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Ultrasonic beam models in anisotropic media

by

Matthias Rudolph

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of

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Major: Engineering Mechanics

Major Professor: Lester W. Schmerr, Jr.

Iowa State University

Ames, Iowa

1999

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Graduate College
Iowa State University

This is to certify that the Doctoral dissertation of

Matthias Rudolph

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Committee Member
Signature was redacted for privacy.

Committee Member
Signature was redacted for privacy.

Committee Member
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Committee Member
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Committee Member
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Major Professor
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For the Major Program
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Wer a sagt muß nicht b sagen.

Er kann auch erkennen,
daß a falsch war.

Bertolt Brecht

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ABSTRACT

This dissertation models the sound fields produced by ultrasonic transducers for immersion inspections where the sound fields pass through general curved interfaces into a general anisotropic material.

The foundations for these models are discussed, including slowness surfaces for anisotropic materials and the reflection/transmission of plane waves at a planar fluid/anisotropic solid interface.

Three beam models are developed. Two of these models – the boundary diffraction wave model and the multi-Gaussian model – rely on the paraxial approximation, resulting in models that are computationally very efficient. As a consequence, those models can be used to conduct parametric studies in a highly effective manner. The third model is a more ‘exact’ model, based on the numerical evaluation of an angular spectrum of plane waves integral with Hopkins’ method. This ‘exact’ model, although computationally less efficient than the paraxial models, can handle the special cases where the paraxial approximation can fail. The beam models are compared to each other for a variety of test cases and limitations of the paraxial models are discussed.
1 INTRODUCTION

Nondestructive Evaluation (NDE) techniques provide information on material properties and the presence of defects in engineered components and structures, and thereby contribute to the integrity and safe operations of those components. Over the last decade software models have been applied to characterize elements of nondestructive evaluation measurement systems [50], and have been used for quantitative design, analysis, and calibration of NDE inspections. There is a broad field of applications for models in NDE. They can provide assistance in the interpretation of measured data, aid in the design or optimization of an inspection technique and are valuable in the validation of an inspection technique to increase reliability. Such software can also provide on-line help for experts during an inspection process. Model-based NDE simulations are also useful in education by providing a student with a highly intuitive visual tool for learning.

Models are especially important in the inspection of modern, complex-shaped components, such as composites, where significant material anisotropy and geometric variations make NDE inspections very difficult. In ultrasonic NDE, for example, material anisotropy can produce sound beam steering, focusing/defocusing effects, and generate anomalous modes. Similarly, curved component interfaces can cause beam distortions, beam 'splitting', etc. In order to understand these complicated effects and quantify their impact on the inspection process, detailed models of how ultrasound is transmitted into and propagates in complex-shaped, anisotropic materials are needed. This thesis will fill this need by developing both approximate and 'exact' ultrasonic beam models capable of predicting the wavefields generated by ultrasonic transducers in general anisotropic
materials. One of these models will also be able to handle interface geometries of a very general nature. These models will help to significantly extend the state-of-the-art for ultrasonic modeling of complex geometry, anisotropic components.

**General Problem Statement**

A typical NDE ultrasonic immersion inspection setup is depicted in a very general form in Fig. 1.1. The transmitting transducer converts the electrical signal from a pulser into mechanical motion on the transducer face. This mechanical motion in turn generates waves which travel through the fluid and are transmitted into the solid. The transmitted elastodynamic waves in the solid interact with a flaw and travel through the interface back to the receiving transducer. This transducer reconverts the mechanical energy into electrical energy which is then amplified by a receiver and typically displayed as a voltage versus time trace on an oscilloscope. During the whole propagation process

![Image of NDE inspection setup](image)

**Figure 1.1** A general setup for a NDE inspection situation
from the transmitting transducer to the receiving transducer the traveling pulse will change its shape and amplitude due to diffraction, scattering and material attenuation. The frequency components of the displayed voltage, $U$, can be given by means of AULD's reciprocity relation as

$$U = \frac{\beta}{S_t} \int_{S_f} (\sigma_{ij} v_j^2 - \sigma_{ij}^2 v_j^2) \cdot n_i \, dS$$

(1.1) where $\sigma_{ij}^\eta, v_j^\eta, (\eta = 1, 2)$, are stresses and velocities for two different fundamental problems. $S_t$ is the area of the transmitting transducer, $S_f$ is the surface of the flaw and $\beta$ is a system 'efficiency factor' [50].

