivatives of the responses of the structure with respect to the sizing design variables, like areas of cross-section, thickness, moment of inertia, etc. In shape design sensitivity analysis, the design sensitivity results provide the derivatives of the responses of the structure with respect to the design variables which are the functions of perturbation of grid points. Shape design sensitivity analysis is apparently more complicated than sizing design sensitivity analysis, but, on the other hand, a set of perturbations at different grid points can also be combined to realize a sizing/property change.

As one of the most effective numerical analysis techniques, the finite element method has made significant progress in design sensitivity analysis and structural optimization [15-17].
Most commercial FEM softwares have the function of sensitivity analysis and optimization. These FEM softwares work well for many engineering problems. Most of these have a method of obtaining at least a good approximate expression for the rate of change of the stiffness matrix with respect to the design variables. Otherwise, the cost of doing shape optimization by simply remeshing and doing repeated FEM analysis would be prohibitive.

Shape design sensitivity analysis can be relatively easier to perform in BEM than in FEM. One reason for this is that shape changes often involve changes in only part of the boundary of a domain. And since BEM involves boundary meshes only, fewer discretization changes are generally needed with BEM sensitivity analysis. Another reason for this is that there is an analytical expression which analytically relates all of the boundary data, i.e. the basic boundary integral equation for the whole system. This equation can be differentiated analytically to get a derivative formula, i.e. the basic equation for the sensitivity analysis, prior to the discretization and numerical integration. The third motivation for developing the BEM in sensitivity analysis is the high accuracy of the method which can mean fewer design iterations.

The early efforts of the boundary element method for sensitivity analysis include the adjoint variable approach, which was borrowed from the FEM, and the implicit differentiation method. The former includes those by Mota Soares et al. [18], Choi and Kwak [19], and Meric [20-21]. The latter includes those by Wu [22], and Kane [23]. The difference between Wu’s work and Kane’s work is that Wu obtained the derivative of the system matrix by a finite difference, and Kane generated the sensitivities by differentiating the system matrix analytically in which the differentiation of the field variable shape functions is needed.

The most direct and elegant sensitivity BIE method, which takes full advantage of the analytical BIE structure is found in Barone and Yang [24-25]. They developed analytical expressions for design sensitivities by differentiating the ordinary boundary integral equations
instead of the coefficient matrix. Only this work is called sensitivity-boundary-integral-equation method in this thesis, because a derivative boundary integral equation does exist explicitly. One aspect of sensitivity calculations via BIE methods, which is very important for reducing the computational costs, is that the coefficient matrix for the sensitivity BIE is exactly the same as that for the corresponding ordinary BIE. Thus, forming and decomposing of the coefficient matrix needs to be done only once if the structural responses and their derivatives are wanted at the same time.

The sensitivity BIE has been of great interest in recent years because of the advantages mentioned above. It has been studied in every aspect and used for many engineering applications [8-14]. However these studies and applications were focused on one kind of boundary integral equations, the conventional BIE (CBIE). No investigation has been done to the author's knowledge, for another kind of BIE, the hypersingular BIE (HBIE), which is necessary for solving problems in a domain involving thin-body structures or cracks, or for overcoming fictitious eigenfrequencies in acoustic or elastic wave problems.

1.2. The Role of the Sensitivity HBIE

In static analysis, the conventional BIE will fail when it is used for problems defined in a domain with thin-body structures or cracks. The problem, analytically, has to do with two surfaces of a crack which occupy the same plane. With a thin-body domain, these surfaces occupy nearly the same plane. In these kinds of problems, the coefficient matrix of the BIE method is singular or nearly-singular because there will be rows which are the same or nearly the same.
Many efforts have been devoted to singular-coefficient-matrix difficulties. Cruse [7], Jia [26], and Smith and Aliabadi [27] used the multidomain method. This method divides the domain along the crack surface to avoid the singular coefficient matrix. The numerical implementation becomes more complicated and the coefficient matrix becomes larger because the nodes on the cutting surfaces are considered twice.

A general and popular method for solving crack problems is to consider another kind of the boundary integral equation, the so-called hypersingular BIE. The HBIE is obtained by taking spatial derivatives of the conventional BIE and using Hooke's law. Degeneracy of the coefficient matrix is avoided by keeping just one side of the crack surfaces in the HBIE [29], or by using CBIE in the one side and HBIE in the other side of the crack or thin-body surfaces [30-31], or by a combination of CBIE and HBIE on both sides of the crack or thin-body surfaces [32]. The hypersingular integrals in the HBIE which are induced by the differentiation are regularized by the singularity-subtraction method and the added-back terms can be converted from surface integrals to line integrals by using Stokes' theorem [29]. Recently, many researchers have contributed to the study and application of the HBIE [33-35].

Similarly, in the sensitivity analysis, the sensitivity CBIE will fail for problems defined in the domain containing thin-body structures or cracks. The sensitivity HBIE is then necessary in solving this kind of problem to avoid degenerate coefficient matrices. While the singularity-subtraction method is still a powerful tool for the regularization of the hypersingular integrals, the added-back terms present more difficulties. A new strategy is employed in the present work to regularize the sensitivity HBIE.

Another important application of the HBIE is in the study of exterior acoustic and elastic wave problems for circumventing the so-called fictitious eigenfrequency difficulty [36]. Among the methods for dealing with this difficulty, Burton and Miller's Composite BIE
formulation has been shown to be the most reliable and effective approach. Therefore, the sensitivity HBIE can also play an important role in the acoustic design sensitivity analysis [38-41] which is a powerful tool in noise reduction, especially for the thin-panel structures in automobile and aircraft industries.

1.3. Present Work

After an overall introduction of the sensitivity analysis and sensitivity BIE, Chapter 2 introduces the sensitivity CBIE for elastostatics. The sensitivity CBIE formulas are derived from the ordinary CBIE via the singularity-subtraction method. The singularity orders of the sensitivity CBIE kernels are proved to be same as those of the ordinary CBIE kernels. Then, the expressions for the sensitivity CBIE kernels, i.e., the derivatives of the ordinary CBIE kernels, are given in the Appendix A. The numerical expressions for the geometric changes are also derived in this Chapter. These expressions will also be used for the sensitivity HBIE in Chapter 3 and Chapter 4. Two numerical examples are presented in the last section of Chapter 2 to show the correctness of the formulas and codes developed in this chapter, and to show the accuracy of the sensitivity CBIE.

The sensitivity CBIE has been used in engineering successfully, for example, in shape optimization [1,2]. But for crack problems, or problems with a thin-body domain, the sensitivity CBIE will break down because of the degeneracy of the coefficient matrix [3]. The sensitivity hypersingular BIE is needed to address these kinds of problems. One kind of sensitivity HBIE, the global formula, is formulated in Chapter 3.

At first it was tried to derive the sensitivity HBIE by a method similar to that used for the sensitivity CBIE, the singularity-subtraction method. However because of the
hypersingularity of the HBIE kernels, that conventional method become too difficult for the derivation of the sensitivity HBIE. A new strategy, starting from the weakly singular form was developed, and this is displayed in Chapter 3 to simplify the derivation. This new strategy can be used for all kinds of BIEs, including CBIE and HBIE, for static problems or dynamic problems, elastic analysis or plastic analysis, etc. However, the expressions for the sensitivity HBIE kernels are considerably more complicated than for the sensitivity CBIE. Regardless, all of these expressions are derived in this chapter, and the details are given in the appendices.

One of the two example problems previously used in Chapter 2, namely, a wedge with increasing length is used again in Chapter 3 to facilitate the comparison of the numerical performance of the global sensitivity HBIE and the sensitivity CBIE.

There are many possible combinations of using BIEs to solve crack problems. Some of them work well, others do not. The first section of Chapter 4 reviews these options, and then discusses one of them in detail, the single surface crack model. The local form of the hypersingular BIE is needed for this single surface model because the identities, which are used in the global formula of the HBIE to treat the hypersingular and strong singular integrals, no longer hold for a non-closed surface. Line integrals are therefore needed in the local formula to treat the singular integrals.

The second section of Chapter 4 contributes to the local formula of the sensitivity HBIE formulation. The new strategy developed in Chapter 3, starting with the derivation from the weakly-singular form, is used again in this chapter to facilitate the regularization of the local formula of the sensitivity HBIE. All of the local sensitivity HBIE formulas are derived in this chapter, and the detailed expressions are given in the Appendices.

A same numerical example, the wedge with increasing is computed once more in the last section of Chapter 4, using the local sensitivity HBIE formulas and codes developed in
this chapter. The results from the local sensitivity HBIE agree very well with those from the ordinary HBIE.

Because of the presence of cracks, meshes for fracture mechanics are usually very fine to describe the steep variations of the displacement and stress distributions around cracks. Thus, fracture mechanics analyses, with a large number of elements, are always computationally intensive.

An alternate approach is investigated in the first part of Chapter 5 to reduce the CPU time required in the computation of the stress intensity factor versus crack size curves. This approach calculates not only the stress intensity factors, but also their derivatives. The CPU time is reduced because the expenses for computing a function value plus a derivative value is less than those for computing two function values. While the hypersingular BIE has been highly successful in modeling crack problems and computing stress intensity factors, the sensitivity HBIE is a powerful tool in modeling crack problems for computing the derivatives of the stress intensity factors. Three numerical examples are presented in the second section of Chapter 5 to show the accuracy of the computation and the effectiveness of the approach, for stress-intensity-factor sensitivity calculations.

The final chapter gives some concluding remarks on this research, and some ideas for the future work in this area.
CHAPTER 2. SENSITIVITY CBIE FOR ELASTOSTATICS

This chapter introduces the sensitivity CBIE for elastostatics. The sensitivity CBIE formulas are derived from the ordinary CBIE via what is called the singularity-subtraction method. The orders of singularity of the sensitivity CBIE kernels are proved to be same as those of the ordinary CBIE kernels. The expressions of the sensitivity CBIE kernels, i.e., the derivatives of the ordinary CBIE kernels, are given in the Appendix A. The numerical expressions of the geometric changes are also derived in this chapter. These expressions will also be used for the sensitivity HBIE in Chapter 3 and Chapter 4. Two numerical examples are presented to show the correctness of the formulas and codes developed in this chapter, and to show the accuracy of the sensitivity CBIE.

2.1. Sensitivity CBIE Formulation

The sensitivity CBIE formulas are derived in this section by using singularity subtraction method. In this method, a manipulation of the added back terms is done to make the resulting formulas weakly singular.

2.1.1. Ordinary CBIE

It is well known that for an elastostatic problem, say, a beam with a cut (Fig. 2.1), the conventional BIE can be written in the form
\[ C_{ij} u_j(P) = \int_S U_{iq}(P,Q) t_j(Q) ds - \int_S T_{iq}(P,Q) u_j(Q) ds \]  
\[ (2.1) \]

where, \( S \) is the boundary of the domain. \( u_j \), the displacements, and \( t_j \), the tractions on the boundary \( S \), are basic variables of the equation. Half of them are known as boundary conditions. Others are unknowns, which will be solved from the equation. If \( r \) is defined as the distance between \( P \), the source point, and \( Q \), the field point, \( U_{iq}(P,Q) \) and \( T_{iq}(P,Q) \) are the singular, fundamental kernels, which have singularities such that

\[ U_{iq}(P,Q) = O\left(\frac{1}{r}\right) \]

\[ T_{iq}(P,Q) = O\left(\frac{1}{r^2}\right). \]

Because of the order of singularity in \( T_{iq} \), the second integral in (2.1) (et. seq.) is interpreted as a Cauchy principal value (CPV) [28]. \( C_{ij} \) are evaluated to be 0.5 when \( i \) equals \( j \), and 0 for other cases if \( P \) is located on a smooth surface.

Figure 2.1 A beam with a semicircular cut
After discretization of the boundary $S$, evaluation of the integrals, and rearrangement, equation (2.1) can be written as a linear system

$$[A]\{x\} = \{c\} \quad (2.2)$$

in which the vector $\{x\}$, which consists of the unspecified surface displacements and tractions, could be obtained as

$$\{x\} = [A]^{-1}\{c\}. \quad (2.3)$$

Considering the singular integrals in equation (2.1) induced by the singularities of kernels, computations are usually done after writing equation (2.1) in the regularized form

$$\int_S U_y(P,Q)t_j(Q)\,ds = \int_S T_y(P,Q)\left[u_j(Q) - u_j(P)\right]\,ds. \quad (2.4)$$

All integrals are at most weakly singular in this form. They can be evaluated via Gaussian quadrature when the integrals over the standard flat elements, to which all integrands are mapped, are written in polar coordinates.

### 2.1.2. Sensitivity CBIE

All physical variables (displacements, $u_j$, tractions, $t_j$, etc.) are determined by equation (2.3), i.e., the ordinary CBIE, for a specific design. When some design variables are changed in the design modification process (taking a single design variable, $b$, as an example for our beam with cut example, Fig. 2.2), the corresponding change of physical variables with respect to $b$, could be predicted by the so called sensitivity conventional BIE.

The sensitivity CBIE formula could be obtained by taking the derivatives of equation (2.1) with respect to the design variable $b$, i.e.,
$C_{ij} \dot{u}_j(P) = \int_s U_{ij}(P,Q) t_j(Q) ds - \int_s T_{ij}(P,Q) \dot{u}_j(Q) ds$ \\
$+ \int_s \dot{U}_{ij}(P,Q) t_j(Q) ds - \int_s \dot{T}_{ij}(P,Q) u_j(Q) ds$ \\
$+ \int_s U_{ij}(P,Q) \dot{t}_j(Q) ds - \int_s T_{ij}(P,Q) \dot{u}_j(Q) ds$ \hspace{1cm} (2.5)

where $\dot{} = d / db$. There are two kinds of new unknowns in equation (2.5), $\dot{u}_j$ and $\dot{t}_j$, i.e.,
the derivatives of $u_j$ with respect to $b$, and the derivatives of $t_j$ with respect to $b$. If an
ordinary CBIE was applied first, $u_j$ and $t_j$ would all be knowns. The last four terms of
equation (2.5) are also known, since they do not involve $\dot{u}_j$ or $\dot{t}_j$. Thus, after known terms
$D_i$ are defined by
\[ D_i = \int_s \dot{U}_q(P,Q) t_j(Q) ds - \int_s \dot{T}_q(P,Q) u_j(Q) ds \]
\[ + \int_s U_q(P,Q) t_j(Q) ds - \int_s T_q(P,Q) u_j(Q) ds \]  
(2.6)
equation (2.5) can be written as
\[ C_{ij} \dot{u}_j(P) = \int_s U_q(P,Q) t_j(Q) ds - \int_s T_q(P,Q) u_j(Q) ds + D_i. \]  
(2.7)
Comparing equation (2.7) and equation (2.1), the sensitivity CBIE can be rearranged to the discretized algebraic form
\[ [A] \{ \dot{x} \} = \{ c \} + \{ D \} \]  
(2.8)
where \([A]\) is the same matrix as in equation (2.2).

