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Multiple-Antenna Interference Channels with Real Interference Alignment and Receive Antenna Joint Processing

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Abstract

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Abstract

In this paper, the degrees of freedom (DoF) regions of constant coefficient multiple antenna interference channels are investigated. First, we consider a K-user Gaussian interference channel with M_k antennas at transmitter k, $1 \leq k \leq K$, and N_j antennas at receiver j, $1 \leq j \leq K$, denoted as a $(K, [M_k], [N_j])$ channel. Relying on a result of simultaneous Diophantine approximation, a real interference alignment scheme with joint receive antenna processing is developed. The scheme is used to obtain an achievable DoF region. The proposed DoF region includes two previously known results as special cases, namely 1) the total DoF of a K-user interference channel with N antennas at each node, $(K, [N], [N])$ channel, is $NK/2$; and 2) the total DoF of a $(K, [M], [N])$ channel is at least $KMN/(M + N)$. We next explore constant-coefficient interference networks with K transmitters and J receivers, all having N antennas. Each transmitter emits an independent message and each receiver requests an arbitrary subset of the messages. Employing the novel joint receive antenna processing, the DoF region for this set-up is obtained. We finally consider wireless X networks where each node is allowed to have an arbitrary number of antennas. It is shown that the joint receive antenna processing can be used to establish an achievable DoF region, which is larger than what is possible with antenna splitting. As a special case of the derived achievable DoF region for constant coefficient X network, the total DoF of wireless X networks with the same number of antennas at all nodes and with joint antenna processing is tight while the best inner bound based on antenna splitting cannot meet the outer bound. Finally, we obtain a DoF region outer bound based on the technique of transmitter grouping.

Keywords: Interference channels; interference alignment; multiple-input multiple-output; degrees of freedom region; X network; Diophantine approximation

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I. INTRODUCTION

Characterizing the capacity region of interference networks is a fundamental problem in information theory. Despite remarkable progress in recent years, the capacity region of interference networks remains unknown in general. Recent work has proposed to use degrees of freedom (DoF) to approximate the capacity region of interference networks. The DoF of a message is its rate normalized by the capacity of single-user additive white Gaussian noise channel, as the signal-to-noise ratio (SNR) tends to infinity. The DoF region quantifies the shape of the capacity region at high SNR; see e.g., [1], [2].

DoF investigations have motivated several fundamental ideas such as interference alignment. With interference alignment, the interference signals at any receiver from multiple transmitters are aligned in the signal space, so that the dimensionality of the interference in the signal space can be minimized. The remaining space is interference free and can be used for the desired signals. Two commonly used alignment schemes are vector alignment and real alignment [3], [4]. In vector alignment, any transmit signal is a linear combination of some vectors in a manner that the coefficients of the linear combination carry useful data. This scheme designs the vectors so that the interferences at each receiver are packed into a common subspace. The orthogonal complement can be used for detecting useful data symbols. In real alignment, the concept of linear independence over the rational numbers replaces the more familiar vector linear independence. A Groshev type theorem is usually used to guarantee the required decoding performance.

A. *DoF of interference channel*

DoF characterizations have been investigated for a variety of wireless networks such as K-user interference channel and wireless X network. In the K-user interference channel, the k-th transmitter has a message intended for the k-th receiver. At receiver k, the messages from transmitters other than the k-th are interference. The DoF region of the K-user interference when all nodes are provided with the same number of antennas is known [5, Corollary 2].

In [6], Gou and Jafar studied the total DoF of the $M \times N$ K-user interference channel where each transmitter has M antennas and each receiver has N antennas. They showed the exact total DoF value is $K\frac{MN}{M+N}$ under the assumption that $R := \frac{\max(M,N)}{\min(M,N)}$ is an integer and $K \geq R$. In [7], Ghasemi et al. employ antenna splitting argument to derive the total DoF $K\frac{MN}{M+N}$ for fixed channels, which is optimal if $K \geq \frac{M+N}{\gcd(M,N)}$ even when R is not an integer. In such antenna splitting arguments, no cooperation is used either at the transmitter side or at the receiver side. The outer bounds of these cases are based on cooperation among groups of transmitters and receivers and employing the DoF outer bound for 2-user multiple-input multiple-output (MIMO) interference channel obtained in [8]. Note that the outer bound discussion is regardless of whether the channel coefficients are constant or time-varying.

A novel genie chains approach for the DoF outer bound of $M \times N$ K-user interference channel has been recently presented in [9]. In this approach, a chain of mapping from genie signals provided at a receiver to the exposed signal spaces of the receiver is served as the genie signals for the next receiver until a genie with an acceptable number of dimensions is obtained. As a result, it is proved that for any $K \geq 4$, the total DoF is outer bounded by $K\frac{MN}{M+N}$ as long as $R \geq \frac{K-2}{K^2-3K+1}$.

The DoF region of MIMO K-user interference channels has not been obtained in general for arbitrary number of antennas except for the 2-user case [8].

B. *DoF of X network*

There is also increasing interest in characterizing DoF region of MIMO X networks. A $K \times J$ MIMO X network consists of K transmitters and J receivers where each transmitter has an independent message for each receiver. Notably, the X networks include interference channels as a special case.

The best known inner bounds on the total DoF of $K \times J$ MIMO time-varying X networks with N antennas at each node are based on:

1) Antenna splitting with no cooperation [10]: The achievable total DoF is attained by decomposing all transmitter and receiver antennas in which we have an $NK \times NJ$ user single-input single-output X network. Therefore, the best total DoF $N\frac{KJ}{K+J-\frac{1}{N}}$ is achieved. However, there is a gap between the inner bound and the DoF outer bound, $N\frac{KJ}{K+J-1}$, implying that a cooperation structure might be needed here.
2) Joint signal processing [11]: Doing joint processing at either transmitter or receiver side, the desired signals at any receiver can be efficiently resolved from the interference. This new insight closes the mentioned gap and the total DoF value $N\frac{KJ}{K+J-1}$ is achieved.

These results offer an opportunity to revise our understanding of antenna splitting technique. Independent processing at each antenna was initially employed to simplify the achievability scheme of K-user MIMO interference channels, which turned out to be optimal in some cases. However, as observed in [11] allowing cooperation among antennas is essential for establishing the desired DoF.

In the class of real interference alignment, the DoF of time-invariant $K \times J$ MIMO X networks has not been studied to the best of our knowledge. Also, except for the two-user case [12], the DoF region of MIMO X networks when each node has an arbitrary number of antennas has not been considered yet.

C. Summary of Results

In this paper, we employ recent results from the field of simultaneous Diophantine approximation for systems of m linear forms in n variables to analyze the performance of joint receive antenna processing. Based on the analysis, we characterize the DoF region of several classes of time-invariant multiple antenna interference networks.

To introduce the main concepts, we first study a time-invariant K-user MIMO Gaussian interference channel with N antennas at each node. We develop a novel real interference alignment scheme for this channel and establish the total DoF for this channel (Theorem 1).

Next, we focus on K-user MIMO Gaussian interference equipped with M antennas at each transmitter and N antennas at each receiver. For this scenario, an achievable DoF region is established (Theorem 2). It is shown that the achieved DoF region includes the previously known results as special cases. We also establish an achievable DoF region for the K-user MIMO Gaussian interference such that each node has an arbitrary number of antennas (Theorem 3).

We then consider $K \times J$ MIMO interference network with general message demands under assumption that all nodes have the same number of antennas. In this model, each transmitter conveys an independent message and each receiver requests an arbitrary subset of messages. With joint receive antenna processing and real interference alignment, the exact DoF region is established (Theorem 4).

We also apply our new scheme to the $K \times J$ MIMO X network and derive an achievable DoF region (Theorem 5), which is shown to be tight under certain circumstances.

Finally, we discuss the outer bound in Section IX. By suitable transmitter grouping argument, we obtain an outer bound on the DoF region for a K user interference channel with M antennas at every transmitter and N antennas at every receiver (Theorem 6).

Notation: Throughout the paper, K, J, M, N, D, and D' are integers and $\mathcal{K} = \{1, \ldots, K\}$, $\mathcal{J} = \{1, \ldots, J\}$, $\mathcal{M} = \{1, \ldots, M\}$, $\mathcal{N} = \{1, \ldots, N\}$. We use k, $\hat{k}$ as transmitter indices, and j, $\hat{j}$ as receiver indices. Superscripts t and r are used for transmitter and receiver antenna indices. Letters i and l are used as the indices of directions and streams (to be specified later), respectively. The set of integers, positive integers, and real numbers are denoted as $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{R}$, respectively. The set of non-negative real numbers is denoted as $\mathbb{R}_+$. For a positive integer Q, we define $\mathbb{Z}_Q \coloneqq \{z | z \in \mathbb{Z}, -Q \leq z \leq Q\}$. We denote the set of directions, a specific direction, and the vector of directions using $\mathcal{T}$, T, and $\mathbf{T}$ respectively. Vectors and matrices are indicated by bold symbols. We use $[M_k]_{k=1}^K$ to denote vector $(M_1, \ldots, M_K)$, and $[d_{j,k}]_{j=1,k=1}^{J,K}$ the $J \times K$ matrix with element $d_{j,k}$ in the (j,k)th position. When there is no confusion, $[M_k]$ is used as an abbreviation for $[M_k]_{k=1}^K$, and $[M]$ is used to denote a vector where all M_k are equal to M. We use $(\cdot)^*$ to denote matrix transpose, $\otimes$ the Kronecker product of two matrices, $\cup$ union of sets, $\|\mathbf{x}\|_\infty$ the infinity norm of vector $\mathbf{x}$, and $\|\mathbf{x}\|_2$ the 2-norm of vector $\mathbf{x}$.

II. Diophantine approximation and joint receive antenna processing

The problem of Diophantine approximation is to approximate real numbers with rational numbers. Let a/b denote a rational approximation to a real number ω. It is useful to identify upper and lower bounds of $|\omega - a/b|$, as a function of b. In addition to approximating a single real number, simultaneous approximations to several rational numbers can be considered. The problem of simultaneous Diophantine approximation is to identify for a given real $n \times m$ matrix $\mathbf{A}$, how small the distance from $\mathbf{Aq}$ to $\mathbb{Z}^n$, in terms of $\mathbf{q} \in \mathbb{Z}^m$, can be made [13].