AULD's relation, Eq.(1.1), holds under very general conditions (basically the entire system must be assumed to be linear) but is difficult to implement because it requires the integration over the flaw surface of complicated stress and velocity fields. However, under the assumption that the incident waves on the flaw could be written as quasi-plane waves, THOMPSON and GRAY [58] showed that AULD's relation could be reduced to the product of a series of terms which characterize the entire system in a very modular manner, namely

$$U(\omega) = \beta(\omega) P(\omega) M(\omega) T_1(\omega) C_1(\omega) A(\omega) T_2(\omega) C_2(\omega).$$

(1.2) Although originally developed for isotropic materials, Eq.(1.2) is also applicable to anisotropic solids as well. In this expression the system efficiency factor, $\beta(\omega)$, combines all the electrical-to-mechanical and mechanical-to-electrical conversion processes present in the system including the pulser/receiver, cabling, transducers, etc. This factor is normally obtained through a well characterized reference experiment [50]. $P(\omega)$ is a propagation term which accounts for the time it takes for a pulse to propagate from the transmitting transducer to the flaw and back to the receiving transducer. $M(\omega)$ accounts for the material attenuation of the ultrasound as it propagates through the various media and $T_1(\omega)$. $T_2(\omega)$ are plane wave transmission coefficients that account for amplitude changes at the fluid-solid interfaces during the transmission and reception
processes, respectively. The term \( A(\omega) \) is the far field plane wave scattering amplitude for the flaw and \( C_1(\omega) \), \( C_2(\omega) \) are diffraction correction terms on transmission and reception, respectively, which account for the fact that the sound beams present are not purely plane waves.

To make effective use of Eq.(1.2) models are needed in particular for the \( T_j(\omega) \) and \( C_j(\omega) \) terms \((j = 1, 2)\) and for the flaw scattering amplitude, \( A(\omega) \). Although extensive modeling work has been done in the past on all of these quantities for isotropic materials, considerably fewer results are available for anisotropic materials. This dissertation will develop models of the sound beam transmitted through curved fluid-solid interfaces into general anisotropic materials that contain specific expressions for the transmission coefficients, \( T_j(\omega) \) and diffraction correction terms \( C_j(\omega) \), needed in Eq.(1.2). We will not consider models needed to determine \( A(\omega) \). The next section will give a brief overview of past work on beam models for both isotropic and anisotropic media.

**Historical Perspective**

Modeling of sound fields in isotropic and anisotropic media has attracted many researchers over the past twenty years. Modeling the reflection and refraction of bounded beams from finite transducers in particular has been approached in three different ways: (1) Methods using the paraxial or Fresnel approximation that lead to expressions that can be rapidly evaluated for the wave fields in the solid, (2) more exact methods, including Green's function methods, Fourier methods or Angular Plane Wave spectrum approaches, and (3) direct numerical methods like the Finite Integration Technique (FIT), the Finite Element Method (FEM) or the Boundary Element Method (BEM).

By combining an angular spectrum of plane wave representation and the paraxial approximation THOMPSON and GRAY [58] originally developed diffraction coefficients for the on-axis response of planar and focused piston transducers radiating through a
planar interface into an isotropic solid. To overcome the difficulty of extending the model to the evaluation of off-axis radiated fields THOMPSON and LOPES [60] later developed Gaussian and Gauss-Hermite (GH) beam models, the latter being based on the work of COOK and ARNOULT [9]. NEWBERRY and THOMPSON [41] extended the Gauss-Hermite beam model to treat radiation through a planar interface into an anisotropic solid. MINACHI [36] extended the GH-beam model to handle ultrasonic piston transducer radiation through a curved anisotropic interface. Essentially these GH-beam models are generalizations of simpler Gaussian beam models in that they use the propagation and transmission rules developed for a single Gaussian [59]. WEN and BREAZEALE [64] developed a 10 Gaussian expansion for a circular, planar piston probe, which made it practical to use Gaussian beam models to represent these piston probes. MINACHI and THOMPSON [35] combined the earlier work on Gaussian beams and the Gaussian superposition to treat piston transducer radiation through planar and curved interfaces into isotropic and anisotropic solids.