Now \( D_i \), as defined by equation (2.6), contains the strong singularities in the kernels \( \dot{U}_q \), \( T_q \) and \( \dot{T}_q \). The integrals involving these kernels exist, however, when the displacements \( u_j \) are sufficiently continuous. A computational form containing only weakly-singular integrals is easily obtained by the singularity subtraction method if \( \dot{U}_q \) is supposed to have the same singularity order as \( U_q \), and \( \dot{T}_q \) is supposed to have the same singularity order as \( T_q \). Hence,
\[ D_i = \int_s \dot{U}_q(P,Q) t_j(Q) ds - \int_s \dot{T}_q(P,Q) [u_j(Q) - u_j(P)] ds \]
\[ + \int_s U_q(P,Q) t_j(Q) ds - \int_s T_q(P,Q) [u_j(Q) - u_j(P)] ds \]
\[ - u_j(P) \left[ \int_s \dot{T}_q(P,Q) ds + \int_s T_q(P,Q) \dot{d}s \right]. \]  
(2.9a)
The integrals in the added back terms, i.e., the last two integrals in equation (2.9a), can be proved to be zero. Explicitly,
Subtracting (2.9b) from equation (2.9a), the weakly singular form of the term $D_i$ can be written as

$$D_i = \int_s \hat{T}_y(P,Q) ds + \int_s T_y(P,Q) ds = \frac{d}{db} \left[ \int_s T_y(P,Q) ds \right]$$

$$= \frac{d}{db} \left[ -\delta_y \right]$$

$$= 0.$$  \hspace{1cm} (2.9b)

and it is noted that the second and the fourth integrals are integrable when $u_j$ is continuous when $Q \to P$.

### 2.1.3. Singularity orders of Sensitivity CBIE Kernels

Equation (2.9) is derived under the tentative assumptions that $\dot{U}_y$ has the same singularity order as $U_y$ and $\dot{T}_y$ has the same singularity order as $T_y$. The truth of these two assumptions can be seen from the derivatives of the kernels by the chain rule

$$\dot{U}_y(P,Q) = U_{y,k}(P,Q) \left( \frac{\partial x_k(Q)}{\partial b} - \frac{\partial x_k(P)}{\partial b} \right)$$  \hspace{1cm} (2.10)

$$\dot{T}_y(P,Q) = \frac{d}{db} \left[ \sigma_{im}(P,Q)n_m(Q) \right]$$

$$= \sigma_{im,k}(P,Q) \left( \frac{\partial x_k(Q)}{\partial b} - \frac{\partial x_k(P)}{\partial b} \right) n_m(Q) + \sigma_{im}(P,Q) \dot{n}(Q)$$  \hspace{1cm} (2.11)

where
The singularity orders seem to be one order higher than those for the ordinary CBIE kernels. If an assumption of continuous design or geometric perturbations is made, i.e., if it is assumed that

\[
\frac{\partial x_k(Q)}{\partial b} - \frac{\partial x_k(P)}{\partial b} = O(r)
\]

(2.12)

the important conclusion can be drawn that the sensitivity CBIE kernels have the singularity orders

\[
\hat{U}_{ij}(P,Q) = O\left(\frac{1}{r}\right)
\]

\[
\hat{T}_{ij}(P,Q) = O\left(\frac{1}{r^2}\right)
\]

that is, they have the same singularity orders with the corresponding ordinary CBIE kernels.

### 2.1.4. Expressions for Sensitivity CBIE Kernels

For 3-D elastostatics, the CBIE kernels are

\[
U_{ij}(P,Q) = \frac{(3 - 4v) \delta_{ij}}{16\pi G(1-v)} \frac{1}{r} + \frac{1}{16\pi G(1-v)} \frac{r_ir_j}{r}
\]

(2.13)

\[
T_{ij}(P,Q) = \frac{-(1 - 2v) \delta_{ij} r_a}{8\pi (1-v) r^2} - \frac{3}{8\pi (1-v) r^3} r_i r_j
\]

\[
+ \frac{(1 - 2v)}{8\pi (1-v) r^3} (r_j n_j - r_i n_i)
\]

(2.14)
The derivatives of the kernels can be obtained as

\[
\dot{U}_y(P, Q) = \frac{\partial U_y}{\partial x_k} \left( \frac{dx_k}{db} (Q) - \frac{dx_k}{db} (P) \right)
\]

\[
\dot{T}_y(P, Q) = \frac{\partial T_y}{\partial x_k} \left( \frac{dx_k}{db} (Q) - \frac{dx_k}{db} (P) \right) + \frac{\partial T_y}{\partial n} n
\]

where, \( \frac{dx}{db} = x \), \( \frac{dy}{db} = y \), and \( \frac{dz}{db} = z \) are the derivatives of the coordinates, or, actually, the changes of the coordinates with respect to the design variable \( b \). These changes will be obtained by comparing the modified design and the original design. They are part of the input data for the sensitivity BIE software systems. Expressions for \( \frac{\partial U_y}{\partial x_k} \) and \( \frac{\partial T_y}{\partial x_k} \) are given in Appendix A.

### 2.2. Numerical Expressions for the Geometric Changes

After the sensitivity CBIE has been obtained in the last section, some expressions for the geometric changes are needed in the implementation. These include the derivatives of the normal \( n \), the Jacobian \( J \), and the differential area \( ds \).

#### 2.2.1. Numerical Expressions for \( n \), \( J \), and \( ds \)

Since the geometry of the surface \( S \) of the body is approximated by shape functions in the BIE, the coordinates of an arbitrary point within an element is expressed as
\[ x_k = \sum_{j=1}^n N_j(\xi, \eta)x_k^{(j)} \]  

(2.17)

where, \( N_j(\xi, \eta) \) are shape functions, and \( x_k^{(j)} \) are nodal coordinates for the \( j \)th node within an element. Thus the normal vector at this point becomes

\[ \vec{n} = \frac{\vec{x}_k \times \vec{x}_n}{|\vec{x}_k \times \vec{x}_n|} = \frac{\vec{n}^*}{J} \]  

(2.18)

where, the denominator

\[ J = |\vec{x}_k \times \vec{x}_n| \]  

(2.19)

is the Jacobian, and the \( k \)th components of \( \vec{x}_k \) and \( \vec{x}_n \) are

\[ x_{k,k} = \sum_{j=1}^n N_j(\xi, \eta)x_k^{(j)} \]

\[ x_{k,n} = \sum_{j=1}^n N_j(\xi, \eta)x_k^{(j)} \]

from (2.17). Finally, the differential area \( ds \) becomes

\[ ds = J d\xi d\eta \]  

(2.20)

2.2.2. Derivatives of \( n, J, \) and \( ds \)

The derivatives of the normal \( n \) with respect to design variable \( b \) can be obtained as

\[ \dot{n}_i = \frac{dn_i}{db} \]

\[ = \sum_{j=1}^n \sum_{k=1}^3 \frac{\partial n_i}{\partial x_k^{(j)}} \frac{dx_k^{(j)}}{db} \]

\[ = \sum_{j=1}^n \sum_{k=1}^3 \frac{\partial n_i}{\partial x_k^{(j)}} x_k^{(j)} \]  

(2.21)
where, $x^{(j)}_k$ is the derivative of the $k$th coordinate of the $j$th node with respect to $b$. In this way, the rate of change of $n_j$ can be calculated when any node has a perturbation in any direction, as may be needed in engineering applications. Similarly, the derivatives of other terms, $J$ and $ds$ can also be obtained as

$$j = \sum_{j=1}^{i} \sum_{k=1}^{J} \frac{\partial J}{\partial x^{(j)}_k} x^{(j)}_k$$ (2.22)

and

$$ds = J \frac{\partial J}{\partial f} \frac{\partial f}{\partial n} \frac{\partial n}{\partial \eta} \eta \dot{\eta}.$$ (2.23)

The quantities $\xi$ and $\eta$ have no derivatives with respect to $b$ because they are local coordinates. At the same time, expressions for $\frac{\partial}{\partial x^{(j)}_k}$ are needed. They are general derivatives of $n$, $J$, and $ds$ with respect to $x$, $y$, and $z$. These are given in Appendix B.

### 2.2.3. Derivatives of coordinates

In previous sections, all of the needed derivatives have been expressed as functions of the derivatives of coordinates with respect to the design variable $b$, $\frac{dx^{(j)}_k}{db}$, or, $x^{(j)}_k$. There are at least two ways of evaluating these coordinate derivatives.

First, an analytical solution could be used. For an engineering problem, the domain, say, a 3-D object, is enclosed by the boundary, a 2-D surface, where a piece-wise depiction of the surface is

$$F(x, y, z, b) = 0$$ (2.24)
with the design variable $b$ as a parameter. From equation (2.24), an exact expression of $\frac{dx_1}{db}$ could be obtained. And then, $\frac{dx_1^{(i)}}{db}$ are actually the nodal values of $\frac{dx_1}{db}$ under this situation.

On the other hand, a completely numerical method could be used. By comparing a modified design and the original design, the coordinate differences for all nodal points can be obtained. These coordinate differences can then be scaled to obtain the coordinate derivatives for all nodes.

While the results from the two methods should be identical, the analytical approach may be very difficult when the shape of the body is complicated. But the numerical method can be used in any situation, and it can be very easy to use with the help of CAD software.

The numerical method is used in this paper.

2.3. Sensitivity CBIE Examples

Two numerical examples are presented in this section to verify the sensitivity CBIE formulas derived in the previous sections and the computer codes developed for the sensitivity CBIE.

2.3.1. Example 1 - A Wedge with Increasing Length

In order to show the accuracy of the sensitivity CBIE, a wedge in tension is taken as an example (Figure 2.3). An analytical solution exists for this problem.

The radius of the wedge is $r = 1.0$. The length is $b = 3.0$. The two flat lateral surfaces and one end of the wedge are constrained in the normal direction ($x$, $y$, or $z$). A distributed loading of $q = 1.0$ is applied at the other end of the wedge along the $z$ direction. When the
Fixed in z
Fixed in x
Fixed in y
Apply $q=1.0$ distributed force

Figure 2.3 A wedge with increasing length
material constants are taken as $E = 1.0$, and $v = 0.3$, the exact solution for the deflection $w$ where $q$ is applied should be $w = 3.0$, and $\frac{dw}{db} = 1.0$ at the loaded end.

The length of the wedge, $b$, is considered as the design variable. When the length $b$ is increased, i.e., when material is added to increase the length of the wedge, the displacements $w$ in the $z$ direction should also increase.

The whole surface is divided into 18 elements, 3 elements each for the two end surfaces, and 4 elements each for the three lateral surfaces. The positions of the sample nodes are shown in the Figure 2.4. The results at the five sample nodes are listed in the Table 2.1.

From the table it can be seen that both CBIE and sensitivity CBIE are very accurate for this relatively simple problem. The displacement errors are below 0.01%, and the derivative errors are below 0.02%.

Figure 2.4 The positions of the selected nodes for example 1
Table 2.1 CBIE results for the wedge in tension

<table>
<thead>
<tr>
<th>Node #</th>
<th>CBIE w</th>
<th>Analytical w</th>
<th>SCBIE dw/db</th>
<th>analytical dw/db</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>3.0000</td>
<td>3.0000</td>
<td>1.0001</td>
<td>1.0000</td>
</tr>
<tr>
<td>24</td>
<td>3.0002</td>
<td>3.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>23</td>
<td>3.0003</td>
<td>3.0000</td>
<td>0.9998</td>
<td>1.0000</td>
</tr>
<tr>
<td>22</td>
<td>3.0002</td>
<td>3.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>21</td>
<td>3.0000</td>
<td>3.0000</td>
<td>1.0001</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

2.3.2. Example 2 - A Beam with a Half Circular Cut

The second example is a little more engineering oriented (Figure 2.5). A beam has a length $L = 6.0$, and a cross section $1.0 \times 1.0$. There is a half-circular cut of radius $b = 0.25$ in the middle top side of the beam. It is fixed in the three directions at one end. The other end of the beam is subjected to a distributed force of $q = 1.0$ in the $z$ direction. The beam is under a combination of tension and bending, because of the cut, even though the loading is just in the $z$ direction. The material constants are $E = 1.0$, and $v = 0.3$ as in Example 1. The radius of the cut, $b$, is taken as the design variable. When $b$ is increased, so that the cross section of the beam becomes smaller, the displacements $w$ at the loaded end should increase.

There is no analytical solution for this problem. The commercial FEA software NASTRAN is used to solve this beam as a 3-D problem; 152 second order HEXA elements are used for the whole body.

For the boundary element model, the whole surface is divided into 188 elements. The mesh around the cut is denser to describe the shape and the displacements and stresses more accurately.
Figure 2.5 A beam with a half circular cut

Fixed in all 3 directions at this end

Radius $b = 0.25$ cut

Apply $q = 1.0$ distributed force at this end
The positions of the selected nodes are shown in the Figure 2.6.

The CBIE and the NASTRAN results for the baseline model (with a cut radius, \( b = 0.25 \)) are tabulated in the Table 2.2. Compared with the NASTRAN results, the CBIE results have differences which are all less than 1.0%.

![Figure 2.6 The positions of the selected nodes for example 2](image)

<table>
<thead>
<tr>
<th>Node #</th>
<th>CBIE ( w )</th>
<th>NASTRAN ( w )</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>7.5021</td>
<td>7.4550</td>
<td>0.63 %</td>
</tr>
<tr>
<td>187</td>
<td>6.9569</td>
<td>6.9290</td>
<td>0.40 %</td>
</tr>
<tr>
<td>71</td>
<td>6.4115</td>
<td>6.4030</td>
<td>0.13 %</td>
</tr>
<tr>
<td>179</td>
<td>5.8661</td>
<td>5.8760</td>
<td>0.17 %</td>
</tr>
<tr>
<td>23</td>
<td>5.3204</td>
<td>5.3500</td>
<td>0.55 %</td>
</tr>
</tbody>
</table>
Next, the CBIE model is modified to have a cut with a radius of $b = 0.26$. The sensitivity CBIE is employed at the same time with $db = 0.01$. Both results are compared in Table 2.3.

In Table 2.3, the second column contains the displacement sensitivities, i.e. the derivatives, $dw$, obtained from the sensitivity CBIE. The baseline-model results, which are in the second column of the Table 2.2, become $w_0$ here. The corresponding sensitivity CBIE results for the displacements, which are shown in the third column of the above table, are obtained as

$$w = w_0 + dw.$$  

<table>
<thead>
<tr>
<th>Node #</th>
<th>SCBIE $dw$</th>
<th>SCBIE $w$</th>
<th>CBIE $w$</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>0.1586</td>
<td>7.6607</td>
<td>7.6672</td>
<td>0.08 %</td>
</tr>
<tr>
<td>187</td>
<td>0.1043</td>
<td>7.0612</td>
<td>7.0656</td>
<td>0.06 %</td>
</tr>
<tr>
<td>71</td>
<td>0.0501</td>
<td>6.4616</td>
<td>6.4638</td>
<td>0.03 %</td>
</tr>
<tr>
<td>179</td>
<td>-.0042</td>
<td>5.8619</td>
<td>5.8619</td>
<td>0.00 %</td>
</tr>
<tr>
<td>23</td>
<td>-.0585</td>
<td>5.2619</td>
<td>5.2598</td>
<td>0.04 %</td>
</tr>
</tbody>
</table>

These results show that the sensitivity CBIE, when there is a design change of 4 %, is very accurate. The biggest difference is under 0.1 %. Remembering that the CBIE results in the fourth column of the above table should also have less than 1.0 % difference when compared with the FEA results, the fifth column of the Table 2.3 doesn’t represent the true error. These are just the differences with the ordinary CBIE results.
CHAPTER 3. SENSITIVITY HBIE FOR ELASTOSTATICS

-- GLOBAL FORMULA

The sensitivity CBIE has been used in engineering successfully, for example, in shape optimization [18,19]. But for crack problems or problems with a thin-body domain (Figure 3.1), the sensitivity CBIE will break down, just as the CBIE does, because of the degeneracy of the main coefficient matrix [30], as mentioned in Chapter 1. The sensitivity hypersingular BIE is needed to address these kinds of problems. With HBIEs, these degeneracy problems are alleviated. One kind of sensitivity HBIE, the global formula, is formulated in this chapter. Another kind, the local formula, will be discussed in the next chapter.