To see how simultaneous Diophantine approximation can be useful in communications, consider a communication receiver that receives a vector of signals, $\mathbf{y}$, in the following form:

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \boldsymbol{\nu} \tag{1}$$

where $\mathbf{A}$ is a real $n \times m$ matrix, and $\mathbf{x} \in \mathbb{R}^{m\times 1}$ contains information symbols to detected, and $\boldsymbol{\nu} \in \mathbb{R}^{n\times 1}$ is additive noise, assumed to contain independent and identically distributed zero-mean Gaussian random variables. If we choose $\mathbf{x} \in \{\lambda \mathbb{Z}_Q^m\}$ where $\mathbb{Z}_Q^m = \{(q_1, \ldots, q_m) | q_i \in \mathbb{Z}_Q, 1 \leq i \leq m\}$ and λ is a positive real number that can be used to control the signal power, then the block error probability for detecting $\mathbf{x}$ is determined by the set of distances $\{\|\mathbf{A}(\mathbf{x} - \mathbf{x}')\|_2 \,\big|\, \mathbf{x}, \mathbf{x}' \in \lambda \mathbb{Z}_Q^m\}$. Therefore, an upper bound on this error probability can be obtained by lower bounding $\|\mathbf{Aq}\|_2$, over non-zero $\mathbf{q} \in \mathbb{Z}^m$.

In this paper, the dimensionality n of $\mathbf{A}$ will be the number of receive antennas. However, the other dimension m is in general much larger than the total number of transmit antennas. The signal $\mathbf{x}$ will contain useful information from the intended transmitters, as well as the interference signals from unintended transmitters. Our strategy will be to select suitably scaled integer lattice constellation for $\mathbf{x}$, create the equivalent matrix $\mathbf{A}$ through transmitter designs that align the interferences at the receivers, and perform joint processing of the entries of $\mathbf{y}$ for detecting $\mathbf{x}$. The fact that signals in $\mathbf{y}$ are jointly processed embodies what we term as joint receive antenna processing.

It is known that for almost every $\mathbf{A}$ in the Lebesgue sense, for any $\delta > 0$, there are at most finitely many $\mathbf{q} \in \mathbb{Z}^m$ with (see e.g., [13, Sec. 1])

$$\|\mathbf{A}\mathbf{q} - \mathbf{p}\|_\infty < \|\mathbf{q}\|_\infty^{-m/n-\delta} \text{ for some } \mathbf{p} \in \mathbb{Z}^n. \tag{2}$$

Therefore, for almost every $\mathbf{A}$, there are at most finite $\mathbf{q}$ such that $\|\mathbf{A}\mathbf{q}\|_\infty < \|\mathbf{q}\|_\infty^{-m/n-\delta}$. If we further restrict $\mathbf{A}$ to be such that elements on at least one row are *rationally independent*, meaning no element can be written as a linear combination of the other elements with rational coefficients, then for large enough Q, $\|\mathbf{A}\mathbf{q}\| > Q^{-m/n-\delta}$ for all non-zero $\mathbf{q} \in \mathbb{Z}_Q^m$. Note that imposing the rational independence requirement only removes a set of $\mathbf{A}$ of zero Lebesgue measure.

In our communication system design, the elements of $\mathbf{A}$ are functionally dependent. We will rely on the result of [13, Theorem 1.2], which we state below as a lemma in a slightly different form that is suitable for its application to communication problems. See Appendix A regarding non-degeneracy of manifolds. The proof of the lemma is provided in Appendix B.

Lemma 1: Let $\mathbf{f}_i$, $i = 1, \ldots, n$ be a non-degenerate map from an open set $U_i \subset \mathbb{R}^{d_i}$ to $\mathbb{R}^m$ and

$$\mathbf{A} : U_1 \times \ldots \times U_n \to \mathcal{M}_{n,m}, \quad (\mathbf{h}_1, \ldots, \mathbf{h}_n) \longmapsto \begin{pmatrix} \mathbf{f}_1(\mathbf{h}_1) \\ \vdots \\ \mathbf{f}_n(\mathbf{h}_n) \end{pmatrix}$$

where $\mathcal{M}_{n,m}$ denotes the space of $n \times m$ real matrices. Then, for almost all $(\mathbf{h}_1, \ldots, \mathbf{h}_n) \in U_1 \times \ldots \times U_n$, for any $\delta > 0$, for all Q large enough, and for all non-zero $\mathbf{q} \in \mathbb{Z}_Q^m$, $\|\mathbf{A}(\mathbf{h}_1, \ldots, \mathbf{h}_n)\mathbf{q}\|_2 \geq Q^{-m/n-\delta}$. $\square$

As far as DoF is concerned, the following lemma will be useful in understanding the basis of our derivation. Its proof is provided in Appendix C.

Lemma 2: For a communication link described by (1), where $\mathbf{A}$ is a matrix as defined in Lemma 1, then for almost all $(\mathbf{h}_1, \ldots, \mathbf{h}_n) \in U_1 \times \ldots \times U_n$, the communication link based on the resulting $\mathbf{A}$ can provide a per-symbol DoF of $n/(m+n)$ and a total DoF of $mn/(m+n)$. $\square$

If the matrix $\mathbf{A}$ represents a point to point MIMO system of m transmit antennas and n receive antennas, then the achieved DoF $mn/(m+n)$ is smaller than the maximum possible DoF $\min(m, n)$. However, if n is the number of receive antennas, and m is the number of simultaneously transmitted symbols using integer lattice, the total achieved DoF is n when m goes to infinity. When using Lemma 2, we will let

$m \to \infty$ so that the gap between the achieved DoF $mn/(m+n)$ based on a integer signaling and the maximum DoF possible $\min(m,n)$ disappears.

III. System model

Consider a MIMO real Gaussian interference network with K transmitters and J receivers. Suppose transmitter k has M_k antennas and receiver j has N_j antennas. At each time, each transmitter, say transmitter k, sends a vector signal $\mathbf{x}_k \in \mathbb{R}^{M_k}$. The channel from transmitter k to receiver j is represented as a matrix

$$\mathbf{H}_{j,k} := [h_{j,k,r,t}]_{r=1,t=1}^{N_j,M_k} \tag{3}$$

where $k \in \mathcal{K}$, $j \in \mathcal{J}$, and $\mathbf{H}_{j,k} \in \mathbb{R}^{N_j \times M_k}$. It is assumed that the channel is constant during all transmissions. Each transmit antenna is subjected to an average power constraint P. The received signal at receiver j can be expressed as

$$\mathbf{y}_j = \sum_{k \in \mathcal{K}} \mathbf{H}_{j,k}\mathbf{x}_k + \boldsymbol{\nu}_j, \quad \forall j \in \mathcal{J} \tag{4}$$

where $\{\boldsymbol{\nu}_j | j \in \mathcal{J}\}$ is the set of independent Gaussian additive noises with real, zero mean, independent, and unit variance entries. Let $\mathbf{H}$ denote the $\sum_{j\in\mathcal{J}} N_j \times \sum_{k\in\mathcal{K}} M_k$ block matrix, whose (j,k)th block of size $N_j \times M_k$ is the matrix $\mathbf{H}_{j,k}$. The matrix $\mathbf{H}$ includes all the channel coefficients.

In view of message demands at receivers, the introduced channel can specialize to three known cases:

1) *The* $(K, J, [M_k], [N_j], [\mathcal{W}_j])$ *interference network with general message demands*: where each receiver, for instance receiver j, requests an arbitrary subsets of transmitted signals as $\mathcal{W}_j = \{k \in \mathcal{K} \,\big|\, \text{receiver } j \text{ requests } \mathbf{x}_k\}$.
2) *The single hop* $(K, J, [M_k], [N_j])$ *wireless X network*: where for each pair $(j,k) \in \mathcal{J} \times \mathcal{K}$, transmitter k conveys an independent message to receiver j.
3) *The* K*-user interference channel*: where $J = K$ and signal $\mathbf{x}_k$, $\forall k \in \mathcal{K}$, is just intended for receiver k. For this model, we use the abbreviation $(K, [M_k], [N_j])$.

In the case of K-user interference channel, the *capacity region* $\mathcal{C}_{IC}(P, K, [M_k], [N_j], \mathbf{H})$ is defined in the usual sense: It contains rate tuples $[R_k(P)]_{k=1}^{K}$ such that reliable transmission from transmitter k to receiver k is possible at rate $R_k - \epsilon$, for any $\epsilon > 0$ and for all $k \in \mathcal{K}$ simultaneously, under the given power constraint P. Reliable transmissions mean that the probability of error can be made arbitrarily small by increasing the encoding block length while keeping the rates and power fixed.

A DoF vector $[d_k]_{k=1}^K$ is said to be *achievable* if for any large enough P, the rates $R_i = 0.5\log(P)d_i$, $i = 1, 2, \dots, K$, are simultaneously achievable by all K users, namely $0.5\log(P)\cdot[d_k]_{k=1}^K \in \mathcal{C}_{IC}(P, K, [M_k], [N_j], \mathbf{H})$. The *DoF region* for a given interference channel $\mathbf{H}$, $\mathcal{D}_{IC}(K, [M_k], [N_j], \mathbf{H})$, is the set of all achievable DoF vectors. The DoF region $\mathcal{D}_{IC}(K, [M_k], [N_j])$ is the largest possible region such that $\mathcal{D}_{IC}(K, [M_k], [N_j]) \subset \mathcal{D}_{IC}(K, [M_k], [N_j], \mathbf{H})$ for almost all $\mathbf{H}$ in the Lebesgue sense. The *total DoF of the K-user interference channel* $\mathbf{H}$ is defined as

$$d_{IC}(K, [M_k], [N_j], \mathbf{H}) = \max_{[d_k]_{k=1}^K \in \mathcal{D}_{IC}(K, [M_k], [N_j], \mathbf{H})} \sum_{k=1}^{K} d_k.$$

The *total DoF* $d_{IC}(K, [M_k], [N_j])$ is defined as the largest possible real number μ such that for almost all (in the Lebesgue sense) real channel matrices $\mathbf{H}$ of size $\sum_{j\in\mathcal{K}} N_j \times \sum_{k\in\mathcal{K}} M_k$, $d_{IC}(K, [M_k], [N_j], \mathbf{H}) \geq \mu$.

Remark 1: The DoF region $\mathcal{D}_X(K, J, [M_k], [N_j])$ for the single hop wireless X network can be defined similarly as for the K-user interference channel except in this case, any DoF point in the DoF region is a matrix of the form $[d_{j,k}]_{j=1,k=1}^{J,K}$. Likewise, the DoF region $\mathcal{D}_G(K, J, [M_k], [N_j], [\mathcal{W}_j])$ for interference network with general message demand can be defined.

IV. Main Results

The main results of our paper regarding achievable DoF regions are presented below. The DoF region outer bound result will be presented in Section IX.