The superposition of Gaussian beams was also used by SPIES [56] to model ultrasonic beam propagation in homogeneous transversely isotropic materials. In SPIES [56] he showed that the inhomogeneity of a weldment can be modeled by dividing the weldment into several layers of respective grain orientations, the approach accounting for the propagation through the isotropic/anisotropic layers and the reflection/refraction processes at the interface. The Gaussian base functions were obtained from relationships previously derived for Gaussian wave packets in transversely isotropic media, as shown in SPIES [54] and NORRIS [42].

Although there has been considerable work done on Gaussian and Gauss-Hermite beam models, most of these models either are restricted to special material cases (certain types of anisotropy, propagation along planes of symmetry, etc.), to special geometrical configurations (plane of incidence aligned with principal directions of the surface, etc.) or both. The general Gaussian wave packets solution developed by NORRIS [42] in principle
can treat general anisotropic materials and general interface geometries, but involve the use of complex and rather cumbersome non-orthogonal coordinate transformations. This dissertation will derive simple expressions for the propagation of a Gaussian beam through a general anisotropic medium which explicitly are parameterized in terms of the interface curvatures and the slopes and curvatures of the slowness surface. Through the use of the Wen and Breazeale [64] expansions, this Gaussian beam solution will be used to represent the wavefield from a circular piston transducer in terms of a new multi Gaussian beam model that is simple in structure, computationally efficient and very general in applicability.

Taking an alternative approach Schmerr, Sedov and Lerch [51] used the Rayleigh-Sommerfeld integral, an angular spectrum of plane wave decomposition and the method of stationary phase to develop a model for the transmitted bulk waves generated in a isotropic solid by a piston transducer radiating at oblique incidence. The total wave field response was decomposed into two parts (1) a direct plane wave component that exists only within the 'main beam' of the transducer and (2) an edge wave that arises from radiation of waves through the interface from the transducer rim. This model is called Boundary Diffraction Wave (BDW) model since the form of this response is the same as obtained in boundary diffraction wave theory [51]. In this dissertation the BDW-model will be extended to treat radiation into general anisotropic media.

All the models just discussed which use the paraxial approximation fail under certain conditions such as in the very near field or at high refracted angles. The second category of beam models mentioned above does not rely on the paraxial approximation and therefore does not suffer from the drawbacks mentioned earlier.

For example, Rose et al.[47] presented a numerical integration scheme, using the scalar Greens function as point sources, for transversely isotropic media exhibiting mild anisotropy. Spies [54] developed, based on the theory of elastic wave propagation in
transversely isotropic media, the Generalized Point Source Synthesis (GPSS) method to model the radiation, propagation and scattering of elastic waves as generated by ultrasonic transducers in those media. GPSS divides the transducer aperture into a grid and locates a point source at each grid point. The transducer beam profile is then obtained as the superposition of these elementary waves. In 1996 SPIES [55] presented a theory to treat scattering of ultrasound at defects, using Kirchhoff theory, in transversely isotropic media, and evaluated the wave fields using GPSS.

AMRANI et al.[2] proposed a method based on the RAYLEIGH integral and geometric optic approximations (second order in phase and first order in amplitude) to deal with ultrasonic broadband transducers radiating onto a planar isotropic interface. The model called Champ son was extended by LHÉMERY [25] to treat curved interfaces. GENGEMBRE et al.[15] extended the model to anisotropic media with planar interfaces. To handle caustics a third order expansion of the phase has been employed.

A different approach employing the Fourier Integral Theory was used by ROBERTS [45] to provide an 'exact' theory for beam transmission through a single planar interface into an anisotropic solid. The spectral integrals were solved by applying the method of steepest descents. Furthermore he presented a numerical scheme for the numerical evaluation of the spectral integrals based on HOPKINS' method [20]. A similar approach was taken by VEZZETTI [61] where he evaluated the Fourier Integrals using perturbation calculations.

For isotropic media LERCH et al.[22], [21] used the edge element method as a convenient way to evaluate the surface integrals appearing when radiating through planar and curved interfaces. The edge element method divides the transducer surface (and interface, if curved) into small area elements and makes a first order approximation on the phase term. The surface integral is then reduced to two finite summations involving terms associated with the edges of the area elements.