Figure 3.1 A beam with a true crack
Initially, an attempt was made to derive the sensitivity HBIE by a method similar to that used for the sensitivity CBIE, namely, the singularity subtraction method. However because of the hypersingularity of the HBIE kernels, that method becomes very difficult to implement for the derivation of the sensitivity HBIE. A new strategy, starting from the weakly-singular form, is established in this chapter to simplify the derivation. This new strategy can be used for all kinds of BIEs, including CBIE and HBIE. It will also be used in the next chapter for the derivation of the sensitivity HBIE local formula. The expressions for the sensitivity HBIE kernels are much more complicated than for the sensitivity CBIE. All of the expressions needed for the sensitivity HBIE are derived in this chapter, and the details are given in the appendices.

One of the two example problems previously used in Chapter 2, a wedge with increasing length is used again in this chapter to facilitate the comparison of the numerical performances of the global sensitivity HBIE and the sensitivity CBIE. Additional examples for which the sensitivity HBIE is applicable, but for which the sensitivity CBIE breaks down, are given in Chapters 4 and 5.

3.1. Hypersingularities in the HBIE and Sensitivity HBIE Formulas

The hypersingular BIE (HBIE) was created to solve thin domain or crack problems [30] in static analysis, or to overcome the fictitious-frequency difficulties in dynamic analysis[29]. While these functions are very useful in solving engineering problems with cracks or thin bodies, the hypersingularity of the kernels presents more difficulties, of both analytical and numerical kind, than does the CBIE. These difficulties are further enlarged for
the sensitivity HBIE because of the additional differentiations with respect to the design variables.

3.1.1. Ordinary HBIE and the Weakly Singular Form

The hypersingular BIE can be obtained by taking spatial gradients of the CBIE (equation 2.1) and multiplying the resulting equation with the elastic modulus tensor $E_{ijkl}$ and the normal vector at the source point $n_{ok}$ [28]. The resulting HBIE, involving tractions and displacements explicitly, has the form

$$ t_j(P) = \int_s K_{ij}(P,Q) t_j(Q) \, ds - \int_s H_{ij}(P,Q) u_j(Q) \, ds $$  \hspace{1cm} (3.1)

where the kernels are

$$ K_{ij}(P,Q) = E_{ikpq} \frac{\partial U_{pq}(P,Q)}{\partial x_{0q}} n_{ok} $$

$$ H_{ij}(P,Q) = E_{ikpq} \frac{\partial T_{pq}(P,Q)}{\partial x_{0q}} n_{ok} $$

and, $U_{pq}(P,Q)$ and $T_{pq}(P,Q)$ are kernels of the CBIE. These kernels are singular of orders

$$ K_{ij}(P,Q) = O\left(\frac{1}{r^3}\right) $$

$$ H_{ij}(P,Q) = O\left(\frac{1}{r^3}\right) $$

that is, one order higher than those in the CBIE. Because of these singularities, the first and second integrals in (3.1) are interpreted (et. seq.) as Cauchy principal value and Hadamard finite parts, respectively [28].

Similar to the CBIE, the HBIE formula (3.1) needs to be regularized before discretization in order to use Gaussian quadrature for evaluating the integrals. The
regularization method is the same as the singularity-subtraction method used with the CBIE. However, one more term in the Taylor series is subtracted such that more "added-back" terms are needed for the integral containing the \( H_{ij}(P,Q) \) kernel. The extra terms are revived because of the higher singularity order in the HBIE. This regularization has been done by Liu and Rizzo [44]. The weakly singular form of the HBIE is

\[
t_j(P) = \int_S [K_{ij}(P,Q) + T_{ij}(P,Q)] t_j(Q) \, ds \\
- \int_S T_{ij}(P,Q) [t_j(Q) - t_j(P)] \, ds \\
- \int_S H_{ij}(P,Q) \left[ u_j(Q) - u_j(P) - \frac{\partial u_j}{\partial \xi_\alpha}(P)(\xi_\alpha - \xi_\alpha) \right] \, ds \\
- E_{\kappa \mu} e_{\alpha q} \frac{\partial u_\alpha}{\partial \xi_\alpha}(P) \int_S [K_{ij}(P,Q) n_\kappa(Q) + T_{ij}(P,Q) n_\kappa(P)] \, ds
\]

where, \( \xi_\alpha, \alpha = 1,2 \) are the local coordinates at the field point, and \( \xi_\alpha, \alpha = 1,2 \) are the local coordinates at the source point. At any point, if the normal vector \( n \) is added to the local coordinate system, a 3-D orthogonal local coordinate system \( \xi - \eta - n \) is formed. A 3-D Jacobian matrix \( J_3 \) is defined as

\[
J_3 = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\
\frac{\partial x}{\partial n} & \frac{\partial y}{\partial n} & \frac{\partial z}{\partial n}
\end{bmatrix}
\]

and \( e_{\alpha q} \) is part of the inverse of the 3-D Jacobian \( J_3 \), i.e. if

\[
J_3^{-1} = \begin{bmatrix}
e_{11} & e_{12} & e_{13} \\
e_{21} & e_{22} & e_{23} \\
e_{31} & e_{32} & e_{33}
\end{bmatrix}
\]

then
\[
\begin{bmatrix}
e_{aq}
\end{bmatrix} = \begin{bmatrix}
e_{11} & e_{12} & e_{13} \\
e_{21} & e_{22} & e_{23}
\end{bmatrix}.
\] (3.5)

The detailed expression for \( e_{aq} \) can be found in the Appendix D.

### 3.1.2. Sensitivity HBIE and the Regularization Difficulties

As in the derivation of the sensitivity CBIE, the sensitivity HBIE could also be obtained by taking derivatives of equation (3.1) with respect to \( b \), thus

\[
\dot{t}_t(P) = \int_5 K_y(P, Q) \dot{t}_t(Q) \, ds - \int_5 H_y(P, Q) \dot{u}_j(Q) \, ds \\
+ \int_5 K_y(P, Q) t_j(Q) \, ds - \int_5 H_y(P, Q) \dot{u}_j(Q) \, ds \\
+ \int_5 K_y(P, Q) t_j(Q) \, ds - \int_5 H_y(P, Q) u_j(Q) \, ds.
\] (3.6)

Now, however, regularization of equation (3.6) via conventional methods is very difficult because of the hypersingularity of the kernels. If the known terms of the equation (3.6) are still defined as

\[
G_i \equiv \int_5 K_y(P, Q) \dot{t}_t(Q) \, ds - \int_5 H_y(P, Q) \dot{u}_j(Q) \, ds \\
+ \int_5 K_y(P, Q) t_j(Q) \, ds - \int_5 H_y(P, Q) \dot{u}_j(Q) \, ds
\] (3.7)

equation (3.6) could still be written as

\[
\dot{t}_t(P) = \int_5 K_y(P, Q) \dot{t}_t(Q) \, ds - \int_5 H_y(P, Q) \dot{u}_j(Q) \, ds + G_i.
\] (3.8)

Similar to the equations (2.10) and (2.11) in Chapter 2, it will be shown below that the sensitivity HBIE kernels have the same singularity orders as those of the ordinary HBIE kernels, i.e.
\[ \dot{K}_g(P,Q) = O\left(\frac{1}{r^2}\right) \]  
\[ \dot{H}_g(P,Q) = O\left(\frac{1}{r^3}\right). \]  

The known terms (3.7) could also be regularized by using the singularity subtraction method as

\[
G_i = \left[ \int_s \dot{K}_g(P,Q) \left[ t_j(Q) - t_j(P) \right] ds \right] 
- \left[ \int_s \dot{H}_g(P,Q) \left[ u_j(Q) - u_j(P) - \frac{\partial u_j}{\partial \xi}(P)(\xi_{\alpha} - \xi_{\alpha}) \right] ds \right] 
+ \left[ \int_s K_g(P,Q) \left[ t_j(Q) - t_j(P) \right] ds \right] 
- \left[ \int_s H_g(P,Q) \left[ u_j(Q) - u_j(P) - \frac{\partial u_j}{\partial \xi}(P)(\xi_{\alpha} - \xi_{\alpha}) \right] ds \right] 
+ G_i^* \tag{3.9}
\]

and

\[
G_i^* = t_j(P) \left[ \int_s \dot{K}_g(P,Q) ds + \int_s K_g(P,Q) \dot{s} \right] 
- \left[ u_j(P) + \frac{\partial u_j}{\partial \xi}(P)(\xi_{\alpha} - \xi_{\alpha}) \right] \left[ \int_s \dot{H}_g(P,Q) ds + \int_s H_g(P,Q) \dot{s} \right] \tag{3.10}
\]

are the added-back terms. These added-back terms are apparently not zero and would be difficult to explicitly determine. Because of this difficulty an alternative easier method for getting the sensitivity HBIE formulas will be given in the next section.
3.2. Deriving Sensitivity HBIE From Weakly Singular Form

A basic observation for the success of the alternate strategy for doing the sensitivity HBIE formulae is that the singularity order of the sensitivity BIE (including sensitivity CBIE and sensitivity HBIE) is no more than that of the ordinary BIE (CBIE and HBIE).

This is straightforward because during the formulation of the sensitivity BIE, the derivatives are taken with respect to the design variable \( b \), but the singularity order involves powers of one over the distance between the field point and the source point \( r \) (3.8a). These powers are unchanged following differentiation with respect to \( b \).

Based on this observation, the alternate strategy for the derivation of the sensitivity BIE should be as follows. Start with the weakly singular form of the ordinary BIEs. The resulting sensitivity BIEs also will be in weakly singular form automatically.

3.2.1. An Illustration of the New Strategy

To illustrate this strategy, reconsider the derivation of the sensitivity CBIE. Start from the weakly singular form of the ordinary CBIE

\[
\int_s U_q(P,Q) t_j(Q) ds = \int_s T_q(P,Q) [u_j(Q) - u_j(P)] ds.
\]

Now take derivatives of this equation with respect to the design variable \( b \), the sensitivity CBIE is obtained as

\[
\int_s U_q(P,Q) \dot{t}_j(Q) ds = \int_s T_q(P,Q) \left[ \dot{u}_j(Q) - \dot{u}_j(P) \right] ds - \int_s \dot{U}_q(P,Q) t_j(Q) ds + \int_s \dot{T}_q(P,Q) [u_j(Q) - u_j(P)] ds
\]

\[
- \int_s U_q(P,Q) \ddot{t}_j(Q) ds + \int_s T_q(P,Q) [u_j(Q) - u_j(P)] \ddot{s}.
\]
This is equivalent to equation (2.7), but all terms are at most weakly singular, therefore, no additional regularization is needed.

### 3.2.2. Sensitivity HBIE From the Alternate Strategy

Using the above strategy, it is relatively easy to get the sensitivity HBIE if the starting point is the weakly singular form of the ordinary HBIE, equation (3.2).

Taking derivatives of equation (3.2) with respect to the design variable \( b \), get

\[
\begin{align}
\dot{t}_j(P) &= \int_S \left[ K_{ij}(P,Q) + T_{ij}(P,Q) \right] \dot{t}_j(Q) ds \\
&- \int_S T_{ij}(P,Q) [\dot{t}_j(Q) - t_j(P)] ds \\
&- \int_S H_{ij}(P,Q) \left[ u_j(Q) - u_j(P) - \frac{\partial u_j}{\partial \xi}(P)(\xi_\alpha - \xi_{0\alpha}) \right] ds \\
&- E_{ijkl} e_{aq} \frac{\partial u_{ij}}{\partial \xi}(P) \int_S \left[ K_{ij}(P,Q)n_k(Q) + T_{ij}(P,Q)n_k(P) \right] ds \\
&- E_{ijkl} e_{aq} \frac{\partial u_{ij}}{\partial \xi}(P) \int_S \left[ K_{ij}(P,Q)n_k(Q) + T_{ij}(P,Q)n_k(P) \right] ds \\
&- E_{ijkl} e_{aq} \frac{\partial u_{ij}}{\partial \xi}(P) \int_S \left[ K_{ij}(P,Q)n_k(Q) + T_{ij}(P,Q)n_k(P) \right] ds \\
&+ \int_S [\dot{K}_{ij}(P,Q) + \dot{T}_{ij}(P,Q)] t_j(Q) ds \\
&- \int_S \dot{T}_{ij}(P,Q) [\dot{t}_j(Q) - t_j(P)] ds \\
&- \int_S \dot{H}_{ij}(P,Q) [u_j(Q) - u_j(P) - \frac{\partial u_j}{\partial \xi}(P)(\xi_\alpha - \xi_{0\alpha})] ds \\
&- E_{ijkl} e_{aq} \frac{\partial u_{ij}}{\partial \xi}(P) \int_S \left[ K_{ij}(P,Q)n_k(Q) + T_{ij}(P,Q)n_k(P) \right] ds \\
&= (3.11)
\end{align}
\]
It can be seen that every term in the equation (3.11) is at most weakly singular. Thus, Gaussian quadrature can be used to evaluate the integrals directly.

3.2.3. Expressions for the Sensitivity HBIE Kernels

In the HBIE for 3-D elastostatic problems, the fundamental solutions, which come from the fundamental solutions of the CBE by taking derivatives with respect to the source point and multiplying the resulting expressions with the elastic modulus tensor and the normal vector at the source point, are:

\[
K_y(P, Q) = \frac{1}{8\pi(1-\nu)^2} \left[ (1-2\nu)\delta_{y\alpha}r_{\alpha0} + 3r_{j\alpha}r_{\alpha0} + (1-2\nu)(r_{j\alpha}n_{\alpha j} - r_{j0\alpha}) \right]
\]

\[
H_y(P, Q) = \frac{\mu}{4\pi(1-\nu)} \left[ 3\nu \frac{r_{\alpha0}r_{\alpha j}}{r^3} + (1-2\nu)\frac{(n_{k\alpha})}{r^3} \right] \delta_{y\alpha} + (1-2\nu)\frac{r_{\alpha0}n_{\alpha j}}{r^3} \\
- (1-4\nu)\frac{n_{\alpha j}}{r^3} + 3(1-2\nu)\frac{r_{\alpha0}r_{\alpha j}n_{\alpha j}}{r^3} + 3\nu \frac{r_{\alpha0}r_{\alpha j}}{r^3} \left[ \gamma r_{j\alpha}n_{\alpha k} - 5r_{j\alpha}r_{\alpha0}r_{\alpha k} \right].
\]
Noting that $K_y(P,Q)$ is a function of $x_k$ and $n_0$, $H_y(P,Q)$ is a function of $x_k$, $n_0$, and $n$, the sensitivity HBIE kernels can be obtained by taking derivatives of $K_y(P,Q)$ and $H_y(P,Q)$ with respect to the design variable $b$.

\[
\begin{align*}
K'_y(P,Q) &= \frac{\partial K_y(P,Q)}{\partial x} \left( \frac{dx}{db}(Q) - \frac{dx}{db}(P) \right) + \frac{\partial K_y(P,Q)}{\partial y} \left( \frac{dy}{db}(Q) - \frac{dy}{db}(P) \right) \\
&\quad + \frac{\partial K_y(P,Q)}{\partial n} \left( \frac{dn}{db}(Q) - \frac{dn}{db}(P) \right) \\
H'_y(P,Q) &= \frac{\partial H_y(P,Q)}{\partial x} \left( \frac{dx}{db}(Q) - \frac{dx}{db}(P) \right) + \frac{\partial H_y(P,Q)}{\partial y} \left( \frac{dy}{db}(Q) - \frac{dy}{db}(P) \right) \\
&\quad + \frac{\partial H_y(P,Q)}{\partial n} \left( \frac{dn}{db}(Q) - \frac{dn}{db}(P) \right)
\end{align*}
\] (3.14) (3.15)

where, $n_0$ is the normal at the source point, $n$ is the normal at the field point. The quantities $\frac{\partial K_y(P,Q)}{\partial x_k}$ and $\frac{\partial H_y(P,Q)}{\partial x_k}$ are derivatives of the ordinary HBIE kernels with respect to Cartesian coordinates. These are given in detail in Appendix C.