Theorem 1: $d_{IC}(K, [N], [N]) = \frac{NK}{2}$. □

This result for constant coefficient channels has been obtained before in [4]. For time-varying channels, the same total DoF was established in [2].

Theorem 2: $d_{IC}(K, [M], [N]) \geq \frac{MN}{M+N}K$. □

This result for constant coefficient channels has been obtained before in [14]. For time-varying channels, the same total DoF was established in [6].

Remark 2: Our proofs for Theorem 1 and Theorem 2 are different from those in [4], [14] because antenna splitting is not employed. Our scheme is more flexible in dealing with cases where the transmit messages do not have the same DoF, in which case antenna splitting is not optimal.

Theorem 3: The DoF region of a $(K, [M_k], [N_j])$ interference channel satisfies $\mathcal{D}_{IC}(K, [M_k], [N_j]) \supset \mathcal{D}_{IC}^{(\text{in})}$ where

$$\mathcal{D}_{IC}^{(\text{in})} := \{[d_k]_{k=1}^K \in \mathbb{R}_+^{K\times 1} \mid \frac{d_k}{N_k} + \max_{\hat{k}\neq k} \frac{d_{\hat{k}}}{M_{\hat{k}}} \leq 1, \forall k \in \mathcal{K}\}. \tag{5}$$

Corollary 1: Setting all $M_K = M$ and $N_j = N$ in Theorem 3, the DoF region of a $(K, [M], [N])$ interference channel satisfies $\mathcal{D}_{IC}(K, [M], [N]) \supset \mathcal{D}_{IC}^{(\text{in})}$ where

$$\mathcal{D}_{IC}^{(\text{in})} := \{[d_k]_{k=1}^{K} \in \mathbb{R}_+^{K\times 1} \,|\, M d_k + N \max_{\hat{k}\neq k} d_{\hat{k}} \leq MN, \forall k \in \mathcal{K}\}. \tag{6}$$

Corollary 2: Let assume $M = N$ in Corollary 1. Employing the outer bound derived in [5], the DoF region of a $(K, [N], [N])$ interference channel is the following

$$\mathcal{D}_{IC}(K, [N], [N]) = \{[d_k]_{k=1}^{K} \in \mathbb{R}_+^{K\times 1} \,|\, d_k + \max_{\hat{k}\neq k} d_{\hat{k}} \leq N, \forall k \in \mathcal{K}\}. \tag{7}$$

Theorem 4: The DoF region of a $(K, J, [N], [N], [\mathcal{W}_j])$ interference network with general message demand is

$$\mathcal{D}_{G}(K, J, [N], [N], [\mathcal{W}_j]) := \{[d_k]_{k=1}^{K} \in \mathbb{R}_+^{K\times 1} \,|\, \sum_{k\in\mathcal{W}_j} d_k + \max_{\hat{k}\in\mathcal{W}_j^c} d_{\hat{k}} \leq N, \forall j \in \mathcal{J}\}. \tag{8}$$

Theorem 5: The DoF region of a $(K, J, [M_k], [N_j])$ X network satisfies $\mathcal{D}_X(K, J, [M_k], [N_j]) \supset \mathcal{D}_X^{(\text{in})}$ where

$$\mathcal{D}_X^{(\text{in})} := \{[d_{j,k}]_{j=1,k=1}^{J,K} \in \mathbb{R}_+^{K\times J} \,|\, \frac{1}{N_j} \sum_{k\in\mathcal{K}} d_{j,k} + \sum_{j\in\mathcal{J},\hat{j}\neq j} \max_{\hat{k}\in\mathcal{K}} \frac{d_{\hat{j},\hat{k}}}{M_{\hat{k}}} \leq 1, \forall j \in \mathcal{J}\}. \tag{9}$$

Corollary 3: As a special case of Theorem 5, the DoF region of a $(K, J, [M], [N])$ X network channel satisfies $\mathcal{D}_X(K, J, [M], [N]) \supset \mathcal{D}_X^{(\text{in})}$ where

$$\mathcal{D}_X^{(\text{in})} := \{[d_{j,k}]_{j=1,k=1}^{J,K} \in \mathbb{R}_+^{K\times J} \,|\, M \sum_{k\in\mathcal{K}} d_{j,k} + N \sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j} \max_{\hat{k}\in\mathcal{K}} d_{\hat{j},\hat{k}} \leq MN, \forall j \in \mathcal{J}\}. \tag{10}$$

Remark 3: The same DoF regions as in Corollary 2 and Theorem 4 for time-varying channel have been obtained before in [5] using vector alignment. It is interesting to note that the DoF region is regardless of whether the channel is time-varying or constant. This indicates that the DoF region for this channel is an inherent spatial property of the channel that is separate from the time or frequency diversity, as has been observed previously [5], [11].

Remark 4: Employing the outer bound derived by [10], the achieved region of Corollary 3 with the condition $M = N$ is tight in the following cases:

1) The total number of receivers is $\mathcal{J} = 2$.
2) $d_{j,k} = d_{j,\hat{k}}$, for all $k, \hat{k} \in \mathcal{K}$ and for all $j \in \mathcal{J}$.

If we set all $d_{j,k} = \frac{N}{K+J-1}$, then we obtain the total DoF $\frac{KJN}{K+J-1}$. The same total DoF has been obtained in [11] for time-varying channel. It is again notable that the total DoF does not depend on the channel variability.

Remark 5: If we set $M = 1$ in Corollary 3, we arrive at the single-input multiple-output X network with N antenna at all receivers. For this model when $K > N$, we establish the total DoF $\frac{NKJ}{K+N(J-1)}$ by fixing all $d_{j,k} = \frac{N}{K+N(J-1)}$ and employing the outer bound of [11]. When $K \leq N$, beamforming and zeroforcing are sufficient to achieve single-user outer bound N.

Remark 6: The achievable DoF regions in Theorems 3–5 are all of the following type: i) there is one inequality for each receiver; ii) the inequality is such that the total DoF of the useful messages, normalized by the number of receive antennas, plus the sum, over the other receivers, of the maximum interference DoF intended for each of these receivers, normalized by the number of transmit antennas, is less than 1.

Remark 7: Theorem 1 follows from Theorem 2 by setting $M = N$ and the outer bound for K-user interference channel that has been obtained before in [2]. Moreover, Theorem 2 follows from Corollary 1 when $d_k = MN/(M + N)$, $\forall k \in \mathcal{K}$.

We conclude from the last remark that the only requirement to establish Theorem 1–2 is proving Theorem 3 (hence Corollary 1). However, we will first prove the achievability of Theorem 1 in Section V, which serves to introduce the real interference alignment scheme, joint antenna processing at the receivers, and the performance analysis based on the results of simultaneous Diophantine approximation on manifolds.

V. TOTAL DOF OF $(K, [N], [N])$ INTERFERENCE CHANNEL

In this section, we examine our new achievability scheme on the $(K, [N], [N])$ interference channel. Theorem 1 is then proved by employing the outer bound in [2]. Our scheme uses real interference alignment such that the dimensions of interferences are aligned as much as possible, leaving more dimensions for useful signals. The dimensions (also named directions) are represented as real numbers that are rationally independent.

ENCODING: Transmitter k sends a vector message $\mathbf{x}_k = (x_k^1, \ldots, x_k^N)^*$ where x_k^t, $\forall t \in \mathcal{N}$ is the signal emitted by antenna t at transmitter k. The signal x_k^t is generated using transmit *directions* in a set $\mathcal{T} = \{T_i \in \mathbb{R} \,|\, 1 \leq i \leq D\}$ as $x_k^t = \mathbf{T}\mathbf{s}_k^t$ where $\mathbf{T} \coloneqq (T_1, \ldots, T_D)$, $\mathbf{s}_k^t \coloneqq (s_{k1}^t, \ldots, s_{kD}^t)^*$, and for all $1 \leq i \leq D$,

$$s_{ki}^t \in \{\lambda q \,|\, q \in \mathbb{Z}, -Q \leq q \leq Q\}. \tag{11}$$

The parameters Q and λ will be designed to satisfy the rate and power constraints.

ALIGNMENT DESIGN: We design transmit directions in such a way that at any receiver antenna, each useful signal occupies a set of directions that are rationally independent of interference directions.

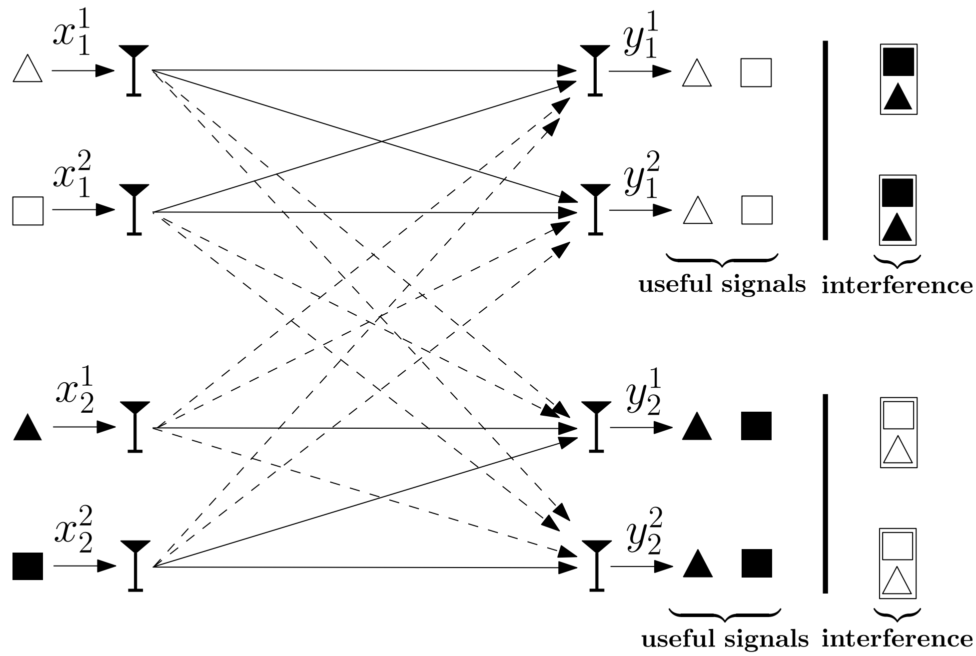

Fig. 1: 2-user Gaussian interference channel with 2 antennas at each transmitter receiver

To illustrate the idea, we use an example as depicted in Figure 1. Messages x_1^1 and x_1^2 are shown by white triangle and square. In a similar fashion, x_2^1 and x_2^2 are indicated with black triangle and square. We are interested in the transmit directions such that at each receiver antenna the interferences, for instance black triangle and square at receiver 1, are aligned while the useful messages, white triangle and square, occupy different set of directions.