The third category of beam models mentioned earlier are those employing direct
numerical methods. LANGENBERG, FELLINGER and MARKLEIN [32], [33] proposed the Elastodynamic Finite Integration Technique (EFIT) to compute wave fields in general inhomogeneous anisotropic media by discretizing CAUCHY’s equation of motion and the deformation rate equation in integral form. The field equations are discretized by applying the Finite Integration Technique (FIT)\(^1\) using a staggered grid. The resulting discrete equations are then solved in a marching on in time sense. The software package is highly developed and can handle complicated inspection situations such as inspection of weldments or concrete [34]. Furthermore additional algorithms were developed by MARKLEIN [31] to treat acoustic wave propagation (AFIT) and piezoelectric wave propagation (PFIT) [32], and by RUDOLPH [49] for nonlinear acoustic wave propagation.

LORD [27], [26] used the Finite Element Technique (FEM) to compute the wave fields in an anisotropic medium. To derive the required finite element formulation, NAVIERS equation for a homogeneous, non-dissipative general anisotropic solid was transformed into a weak form by the weighted residual scheme. The FEM-code can handle complex geometries containing material inhomogeneities such as flaws or inclusions. However, the method is computationally intensive particularly if a 3D solution is necessary.

Finally, another direct numerical method is the Boundary Element Method (BEM). ACHENBACH et al.[17] extended the BEM method to compute the wave fields in a transversely anisotropic half-space. Using an integral representation for the displacement components the problem was first reduced to a system of singular integral equations for the displacements on the interface. Using a simplified form of the Green’s function the system of singular integral equations was solved by the Boundary Element Method over a truncated area. The required Green’s functions for the solid were computed numerically.

\(^1\)Originally developed for the electromagnetic wave propagation by WEILAND [63]


**Dissertation Outline**

As mentioned before, this dissertation is concerned with modeling the interaction of transducer generated sound fields with anisotropic solids. To lay the foundation for developing the beam models chapter 2 reviews the notation used and the governing equations for wave propagation in an anisotropic solid. A major emphasis of this chapter is on the constitutive equations which relate the strain and the stress through the elastic stiffness tensor. Since the elastic stiffness tensor is normally given in reference to a particular material coordinate system, transformations will be derived to transform the tensor into other coordinate systems. Finally the important CHRISTOFFEL equation is derived describing plane wave propagation in an anisotropic solid.

Chapter 3 will show how to solve CHRISTOFFEL’s equation numerically, leading to the slowness surfaces for anisotropic media. Obtaining a slowness surface requires the solution of an eigenvalue problem so special attention will be given to the solution procedure for this problem. The concept of the group velocity will be also discussed and related to the physical behavior of the plane wave propagation. Since the ultrasonic beam models derived in the later chapters use the slopes and the curvatures of the slowness surface, chapter 3 will also show how to obtain those parameters.

Chapter 4 is concerned with the reflection and refraction of an obliquely incident plane wave on an interface between a fluid and a general anisotropic half space. The chapter follows mostly the approach of ROKHLIN [46]: but uses the method of CHEN [7] to compute the polarization vectors, a key problem in solving for the reflection and transmission coefficients. Mostly numerical solution procedures are required for the general anisotropic case. However for some special materials and symmetries explicit transmission and reflection coefficient expressions can be obtained. These cases are used to help validate the more general numerical scheme.

Having laid the foundations in chapters 2-4, chapter 5 develops a boundary diffrac-
tion wave (BDW) model for an arbitrarily shaped plane piston transducer radiating though a planar interface into a general anisotropic solid. The model uses the paraxial approximation and is a generalization of similar models developed for isotropic materials by SCHMERR [50]. This BDW model represents the radiated field as a superposition of direct and edge wave terms. Furthermore the model relies on the paraxial approximation to obtain expressions for the radiated field in terms of simple line integrals that can be easily evaluated. This model can handle planar transducers of various shapes and can serve as the basis of fast simulations. However this model fails at points on the slowness surface where the curvatures are large or for points in the solid where a focal point is developed. Those limitations will be removed by the multi-Gaussian beam model developed in chapter 6. Again explicit expressions are derived for special cases and results are compared for the isotropic case. More detailed results for anisotropic materials are analyzed later in chapter 9, where comparisons are made with the other beam models.