3.2.4. Expressions for the Derivative of Inverse Jacobian

There is a special derivative term in the equation (3.11), $\epsilon^*_{aq}$. As stated in the equations (3.3)-(3.5), $\epsilon_{aq}$ is part of the inverse of the 3-D Jacobian matrix $J_3^{-1}$. So, the expression for $\epsilon^*_{aq}$ involves the derivatives of the inverse of the 3-D Jacobian matrix (3.3).
Because the 3-D Jacobian matrix $J_3$ in the boundary element method has been expressed as a
function of the local derivatives of the shape functions, the direct differentiation of its inverse matrix is very complicated. However, for the invertable matrix $J_3$ and its inverse $J_3^{-1}$

$$J_3 J_3^{-1} = I$$

(3.16)

where, $I$ is the unit matrix. Thus, taking the derivative of (3.16) to get

$$J_3 \dot{J}_3^{-1} + \dot{J}_3 J_3^{-1} = 0$$

(3.17)

it follows that

$$\dot{J}_3^{-1} = -J_3^{-1} \dot{J}_3 J_3^{-1}.$$  

(3.18)

See Appendix D for details.

### 3.3. Global Sensitivity HBIE Example

To compare the numerical performance between the sensitivity CBIE and sensitivity HBIE, a numerical example, the wedge with increasing length is used here. This example also serves to verify the sensitivity HBIE formulas and computer codes.

#### 3.3.1. Example - A Wedge with Increasing Length

The problem is completely the same as that in Chapter 2 (Figure 2.3), but the nodal numbers are different. Figure 3.2 shows the position of the sample nodes. The results at the sample nodes are listed in the Table 3.1.

The table shows that the HBIE and the sensitivity HBIE are also very accurate. The displacement errors are less than 0.02 %, and the derivative errors are less than 0.16 %.
Figure 3.2 The position of the selected nodes for example 1

Table 3.1 Global HBIE results for the wedge in tension

<table>
<thead>
<tr>
<th>Node #</th>
<th>HBIE w</th>
<th>Analytical w</th>
<th>SHBIE dw/db</th>
<th>analytical dw/db</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>3.0005</td>
<td>3.0000</td>
<td>1.0016</td>
<td>1.0000</td>
</tr>
<tr>
<td>38</td>
<td>2.9999</td>
<td>3.0000</td>
<td>1.0006</td>
<td>1.0000</td>
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<tr>
<td>34</td>
<td>2.9998</td>
<td>3.0000</td>
<td>1.0002</td>
<td>1.0000</td>
</tr>
<tr>
<td>28</td>
<td>2.9998</td>
<td>3.0000</td>
<td>1.0002</td>
<td>1.0000</td>
</tr>
<tr>
<td>31</td>
<td>2.9999</td>
<td>3.0000</td>
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<td>1.0000</td>
</tr>
<tr>
<td>27</td>
<td>3.0005</td>
<td>3.0000</td>
<td>1.0016</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
CHAPTER 4. SENSITIVITY HBIE FOR ELASTOSTATICS
-- LOCAL FORMULA

Now that both the sensitivity form of CBIE and HBIE are formulated, as in the previous two chapters, there are many possible combinations of using these formulas to solve crack problems. Some of the combinations work well, others do not. The first section of this chapter reviews these options, and then discusses one of them in detail, the single-surface crack model. A local-formula version of the hypersingular BIE is needed for this single surface model because the identities, which are used in the global formula of the HBIE to treat the hypersingular and strong singular integrals, no longer hold for a non-closed surface. Note that a single surface model of a crack is a non-closed surface. Line integrals around the crack are therefore needed to treat the singular integrals. The local formula and line integral treatment are developed in the second section of this chapter for the sensitivity HBIE.

The new strategy developed in Chapter 3, starting the derivation from the weakly singular form, is used again in this chapter to facilitate the regularization of the local formula of the sensitivity HBIE. All of the formulas are derived in this chapter, and the detailed expressions are given in the Appendices.

A numerical example, the wedge with increasing is examined once more in the last section using the local sensitivity HBIE formulas and codes developed in this chapter. The results from the local sensitivity HBIE are shown to agree very well with the ordinary HBIE and NASTRAN solutions.
4.1. Single Surface Crack Model and Local Formula

For a domain with a notch (Figure 4.1), the accuracy of the numerical results will decrease when the width \( w \) becomes smaller with respect to the depth \( b \), and the results become nonsense when \( w \) goes to zero (i.e., a true crack).

4.1.1. Crack Model Options

For a true crack, label the two crack surfaces \( S^+ \) and \( S^- \) (Figure 4.2), and define

\[
\Delta u \equiv u_{s^+} - u_{s^-}
\]

as the difference of displacements at the two nodes on the opposite crack surfaces, and

\[
\Sigma t \equiv t_{s^+} + t_{s^-}
\]

Figure 4.1 A notch with width \( W \) and depth \( b \)
as the total traction there. It is known that:

1). The CBIE using $u$ and $t$ on both $S^+$ and $S^-$ is degenerate;

2). The CBIE using $\Delta u$ and $\Sigma t$ on $S^+$ only is degenerate;

3). The HBIE using $u$ and $t$ on both $S^+$ and $S^-$ is degenerate;

4). The HBIE using $\Delta u$ and $\Sigma t$ on $S^+$ only is possible [29];

5). The CBIE using $u$ and $t$ on $S^+$ plus HBIE using $u$ and $t$ on $S^-$ is possible [30]; this strategy is also called the dual boundary element method by Portela and Aliabadi [31];

6). A Linear combination of CBIE and HBIE using $u$ and $t$ on both $S^+$ and $S^-$ is possible [32].

Among the three possible combinations, the last one, i.e., the 6th option uses CBIE and HBIE on both crack surfaces. The whole boundary is still a closed surface. All of the

![Figure 4.2 A true crack and the two crack surfaces](image-url)
integral identities hold. The global version of the CBIE and HBIE, the global version of the sensitivity CBIE, as well as the global sensitivity HBIE developed in the Chapter 3 could be used. If the coefficient matrix formed by CBIE is $A_c$, and that by HBIE, $A_H$, the combination of the two

$$A = \alpha A_c + \beta A_H$$

will not be degenerate; $\alpha$ and $\beta$ are two constants.

The 5th option above uses CBIE or HBIE on the different crack surfaces. For each method, just one crack surface is considered. However, the whole boundary becomes several non-closed surfaces. The integral identities no longer hold. Thus, as mentioned earlier, a local formula of the CBIE, HBIE, sensitivity CBIE, and sensitivity HBIE would be needed.

In this thesis, the 4th option is used, that is, a single surface crack model. The $\Delta u$ and $\Sigma t$, instead of $u$ and $t$ are used as basic variables. The details are given below. Again because of the non-closed surface, the local formulas of the HBIE and sensitivity HBIE are needed.

### 4.1.2. Single Surface Crack Model

Considering the two surfaces of the crack and the remainder surface which is not related to the crack (Figure 4.2), the HBIE formula, equation (3.1) becomes

$$t_j(P) = \int_{\gamma} \left[ K_y(P,Q^+) t_j(Q^+) - H_y(P,Q^+) u_j(Q^+) \right] ds(Q^+)$$

$$+ \int_{\gamma} \left[ K_y(P,Q^-) t_j(Q^-) - H_y(P,Q^-) u_j(Q^-) \right] ds(Q^-)$$

$$+ \int_{\gamma^*} \left[ K_y(P,Q^+) t_j(Q^+) - H_y(P,Q^+) u_j(Q^+) \right] ds(Q^+)$$

Since the fundamental solutions of the HBIE have the properties

$$K_y(P,Q^+) = K_y(P,Q^-)$$

$$H_y(P,Q^+) = -H_y(P,Q^-)$$
equation (4.1) becomes

\[
\begin{align*}
    t_j(P) &= \int_{\gamma^*} \left[ K_q(P, Q^*) \Sigma_j(Q^*) - H_q(P, Q^*) \Delta u_j(Q^*) \right] ds(Q^*) \\
    &+ \int_{\gamma^*} \left[ K_q(P, Q^8) t_j(Q^8) - H_q(P, Q^8) u_j(Q^8) \right] ds(Q^8)
\end{align*}
\] (4.2)

As with other hypersingular integral equation, a regularization is necessary before computing with (4.2). However, as mentioned earlier, regularization involving line integrals is needed.

### 4.1.3. Line Integrals for Local Regularization

At the first glance, the local regularization seems much simpler than the global one. The two integrals in the equation (3.1) are considered element by element, that is

\[
\begin{align*}
    t_j(P) &= \int_{\Delta S_0} K_q(P, Q) t_j(Q) ds - \int_{\Delta S_0} H_q(P, Q) u_j(Q) ds \\
    &+ \int_{\Delta S} K_q(P, Q) t_j(Q) ds - \int_{\Delta S} H_q(P, Q) u_j(Q) ds
\end{align*}
\] (4.3)

where $\Delta S$ is the boundary formed by singular elements which have the collocation point $P$ as one of the nodal points. The first two integrals are not singular. For the singular integrals, the singularity subtraction method is applied locally

\[
\begin{align*}
    \int_{\Delta S} H_q(P, Q) u_j(Q) ds &= \int_{\Delta S} H_q(P, Q) \left[ u_j(Q) - u_j(P) - \frac{\partial u_j}{\partial \xi_0}(P)(\xi_{0a} - \xi_{0a}) \right] ds \\
    &+ u_j(P) \int_{\Delta S} H_q(P, Q) ds \\
    &+ \frac{\partial u_j}{\partial \xi_0}(P) \int_{\Delta S} H_q(P, Q)(\xi_{0a} - \xi_{0a}) ds
\end{align*}
\] (4.4)

and
The first integral on the right-hand-side of equation (4.4), and the first and the second integrals on the right-hand-side of equation (4.5) are now weakly singular. But the added-back terms, that is, the second and the third integrals on the right-hand-side of equation (4.4), and the third integral in the right-hand-side of equation (4.5) are still strongly-singular or hypersingular integrals. Here, a line integral approach is taken to evaluate these integrals. Details on this approach are given in [45].

The first line integral, comes from the second term on the right side of equation (4.4) according to

\[ L_1 = u_j(P) \cdot \int_{\partial S} H_j(P, Q) ds = u_j(P)H_1(i, j) \]  

(4.6)

where

\[ H_1(i, j) = \sum E_{\alpha \beta \gamma} \epsilon_{\alpha \beta \gamma} n_\alpha \int_{\Sigma} \Sigma_{\gamma \xi} dx \]  

(4.7)

and

\[ \Sigma_{\gamma \xi} = E_{\mu \nu \gamma} \frac{\partial U_\gamma}{\partial x_\mu} . \]

The second line integral, comes from the third term on the right side of equation (4.4) according to

\[ L_2 = \frac{\partial u_i}{\partial \xi_\alpha} \cdot \int_{\partial S} H_i(P, Q)(\xi_\alpha - \xi_0 \alpha) ds(Q) \]

= \frac{\partial u_i}{\partial \xi_\alpha} \left[ H_2(i, j, \alpha) + H_3(i, j, \alpha) \right] \]  

(4.8a)
where

\[
H_2(i, j, \alpha) = e_{\alpha \mu} E_{\text{kin}} n_{0k} \epsilon_{\mu \nu \lambda} \int_C (x_q - x_{0q}) \Sigma_{\gamma \delta} dx_r \\
- e_{\alpha \mu} E_{\text{kin}} E_{\mu \nu \lambda} n_{0k} \epsilon_{\mu \nu \lambda} \int_C U_{\gamma \delta} dx_r \\
+ e_{\alpha m} E_{\text{kin}} n_{0k} \int_{\Delta S} T_q(P, Q) ds \\
- e_{\alpha m} E_{\text{kin}} n_{0k} \int_{\Delta S} T_q(P, Q) ds.
\]

(4.9)

Note that the integral

\[
H_3(i, j, \alpha) = e_{\alpha \mu} E_{\text{kin}} \int_{\Delta S} \left[ K_{\mu \nu Q}(P, Q) n_k + T_{\mu}(P, Q)n_{0k} \right] ds
\]

(4.10)
is in weakly-singular form already. Another term defined as

\[
L_2 = \frac{\partial u_j}{\partial x_{\alpha}} (P) H_2(i, j, \alpha)
\]

(4.8b)
is introduced here to make a later derivation convenient. The last two integrals in equation (4.9) will be evaluated using expression (4.12) below.