TRANSMIT DIRECTIONS: Our scheme requires all directions of set $\mathcal{T}$ to be in the following form

$$T = \prod_{j \in \mathcal{K}} \prod_{k \in \mathcal{K}, k \neq j} \prod_{r \in \mathcal{N}} \prod_{t \in \mathcal{N}} (h_{j,k,r,t})^{\alpha_{j,k,r,t}} \tag{12}$$

where $0 \leq \alpha_{j,k,r,t} \leq n-1$, $\forall j \in \mathcal{K}$, $k \in \mathcal{K}$, $k \neq j$, $r \in \mathcal{N}$, $t \in \mathcal{N}$. It is easy to see that the total number directions is

$$D = n^{K(K-1)N^2}. \tag{13}$$

We assume that directions in $\mathcal{T}$ are indexed from 1 to D. The exact indexing order is not important here. Note that in the single-input single-output (SISO) case, the proposed transmission scheme coincides with the scheme in [4].

ALIGNMENT ANALYSIS: Our design proposes that at each antenna of receiver j, $j \in \mathcal{K}$, the set of messages $\{x_k^t \,\big|\, k \in \mathcal{K}, k \neq j, t \in \mathcal{N}\}$ are aligned. To verify, consider all x_k^t, $k \neq j$ that are generated in directions of set $\mathcal{T}$. These symbols are interpreted as the interferences for receiver j. Let

$$D' = (n+1)^{K(K-1)N^2}. \tag{14}$$

and define a set $\mathcal{T}' = \{T'_i \in \mathbb{R} \,\big|\, 1 \leq i \leq D'\}$ such that all T'_i are in from of T as in (12) but with a small change as follows

$$0 \leq \alpha_{j,k,r,t} \leq n. \tag{15}$$

Clearly, all x_k^t, $k \neq j$ arrive at antenna r of receiver j in the directions of $\{(h_{j,k,r,t})T \,\big|\, k \in \mathcal{K}, k \neq j, t \in \mathcal{N}, T \in \mathcal{T}\}$ which is a subset of $\mathcal{T}'$.

This confirms that at each antenna of any receiver, all the interferences only contain the directions from $\mathcal{T}'$. These interference directions can be described by a vector $\mathbf{T}' \coloneqq (T'_1, \ldots, T'_{D'})$.

DECODING SCHEME: In this part, we first rewrite the received signals. Then, we prove the achievability part of Theorem 1 using Lemma 2 based on joint antenna processing.

The received signal at receiver j is represented by

$$\mathbf{y}_j = \underbrace{\mathbf{H}_{j,j}\mathbf{x}_j}_{\text{the useful signal}} + \underbrace{\sum_{k \in \mathcal{K}, k \neq j} \mathbf{H}_{j,k}\mathbf{x}_k}_{\text{interference}} + \boldsymbol{\nu}_j. \tag{16}$$

Let us define

$$\mathbf{B} \coloneqq \begin{pmatrix} \mathbf{T} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{T} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{T} \end{pmatrix}, \quad \mathbf{s}_k \coloneqq \begin{pmatrix} \mathbf{s}_k^1 \\ \mathbf{s}_k^2 \\ \vdots \\ \mathbf{s}_k^N \end{pmatrix}, \quad \mathbf{u}_k \coloneqq \frac{\mathbf{s}_k}{\lambda}, \tag{17}$$

such that $\mathbf{B}$ is an $N \times ND$ matrix with $(N-1)D$ zeros at each row. Using above definitions, $\mathbf{y}_j$ can be rewritten as

$$\mathbf{y}_j = \lambda \left(\mathbf{H}_{j,j}\mathbf{B}\mathbf{u}_j + \sum_{k \in \mathcal{K}, k \neq j} \mathbf{H}_{j,k}\mathbf{B}\mathbf{u}_k \right) + \boldsymbol{\nu}_j. \tag{18}$$

The elements of $\mathbf{u}_k$ are integers between $-Q$ and Q, cf. (11).

We rewrite

$$\mathbf{H}_{j,j}\mathbf{B}\mathbf{u}_j = (\mathbf{H}_{j,j} \otimes \mathbf{T})\,\mathbf{u}_j = \begin{pmatrix} h_{j,j,1,1}\mathbf{T} & h_{j,j,1,2}\mathbf{T} & \ldots & h_{j,j,1,N}\mathbf{T} \\ h_{j,j,2,1}\mathbf{T} & h_{j,j,2,2}\mathbf{T} & \ldots & h_{j,j,2,N}\mathbf{T} \\ \vdots & \vdots & \ddots & \vdots \\ h_{j,j,N,1}\mathbf{T} & h_{j,j,N,2}\mathbf{T} & \ldots & h_{j,j,N,N}\mathbf{T} \end{pmatrix} \mathbf{u}_j \coloneqq \begin{pmatrix} \mathbf{T}_j^1 \\ \mathbf{T}_j^2 \\ \vdots \\ \mathbf{T}_j^N \end{pmatrix} \mathbf{u}_j \tag{19}$$

where $\forall r \in \mathcal{N}$, $\mathbf{T}_j^r$ is the r^{th} row of $\mathbf{H}_{j,j}\mathbf{B}$. Also,

$$\sum_{k\in\mathcal{K},k\neq j}\mathbf{H}_{j,k}\mathbf{B}\mathbf{u}_k=\sum_{k\in\mathcal{K},k\neq j}\left(\mathbf{H}_{j,k}\otimes\mathbf{T}\right)\mathbf{u}_k=\begin{pmatrix}\sum_{k\in\mathcal{K},k\neq j}\sum_{t\in\mathcal{N}}\left(h_{j,k,1,t}\mathbf{T}\mathbf{u}_k^t\right)\\ \sum_{k\in\mathcal{K},k\neq j}\sum_{t\in\mathcal{N}}\left(h_{j,k,2,t}\mathbf{T}\mathbf{u}_k^t\right)\\ \vdots\\ \sum_{k\in\mathcal{K},k\neq j}\sum_{t\in\mathcal{N}}\left(h_{j,k,N,t}\mathbf{T}\mathbf{u}_k^t\right)\end{pmatrix}\overset{(a)}{=}\begin{pmatrix}\mathbf{T}'\mathbf{u}_j'^1\\ \mathbf{T}'\mathbf{u}_j'^2\\ \vdots\\ \mathbf{T}'\mathbf{u}_j'^N\end{pmatrix}\tag{20}$$

where $\forall r \in \mathcal{N}$, $\mathbf{u}_j'^r$ is a column vector with D' integer elements (some of the entries are zero), and (a) follows since the set $\mathcal{T}'$ contains all directions of the form $(h_{j,k,r,t})\,T$ where $k \neq j$; cf. the definition of $\mathcal{T}'$.

Considering (19) and (20), we are able to equivalently denote $\mathbf{y}_j$ as

$$\mathbf{y}_j=\lambda\begin{pmatrix}\mathbf{T}_j^1 & \mathbf{T}' & \mathbf{0} & \dots & \mathbf{0}\\ \mathbf{T}_j^2 & \mathbf{0} & \mathbf{T}' & \dots & \mathbf{0}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathbf{T}_j^N & \mathbf{0} & \mathbf{0} & \dots & \mathbf{T}'\end{pmatrix}\begin{pmatrix}\mathbf{u}_j\\ \mathbf{u}_j'^1\\ \vdots\\ \mathbf{u}_j'^N\end{pmatrix}+\boldsymbol{\nu}_j.\tag{21}$$

It should be pointed out $\mathbf{T}_j^r$ represents the useful directions at antenna r of receiver j. The elements in $\mathbf{T}'$ represent the interference directions, which is common to all antennas at all receivers.

We finally left multiply $\mathbf{y}_j$ by an $N \times N$ weighting matrix

$$\mathbf{W}=\begin{pmatrix}1 & \gamma_{12} & \dots & \gamma_{1N}\\ \gamma_{21} & 1 & \dots & \gamma_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ \gamma_{N1} & \gamma_{N2} & \dots & 1\end{pmatrix}\tag{22}$$

such that all indexed γ can be chosen randomly, and independently from any continuous distribution, say, uniformly from the interval $[\frac{1}{2}, 1]$. This process causes the zeros in (21) to be filled by non-zero directions.

After multiplying $\mathbf{W}$, the noiseless received constellation belongs to a lattice generated by the $N \times N(D + D')$ matrix

$$\mathbf{A}=\mathbf{W}\begin{pmatrix}\mathbf{T}_j^1 & \mathbf{T}' & \mathbf{0} & \dots & \mathbf{0}\\ \mathbf{T}_j^2 & \mathbf{0} & \mathbf{T}' & \dots & \mathbf{0}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathbf{T}_j^N & \mathbf{0} & \mathbf{0} & \dots & \mathbf{T}'\end{pmatrix}.\tag{23}$$

The above matrix has a significant property that allows us to use Lemma 1. More precisely, Lemma 1 requires each row of $\mathbf{A}$ to be a non-degenerate map from a subset of channel coefficients to $\mathbb{R}^{N(D+D')}$. The non-degeneracy is established because (cf. Appendix A):

1) all elements of $\mathbf{T}'$ and $\mathbf{T}_j^t$, $\forall t \in \mathcal{N}$ are analytic functions of the channel coefficients;
2) all the directions in $\mathbf{T}'$ and $\mathbf{T}_j^t$, $\forall t \in \mathcal{N}$ together with 1 are linearly independent over $\mathbb{R}$;
3) all indexed γ in $\mathbf{W}$ have been chosen randomly and independently.

Since $\|\mathbf{q}\|_\infty \leq (K-1)NQ$, for any $\delta > 0$ and large enough Q, the distance between any two points of the received constellation (without considering noise) is lower bounded via Lemma 1 by

$$\lambda\big(\, 2(K-1)NQ\,\big)^{-(D+D')-\delta}. \tag{24}$$

We now focus our attention on the design of λ and Q to complete the coding scheme. The parameter λ controls the input power of transmitter antennas. The average power of antenna t at transmitter k is computed as

$$P = E[(x_k^t)^2] = E[(\mathbf{T}\mathbf{s}_k^t)^2] = \sum_{i=1}^{D} {T_i}^2 E[(s_{ki}^t)^2] \leq \lambda^2 Q^2 \sum_{i=1}^{D} {T_i}^2 := \lambda^2 Q^2 \nu^2 \tag{25}$$

where the inequality follows from equation (11) and $\nu^2 := \sum_{i=1}^{D} {T_i}^2$. Thus, the only requirement to satisfy the power constraint is $\lambda \leq \frac{P^{\frac{1}{2}}}{Q\nu}$. It is sufficient to choose

$$\lambda = \frac{\zeta P^{\frac{1}{2}}}{Q}, \tag{26}$$

where $\zeta = \frac{1}{\nu}$.