Chapter 6 and chapter 7 develop a multi-Gaussian beam model which will remove some of the limitations of the boundary diffraction wave model. Unlike the BDW model the multi-Gaussian beam model is designed to handle both general curved interfaces as well as a general anisotropic solid. The multi-Gaussian beam model does not fail for points in the solid where a focal point is developed. However, since both BDW and multi-Gaussian models rely on the paraxial approximation they will fail for points on the interface where the surface curvature may be rapidly varying and for points on the slowness surface where the curvature is rapidly changing.

Because of these limitations of paraxial models, it is important to have more 'exact' models to evaluate when paraxial models fail. Therefore chapter 8 derives a more exact beam model based on the angular plane wave spectrum approach. This model will treat the radiation of a planar transducer through a planar interface into a general anisotropic solid. HOPKINS' method [20] will be used to numerically evaluate the spectral integrals.
This more exact model, however, is considerably slower than the paraxial models.

In chapter 9 results will be shown for various test problems. The models are compared to each other for a variety of test cases and the accuracy of the paraxial models is examined. Examples are shown for planar and curved interfaces, isotropic and anisotropic materials. Beam profiles will be used to visualize the effects of beam steering and focusing/defocusing.

Finally, in chapter 10, a brief discussion of the content and results of the dissertation is given, along with some suggestions for future directions.

There are also five appendices that contain detailed derivations and information on material properties that support the overall dissertation.
2 NOTATION AND GOVERNING EQUATIONS FOR AN ANISOTROPIC SOLID

Notation

In this thesis only right handed coordinate systems are considered. The Cartesian coordinates \((x, y, z)\) are denoted by subscripted variables \((x_1, x_2, x_3)\) and the corresponding base vectors by \((\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\) as shown in Fig. 2.1. In later chapters, we will find it necessary to deal with a variety of different Cartesian coordinate systems and we will also need to associate quantities with different types of waves. Thus we will adopt the following notation:

1. If necessary, the components of a quantity measured in a particular coordinate system will be indicated by a 'coordinate superscript' on that quantity that explicitly identifies the coordinate system. These same superscript will also be used, when necessary, on vector quantities that are coordinate dependent such as Cartesian base vectors.

2. If the type of wave under consideration of a particular quantity is needed, it will be indicated by use of a 'wave type superscript'. When both coordinate superscript and wave type superscript are needed, the wave type superscript will be separated by a semicolon from the coordinate superscript.

Thus, for example, a material stiffness tensor quantity, \(C_{ijkl}\), as measured in a material coordinate system \((x_1^m, x_2^m, x_3^m)\) whose coordinate unit vectors are \((\mathbf{e}_1^m, \mathbf{e}_2^m, \mathbf{e}_3^m)\) would be
Figure 2.1  Coordinate system with base unit vectors

written as $C_{ijkl}^m$. Similarly, the components of a wave number vector, $\mathbf{k}$, written in terms of a particular 'ray-based' coordinate system $(x_1^r, x_2^r, x_3^r)$ for a wave of type $\alpha$ would be written as $k_i^{r\alpha}$.

This thesis deals with both solid and fluid media. To distinguish a quantity measured in a fluid from one measured in the solid, an extra 'f' will be added as a 'material subscript' for the fluid only. Since only waves of compressional (P) type are present in a fluid, the 'wave type superscript' will be omitted for a quantity measured in the fluid as redundant. For example, the components of a wave number vector for a P-wave in a fluid as measured in a set of coordinates $(x_1^f, x_2^f, x_3^f)$ would be written as $k_i^f$ whereas a wave number measured in this same coordinate system for a wave of type $\alpha$ in the solid would be given by $k_i^{f\alpha}$.