The third line integral, the last one in the right-hand-side of equation (4.5) is relatively simple,

\[
L_3 = t_j(P) \int_{\Delta S} T_{\mu}(P, Q) ds = t_j(P) \cdot T_j(j, i)
\]

(4.11)

where

\[
T_j(i, j) = \delta_{\eta \eta} \cdot S + \frac{1}{4\pi} \epsilon_{\eta \eta \eta} \int_C \frac{1}{r} dx_k + \frac{1}{8\pi(1 - v)} \epsilon_{\eta \eta \eta} \int_C r_{ik} dx_l
\]

(4.12)

and

\[
S = \int_{\Delta S} \frac{\partial G}{\partial n} (P, Q) ds
\]

(4.13)
is the solid angle term, which is not singular.
Substituting equations (4.6) - (4.13) into equations (4.4) and (4.5), and then into the HBIE (4.3), the local regularization, via line integrals, gives a weakly-singular form of the HBIE as

\begin{align*}
t_j(P) &= \int_{S} K_{q}(P, Q) t_j(Q) \, ds - \int_{S} H_{q}(P, Q) u_j(Q) \, ds \\
&\quad + \int_{S} \left[ K_{q}(P, Q) + T_{ji}(P, Q) \right] t_j(Q) \, ds \\
&\quad - \int_{S} T_{ji}(P, Q) [t_j(Q) - t_j(P)] \, ds \\
&\quad - \int_{S} H_{q}(P, Q) \left[ u_j(Q) - u_j(P) - \frac{\partial u_j}{\partial \xi_\alpha}(P) (\xi_\alpha - \xi_0^\alpha) \right] \, ds \\
&\quad - E_{\mu \nu \rho} e_{\nu \alpha} \frac{\partial u_j}{\partial \xi_\alpha}(P) \int_{S} \left[ K_{q}(P, Q) n_k(Q) + T_{ji}(P, Q) n_k(P) \right] \, ds \\
&\quad - t_j(P) \int_{S} T_{ji}(P, Q) \, ds - u_j(P) E_{i \lambda m \nu} \delta_{\lambda m} n_{\nu k} \Sigma_{\nu \alpha} \, dx_r \\
&\quad - \frac{\partial u_j}{\partial \xi_\alpha}(P) e_{\alpha \nu \lambda} n_{\nu k} \delta_{\lambda m} (x_q - x_{0q}) \Sigma_{\nu \alpha} \, dx_r \\
&\quad + \frac{\partial u_j}{\partial \xi_\alpha}(P) e_{\alpha \nu \lambda} E_{\nu \mu \nu} n_{\nu k} \delta_{\nu \alpha} U_{\gamma \mu} \, dx_r \\
&\quad - \frac{\partial u_j}{\partial \xi_\alpha}(P) e_{\alpha \nu \lambda} E_{\nu \lambda m \nu} n_{\nu k} \int_{S} T_{ij}(P, Q) \, ds \\
&\quad + \frac{\partial u_j}{\partial \xi_\alpha}(P) e_{\alpha \nu \lambda} E_{\mu \nu \lambda} n_{\nu k} \int_{S} T_{ji}(P, Q) \, ds.
\end{align*}
4.2. Local Sensitivity HBIE Formulation

Comparing equations (4.14) and (3.1.2), it can be seen that the local formula is similar to the global formula. The only difference is the extra line integrals in the local formula. Therefore the local formula of the sensitivity HBIE can be built on the global formula as a base with additional consideration of the line integrals. In other words, the main additional work needed for the local formula of the sensitivity HBIE is to take the derivatives of the line integrals. This is done next.

4.2.1. Derivative Expressions for Line Integrals

Taking a derivative of equation (4.6) with respect to \( b \), get the derivative expression for the first line integral

\[
\dot{L}_1 = \dot{u}_i(P) \cdot H_1(i, j) + u_j(P) \cdot \dot{H}_1(i, j) \tag{4.15}
\]

where

\[
\dot{H}_1(i, j) = E_{klm} e_{mn} \cdot \dot{\phi}_c \sum_{\gamma} y_{\gamma} dx_r + E_{klm} e_{mn} n_{\alpha} \bullet \phi_c \sum_{\gamma} y_{\gamma} dx_r + E_{klm} e_{mn} n_{\alpha} \bullet \phi_c \sum_{\gamma} y_{\gamma} dx_r \tag{4.16}
\]

Similarly, taking a derivative of equation (4.8b) with respect to \( b \), get the expression for part of the second line integral

\[
\dot{L}_2 = \frac{\partial u_i}{\partial \xi_\alpha}(P) H_2(i, j, \alpha) + \frac{\partial u_j}{\partial \xi_\alpha}(P) \dot{H}_2(i, j, \alpha) \tag{4.17}
\]

where
Through similar derivation process, for the third line integral (4.11), the derivative becomes

\[
\dot{L}_3 = \dot{t}_j(P) \cdot T_i(l, j) + t_j(P) \cdot \dot{T}_i(l, i)
\]

and

\[
\dot{T}_i(l, j) = \delta_j S + \frac{1}{4\pi} \varepsilon_{ijk} \frac{-1}{r^3} dx_k + \frac{1}{8\pi(1-v)} \varepsilon_{ijk} \frac{r^2}{r^3} dx_k \\
+ \frac{1}{4\pi} \varepsilon_{ijk} \frac{1}{r^3} dx_k + \frac{1}{8\pi(1-v)} \varepsilon_{ijk} \frac{r^2}{r^3} dx_k.
\]
4.2.2. Derivative of the 'Solid Angle' Term

The term $\tilde{S}$ in the equation (4.20) is the derivative of the so called solid angle. The solid angle is not singular. So, for the true crack problem, it's not necessary to transform this form into line integrals. But for very narrow notch problems, the solid angle also need to be transformed into line integrals because the source point is not in the surface. Under this situation, $\tilde{S}$ will be very complicated because the solid angle consists of several line integrals. Fortunately, when non-conforming elements are used, the solid angle itself is always a constant

$$S = \frac{1}{2}$$

because the source point is on a smooth surface. In this case

$$\tilde{S} = 0.$$  

4.3. Local Sensitivity HBIE Example

The same simple example, the wedge with increasing length is computed once more using the local sensitivity HBIE formulas and codes developed in this chapter. The results from the local sensitivity HBIE are then compared to the HBIE and NASTRAN results to show the performance of the sensitivity HBIE local formula.
4.3.1. Example - A Wedge with Increasing Length

The problem is completely the same as in Chapter 2 (Figure 2.3). The position of the sample nodes are the same as in Chapter 3 (Figure 3.2). The numerical results are tabulated in the Table 4.1.

Table 4.1 shows that the local formula HBIE and the local formula sensitivity HBIE are also very accurate. The displacement errors and the displacement sensitivity errors are all less than 0.06%.

<table>
<thead>
<tr>
<th>Node #</th>
<th>HBIE $w$</th>
<th>Analytical $w$</th>
<th>SHBIE $dw/db$</th>
<th>analytical $dw/db$</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>3.0015</td>
<td>3.0000</td>
<td>1.0003</td>
<td>1.0000</td>
</tr>
<tr>
<td>38</td>
<td>3.0017</td>
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<td>1.0000</td>
</tr>
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<tr>
<td>27</td>
<td>3.0015</td>
<td>3.0000</td>
<td>1.0003</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
CHAPTER 5. APPLICATION TO FRACTURE MECHANICS

Because of the presence of cracks, boundary-element meshes for fracture mechanics are usually very fine to describe the steep variations of the displacement and stress distributions around cracks. Thus, fracture mechanics analyses, with a large number of elements, are usually very computationally intensive.

An alternate approach is investigated in the first part of this chapter to reduce the CPU time required in the computation of the stress intensity factor versus crack size curves. This approach calculates not only the stress intensity factors, but also their derivatives with respect to variables defining the crack size. The CPU time is reduced because the expenses for computing a function value plus a derivative value is generally less than those for computing two function values. While Hypersingular BIEs have been highly successful in modeling crack problems and computing stress intensity factors, sensitivity HBIEs are powerful tools in this approach for computing the derivatives of the stress intensity factors. Three numerical examples are presented in the second section of this chapter to show the accuracy of the computation and the effectiveness of the approach.

5.1. Stress Intensity Factor Computations

Stress-intensity-factor versus crack-size curves (Figure 5.1) are useful in engineering applications. Generally, obtaining such curves requires intensive computational work. Every
point on the curve needs a full computation of the problem, and every computation is a complicated analysis for a domain with crack. Interpolation between data points will save much computing time, but this can be done even better with another option.

5.1.1. An Option with the Computation of Derivative Values

In sensitivity analysis, the computational expenses for obtaining one function value AND one derivative value is generally less than those for obtaining two function values. Two principal reasons contribute to this conclusion. The first is that when the function values \( \{x\} \) are solved from equation (2.2), (repeated here)

\[
[A]\{x\} = \{c\}
\]  

(2.2)

the coefficient matrix \( [A] \), which is NxN in size is stored and then is reused later when the derivative values \( \{x'\} \) are solved from equation (2.8), (repeated here)

![Figure 5.1 Stress Intensity Factor (K) Versus Crack Size (b) Curve](image)

Figure 5.1 Stress Intensity Factor (K) Versus Crack Size (b) Curve
because, as emphasized before, the two matrices are identical. No additional CPU time for forming the coefficient matrix in the derivative computations is required. A more important point is that, in practical computations, not the coefficient matrix $[A]$ itself, but the LU decomposed $[A]$ is stored, so even the decomposition needs to be done only once. Thus, solving equation (2.2) twice to obtain a sensitivity is an operation the CPU time for which is of order $2 \times N^3$. On the other hand, solving equation (2.2) once and then using the already-formed and stored LU decomposition of $[A]$ to solve equation (2.8), is an operation of order $N^3 + N^2$. The second reason is that most of the computational work for forming of the term $\{D\}$ in equation (2.8) is needed only in the regions related to the varying the radius of the crack $b$. For example, for a circular bar with penny-shaped crack, if $b$ is considered as the design variable, most of the work in forming of the term $\{D\}$ is needed just on the crack surfaces. Generally speaking, in sensitivity analysis problems, the regions related to the variable $b$ are always very small compared to the whole domain. Thus CPU time is saved accordingly.

After both the function values and the derivative values are obtained, a Hermite type interpolation can be employed to draw out the curves (Figure 5.2).

### 5.1.2. Calculation of Stress Intensity Factors via Displacements

There are many ways to calculate the stress intensity factors. One of them is to calculate the stress intensity factors via the displacements at the points on the two crack surfaces near the crack tip. From fracture mechanics [42], the displacements around a crack tip (Figure 5.3) can be formulated as functions of the stress intensity factors.
Formula (5.1) can be inverted by considering the difference of the displacements at two points near the crack tip, which have the same coordinates but on the opposite surfaces of the crack. The inverted formulas are

\[ K_i = \frac{G}{2(1-v)} \lim_{r \to 0} \sqrt{\frac{\pi}{2r}} (u_2|_{\theta=\pm 180^\circ} - u_2|_{\theta=\pm 180^\circ}) \]

\[ K_{ii} = \frac{G}{2(1-v)} \lim_{r \to 0} \sqrt{\frac{\pi}{2r}} (u_1|_{\theta=\pm 180^\circ} - u_1|_{\theta=\pm 180^\circ}) \]

\[ K_{iii} = \frac{G}{2} \lim_{r \to 0} \sqrt{\frac{\pi}{2r}} (u_3|_{\theta=\pm 180^\circ} - u_3|_{\theta=\pm 180^\circ}) \]

Figure 5.2 The K versus b curve by using slopes
Figure 5.3 Coordinate system at the crack tip

Therefore, the computation procedure is to solve the problem and get the displacements on the crack surface. In the single-surface crack model, i.e., when the HBIE local formula is employed, the solutions on the crack surface, $\Delta u_k$, are automatically the difference of the displacements between the two surfaces of the crack. The stress intensity factors can be obtained by substituting these $\Delta u_k$ into equation (5.2) as $u_k|_{\theta=\pm90^\circ} - u_k|_{\theta=\pm90^\circ}$. Because of the limit process, equation (5.1) and (5.2) are effective only for the region near the crack tip. For the present work, data points are always taken within one tenth of the crack length from the crack tip.
5.2. Numerical Results

Three crack problem examples are presented in this section, a circular bar with penny-shaped crack where the bar is loaded in tension, in bending, and in torsion. These problems are selected mainly because analytical solutions exist for them, so that the accuracy of the BIE solution can be verified. However, the derived formulas and the developed computer codes are sufficiently general to be used for the solution of more complicated problems. Only obviously new geometrical, material property, and loading data are required for each new problem.

5.2.1. Example 1 - A Circular Bar with Crack in Tension

A circular bar with a penny-shaped crack (Figure 5.4) is taken as the first example for the computations of stress intensity factors. The theoretical solutions for this bar have been determined (Tada et al. [43]). As the theoretical solutions are for a bar with infinite length, the length of this bar is taken to be \( L = 6.0 \) while the radius of the bar is \( R = 1.0 \). The whole structure is divided into 172 elements, including the 68 elements on the crack surface Figure 5.5). One end of the bar is totally fixed, and a uniformly distributed tension load of \( q = 1.0 \) is applied at the other end.

The radius of the crack, \( b \), is taken as the design variable. Six data points are considered for making a stress-intensity-factor versus crack-size curve, that is, \( b = 0.01, 0.05, 0.1, 0.25, 0.5, \) and \( 0.75 \). At each point, the stress intensity factor is calculated by

\[
K_I = \frac{G}{2(1-v)} \sqrt{\pi \Delta u_2}
\]

and the derivative of the stress intensity factor with respect to \( b \) is given by
Figure 5.4 A circular bar with penny-shaped crack

\[
\dot{K}_I = \frac{G}{2(1-v)} \sqrt{\frac{\pi}{2r}} \left( \Delta u_2 - \frac{1}{2r} \frac{\dot{r}}{r} \Delta u_2 \right)
\]  

(5.4)

where, \( \Delta u_2 \) is evaluated by the sensitivity HBIE, but \( \dot{r} \) needs to be determined from (see Figure 5.6)

\[
r = b - R
\]  

(5.5)

as

\[
\dot{r} = 1 - \dot{R}
\]  

(5.6)
Figure 5.5 The boundary element mesh for the crack surface
if an assumption is made that the increased length of the crack is distributed evenly along the radius direction, in other words, if it is supposed that \( \dot{R} \) is a linear function of \( R \), and, \( \dot{R} = 0 \) at \( R = 0 \), \( \dot{R} = 1 \) at \( R = b \), the \( \dot{R} \) will simply be

\[
\dot{R} = \frac{R}{b}
\]  \hspace{1cm} (5.7)

The numerical results are shown in Figure 5.7. The dotted line represents the theoretical solution from Tada [43]. The circles show the stress intensity factor values.

![Figure 5.6 The relationship among \( r, R, \) and \( b \)](image)
calculated from the ordinary HBIE, and the short straight line sections show the slopes obtained from the sensitivity HBIE. Reasonable agreement can also be seen from the Table 5.1 and Table 5.2.

The results for $b = 0.5$ and $b = 0.75$ have relatively big errors. That is because the same mesh configurations were used for all situations. Actually, a finer mesh should be used when $b$ becomes larger.

Table 5.1. $K_{r}$ of the Bar in Tension

<table>
<thead>
<tr>
<th>$b/R$</th>
<th>$K_{r}$, Theoretical</th>
<th>$K_{r}$, HBIE</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.1128</td>
<td>0.1138</td>
<td>0.80%</td>
</tr>
<tr>
<td>0.05</td>
<td>0.2523</td>
<td>0.2544</td>
<td>0.82%</td>
</tr>
<tr>
<td>0.10</td>
<td>0.3570</td>
<td>0.3600</td>
<td>0.82%</td>
</tr>
<tr>
<td>0.25</td>
<td>0.5694</td>
<td>0.5680</td>
<td>0.24%</td>
</tr>
<tr>
<td>0.50</td>
<td>0.8624</td>
<td>0.8647</td>
<td>0.27%</td>
</tr>
<tr>
<td>0.75</td>
<td>1.3413</td>
<td>1.3528</td>
<td>0.85%</td>
</tr>
</tbody>
</table>

Table 5.2. $\dot{K}_{r}$ of the Bar in Tension

<table>
<thead>
<tr>
<th>$b/R$</th>
<th>$\dot{K}_{r}$, Theoretical</th>
<th>$\dot{K}_{r}$, HBIE</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>5.6419</td>
<td>5.6766</td>
<td>0.60%</td>
</tr>
<tr>
<td>0.05</td>
<td>2.5245</td>
<td>2.5399</td>
<td>0.61%</td>
</tr>
<tr>
<td>0.10</td>
<td>1.7915</td>
<td>1.8027</td>
<td>0.62%</td>
</tr>
<tr>
<td>0.25</td>
<td>1.2017</td>
<td>1.1923</td>
<td>0.79%</td>
</tr>
<tr>
<td>0.50</td>
<td>1.2933</td>
<td>1.3207</td>
<td>2.12%</td>
</tr>
<tr>
<td>0.75</td>
<td>3.1152</td>
<td>2.7537</td>
<td>11.6%</td>
</tr>
</tbody>
</table>
Figure 5.7 Stress intensity factors and their derivatives for a circular bar with penny-shaped crack in tension
5.2.2. Example 2 - A Circular Bar with Crack in Bending

The same bar with the same mesh is taken for this problem. One end of the bar is fixed. At the other end, a linear distribution of traction is applied forming a pure bending couple.