Let $P_0 = \lambda Q = P/\nu^2$. By Lemma 2, each symbol s_{ki}^t can achieve a rate of $d_0 \log(P_0)$ for large P_0, where $d_0 = N/[N + N(D + D'] = 1/(1 + D + D')$. Since there are totally ND useful symbols from each transmitter, the total achievable rate, as normalized by $\log(P_0)$ for each transmitter is

$$\frac{ND}{D + D' + 1} = \frac{Nn^{K(K-1)N^2}}{n^{K(K-1)N^2} + (n+1)^{K(K-1)N^2} + 1} \tag{27}$$

and as n increases, it converges to $\frac{N}{2}$. Since P and P_0 are different by a multiplication factor ν^2, when the rate is normalized by $\log(P)$ instead, as required in the definition of DoF, the same limit of $N/2$ will result as the per user DoF, as $P \to \infty$. The total DoF of the K users is therefore $NK/2$, which meets the outer bound [2]. This finishes the proof of the achievability of the total DoF. When combined with the corresponding outer bound, the theorem is proved.

VI. K-USER INTERFERENCE CHANNEL AND INNER BOUND ON DOF REGION

For simplicity, we will first prove Corollary 1 in this section. Then utilizing the presented proof, Theorem 3 will be established.

Consider a $(K, [M], [N])$ MIMO interference channel. We prove that for any $[d_k]_{k=1}^K \in \mathcal{D}_{IC_1}^{(\text{in})}$, $[d_k]_{k=1}^K$ is achievable.

Assume that it is possible to find an integer ρ such that $\forall k \in \mathcal{K}$, $\bar{d}_k = \rho \frac{d_k}{M}$ is a non-negative integer. The signal x_k^t is divided into $\bar{d}_k$ streams. For stream l, $l \in \{1, \ldots, \max_{k \in \mathcal{K}} \bar{d}_k\}$, we use directions $\{T_{l1}, \ldots, T_{lD}\}$ of the following form

$$T_l = \prod_{j \in \mathcal{K}} \prod_{k \in \mathcal{K}, k \neq j} \prod_{r \in \mathcal{M}} \prod_{t \in \mathcal{N}} (h_{j,k,r,t} \delta_l)^{\alpha_{j,k,r,t}} \tag{28}$$

where $0 \leq \alpha_{j,k,r,t} \leq n-1$ and δ_l is a design parameter that is chosen randomly, independently, and uniformly from the interval $[\frac{1}{2}, 1]$. Let $\mathbf{T}_l \coloneqq (T_{l1}, \ldots, T_{lD})$. Note that, at any antenna of transmitter k, the constants $\{\delta_l\}$ cause the streams to be placed in $\bar{d}_k$ different sets of directions. Indeed the constants $\{\delta_l\}$ play the role analogous to the base vectors $\mathbf{w}_i$ in [5]. The alignment scheme is the same as before, considering the fact that at each antenna of receiver j, the useful streams occupy $M\bar{d}_j$ separate sets of directions. The interferences are also aligned at most in $\max_{k \in \mathcal{K}, k \neq j} \bar{d}_k$ sets of directions independent from useful directions.

By design, x_k^t is emitted in the following form

$$x_k^t = \sum_{l=1}^{\bar{d}_k} \delta_l \sum_{i=1}^{D} T_{li} s_{kli}^t = \mathbf{T}_k \mathbf{s}_k^t \tag{29}$$

where

$$\mathbf{T}_k \coloneqq (\delta_1 \mathbf{T}_1, \ldots, \delta_{\bar{d}_k} \mathbf{T}_{\delta_{\bar{d}_k}}), \quad \mathbf{s}_k^t \coloneqq (s_{k11}^t, \ldots, s_{k\bar{d}_k D}^t)^*, \tag{30}$$

and all s_{kli}^t belong to the set defined in (11).

Pursuing the same steps of the previous section for receiver j, $\mathbf{B}$ becomes an $M \times MD\bar{d}_j$ matrix as

$$\begin{pmatrix} \mathbf{T}_j & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_j & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{T}_j \end{pmatrix} \tag{31}$$

and $\mathbf{A}$ will have N rows and $MD\bar{d}_j + ND' \max_{k \in \mathcal{K}, k \neq j} \bar{d}_k$ columns. To be more precise, matrix $\mathbf{A}$ has the same form as (23) noting that $\mathbf{T}_j^r$ and $\mathbf{T}'$ are now vectors with $MD\bar{d}_j$ and $D' \max_{k \in \mathcal{K}, k \neq j} \bar{d}_k$ elements, respectively.

Remark 8: As it has been proved in the previous section, the dimensions of matrix $\mathbf{A}$ inherits two characteristics as follows:

1) The number of columns is the number of all available directions at the receiver.
2) For large n, the number of rows over the number of columns specifies the achievable DoF per direction.

Let G_j denote the number of columns of $\mathbf{A}$. For any DoF points in $\mathcal{D}_{IC}^{(\text{in})}$ satisfying Corollary 1, we have

$$G_j = MD\bar{d}_j + ND' \max_{k\in\mathcal{K}, k\neq j} \bar{d}_k \leq \frac{\rho}{M} NMD' = \rho ND' \tag{32}$$

and as n increases, the DoF of the signal $\mathbf{x}_j$ intended for receiver j, $\forall j \in \mathcal{K}$ is at least

$$\lim_{n\to\infty} MD\bar{d}_j \frac{N}{G_j} \geq \lim_{n\to\infty} MD\bar{d}_j \frac{N}{\rho ND'} = \lim_{n\to\infty} \frac{M}{\rho} \frac{\bar{d}_j n^{K(K-1)N^2}}{(n+1)^{K(K-1)N^2}} = \frac{M}{\rho}\bar{d}_j = d_j \tag{33}$$

where $\frac{N}{\rho ND'}$ is the DoF per direction for large D'. This proves Corollary 1.

As a special case, it is easy to see when all d_k are equal, the total achievable DoF is $\frac{MN}{M+N}K$. Moreover, when $M = N$, the achievable DoF region is tight, cf. Remark 11.

To establish Theorem 3, we follow the proof of Corollary 1 with a small change in assumption, which is $\bar{d}_k = \rho \frac{d_k}{M_k}$. As a result, $\mathbf{A}$ becomes N_j by $M_k D\bar{d}_j + N_j D' \max_{k\in\mathcal{K}, k\neq j} \bar{d}_k$ matrix. Therefore, for any DoF points in $\mathcal{D}_{IC}^{(\text{in})}$ satisfying Theorem 3, we have

$$G_j = M_K D\bar{d}_j + N_j D' \max_{k\in\mathcal{K}, k\neq j} \bar{d}_k \leq \rho N_j D' \tag{34}$$

and the DoF of signal x_j is finally obtained as

$$\lim_{n\to\infty} M_k D\bar{d}_j \frac{N_j}{\rho N_j D'} = \lim_{n\to\infty} d_j \frac{n^{K(K-1)N^2}}{(n+1)^{K(K-1)N^2}} = d_j. \tag{35}$$

VII. Interference Network with General Message Demands

Consider a $(K, J, [N], [N], [\mathcal{W}_j])$ single hop interference network with general message demand. Transmitter k emits independent message $\mathbf{x}_k$, and receiver j requests an arbitrary subset of messages denoted by $\mathcal{W}_j$. We follow the same definitions and steps of Section VI considering stream l, uses directions of the following form

$$T_l = \prod_{j\in\mathcal{J}} \prod_{k\in\mathcal{W}_j^c} \prod_{r\in\mathcal{N}} \prod_{t\in\mathcal{N}} (h_{j,k,r,t}\delta_l)^{\alpha_{j,k,r,t}} \tag{36}$$

where $0 \leq \alpha_{j,k,r,t} \leq n-1$, $\mathcal{W}_j^c := \{k \in \mathcal{K} \,\big|\, k \notin \mathcal{W}_j\}$, and δ_l is a design parameter chosen as before. Notice that the directions has been designed in such a manner that at any receiver, for example receiver j, while the useful signal subspace is separated from the interference subspace, all interferences caused by $\mathbf{x}_k$, $k \in \mathcal{W}_j$ are aligned. As a result, matrix $\mathbf{A}$ at receiver j will have N rows and $ND \sum_{k \in \mathcal{W}_j} \bar{d}_k + ND' \max_{\hat{k} \in \mathcal{W}_j^c} \bar{d}_{\hat{k}}$ columns. Thus, for any DoF point in $\mathcal{D}_G^{(\mathrm{in})}$ satisfying Theorem 4, G_j is upper bounded by $\rho N D'$ and d_k, $k \in \mathcal{W}_j$, is achieved similar to (33). The proof of the converse is the same as in [5].