To distinguish scalars, vectors and tensors the following rule will apply. Scalars will always be printed in non-capitalized, non-bold italic letters, vectors are bold, non-capitalized letters and tensors (or matrices) are represented by bold, capitalized letters. Unit vectors will always be indicated by a 'hat' on top of the vector, e.g. the polarization vector

$$\mathbf{d} = d_1 \mathbf{\hat{e}}_1 + d_2 \mathbf{\hat{e}}_2 + d_2 \mathbf{\hat{e}}_3. \quad (2.1)$$
The indicial notation is frequently used to represent vectors, tensors and their mathematical operations. Furthermore EINSTEIN’s summation convention is always assumed, which implies that whenever an index appears twice a summation with the index running through the integers 1, 2, 3 has to be taken. For example a wave number vector can be written using the summation convention as

$$\mathbf{k} = \sum_{i=1}^{3} k_i \hat{e}_i = k_i \hat{e}_i.$$  \hfill (2.2)

Special tensors used in the thesis are 1) the KRONECKER Delta, denoted by $\delta_{ij}$, and defined by

$$\delta_{ij} = \begin{cases} 1 & : i = j \\ 0 & : i \neq j \end{cases}.$$ \hfill (2.3)

2) the permutation symbol $\epsilon_{ijk}$ defined by

$$\epsilon_{ijk} = \begin{cases} +1 & \text{form an even permutation of 1.2.3.} \\ -1 & \text{form an odd permutation of 1.2.3.} \\ 0 & \text{do not form a permutation of 1.2.3.} \end{cases}.$$ \hfill (2.4)

and. 3) the identity matrix $\mathbf{I}$ having ones on the main diagonal and otherwise zeros.

The cross product of two vectors $\mathbf{a}, \mathbf{b}$ will be written in coordinate free form as

$$\mathbf{a} \times \mathbf{b},$$ \hfill (2.5)

or in index notation

$$a_i b_j \epsilon_{ijk} \hat{e}_k.$$ \hfill (2.6)

The dot product of the same vectors read in coordinate free notation

$$\mathbf{a} \cdot \mathbf{b},$$ \hfill (2.7)

and in index notation

$$a_i b_i.$$ \hfill (2.8)
Cauchy's Equations of Motion

When time dependent forces and/or moments act on a deformable body, that body will be set into motion. From the balance of linear momentum, it can be shown that time dependent stresses, $\sigma_{ij}$, and displacements, $u_i$, of the body are related through Cauchy's equation of motion, given by

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho_0 \ddot{u}_i \quad (2.9)$$

(under the assumption of small displacements) where $\rho_0$ is the density of the solid, which is taken here as a constant, and $\ddot{u}_i = \partial^2 u_i / \partial t^2$. It can also be shown, based on the balance of angular momentum, that the stresses must be symmetric, i.e. $\sigma_{ij} = \sigma_{ji}$.

Constitutive Equations

Cauchy's equations of motion are valid for any type of material. If we assume that the material is a homogeneous, linear elastic material, then the stresses in Cauchy's equations can be related to the strains, $\epsilon_{ij}$, through a generalized Hooke's law in the form

$$\sigma_{ij} = C_{ijkl}^m \epsilon_{kl} \quad (2.10)$$

In the latter equation $C_{ijkl}^m$ denotes the components of the fourth order elastic stiffness tensor - again assumed constant throughout the solid - and $\epsilon_{kl}$ the strain tensor components. One can think of the $C_{ijkl}^m$ components as representing microscopic spring constants and these components are called elastic stiffness constants. The stiffness constants have small values for easily deformable materials and large values for very rigid materials. Since the constitutive relation contains nine equations and each equation contains nine strain variables, the total number of $C_{ijkl}^m$ constants is 81. However those 81 constants are not independent of each other, and the symmetry requirements

$$C_{ijkl}^m = C_{jikl}^m = C_{ijlk}^m = C_{jilk}^m \quad (2.11)$$
reduce the number of independent constants to 36. The additional symmetry requirement

\[ C_{ijkl}^m = C_{klij}^m, \quad (2.12) \]

which is a consequence of the existence of a strain energy density function, further reduces the number of independent constants to 21 [43]. This is the maximum number of constants for any material and is associated with the triclinic crystal class. Usually - due to the symmetry nature of the material - additional restrictions can be imposed leading to a further reduction of the independent constants. Tab. 2.1 lists the number of independent constants for certain symmetry classes. If we expand out the general stress-strain relationship for an anisotropic material explicitly, and use the symmetry properties of the strain tensor we have