The stress intensity factor and their derivatives results at five data points are determined, that is, \( b = 0.05, 0.1, 0.3, 0.5, \) and \( 0.7 \). The stress intensity factors calculated are also \( K_I \)'s. Thus, the formulas are completely the same as those in tension. The results are shown in Figure 5.8, Table 5.3, and Table 5.4.

<table>
<thead>
<tr>
<th>( b/R )</th>
<th>( K_I, ) Theoretical</th>
<th>( K_I, ) HBIE</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.0087</td>
<td>0.0086</td>
<td>1.41%</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0247</td>
<td>0.0243</td>
<td>1.42%</td>
</tr>
<tr>
<td>0.30</td>
<td>0.1285</td>
<td>0.1267</td>
<td>1.42%</td>
</tr>
<tr>
<td>0.50</td>
<td>0.2816</td>
<td>0.2770</td>
<td>1.61%</td>
</tr>
<tr>
<td>0.70</td>
<td>0.5100</td>
<td>0.4954</td>
<td>2.86%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( b/R )</th>
<th>( \dot{K}_I, ) Theoretical</th>
<th>( \dot{K}_I, ) HBIE</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.2618</td>
<td>0.2581</td>
<td>1.42%</td>
</tr>
<tr>
<td>0.10</td>
<td>0.3703</td>
<td>0.3651</td>
<td>1.42%</td>
</tr>
<tr>
<td>0.30</td>
<td>0.6458</td>
<td>0.6359</td>
<td>1.54%</td>
</tr>
<tr>
<td>0.50</td>
<td>0.9000</td>
<td>0.8804</td>
<td>2.18%</td>
</tr>
<tr>
<td>0.70</td>
<td>1.5154</td>
<td>1.3732</td>
<td>9.38%</td>
</tr>
</tbody>
</table>
Figure 5.8 Stress intensity factors and their derivatives for a circular bar with penny-shaped crack in bending
5.2.3. Example 3 - A Circular Bar with Crack in Torsion

The evaluation of the stress intensity factors involves an extra step even for the same bar with the same mesh, if a torsion load is applied, because tangential displacements are needed. The stress intensity factor can be expressed as simply as

\[ K_{III} = \frac{G}{2} \frac{\pi}{\sqrt{2r}} \Delta u_t. \]  \hspace{1cm} (5.8)

To get the tangential displacement, the relationship (Figure 5.9)

Figure 5.9 Displacement components in two coordinate systems
\[ u_r = d \cdot \sin \theta \]  \hspace{1cm} (5.9)

is used, where

\[ \sin \theta = \left| \sin \beta \cos \alpha - \cos \beta \sin \alpha \right| \]  \hspace{1cm} (5.10)

and

\[
\begin{align*}
\sin \beta &= \frac{v}{d}, & \cos \beta &= \frac{u}{d} \\
\sin \alpha &= \frac{y}{R_0}, & \cos \alpha &= \frac{x}{R_0} .
\end{align*}
\]  \hspace{1cm} (5.11)

Thus

\[ u_r = \frac{1}{R_0} |v x - u y| \]  \hspace{1cm} (5.12)

or

\[ \Delta u_r = \frac{1}{R_0} \left| \Delta v x - \Delta u y \right| . \]  \hspace{1cm} (5.13)

Under the assumption made previously, the increasing size of the crack is distributed evenly along the radius direction. Thus, the derivatives of \( \Delta u_r \) can be obtained simply as

\[
\Delta u_r = \frac{1}{R_0} \left| \Delta v x - \Delta u y \right| .
\]  \hspace{1cm} (5.14)

if it is realized that \( \frac{x}{R_0} \) and \( \frac{y}{R_0} \) are actually constants at any moment during the deformation.

The results still agree well with the theoretical solution (see Figure 5.10, Table 5.5, and Table 5.6).
Figure 5.10 Stress intensity factors and their derivatives for a circular bar with penny-shaped crack in torsion
Table 5.5. $K_{in}$ of the Bar in Torsion

<table>
<thead>
<tr>
<th>$b/R$</th>
<th>$K_{in}$, Theoretical</th>
<th>$K_{in}$, HBIE</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.0087</td>
<td>0.0086</td>
<td>1.70%</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0247</td>
<td>0.0243</td>
<td>1.74%</td>
</tr>
<tr>
<td>0.30</td>
<td>0.1284</td>
<td>0.1261</td>
<td>2.06%</td>
</tr>
<tr>
<td>0.50</td>
<td>0.2787</td>
<td>0.2734</td>
<td>1.88%</td>
</tr>
<tr>
<td>0.70</td>
<td>0.4854</td>
<td>0.4751</td>
<td>2.11%</td>
</tr>
</tbody>
</table>

Table 5.6. $\dot{K}_{in}$ of the Bar in Torsion

<table>
<thead>
<tr>
<th>$b/R$</th>
<th>$\dot{K}_{in}$, Theoretical</th>
<th>$\dot{K}_{in}$, HBIE</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.2618</td>
<td>0.2573</td>
<td>1.73%</td>
</tr>
<tr>
<td>0.10</td>
<td>0.3703</td>
<td>0.3639</td>
<td>1.74%</td>
</tr>
<tr>
<td>0.30</td>
<td>0.6434</td>
<td>0.6318</td>
<td>1.80%</td>
</tr>
<tr>
<td>0.50</td>
<td>0.8637</td>
<td>0.8449</td>
<td>2.18%</td>
</tr>
<tr>
<td>0.70</td>
<td>1.2832</td>
<td>1.2575</td>
<td>2.00%</td>
</tr>
</tbody>
</table>
CHAPTER 6. CONCLUSIONS

The sensitivity CBIE has been developed for several years and has been successfully used in engineering such as in design sensitivity analysis. Particular applications include shape optimization, flaw detecting, acoustic design for noise control, etc. It is very difficult to use the sensitivity CBIE for problems with cracks or thin-body domains because of the degeneracy of the coefficient matrix. The sensitivity HBIE is a powerful and efficient tool for solving these kinds of problems. This thesis derives the sensitivity HBIE formulas and develops the associated computer codes. No similar work has been done to the knowledge of the author.

Formulas for the sensitivity HBIE are derived in this thesis in two forms, the global formula and the local formula. The derivation work with either formula is facilitated by realizing that the sensitivity BIE formulas have the same singularity order as the ordinary BIE. No extra regularization is needed if the derivation of the sensitivity BIE is started from the weakly-singular form of the corresponding ordinary BIE formulas. This strategy can also be used to reduce the effort in the derivation of other kinds of sensitivity BIE formulas, such as those in acoustics, elastodynamics, nonlinear BIEs, etc.

The two forms of the sensitivity HBIEs developed in this thesis can be used with several optional methods to solve crack problems or problems with thin body domains. One of the options, the single-surface-crack model, which uses the sensitivity HBIE only, is used here successfully for true-crack problems. The other two options can also be exercised using either the global formula or the local formula of the sensitivity HBIE. Those options should be very useful for open-crack problems or thin-body problems.
Numerical results from some stress-intensity-factor computations for structures with cracks are shown to verify the correctness of the formulas and codes developed in this thesis. It is shown that this is a good way to get stress-intensity-factor versus crack-size curves effectively. Computation time is reduced because the computational expenses for obtaining one function value plus one derivative value is less than the time for obtaining two function values. The feature of high accuracy of the sensitivity CBIE and HBIE can be especially valuable for shape optimization studies in which high-accuracy sensitivity results could reduce high computation costs associated with repeated analysis.

Sensitivity BIEs, as shown in this thesis, give rise to computations of displacement sensitivities and stress-intensity sensitivities. More generally, it is very easy with the sensitivity BIEs, to calculate the derivatives of tractions and stresses with respect to the change of any design variable in any direction. This would be convenient for other kinds of engineering applications, such as shape optimization problems and acoustic design sensitivity analysis.

There is a single design variable appearing in the examples given here for simplicity. The basic ideas are the same when this technique needs to be extended to problems with more than one design variable. All of the formulas are the same. The computer codes will have little difference in data structures. The input data for the design variable perturbations will become a matrix instead of a vector, and so will the resulting design-sensitivity results.
APPENDIX A. EXPRESSIONS FOR SENSITIVITY CBIE KERNELS

For 3-D elastostatics, the CBIE kernels are given by (2.13) and (2.14), (repeated here)

\[ U_y(P,Q) = \frac{(3-4v)\delta_y}{16\pi G(1-v)r} + \frac{1}{16\pi G(1-v)} \frac{r \cdot r}{r} \]  

\[ T_y(P,Q) = \frac{-(1-2v)\delta_y}{8\pi (1-v)} \frac{r \cdot r}{r^2} - \frac{3}{8\pi (1-v)} \frac{r \cdot r}{r^2} \cdot \frac{r \cdot r}{r} + \frac{(1-2v)}{8\pi (1-v)} \frac{1}{r^2} (r \cdot n_i - r \cdot n_i). \]  

(2.14)

The sensitivity CBIE kernels, which are the derivatives of the CBIE kernels with respect to the design variable \( b \), can be obtained from (2.15) and (2.16) (repeated here)

\[ \dot{U}_y(P,Q) = \frac{\partial U_y}{\partial x_k} \left( \frac{dx_k}{db} (Q) - \frac{dx_k}{db} (P) \right) \]  

\[ \dot{T}_y(P,Q) = \frac{\partial T_y}{\partial x_k} \left( \frac{dx_k}{db} (Q) - \frac{dx_k}{db} (P) \right) + \frac{\partial T_y}{\partial n} \]  

(2.16)

After the derivatives of the coordinates, \( \frac{dx}{db} = x \), \( \frac{dy}{db} = y \), and \( \frac{dz}{db} = z \) are given as input data, the derivatives of the CBIE kernels with respect to \( x_k \), \( \frac{\partial U_y}{\partial x_k} \) and \( \frac{\partial T_y}{\partial x_k} \) need to be derived.

First, some basic derivatives are needed, which are
\[ \frac{\partial}{\partial x_k} r_{ij} = \frac{1}{r} (\delta_{ik} - r_j r_k) \]  
(A.1)

\[ r^* = r_j n_i \]  
(A.2)

and

\[ \frac{\partial}{\partial x_k} r_n = \frac{\partial}{\partial x_k} \left( \frac{\partial r}{\partial n} \right) = \frac{\partial}{\partial n} \left( \frac{\partial r}{\partial x_k} \right) = r_n n_i \]

that is

\[ \frac{\partial}{\partial x_k} r_n = \frac{1}{r} (n_k - r_n r_k). \]  
(A.3)

Then, the following terms may be defined

\[ U_{1_k} = \frac{\partial}{\partial x_k} \left( \frac{1}{r} \right) = -\frac{1}{r^2} r_k \]  
(A.4)

\[ U_{2_k} = \frac{\partial}{\partial x_k} \left[ \frac{1}{r} r_j r_j \right] = \frac{\partial}{\partial x_k} \left( \frac{1}{r} \right) r_j r_j + \frac{1}{r} \frac{\partial r_j}{\partial x_k} r_j + \frac{1}{r} r_j \frac{\partial r_j}{\partial x_k} \]

\[ = -\frac{1}{r^2} r_k r_j r_j + \frac{1}{r^2} (\delta_{jk} - r_j r_k) r_j + \frac{1}{r^2} (\delta_{jk} - r_j r_k) r_j. \]

\[ U_{2_k} \] is simplified to

\[ U_{2_k} = \frac{1}{r^2} (\delta_{jk} r_j + \delta_{jk} r_j - 3r_j r_j r_k). \]  
(A.5)

Similarly,

\[ T_{1_k} = \frac{\partial}{\partial x_k} \left[ \frac{r_n}{r^2} \right] = \frac{\partial}{\partial x_k} \left( \frac{1}{r^2} \right) r_n + \frac{1}{r^2} \frac{\partial r_n}{\partial x_k} \]

\[ = -\frac{2}{r^3} (r_n r_k) + \frac{1}{r^3} (n_k - r_n r_k) \]

which may be written
\[ T_{1,k} = \frac{1}{r^3} (n_k - 3r_n r_k). \] (A.6)

Also
\[ T_{2,k} \equiv \frac{\partial}{\partial x_k} \left[ \frac{r_n}{r^2} r_i r_j \right] = \frac{\partial}{\partial x_k} \left( \frac{r_n}{r^2} \right) r_j r_j + \frac{r_n}{r^2} \frac{\partial r_j}{\partial x_k} \cdot r_j + \frac{r_n}{r^2} \cdot r_j \cdot \frac{\partial r_j}{\partial x_k} \]
\[ = \frac{1}{r^3} (n_k - 3r_n r_k) r_j r_j + \frac{r_n}{r^2} (r_{nk} r_j + r_{jk} r_n) \]
becomes
\[ T_{2,k} \equiv \frac{r_n}{r^3} (\delta_{nk} r_j + \delta_{jk} r_n - 5r_n r_k) + \frac{1}{r^3} r_j r_j n_k \] (A.7)

and
\[ T_{3,k} \equiv \frac{\partial}{\partial x_k} \left[ \frac{1}{r^2} (r_n r_j - r_j n_i) \right] \]
\[ = \frac{\partial}{\partial x_k} \left( \frac{1}{r^2} \right) r_j n_j + \frac{1}{r^2} \cdot r_j \cdot n_j - \frac{\partial n_j}{\partial x_k} \left( \frac{1}{r^2} \right) \cdot r_j n_j - \frac{1}{r^2} \cdot \frac{\partial r_j}{\partial x_k} \cdot n_j \]
becomes
\[ T_{3,k} = \frac{1}{r^3} \left( 3r_n r_j r_k - 3r_j n_j r_k + \delta_{nk} n_j - \delta_{jk} n_i \right). \] (A.8)

Then \( \frac{\partial U_{\tilde{q}}}{\partial x_k} \) and \( \frac{\partial T_{\tilde{q}}}{\partial x_k} \) become
\[ \frac{\partial U_{\tilde{q}}}{\partial x_k} = \frac{(3-4v)\delta_q U_{1_k} + U_{2_k}}{16\pi G(1-v)} \] (A.9)
\[ \frac{\partial T_{\tilde{q}}}{\partial x_k} = \frac{1}{8\pi(1-v)} \left( -(1-2v)\delta_q T_{1_k} - 3T_{2_k} + (1-2v)T_{3,k} \right) \] (A.10)

and
\[ \frac{\partial T_q}{\partial n} = \frac{-(1-2\nu)\delta_g r_n^*}{8\pi(1-\nu) r^2} + \frac{3 r_n^*}{8\pi(1-\nu) r^2} r_j r_i^* \\
+ \frac{(1-2\nu)}{8\pi(1-\nu) r^2} (r_j n_j^* - r_j n_i^*) \]
APPENDIX B. EXPRESSIONS FOR GEOMETRIC CHANGES

In Chapter 2, the coordinates of an arbitrary point within an element has been expressed as

\[ x_i = \sum_{j=1}^{n} N_j(\xi, \eta) x_i^{(j)}. \tag{2.17} \]