VIII. WIRELESS X NETWORKS

Consider a $(K, J, [M], [N])$ Gaussian X network. For each pair $(j, k) \in \mathcal{J} \times \mathcal{K}$, transmitter k sends an $M \times 1$ vector message $\mathbf{x}_{j,k} = (x_{j,k}^1, \ldots, x_{j,k}^M)^*$ to receiver j. Consequently, the signal emitted by transmitter k is in the following form

$$\mathbf{x}_k = \sum_{j \in \mathcal{J}} \mathbf{x}_{j,k}. \tag{37}$$

We assume that it is possible to find an integer ρ such that for all $j \in \mathcal{J}$ and all $k \in \mathcal{K}$, $\bar{d}_{j,k} = \rho \frac{d_{j,k}}{M}$ is a non-negative integer. Message $x_{j,k}^t$ is divided into $\bar{d}_{j,k}$ streams such that each stream, say stream $l \in \{1, \ldots, \max_{k \in \mathcal{K}} \bar{d}_{j,k}\}$, uses directions in set $\mathcal{T}_{j,l} = \{T_{j,l,i} \in \mathbb{R} \,\big|\, 1 \le i \le D\}$. All $T_{j,l,i}$ are generated in the following form

$$T_{j,l} = \prod_{\hat{j} \in \mathcal{J}, \hat{j} \neq j} \prod_{\hat{k} \in \mathcal{K}} \prod_{\hat{r} \in \mathcal{N}} \prod_{\hat{t} \in \mathcal{M}} \left(h_{\hat{j},\hat{k},\hat{r},\hat{t}} \delta_{j,l} \right)^{\alpha_{\hat{j},\hat{k},j,\hat{r},\hat{t},l}} \tag{38}$$

where $0 \le \alpha_{\hat{j},\hat{k},j,\hat{r},\hat{t},l} \le n-1$ and $\delta_{j,l}$ is a design parameter that is chosen randomly, independently, and uniformly from the interval $[\frac{1}{2}, 1]$. Define $\mathbf{T}_{j,l} \coloneqq (T_{j,l,1}, \ldots, T_{j,l,D})$. The signal $x_{j,k}^t$ is generated as

$$x_{j,k}^t = \sum_{l=1}^{\bar{d}_{j,k}} \delta_{j,l} \sum_{i=1}^{D} T_{j,l,i} s_{j,k,l,i}^t = \mathbf{U}_{j,k} \mathbf{s}_{j,k}^t \tag{39}$$

where

$$\mathbf{U}_{j,k} = (\delta_{j,1} \mathbf{T}_{j,1}, \ldots, \delta_{j,\bar{d}_{j,k}} \mathbf{T}_{j,\bar{d}_{j,k}}), \tag{40}$$

$$\mathbf{s}_{j,k}^t = (s_{j,k,1,1}^t, \ldots, s_{j,k,\bar{d}_{j,k},D}^t)^*, \tag{41}$$

and all $s_{j,k,l,i}^t$ are members of the set in (11).

ALIGNMENT DESIGN: Suppose we are at receiver j. The design of transmit directions guarantees that at any antenna of receiver j, the useful signals are placed in K separate sets of directions. Each set has $D\bar{d}_{j,k}$, $k \in \mathcal{K}$ directions. The interferences are also put in $J-1$ different sets of directions, each containing all signals intended for receiver $\hat{j}$, $\hat{j} \in \mathcal{J}$, $\hat{j} \neq j$ with at most $D' \max_{k \in \mathcal{K}} \bar{d}_{\hat{j},k}$ directions.

Let us explain the above mentioned argument for a $(3, 3, [1], [2])$ Gaussian X network. This system is depicted in Figure 2. Each transmitter conveys an independent message to each receiver. We have assumed that white square, triangle, and circle are the useful signals for the first receiver. Similarly, black and gray nodes show the signals intended for receiver 2 and 3, respectively. The transmission scheme is such that at any antenna of receiver 1:

- The interferences, black square triangle and circle, are aligned. The gray signals are also aligned.
- The useful signals, white square triangle and circle, are not aligned.

Hence, at each receive antenna of first user, we have the sum of five terms made by three useful signals and two sets of aligned signals. The set of directions used for each term is separate from others in sense of rational independence. A similar statement is also valid for other receivers. We prove Theorem 3 provided that the described alignment scheme is successful.

ALIGNMENT VERIFICATION: The proposed transmit directions guarantee that the interferences created by messages intended for the same receiver are aligned at all other receivers. To see this, let us define $\mathcal{T}'_{j,l} = \{T'_{j,l,i} \in \mathbb{R} \,\big|\, 1 \leq i \leq D'\}$ such that all $T'_{j,l,i}$ are in the form of (38) but with $0 \leq \alpha_{\hat{j},\hat{k},j,\hat{r},\hat{t},l} \leq n$. We use $\mathbf{T}'_{j,l}$ to denote vector $(T'_{j,l,1}, \ldots, T'_{j,l,D'})$. According to (39), the l^{th} stream of message $x^t_{j,k}$ is transmitted in directions of the form $\delta_{j,l} T_{j,l}$. This stream arrives at antenna r of receiver $\hat{j}$, $\hat{j} \neq j$, in directions of the form $\left(h_{\hat{j},k,r,t}\delta_{j,l}\right) T_{j,l}$, which are obviously in set $\mathcal{T}'_{j,l}$. Since $\mathcal{T}'_{j,l}$ does not depend on indices k and r, cf. (38), at any antenna of receiver $\hat{j}$, $\hat{j} \neq j$, all directions created by the streams intended for receiver j are subset of $\mathcal{T}'_{j,l}$, $\forall l \in \{1, \ldots, \max_{k \in \mathcal{K}} \bar{d}_{j,k}\}$ and occupy at most $D' \max_{k \in \mathcal{K}} \bar{d}_{j,k}$ dimensions. We denote these directions as a vector $\mathbf{T}'_j := (\mathbf{T}'_{j,1}, \ldots, \mathbf{T}'_{j,\max_{k\in\mathcal{K}} \bar{d}_{j,k}})$.

DECODING SCHEME: The received signal at receiver j can be divided into two parts, the useful signals and interference, of the following form

$$\mathbf{y}_j = \sum_{k \in \mathcal{K}} \mathbf{H}_{j,k} \mathbf{x}_{j,k} + \sum_{k \in \mathcal{K}} \sum_{\hat{j} \in \mathcal{J}, \hat{j} \neq j} \mathbf{H}_{j,k} \mathbf{x}_{\hat{j},k} + \boldsymbol{\nu}. \tag{42}$$

For notational convenience, let $\mathbf{s}_{j,k} := (\mathbf{s}^1_{j,k}, \ldots, \mathbf{s}^M_{j,k})^*$ and $\mathbf{u}_j := \frac{1}{\lambda}(\mathbf{s}_{j,1}, \ldots, \mathbf{s}_{j,K})^*$ with integer elements

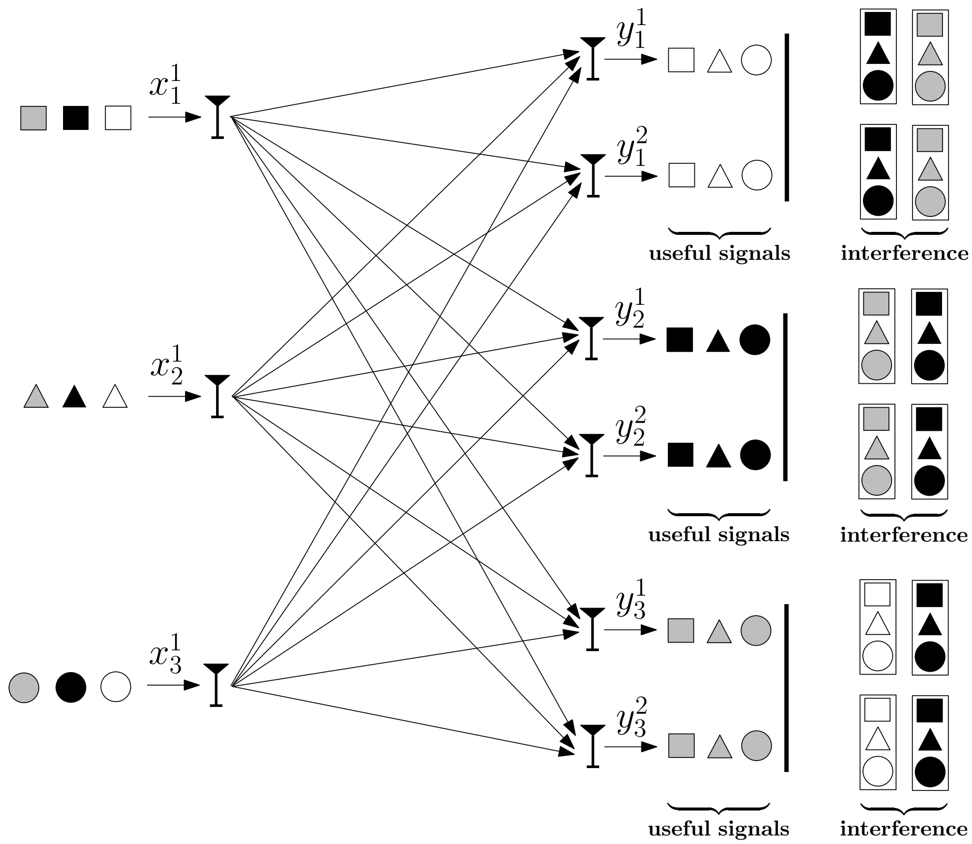

Fig. 2: $(3 \times 3, 1, 2)$ Gaussian X network channel

between $-Q$ and Q. Then, we can rewrite the useful signals as follows

$$\sum_{k\in\mathcal{K}} \mathbf{H}_{j,k}\mathbf{x}_{j,k} = \sum_{k\in\mathcal{K}} \mathbf{H}_{j,k} \begin{pmatrix} x_{j,k}^1 \\ x_{j,k}^2 \\ \vdots \\ x_{j,k}^M \end{pmatrix} \overset{(b)}{=} \sum_{k\in\mathcal{K}} \begin{pmatrix} h_{j,k,1,1}\mathbf{U}_{j,k} & h_{j,k,1,2}\mathbf{U}_{j,k} & \dots & h_{j,k,1,N}\mathbf{U}_{j,k} \\ h_{j,k,2,1}\mathbf{U}_{j,k} & h_{j,k,2,2}\mathbf{U}_{j,k} & \dots & h_{j,k,2,N}\mathbf{U}_{j,k} \\ \vdots & \vdots & \ddots & \vdots \\ h_{j,k,N,1}\mathbf{U}_{j,k} & h_{j,k,N,2}\mathbf{U}_{j,k} & \dots & h_{j,k,N,N}\mathbf{U}_{j,k} \end{pmatrix} \mathbf{s}_{j,k} \tag{43}$$

$$:= \sum_{k\in\mathcal{K}} \begin{pmatrix} \mathbf{U}_{j,k}^1 \\ \mathbf{U}_{j,k}^2 \\ \vdots \\ \mathbf{U}_{j,k}^N \end{pmatrix} \mathbf{s}_{j,k} = \lambda \begin{pmatrix} \mathbf{U}_{j,1}^1 & \mathbf{U}_{j,2}^1 & \dots & \mathbf{U}_{j,K}^1 \\ \mathbf{U}_{j,1}^2 & \mathbf{U}_{j,2}^2 & \dots & \mathbf{U}_{j,K}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{U}_{j,1}^N & \mathbf{U}_{j,2}^N & \dots & \mathbf{U}_{j,K}^N \end{pmatrix} \mathbf{u}_j \tag{44}$$

where $\mathbf{U}_{j,k}^r := (h_{j,k,r,1}\mathbf{U}_{j,k}, h_{j,j,r,2}\mathbf{U}_{j,k}, \dots, h_{j,j,r,N}\mathbf{U}_{j,k})$, $\forall j \in \mathcal{J}$, $k \in \mathcal{K}$, $r \in \mathcal{N}$. Using the definition in (39), (b) follows. We take into account that none of $\mathcal{T}'_{\hat{j},l}$, $\hat{j} \neq j$, contains generators $\{(h_{j,k,r,t}\delta_{j,l}) \,\big|\, k \in \mathcal{K}, r \in \mathcal{N}, t \in \mathcal{M}\}$. Hence, the directions in all $\mathbf{U}_{j,k}^r$ and $\mathbf{T}'_{\hat{j}}$, $\hat{j} \neq j$ are rationally independent.