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{bmatrix} =
\begin{bmatrix}
C_{1111}^m & C_{1122}^m & C_{1133}^m & C_{1123}^m + C_{1132}^m & C_{1113}^m + C_{1131}^m & C_{1112}^m + C_{1121}^m \\
C_{2211}^m & C_{2222}^m & C_{2233}^m & C_{2223}^m + C_{2232}^m & C_{2213}^m + C_{2231}^m & C_{2212}^m + C_{2221}^m \\
C_{3311}^m & C_{3322}^m & C_{3333}^m & C_{3323}^m + C_{3332}^m & C_{3313}^m + C_{3331}^m & C_{3312}^m + C_{3321}^m \\
C_{2311}^m & C_{2322}^m & C_{2333}^m & C_{2323}^m + C_{2332}^m & C_{2313}^m + C_{2331}^m & C_{2312}^m + C_{2321}^m \\
C_{1311}^m & C_{1322}^m & C_{1333}^m & C_{1323}^m + C_{1332}^m & C_{1313}^m + C_{1331}^m & C_{1312}^m + C_{1321}^m \\
C_{1211}^m & C_{1222}^m & C_{1233}^m & C_{1223}^m + C_{1232}^m & C_{1213}^m + C_{1231}^m & C_{1212}^m + C_{1221}^m
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{23} \\
\varepsilon_{13} \\
\varepsilon_{12}
\end{bmatrix}. \quad (2.13)
\]

However, if we use the symmetry properties of the stiffness matrix (\( C_{ijkl}^m = C_{ijlk}^m \)) the summed terms appearing in Eq.(2.13) can be combined, leaving factors of two in
some of the elastic constant terms. And, if in addition we replace the tensor shear strain components by their equivalent engineering strain components, i.e. we set

\[ \epsilon_{kl} = \frac{1}{2} \gamma_{kl}, \quad k \neq l, \quad (2.14) \]

the factors of one half appearing in Eq.(2.14) can be used to cancel the factors of two appearing in the stiffness matrix, leaving

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{pmatrix} =
\begin{pmatrix}
C_{1111}^{m} & C_{1122}^{m} & C_{1133}^{m} & C_{1123}^{m} & C_{1113}^{m} & C_{1112}^{m} \\
C_{2211}^{m} & C_{2222}^{m} & C_{2233}^{m} & C_{2223}^{m} & C_{2213}^{m} & C_{2212}^{m} \\
C_{3311}^{m} & C_{3322}^{m} & C_{3333}^{m} & C_{3323}^{m} & C_{3313}^{m} & C_{3312}^{m} \\
C_{2311}^{m} & C_{2322}^{m} & C_{2333}^{m} & C_{2323}^{m} & C_{2313}^{m} & C_{2312}^{m} \\
C_{1311}^{m} & C_{1322}^{m} & C_{1333}^{m} & C_{1323}^{m} & C_{1313}^{m} & C_{1312}^{m} \\
C_{1211}^{m} & C_{1222}^{m} & C_{1233}^{m} & C_{1223}^{m} & C_{1213}^{m} & C_{1212}^{m}
\end{pmatrix}
\begin{pmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
\gamma_{23} \\
\gamma_{13} \\
\gamma_{12}
\end{pmatrix}, \quad (2.15)
\]

For computational purposes it is advantageous to adopt a contracted index notation by defining a 6x6 matrix, \( \{C_{IJ}^{m}\}_{I,J=1,6} \), where

\[
I = \begin{cases} 
    i & : i = j \\
    9 - (i + j) & : i \neq j 
\end{cases}, \quad (2.16)
\]

\[
J = \begin{cases} 
    k & : k = l \\
    9 - (k + l) & : k \neq l 
\end{cases}, \quad (2.17)
\]

With the previously defined index-substitution rules it is possible to write Eq.(2.10) in its contracted notation as

\[ \sigma_I = C_{IJ}^{m} \epsilon_J, \quad (2.18) \]

with

\[ \sigma_I = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}\}^T, \quad (2.19) \]

and

\[ \epsilon_J = \{\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \gamma_{23}, \gamma_{13}, \gamma_{12}\}^T. \quad (2.20) \]
Expanded explicitly for a triclinic material, Eq. (2.18) gives