The normal vector and the Jacobian at this point are

\[ \vec{n} = \frac{\vec{x}_\xi \times \vec{x}_\eta}{|\vec{x}_\xi \times \vec{x}_\eta|} = \frac{\vec{n}^*}{J} \tag{2.18} \]

and

\[ J = |\vec{x}_\xi \times \vec{x}_\eta| \tag{2.19} \]

where

\[ \vec{n}^* = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sum_{j=1}^{n} N_{j\xi} x_i^{(j)} & \sum_{j=1}^{n} N_{j\xi} y_i^{(j)} & \sum_{j=1}^{n} N_{j\xi} z_i^{(j)} \\ \sum_{j=1}^{n} N_{j\eta} x_i^{(j)} & \sum_{j=1}^{n} N_{j\eta} y_i^{(j)} & \sum_{j=1}^{n} N_{j\eta} z_i^{(j)} \end{vmatrix}. \tag{B.1a} \]

Written in component form
\[ n^*(1) = \left( \sum_{j=1}^{n} N_{i,k} y^{(j)} \right) \left( \sum_{j=1}^{n} N_{i,n} z^{(j)} \right) - \left( \sum_{j=1}^{n} N_{i,k} z^{(j)} \right) \left( \sum_{j=1}^{n} N_{i,n} y^{(j)} \right) \]

\[ n^*(2) = \left( \sum_{j=1}^{n} N_{i,k} z^{(j)} \right) \left( \sum_{j=1}^{n} N_{i,n} x^{(j)} \right) - \left( \sum_{j=1}^{n} N_{i,k} x^{(j)} \right) \left( \sum_{j=1}^{n} N_{i,n} z^{(j)} \right) \]  \hspace{1cm} (B.1b)

\[ n^*(3) = \left( \sum_{j=1}^{n} N_{i,k} x^{(j)} \right) \left( \sum_{j=1}^{n} N_{i,n} y^{(j)} \right) - \left( \sum_{j=1}^{n} N_{i,k} y^{(j)} \right) \left( \sum_{j=1}^{n} N_{i,n} x^{(j)} \right) \]

and

\[ J = \sqrt{\mathbf{n}^* \cdot \mathbf{n}^*} = \sqrt{n^*(1)^2 + n^*(2)^2 + n^*(3)^2} \]  \hspace{1cm} (B.2a)

or

\[ J = \text{SQRT} \left\{ \left[ \left( \sum_{j=1}^{n} N_{i,k} y^{(j)} \right) \left( \sum_{j=1}^{n} N_{i,n} z^{(j)} \right) - \left( \sum_{j=1}^{n} N_{i,k} z^{(j)} \right) \left( \sum_{j=1}^{n} N_{i,n} y^{(j)} \right) \right]^2 \right. \\
+ \left[ \left( \sum_{j=1}^{n} N_{i,k} z^{(j)} \right) \left( \sum_{j=1}^{n} N_{i,n} x^{(j)} \right) - \left( \sum_{j=1}^{n} N_{i,k} x^{(j)} \right) \left( \sum_{j=1}^{n} N_{i,n} z^{(j)} \right) \right]^2 \right\} \]  \hspace{1cm} (B.2b)

From these definitions, the derivatives of the normal \( n \), and the Jacobian \( J \) with respect to the design variable \( b \) can been obtained by using the chain rule as

\[ \dot{n}_i = \sum_{j=1}^{n} \sum_{k=1}^{3} \frac{\partial n}{\partial x_k^{(j)}} \dot{x}_k^{(j)} \]  \hspace{1cm} (2.21)

and

\[ \dot{J} = \sum_{j=1}^{n} \sum_{k=1}^{3} \frac{\partial J}{\partial x_k^{(j)}} \dot{x}_k^{(j)} \]  \hspace{1cm} (2.22)

where
\[
\frac{\partial J}{\partial x^{(j)}} = n(2) \frac{\partial n^*(2)}{\partial x^{(j)}} + n(3) \frac{\partial n^*(3)}{\partial x^{(j)}}
\]

\[
\frac{\partial J}{\partial y^{(j)}} = n(1) \frac{\partial n^*(1)}{\partial y^{(j)}} + n(3) \frac{\partial n^*(3)}{\partial y^{(j)}}
\]

\[
\frac{\partial J}{\partial z^{(j)}} = n(1) \frac{\partial n^*(1)}{\partial z^{(j)}} + n(2) \frac{\partial n^*(2)}{\partial z^{(j)}}
\]

with

\[
\frac{\partial n^*(1)}{\partial x^{(j)}} = 0
\]

\[
\frac{\partial n^*(2)}{\partial x^{(j)}} = N_{j,\eta} \left( \sum_{k=1}^{n} N_{k,\xi} z^{(k)} \right) - N_{j,\eta} \left( \sum_{k=1}^{n} N_{k,\xi} z^{(k)} \right)
\]

\[
\frac{\partial n^*(3)}{\partial x^{(j)}} = N_{j,\xi} \left( \sum_{k=1}^{n} N_{k,\eta} x^{(k)} \right) - N_{j,\eta} \left( \sum_{k=1}^{n} N_{k,\xi} x^{(k)} \right)
\]

\[
\frac{\partial n^*(1)}{\partial y^{(j)}} = N_{j,\xi} \left( \sum_{k=1}^{n} N_{k,\eta} y^{(k)} \right) - N_{j,\xi} \left( \sum_{k=1}^{n} N_{k,\xi} y^{(k)} \right)
\]

\[
\frac{\partial n^*(2)}{\partial y^{(j)}} = 0
\]

\[
\frac{\partial n^*(3)}{\partial y^{(j)}} = N_{j,\eta} \left( \sum_{k=1}^{n} N_{k,\xi} x^{(k)} \right) - N_{j,\xi} \left( \sum_{k=1}^{n} N_{k,\eta} x^{(k)} \right)
\]

\[
\frac{\partial n^*(1)}{\partial z^{(j)}} = N_{j,\eta} \left( \sum_{k=1}^{n} N_{k,\xi} y^{(k)} \right) - N_{j,\xi} \left( \sum_{k=1}^{n} N_{k,\eta} y^{(k)} \right)
\]

\[
\frac{\partial n^*(2)}{\partial z^{(j)}} = N_{j,\xi} \left( \sum_{k=1}^{n} N_{k,\eta} x^{(k)} \right) - N_{j,\eta} \left( \sum_{k=1}^{n} N_{k,\xi} x^{(k)} \right)
\]

\[
\frac{\partial n^*(3)}{\partial z^{(j)}} = 0
\]

(B.3)

(B.4-1)

(B.4-2)

(B.4-3)

(B.4-4)

(B.4-5)

(B.4-6)

(B.4-7)

(B.4-8)

(B.4-9)
and

\[
\frac{\partial n(1)}{\partial x(i)} = -\frac{n(1)}{J} \frac{\partial J}{\partial x(i)} \tag{B.5-1}
\]

\[
\frac{\partial n(1)}{\partial y(i)} = \frac{1}{J} \left[ \frac{\partial n^*(1)}{\partial y(i)} - n(1) \frac{\partial J}{\partial y(i)} \right] \tag{B.5-2}
\]

\[
\frac{\partial n(1)}{\partial z(i)} = \frac{1}{J} \left[ \frac{\partial n^*(1)}{\partial z(i)} - n(1) \frac{\partial J}{\partial z(i)} \right] \tag{B.5-3}
\]

\[
\frac{\partial n(2)}{\partial x(i)} = \frac{1}{J} \left[ \frac{\partial n^*(2)}{\partial x(i)} - n(2) \frac{\partial J}{\partial x(i)} \right] \tag{B.5-4}
\]

\[
\frac{\partial n(2)}{\partial y(i)} = -\frac{n(2)}{J} \frac{\partial J}{\partial y(i)} \tag{B.5-5}
\]

\[
\frac{\partial n(2)}{\partial z(i)} = \frac{1}{J} \left[ \frac{\partial n^*(2)}{\partial z(i)} - n(2) \frac{\partial J}{\partial z(i)} \right] \tag{B.5-6}
\]

\[
\frac{\partial n(3)}{\partial x(i)} = \frac{1}{J} \left[ \frac{\partial n^*(3)}{\partial x(i)} - n(3) \frac{\partial J}{\partial x(i)} \right] \tag{B.5-7}
\]

\[
\frac{\partial n(3)}{\partial y(i)} = \frac{1}{J} \left[ \frac{\partial n^*(3)}{\partial y(i)} - n(3) \frac{\partial J}{\partial y(i)} \right] \tag{B.5-8}
\]

\[
\frac{\partial n(3)}{\partial z(i)} = -\frac{n(3)}{J} \frac{\partial J}{\partial z(i)} \tag{B.5-9}
\]

Finally, the derivative of the differential area \( ds \) with respect to the design variable \( b \) can be expressed as

\[
\dot{ds} = J \, d\xi \, d\eta \tag{2.23}
\]
APPENDIX C. EXPRESSIONS FOR SENSITIVITY HBIE KERNELS

For 3-D elastostatics, the HBIE kernels are as (3.12) and (3.13)

\[ K_{ij}(P, Q) = \frac{1}{8\pi (1 - \nu) r^2} \left[ (1 - 2\nu) \delta_{ij} r_{a0} + 3 r_{j} r_{a} r_{n0} + (1 - 2\nu)(r_{j} n_{a0} - r_{a} n_{0j}) \right] \]  
\[ (3.12) \]

\[ H_{ij}(P, Q) = \frac{\mu}{4\pi (1 - \nu)} \left\{ 3 v \frac{r_{a0} r_{n0j}}{r^3} + (1 - 2\nu) \frac{n_{k} n_{a0j}}{r^3} \delta_{ij} + (1 - 2\nu) \frac{n_{i} n_{0j}}{r^3} - (1 - 4\nu) \frac{n_{0i} n_{nj}}{r^3} + 3(1 - 2\nu) \frac{r_{a0} r_{ij} n_{0j}}{r^3} + 3v \frac{r_{a0} r_{j} n_{ij}}{r^3} \right\} \]  
\[ (3.13) \]

Notice that \( K_{ij}(P, Q) \) is a function of \( x_k \) and \( n_0 \), \( H_{ij}(P, Q) \) is a function of \( x_k \), \( n_0 \), and \( n \). The sensitivity HBIE kernels can be obtained by taking derivatives of \( K_{ij}(P, Q) \) and \( H_{ij}(P, Q) \) with respect to the design variable \( b \). These derivatives have the form

\[ K_{ij}(P, Q) = \frac{\partial K_{ij}(P, Q)}{\partial x_k} \left( \frac{dx_k}{db} (Q) - \frac{dx_k}{db} (P) \right) + \frac{\partial K_{ij}(P, Q)}{\partial n_0} \frac{dn_0}{db} \]  
\[ (3.14) \]

\[ H_{ij}(P, Q) = \frac{\partial H_{ij}(P, Q)}{\partial x_k} \left( \frac{dx_k}{db} (Q) - \frac{dx_k}{db} (P) \right) + \frac{\partial H_{ij}(P, Q)}{\partial n_0} \frac{dn_0}{db} + \frac{\partial H_{ij}(P, Q)}{\partial n} \frac{dn}{db} \]  
\[ (3.15) \]
where, \( n_0 \) is the normal at the source point, \( n \) is the normal at the field point. The quantities
\[
\frac{\partial K^q(P,Q)}{\partial x_k} \quad \text{and} \quad \frac{\partial H^q(P,Q)}{\partial x_k}
\]
are derivatives of the ordinary HBIE kernels with respect to the Cartesian coordinates.

Alternatively, considering that \( K^q(P,Q) \) is similar to the \( T^q(P,Q) \) in the CBIE, all of the formulas for \( K^q(P,Q) \) could be obtained simply by replacing \( n \) with \( n_0 \) and being aware of the differences among the signs of the terms. That is

\[
\dot{K}^q = \sum_{k=1}^{3} \frac{(1-2v)\delta_{kj}K_1(k) + 3K_2(k) + (1-2v)K_3(k)}{8\pi(1-v)} \left[ \dot{x}_k(Q) - \dot{x}_k(P) \right]
\]

\[\text{+} \quad \frac{1}{8\pi(1-v)r^2} \left\{ r_{n0} \left[ (1-2v)\delta_{nj} + 3r_jr_j \right] + (1-2v) \left( r_i n^*_{nj} - r_j n^*_{ni} \right) \right\} \]

(\text{C.1})

where

\[
K_1(k) = \frac{\partial}{\partial x_k} \left( \frac{r_{n0}}{r^2} \right)
\]

\[= \frac{1}{r^3} (n_{0k} - 3r_{n0}r_k) \quad \text{(C.2)}\]

\[
K_2(k) = \frac{\partial}{\partial x_k} \left( \frac{r_{n0}}{r^2} r_jr_j \right)
\]

\[= \frac{r_{n0}}{r} \left( \delta_{jk}r_j + \delta_{jk}r_k - 5r_jr_j' \right) + \frac{1}{r^3} r_jr_j n_{0k} \quad \text{(C.3)}\]

\[
K_3(k) = \frac{\partial}{\partial x_k} \left[ \frac{1}{r^2} (r_jn_{0j} - r_jn_{0i}) \right]
\]

\[= \frac{1}{r^3} (3r_jr_k n_{0i} - 3r_jr_k n_{0j} + \delta_{jk}n_{0j} - \delta_{jk}n_{0i}) \quad \text{(C.4)}\]
\[ r_{n0} = r_j n_{0i} \quad (C.5) \]

\[ n_{0i} = \frac{dn_{0i}}{db} \quad (C.6) \]

and, \( n_{0i} \) are the directional cosines of the normal vector at the source point.