The interference part can be written as

$$\sum_{k\in\mathcal{K}}\sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j}\mathbf{H}_{j,k}\mathbf{x}_{\hat{j},k} = \sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j}\sum_{k\in\mathcal{K}}\mathbf{H}_{j,k}\begin{pmatrix} x^1_{\hat{j},k} \\ x^2_{\hat{j},k} \\ \vdots \\ x^M_{\hat{j},k}\end{pmatrix} \overset{(c)}{=} \sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j}\sum_{k\in\mathcal{K}}\mathbf{H}_{j,k}\begin{pmatrix} \mathbf{U}_{\hat{j},k}\mathbf{s}^1_{\hat{j},k} \\ \mathbf{U}_{\hat{j},k}\mathbf{s}^2_{\hat{j},k} \\ \vdots \\ \mathbf{U}_{\hat{j},k}\mathbf{s}^M_{\hat{j},k}\end{pmatrix}$$

$$= \sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j}\begin{pmatrix} \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{M}}\left(h_{j,k,1,t}\mathbf{U}_{\hat{j},k}\mathbf{s}^t_{\hat{j},k}\right) \\ \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{M}}\left(h_{j,k,2,t}\mathbf{U}_{\hat{j},k}\mathbf{s}^t_{\hat{j},k}\right) \\ \vdots \\ \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{M}}\left(h_{j,k,N,t}\mathbf{U}_{\hat{j},k}\mathbf{s}^t_{\hat{j},k}\right)\end{pmatrix} \overset{(d)}{=} \sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j}\lambda\begin{pmatrix} \mathbf{T}'_{\hat{j}}\mathbf{u}'^1_{\hat{j}} \\ \mathbf{T}'_{\hat{j}}\mathbf{u}'^2_{\hat{j}} \\ \vdots \\ \mathbf{T}'_{\hat{j}}\mathbf{u}'^N_{\hat{j}}\end{pmatrix} \tag{45}$$

where for all $r\in\mathcal{N}$, $\mathbf{u}'^r_{\hat{j}}$ is a column vector with integer elements. Equivalence relation (c) follows from (39). The equality (d) is due to alignment by our design. It is convenient to represent equation (45) as

$$\lambda\begin{pmatrix} \mathbf{I}_j\mathbf{z}^1_j \\ \mathbf{I}_j\mathbf{z}^2_j \\ \vdots \\ \mathbf{I}_j\mathbf{z}^N_j\end{pmatrix} \tag{46}$$

where $\mathbf{I}_j \coloneqq (\mathbf{T}'_1,\dots,\mathbf{T}'_{j-1},\mathbf{T}'_{j+1},\dots,\mathbf{T}'_J)$ and $\mathbf{z}^r_j \coloneqq (\mathbf{u}'^r_1,\dots,\mathbf{u}'^r_{j-1},\mathbf{u}'^r_{j+1},\dots,\mathbf{u}'^r_J)$ for all $t\in\mathcal{N}$.

Using (44) and (46), received signal $\mathbf{y}_j$ is represented by

$$\lambda\begin{pmatrix} \mathbf{U}^1_{j,1} & \mathbf{U}^1_{j,2} & \dots & \mathbf{U}^1_{j,K} & \mathbf{I}_j & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{U}^2_{j,1} & \mathbf{U}^2_{j,2} & \dots & \mathbf{U}^2_{j,K} & \mathbf{0} & \mathbf{I}_j & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{U}^N_{j,1} & \mathbf{U}^N_{j,2} & \dots & \mathbf{U}^N_{j,K} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{I}_j\end{pmatrix}\begin{pmatrix} \mathbf{u}_j \\ \mathbf{z}^1_j \\ \vdots \\ \mathbf{z}^N_j\end{pmatrix} + \boldsymbol{\nu}_j. \tag{47}$$

Analogous to achievability proof of Theorem 1, we left multiply $\mathbf{y}_j$ by an $N\times N$ weighting matrix. Then, $\mathbf{A}$ in (23) becomes an $N\times(MD\sum_{k\in\mathcal{K}}\bar{d}_{j,k} + ND'\sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j}\max_{k\in\mathcal{K}}\bar{d}_{\hat{j},k})$ matrix such that the non-degeneracy conditions is satisfied.

For any DoF point in $\mathcal{D}^{(\text{in})}_{XC_1}$ that satisfies Theorem 3, the total directions G_j of the useful signals and the interferences at receiver j is

$$G_j = MD\sum_{k\in\mathcal{K}}\bar{d}_{j,k} + ND'\sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j}\max_{k\in\mathcal{K}}\bar{d}_{\hat{j},k} \le \rho ND'. \tag{48}$$

Thus, as n increases, the DoF of $\mathbf{x}_{j,k}$, $\forall j\in\mathcal{J}$, $k\in\mathcal{K}$, is at least

$$\lim_{n\to\infty} MD\bar{d}_{j,k}\frac{N}{\rho ND'} = \lim_{n\to\infty}\frac{M}{\rho}\frac{\bar{d}_{j,k}n^{K(K-1)N^2}}{(n+1)^{K(K-1)N^2}} = \frac{M}{\rho}\bar{d}_{j,k} = d_{j,k}, \tag{49}$$

which establishes Theorem 3.

The provided scheme for the $(K, J, [M], [N])$ Gaussian X network channel can be applied to a more general case where each transmitter/receiver has an arbitrary number of antennas. Let us assume that transmitter k has M_k antennas and receiver j has N_j antennas. To prove Theorem 5, we follow the same procedure of this section for receiver j considering the integer ρ is changed such that $\bar{d}_{j,k} = \rho\frac{d_{j,k}}{M_k}$, $\forall k \in \mathcal{K}$, $j \in \mathcal{J}$. Accordingly, $\mathbf{A}$ becomes an $N \times (D\sum_{k\in\mathcal{K}} M_k\bar{d}_{j,k} + N_jD'\sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j}\max_{k\in\mathcal{K}}\bar{d}_{\hat{j},k})$ matrix. Hence, the total number of useful and interference directions at receiver j is

$$G_j = D\sum_{k\in\mathcal{K}} M_k\bar{d}_{j,k} + N_jD'\sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j}\max_{k\in\mathcal{K}}\bar{d}_{\hat{j},k} \tag{50}$$

and $G_j \leq \rho N_j D'$ for any DoF point in $\mathcal{D}^{(\text{in})}_{XC_2}$ satisfying Theorem 5. As a result, for large enough n, the DoF of signal $\mathbf{x}_{j,k}$ is attained as

$$\lim_{n\to\infty} M_k D\bar{d}_{j,k}\frac{N_j}{\rho N_j D'} = \lim_{n\to\infty}\frac{M_k}{\rho}\frac{\bar{d}_{j,k}n^{K(K-1)N^2}}{(n+1)^{K(K-1)N^2}} = \frac{M_k}{\rho}\bar{d}_{j,k} = d_{j,k} \tag{51}$$

for all $j \in \mathcal{J}$ and $k \in \mathcal{K}$. This completes the proof.

IX. OUTER BOUND DISCUSSION

Although our focus in this paper is on the new receive antenna joint processing, we present a brief discussion on existing outer bounds of interference networks. Note that all outer bounds are general as it applies to interference networks regardless of whether the channel coefficients are time varying or constant. We also present a new outer bound on the DoF region based on a known technique of transmitter grouping.

Ghasemi et al. in [7] show that the total DoF of $(K, [M], [N])$ MIMO Gaussian interference channel is outer bounded by $K\frac{MN}{M+N}$ when $K \geq \frac{M+N}{\text{gcd}(M,N)}$. To establish this result, first consider an $(L, [M], [N])$ MIMO interference channel where $L \leq K$. For this scenario, the L users are divided into two arbitrary disjoint sets of size L_1 and L_2 such that $L = L_1 + L_2$. The full cooperation among transmitters in each set is assumed and similarly for each set of receivers. Accordingly, the 2-user MIMO interference channel with L_1M, L_2M antenna at transmitters and L_1N, L_2N antennas at receivers is obtained. Using the DoF region of 2-user MIMO interference channel [8], the DoF is finally outer bounded.

It is also shown that for $K \leq \frac{\max(M,N)}{\min(M,N)}+1$, the total DoF outer bound is $\min(M,N)\min(K, \frac{\max(M,N)}{\min(M,N)})$. However, the DoF characterization for the remaining region $\lfloor\frac{\max(M,N)}{\min(M,N)}\rfloor + 1 < K < \frac{M+N}{\text{gcd}(M,N)}$ has not been established due to the complexity of convex optimizations over integers. To understand the origin of this problem, we next examine the mentioned scheme when L_2M has the minimum difference from L_1N and we extend the result to obtain an outer bound on the DoF region.

The key to establishing the outer bound on $(K, [M], [N])$ interference channel is to consider a set of g receivers as a group. For this receiver set, the corresponding transmitters emitting useful signals are assumed to be cooperative as one set. Hence, the rest of transmitters only create interference. We then pick a subset of the remaining transmitters such that their total number of antennas is the closest to the number of antennas of the receiver set, namely gN. Such grouping creates a two users MIMO interference channel to which the known DoF region will be applied.

Consider an arbitrary subset of receivers $G_{R_1} \subseteq \mathcal{K}$ with cardinality g. Let $G_{T_1} = G_{R_1}$. The set G_{T_1} contains indices of transmitters whose signals are useful for the receivers in G_{R_1}. We define another subset of transmitters, $G_{T_2} \subseteq \mathcal{K} \setminus G_{T_1}$, such that

1) The cardinality of G_{T_2} is $\min\{K-g, \lfloor \frac{gN}{M} \rfloor\}$.
2) Set G_{T_2} maximizes $\sum_{k \in G_{T_2}} d_k$.

The corresponding receivers of G_{T_2} are shown by set G_{R_2}. We then remove all the remaining users with indices in $\mathcal{K} \setminus \{G_{T_1} \cup G_{T_2}\}$.