For a more explicit expression for \( H_{ij}(P,Q) \), certain expressions should be defined first:

\[ H_{ij}(k) \equiv \frac{\partial}{\partial x_k} \left( \frac{1}{r^3} \right) \]

\[ = -\frac{3}{r^4} r_k \quad (C.7) \]

\[ \frac{\partial r_{n0}}{\partial x_k} = \frac{\partial}{\partial x_k} \left( \frac{\partial r}{\partial n_0} \right) = \frac{\partial}{\partial n_0} \left( \frac{\partial r}{\partial x_k} \right) = r_{ki} n_{0i} = \frac{1}{r} (n_{0k} - r_k r_{n0}) \quad (C.8) \]

Then, the items

\[ \frac{\partial}{\partial x_k} \left( \frac{r_{n0} r_{n\alpha}}{r^3} \right) = \frac{\partial}{\partial x_k} \left( \frac{1}{r^3} \right) \cdot r_{n0} r_{n\alpha} + \frac{1}{r^3} \cdot \frac{\partial r_{n0}}{\partial x_k} \cdot r_{n\alpha} + \frac{1}{r^3} \cdot r_{n0} \cdot \frac{\partial r_{n\alpha}}{\partial x_k} \]

\[ = -\frac{3}{r^4} r_k r_{n0} r_{n\alpha} + \frac{1}{r^3} \cdot \frac{1}{r} (n_{0k} - r_{n0} r_{n\alpha}) \cdot r_{n\alpha} + \frac{1}{r^3} r_{n0} \cdot \frac{1}{r} (n_{k} - r_k r_{n\alpha}) \]

may be written. Similarly,

\[ H_{1}(k) \equiv \frac{\partial}{\partial x_k} \left( \frac{r_{n0} r_{\alpha i}}{r^3} \right) \]

\[ = \frac{1}{r^3} (n_k r_{n0} + n_{0k} r_{n\alpha} - 5 r_k r_{n0} r_{\alpha i}) \quad (C.9) \]

\[ \frac{\partial}{\partial x_k} \left( \frac{r_{n0} r_{\alpha i}}{r^3} \right) = \frac{\partial}{\partial x_k} \left( \frac{1}{r^3} \right) \cdot r_{n0} r_{\alpha i} + \frac{1}{r^3} \cdot \frac{\partial r_{n0}}{\partial x_k} \cdot r_{\alpha i} + \frac{1}{r^3} r_{n0} \cdot \frac{\partial r_{\alpha i}}{\partial x_k} \]

\[ = -\frac{3}{r^4} r_k r_{n0} r_{\alpha i} + \frac{1}{r^3} \cdot \frac{1}{r} (n_{0k} - r_{n0} r_{n\alpha}) \cdot r_{\alpha i} + \frac{1}{r^3} r_{n0} \cdot \frac{1}{r} (\delta_{ik} - r_k r_{n\alpha}) r_{\alpha i} \]
and

\[ H_2(k) = \frac{\partial}{\partial x_k} \left( \frac{r_{\alpha\beta} r_{ij} n_{ij}}{r^3} \right) \]

\[ = \frac{n_{ij}}{r^4} \left( \delta_{\alpha\beta} r_{n\alpha} + n_{n\beta} r_{ij} - 5r_{\alpha\beta} r_{n\alpha} r_{ij} \right) \tag{C.10} \]

\[ H_3(k) = \frac{\partial}{\partial x_k} \left( \frac{r_{\alpha\gamma} r_{ij} n_{ij}}{r^3} \right) \]

\[ = \frac{n_{ij}}{r^4} \left( \delta_{\alpha\gamma} r_{n\alpha} + n_{n\gamma} r_{ij} - 5r_{\alpha\gamma} r_{n\alpha} r_{ij} \right) \tag{C.11} \]

\[ H_4(k) = \frac{\partial}{\partial x_k} \left( \frac{r_{\alpha\xi} r_{ij} n_{ij}}{r^3} \right) \]

\[ = \frac{n_{ij}}{r^4} \left( \delta_{\alpha\xi} r_{n\alpha} + n_{n\xi} r_{ij} - 5r_{\alpha\xi} r_{n\alpha} r_{ij} \right) \tag{C.12} \]

\[ H_5(k) = \frac{\partial}{\partial x_k} \left( \frac{r_{\gamma\gamma} r_{ij} n_{ij}}{r^3} \right) \]

\[ = \frac{n_{ij}}{r^4} \left( \delta_{\gamma\gamma} r_{n\gamma} + n_{n\gamma} r_{ij} - 5r_{\gamma\gamma} r_{n\gamma} r_{ij} \right) \tag{C.13} \]

\[ \frac{\partial}{\partial x_k} \left[ \frac{r_{ij}}{r^3} \cdot (n_{n\alpha}) \right] = \frac{\partial}{\partial x_k} \left( \frac{1}{r^3} \right) r_{ij} (n_{n\alpha}) + \frac{1}{r^3} \frac{\partial r_{ij}}{\partial x_k} (n_{n\alpha}) + \frac{1}{r^3} r_{ij} \frac{\partial}{\partial x_k} (n_{n\alpha}) \]

\[ = \frac{-3}{r^4} r_{\gamma k} r_{ij} (n_{n\alpha}) + \frac{1}{r^3} \frac{1}{r} \left( \delta_{\gamma k} - r_{\gamma k} \right) r_{ij} (n_{n\alpha}) \]

\[ + \frac{1}{r^3} r_{ij} \frac{1}{r} \left( \delta_{\gamma k} - r_{\gamma k} \right) (n_{n\alpha}) \]

\[ H_6(k) = \frac{\partial}{\partial x_k} \left[ \frac{r_{ij}}{r^3} \cdot (n_{n\alpha}) \right] \]

\[ = \frac{(n_{n\alpha})}{r^4} \left( r_{ij} \delta_{\gamma k} + r_{ij} \delta_{\alpha k} - 5r_{ji} r_{ij} k \right) \]
\[
\frac{\partial}{\partial x_k} \left( \frac{r_{a0} r_a r_j r_j}{r^3} \right) = \frac{\partial}{\partial x_k} \left( \frac{1}{r^3} \right) r_{a0} r_a r_j r_j + \frac{\partial r_{a0}}{\partial x_k} r_a r_j r_j + \frac{1}{r^3} r_{a0} \frac{\partial r_j}{\partial x_k} r_j r_j
\]
\[
+ \frac{1}{r^3} r_{a0} r_a r_j r_j + \frac{1}{r^3} r_{a0} r_a \frac{\partial r_j}{\partial x_k} r_j
\]
\[
= -\frac{3}{r^2} r_{a0} r_a r_j r_j + \frac{1}{r^3} \left( n_{a0} - n_j r_{a0} \right) r_a r_j r_j + \frac{1}{r^3} r_{a0} \frac{1}{r} \left( n_j - n_j r_{a0} \right) r_j r_j
\]
\[
+ \frac{1}{r^3} r_{a0} r_a \frac{1}{r} \left( \delta_{jk} - r_j r_j \right) r_j + \frac{1}{r^3} r_{a0} r_a \frac{1}{r} \left( \delta_{jk} - r_j r_j \right)
\]
and
\[
H_y(k) = \frac{\partial}{\partial x_k} \left( \frac{r_{a0} r_a r_j r_j}{r^3} \right)
\]
\[
= \frac{1}{r^3} \left( r_j r_j r_j n_{a0} + r_j r_j r_{a0} n_k + r_j \delta_{jk} r_{a0} r_{a0} + r_j \delta_{jk} n_{a0} r_{a0} + 7 r_j r_j r_{a0} r_{a0} \right)
\]

Then, replacing \( r_{a0} \) with \( \dot{r}_{a0} \) and \( n_{a0} \) with \( \dot{n}_{a0} \) in the expression for \( H_y(P, Q) \), it follows that

\[
\frac{\partial H_y}{\partial n_{a0}} \cdot \dot{n}_{a0} = \frac{\mu}{4\pi(1-\nu)} \left[ \left( \frac{3}{r^3} \frac{\dot{r}_{a0} r_a}{r^3} + (1-2\nu) \frac{\left( n_{a0} \dot{n}_{a0} \right)}{r^3} \right) \delta_j + (1-2\nu) \frac{n_j n_{a0}}{r^3} \right]
\]
\[
- (1-4\nu) \frac{n_{a0} n_j}{r^3} + 3(1-2\nu) \frac{r_{a0} r_j n_j}{r^3} + 3\nu \frac{r_{a0} r_j n_j}{r^3}
\]
\[
+ 3\nu \frac{r_j n_j n_{a0}}{r^2} + 3(1-2\nu) \frac{r_{a0} r_j n_j}{r^3} + 3 \left[ \frac{r_j n_j (n_{a0})}{r^3} - 5 \frac{r_j r_j n_{a0} r_{a0}}{r^3} \right]
\]

Similarly, replacing \( r_a \) with \( \dot{r}_a \) and \( n_j \) with \( \dot{n}_j \) in the expression for \( H_y(P, Q) \), it follows that
\[
\frac{\partial H_y}{\partial n_k} \cdot \hat{n}_k = \frac{\mu}{4\pi(1-v)} \left\{ 3v \left( r_{a0} \frac{r_a}{r^3} + (1-2v) \left( \frac{n_k n_{0l}}{r^3} \right) \right) \delta_y + (1-2v) \left( \frac{n_i n_{0j}}{r^3} \right) \\
- (1-4v) \left( \frac{n_{0i} n_j}{r^3} \right) + 3(1-2v) \left( \frac{r_{a0} r_{ij} n_i}{r^3} + 3v \frac{r_{a0} r_{ij} n_i}{r^3} \right) \\
+ 3v \left( \frac{r_{a} r_{ij} n_{0j}}{r^3} + 3(1-2v) \frac{r_{a} r_{ij} n_{0i}}{r^3} + 3 \left( \frac{r_{a} r_{ij} (n_k n_{0k})}{r^3} - 5 \frac{r_{a} r_{ij} r_{a0} r_{a}}{r^3} \right) \right) \right\}.
\]

Finally, the required expression for \( H_y(P,Q) \) is

\[
H_y(P,Q) = \frac{\mu}{4\pi(1-v)} \left\{ 3v H_y(k) + (1-2v) H_6(k) (n_m n_{0m}) \delta_y + (1-2v) H_6(k) n_i n_{0j} \\
- (1-4v) H_6(k) n_i n_j + 3(1-2v) H_5(k) + 3v H_5(k) + 3v H_5(k) \\
+ 3(1-2v) H_5(k) + 3 \left[ v H_6(k) - 5H_5(k) \right] \left[ x_i(Q) - x_i(P) \right] \right\} \\
+ \frac{\mu}{4\pi(1-v)} \left\{ 3v \left( r_{a0} \frac{r_a}{r^3} + (1-2v) \left( \frac{n_k n_{0k}}{r^3} \right) \right) \delta_y + (1-2v) \left( \frac{n_i n_{0j}}{r^3} \right) \\
- (1-4v) \left( \frac{n_{0i} n_j}{r^3} \right) + 3(1-2v) \left( \frac{r_{a0} r_{ij} n_i}{r^3} + 3v \frac{r_{a0} r_{ij} n_i}{r^3} \right) \\
+ 3v \left( \frac{r_{a} r_{ij} n_{0j}}{r^3} + 3(1-2v) \frac{r_{a} r_{ij} n_{0i}}{r^3} + 3 \left( \frac{r_{a} r_{ij} (n_k n_{0k})}{r^3} - 5 \frac{r_{a} r_{ij} r_{a0} r_{a}}{r^3} \right) \right) \right\}.
\]
APPENDIX D. THE DERIVATIVE EXPRESSION FOR INVERSE JACOBIAN

Jacobian Matrix and \( e_{aq} \)

In the weakly singular form HBIE, a special term, \( e_{aq} \), is used when the hypersingular integrals are processed (see equation (3.2)). This term is defined as the inverse of the Jacobian matrix.

In the boundary element method, the boundary of the domain is divided into elements. Every element is defined by its nodes with nodal coordinates \( x_k^{(i)} \) and the shape functions \( N_j(\xi, \eta) \), so that the coordinates of an arbitrary point within this element can be expressed as

\[
x_k = \sum_{j=1}^{n} N_j(\xi, \eta)x_k^{(i)}.
\]

This represents a coordinate transformation from the global coordinate system \( X - Y - Z \) to the local coordinate system \( \xi - \eta \). The corresponding Jacobian matrix for this transformation should be

\[
J_2 = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta}
\end{bmatrix}.
\]

Generally, \( J_2 \) is formed numerically in the computation. From equation (2.17), the terms of \( J_2 \) could be evaluated as
This seems to be a transformation from a 3-D domain to a 2-D domain. The inverse of $J_3$ seems impossible because it is, at least, not square. But, if the normal vector $n$ is also considered in the local coordinate system to form a $\xi - \eta - n$ coordinate system, it is actually a formal 3-D coordinate transformation with the Jacobian matrix

$$J_3 = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\
\frac{\partial x}{\partial n} & \frac{\partial y}{\partial n} & \frac{\partial z}{\partial n}
\end{bmatrix}$$

or

$$J_3 = \begin{bmatrix}
\sum_{j=1}^{n} N_{j,\xi}(\xi,\eta)x^{(j)} & \sum_{j=1}^{n} N_{j,\eta}(\xi,\eta)y^{(j)} & \sum_{j=1}^{n} N_{j,n}(\xi,\eta)z^{(j)} \\
\sum_{j=1}^{n} N_{j,\xi}(\xi,\eta)x^{(j)} & \sum_{j=1}^{n} N_{j,\eta}(\xi,\eta)y^{(j)} & \sum_{j=1}^{n} N_{j,n}(\xi,\eta)z^{(j)} \\
n_1 & n_2 & n_3
\end{bmatrix}$$

where, the expressions for $n_k$ can be found in the Appendix B.

While it is very complicated to invert $J_3$ analytically, it is very easy to perform a numerical inverse. Suppose the result of the numerical inverse of $J_3$ is denoted as

$$J_3^{-1} = \begin{bmatrix}
e_{11} & e_{21} & e_{31} \\
e_{12} & e_{22} & e_{32} \\
e_{13} & e_{23} & e_{33}
\end{bmatrix}$$
Now define a matrix \([e_{aq}]\), which is the transpose of the first two columns of (3.4), i.e.

\[
[e_{aq}] = \begin{bmatrix}
e_{11} & e_{12} & e_{13} \\
e_{21} & e_{22} & e_{23}
\end{bmatrix}
\]  

(3.5)

where, \(\alpha = 1, 2\), is correspondent to the local coordinates \(\xi, \eta\), and \(q = 1, 3\), is correspondent to the global coordinates \(x, y\), and \(z\). This is the term introduced in equation (3.6).

**The Derivative of \(e_{aq}\)**

In the sensitivity HBIE, the derivative of \(e_{aq}\) with respect to the design variable \(b\) is needed (equation (3.11)). It has been seen that \(J_2\) and \(J_3\) are all obtained numerically. \(e_{aq}\) is also obtained numerically. It is impossible to take derivative of those numbers. On the other hand, for the invertable matrix \(J_3\) and its inverse \(J_3^{-1}\)

\[
J_3 J_3^{-1} = I
\]

(3.16)

where, \(I\) is an unit matrix. Taking derivative of (3.16) to get

\[
J_3 \dot{J}_3^{-1} + \dot{J}_3 J_3^{-1} = 0
\]

(3.17)

and then

\[
\dot{J}_3^{-1} = -J_3^{-1} \dot{J}_3 J_3^{-1}.
\]

(3.18)

Equation (3.18) means that a numerical procedure do exist for obtaining \(\dot{J}_3^{-1}\) if \(J_3^{-1}\) and \(\dot{J}_3\) could be obtained in advance.

Obtaining \(J_3^{-1}\) is just a usual numerical procedure, and by taking derivatives of \(J_3\) with respect to the design variable \(b\), \(\dot{J}_3\) takes the form
where, $x_k^{(j)}$ are the nodal values of the derivatives of the coordinates with respect to $b$, which are given as the input data; $n_k$ are the derivatives of the normal vector with respect to $b$, which can be found in the Appendix B.

It is impossible for the $J_3^{-1}$ to be given explicitly because that is a numerical procedure. But, if it is denoted that

\[
J_3^{-1} = \begin{bmatrix}
\dot{e}_{11} & \dot{e}_{12} & \dot{e}_{13} \\
\dot{e}_{21} & \dot{e}_{22} & \dot{e}_{23} \\
\dot{e}_{31} & \dot{e}_{32} & \dot{e}_{33}
\end{bmatrix}
\]  

(D.5)

then the \( [\dot{e}_{a\eta}] \) can be obtained as

\[
[\dot{e}_{a\eta}] = \begin{bmatrix}
\dot{e}_{11} & \dot{e}_{12} & \dot{e}_{13} \\
\dot{e}_{21} & \dot{e}_{22} & \dot{e}_{23}
\end{bmatrix}.
\]  

(D.6)
BIBLIOGRAPHY