Theorem 6: For the aforementioned G_{T_1}, G_{T_2}, and g, the following equations define a DoF region outer bound for the $(K, [M], [N])$ interference channel:

$$\sum_{k \in G_{T_1}} d_k \leq g \min(M, N) \tag{52}$$

$$\sum_{\hat{k} \in G_{T_2}} d_{\hat{k}} \leq \min\{K-g, \lfloor \frac{gN}{M} \rfloor\} \min(M, N) \tag{53}$$

$$\sum_{k \in G_{T_1}} d_k + \sum_{\hat{k} \in G_{T_2}} d_{\hat{k}} \leq gN. \tag{54}$$

Proof: In [8], it is proved that the DoF region for a 2-user MIMO Gaussian interference channel with M_1, M_2 antennas at transmitters and N_1, N_2 antennas at the corresponding receivers is

$$d_1 \leq \min(M_1, N_1), \quad d_2 \leq \min(M_2, N_2)$$

$$d_1 + d_2 \leq \min\{M_1 + M_2, N_1 + N_2, \max(M_1, N_2), \max(M_2, N_1)\} \tag{55}$$

Using this result when G_{T_1}, G_{R_1} are viewed as the first user and G_{T_2}, G_{R_2} as the second user, we arrive at (52)–(54). □

Remark 9: By considering all $1 \leq g \leq K$ and for each g all possible $G_{T_1} \subseteq \mathcal{K}$ with cardinality g, the outer bound can be optimized.

As a special case, if we set all d_k equal to d, we have

$$gd + \min\{K - g, \lfloor \frac{gN}{M} \rfloor\} d \leq gN \tag{56}$$

for all $g \in \mathcal{K}$. The above inequality can be represented as

$$d \leq \frac{gN}{\min\{K, \lfloor \frac{g(N+M)}{M} \rfloor\}}. \tag{57}$$

Therefore, the outer bound for the total DoF is obtained as

$$\min_{g \in \mathcal{K}} \frac{gNK}{\min\{K, \lfloor \frac{g(N+M)}{M} \rfloor\}}. \tag{58}$$

For $K \geq \frac{M+N}{\text{gcd}(M,N)}$, we are able to choose $g = \frac{M}{\text{gcd}(M,N)}$ resulting in the same number of antennas at transmitters in G_{T_2}, and at receivers in G_{R_1}. Subsequently, the total DoF is upper bounded by $\frac{MN}{M+N}K$, which is achievable according to Theorem 2.

It can be seen that having an identical number of antennas at the receive side of user 1 and transmit side of user 2 is important for establishing the optimality of total DoF. In other words, the desired outer bound occurs when the receivers of group user 1 with gN antennas are able to successfully decode interferences created by gN antennas. Such requirement can be satisfied if $K \geq \frac{M+N}{\text{gcd}(M,N)}$.

Remark 10: Zero-forcing always allows us to achieve the total DoF $\min\{\max(M, N), K \min(M, N)\}$, which is indeed tight when $K < \frac{M+N}{\min(M,N)}$, cf. [7].

Remark 11: In the case $M = N$, it is optimal to set $g = 1$. Therefore, the DoF region is upper bounded by

$$d_k + \max_{\hat{k} \in \mathcal{K}, \hat{k} \neq k} d_{\hat{k}} \leq N \tag{59}$$

for all $k \in \mathcal{K}$.

To improve outer bounds associated with grouping approach, a new method in [9] called genie chains is proposed where a receiver is provided with a subspace of signals (part of transmitted symbols) as a genie. As a result of this approach, the total DoF $\frac{MN}{M+N}$ is obtained for the wider range of $\frac{M}{N} \geq \frac{K-2}{K^2-3K+1}$.

In MIMO X network channel, a general outer bound has been obtained in [10]. It is shown that the sum of all the DoFs of the messages associated with transmitter k and receiver j is upper bounded by $\max(M_k, N_j)$. Despite the assurance that the total DoF outer bound is achieved for the single antenna X network, the characterization for the case of MIMO seems to be challenging.

X. Conclusions and Future Works

We developed a new real interference alignment scheme for multiple-antenna interference networks that employed joint receiver antenna processing. The scheme utilized a result on simultaneous Diophantine approximation and aligned all interferences at each receive antenna. We were able to derive several new DoF region results, as summarized in the theorems.

It is desirable to extend the result of the paper to a multiple-antenna interference network with K transmitters and J receivers where each transmitter sends an arbitrary number of messages, and each receiver may be interested in an arbitrary subset of the transmitted messages. The asymptotic alignment schemes have been successfully used to achieve the optimal DoF for both SISO and MIMO wireless networks for time-varying channels. It is interesting to translate these result to the constant channels under real interference alignment framework and find the connection between real and vector interference alignment. It is also possible that one can improve the existing outer bounds so that the optimality of the achieved DoF regions are generally proved.

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Appendix

A. Nondegenerate manifolds

One important notion in studying Diophantine approximation on manifolds is the so called nondegeneracy, which we briefly review the useful definitions and facts; see [15], [16] for more discussion.

A smooth map $\mathbf{f}$ from $U \subset \mathbb{R}^d$ to $\mathbb{R}^m$ is called *l-nondegenerate* at $\mathbf{x} \in U$ if partial derivatives of $\mathbf{f}$ at $\mathbf{x}$ up to order l span $\mathbb{R}^m$. The mapping $\mathbf{f}$ is called *non-degenerate* if for almost every $\mathbf{x} \in U$ it is l-nondegenerate for some l. The non-degeneracy of a manifold guarantees that the manifold can not be approximated by a hyperplane "too well"; see [16, Lemma 1].

A set of functions are *linearly independent over* $\mathbb{R}$ if none of the functions can be represented by a linear combination of the other functions with real coefficients. If the functions $f_1, \ldots, f_n$ are analytic, and $1, f_1, \ldots, f_n$ are linearly independent over $\mathbb{R}$ in a domain U, all points of $\mathcal{M} = \mathbf{f}(U)$ are nondegenerate.

B. Proof of Lemma 1

In the following, we will need the concept of *strongly extremal*, *very well multiplicative approximable* (VWMA), and *very well approximable* (VWA). For definitions of these concepts, we refer the reader to

[13, Sec. 1].

Based on [13, Thoerem 1.2], the pushforward of Lebesgue measure on $U_1 \times \ldots \times U_n$ by $\mathbf{A}$ is strongly extremal. That is, for almost all $(\mathbf{h}_1, \ldots, \mathbf{h}_n)$, $\mathbf{A}(\mathbf{h}_1, \ldots, \mathbf{h}_n)$ is not VWMA, which in turn implies that $\mathbf{A}$ is not VWA. The fact A is not VWA means that there are at most finitely many $\mathbf{q} \in \mathbb{Z}^m$ with

$$\|\mathbf{A}\mathbf{q} - \mathbf{p}\|_\infty < \|\mathbf{q}\|_\infty^{-m/n-\delta} \text{ for some } \mathbf{p} \in \mathbb{Z}^n \tag{60}$$

We require $\mathbf{A}$ to have at least one row whose elements are rationally independent, so that for any non-zero $\mathbf{q}$, $\|\mathbf{A}\mathbf{q}\|_\infty > 0$. For such $\mathbf{A}$ and for all the $\mathbf{q}$ such that (60) does not hold, knowing that there are at most finitely many of such $\mathbf{q}$, it is possible to choose Q large enough such that $\|\mathbf{A}\mathbf{q}\|_\infty > Q^{-m/n-\delta}$. As a result, for large enough Q, for all $\mathbf{q} \in \mathbb{Z}_Q^m$, we have $\|\mathbf{A}\mathbf{q}\|_\infty > Q^{-m/n-\delta}$. Since the 2-norm is at least as large as the infinity norm, the desired result is obtained. □

C. Proof of Lemma 2

The proof is similar to that in [4]. The difference here is that it does not resort to the Fano's inequality. Without loss of generality, we fix the average power per symbol to be P_0 and set the per-element noise variance to 1. Let $w := m/n$, which measures the ratio of the width and height of matrix $\mathbf{A}$. Fix $0 < \epsilon < 1$ and $0 < \delta < \frac{\epsilon(1+w)}{1-\epsilon}$. For large enough P_0, we select $\mathbf{x} \in \lambda\mathbb{Z}_Q^m$, where

$$Q = {P_0}^{\frac{1-\epsilon}{2(1+w)}}, \quad \lambda = \frac{{P_0}^{1/2}}{Q} = {P_0}^{\frac{w+\epsilon}{2(1+w)}} \tag{61}$$

From Lemma 1, we know that for almost all $\mathbf{A}$, and for all $\mathbf{x}, \mathbf{x}' \in \mathbb{Z}_Q^m$, such that $\mathbf{x} \neq \mathbf{x}'$, we have

$$\|\mathbf{A}(\mathbf{x} - \mathbf{x}')\|_2 > d_{\text{min}} := \lambda(2Q)^{-m/n-\delta} = 2^{-m/n-\delta} {P_0}^{\frac{\epsilon}{2} - \frac{\delta(1-\epsilon)}{2(1+w)}}. \tag{62}$$

By the choice of δ, the pairwise distance in (62) grows with P as $P \to \infty$. The pairwise error probability is therefore upper bounded by

$$\int_{d_{\text{min}}/2}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-t^2/2) dt \leq \exp(-d_{\text{min}}^2/8) \tag{63}$$

where the Chernoff bound for the Gaussian Q-function has been applied. Employing the union bound, we can upper bound the average probability of error as

$$P_e < (2Q+1)^m \exp(-d_{\text{min}}^2/8) \tag{64}$$

$$< (3Q)^m \exp(-d_{\text{min}}^2/8) \tag{65}$$

$$= 3^m \exp\left[m\frac{1-\epsilon}{2(1+w)} \log P_0 - \frac{1}{8} \cdot 2^{-\frac{2m}{n} - 2\delta} {P_0}^{\epsilon - \frac{\delta(1-\epsilon)}{2(1+w)}}\right]. \tag{66}$$

By the choice of δ, the exponent of P_0, namely $\epsilon - \frac{\delta(1-\epsilon)}{2(1+w)}$ is positive. Also for large P_0, the polynomial term dominates the $\log(P_0)$ term in the exponent. As a result, the upper bound goes to zero as $P_0 \to \infty$. The achieved DoF per symbol is

$$\lim_{P_0 \to \infty} \frac{\log(2Q+1)}{0.5\log(P_0)} = \frac{1-\epsilon}{1+w}. \tag{67}$$

Since ϵ can be made arbitrarily small, the per-symbol DoF of $1/(1+w) = n/(m+n)$ can be achieved. The total achieved DoF is $mn/(m+n)$. □

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