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{pmatrix}
= \begin{bmatrix}
C_{11}^m & C_{12}^m & C_{13}^m & C_{14}^m & C_{15}^m & C_{16}^m \\
C_{12}^m & C_{22}^m & C_{23}^m & C_{24}^m & C_{25}^m & C_{26}^m \\
C_{13}^m & C_{23}^m & C_{33}^m & C_{34}^m & C_{35}^m & C_{36}^m \\
C_{14}^m & C_{24}^m & C_{34}^m & C_{44}^m & C_{45}^m & C_{46}^m \\
C_{15}^m & C_{25}^m & C_{35}^m & C_{45}^m & C_{55}^m & C_{56}^m \\
C_{16}^m & C_{26}^m & C_{36}^m & C_{46}^m & C_{56}^m & C_{66}^m
\end{bmatrix}
\begin{pmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
\gamma_{23} \\
\gamma_{13} \\
\gamma_{12}
\end{pmatrix}.
\] (2.21)

Appendix A lists the matrix-equations and the stiffness matrices for different symmetry classes and different materials. The stiffness constants $C_{ij}^m$ for crystalline materials are normally given with respect to a crystal coordinate\textsuperscript{1}, system $(x_1^m, x_2^m, x_3^m)$. This however might not be the most practical choice of the coordinate system – this will be important in the next chapter – and it is therefore necessary to consider how the stiffness constants are transformed into another coordinate system. Figure 2.2 for example, shows a material coordinate system and several rotated coordinate systems defined by the rotation angles $\xi, \eta$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{coordinate_transformations.png}
\caption{Coordinate transformations}
\end{figure}

\textsuperscript{1}Sometimes called material coordinate system
The transformation of the fourth rank stiffness tensor \( C_{ijkl} \) from a material coordinate system \( (x_1^m, x_2^m, x_3^m) \) into a different coordinate system \( (x_1, x_2, x_3) \) is defined by the transformation law [52]

\[
C_{mnop} = Q_{mi}Q_{nj}Q_{ok}Q_{pl}C_{ijkl}. \tag{2.22}
\]

The components, \( Q_{ij} \), are the direction cosines between \( \hat{e}_i \) and \( \hat{e}_j^m \) and can be written as [4]

\[
Q_{ij} = \cos(\hat{e}_i, \hat{e}_j^m) = \hat{e}_i \cdot \hat{e}_j^m. \tag{2.23}
\]

Under the definition of the fourth rank tensor

\[
\mu_{ijkl} = Q_{ik}Q_{jl}, \tag{2.24}
\]

the transformation given in Eq.(2.22) can be written as

\[
C_{mnop} = M_{mnij}C_{ijkl}M_{opkl}. \tag{2.25}
\]

To obtain a reduced form of this equation we can follow the same procedure we used in the derivation of the contracted stiffness matrix tensor. First, we consider the terms in Eq.(2.25) involving \( M_{mnij} \) only. Omitting those subscripts that are not involved with this matrix we can write Eq.(2.25) as

\[
C_{m..} = M_{mnij}C_{ijkl}M_{opkl}. \tag{2.26}
\]

which now is a relationship between two symmetrical matrices \( C_{m..} \) and \( C_{ij..} \) analogous to Eq.(2.10). Thus, Eq.(2.26) can also be expressed in a contracted form terms of a 6x6 matrix \( \mathbf{M} \) given by

\[
C_{M..} = M_{kl}C_{i..}M_{..l}. \tag{2.27}
\]

In the same fashion, Eq.(2.25) gives

\[
C_{..op} = M_{..op}C_{..kl}M_{opkl}. \tag{2.28}
\]
which is again a relationship between two symmetrical matrices that can be reduced to

\[ C_{..p} = M .. C_{M K}^m M_{PK}. \quad (2.29) \]

Combining Eqs. (2.27) and (2.29), we obtain

\[ C_{MP} = M_{MI} C_{IK}^m M_{PK}. \quad (2.30) \]

The form of the 6x6 \( M \) matrix in terms of the \( M_{ijkl} \) components, therefore, is the same as given previously for the 6x6 stiffness matrix in terms of the \( C_{ijkl}^m \) (Eq.(2.13)), namely

\[
M = \begin{bmatrix}
M_{1111} & M_{1122} & M_{1133} & M_{1123} + M_{1132} & M_{1113} + M_{1131} & M_{1112} + M_{1121} \\
M_{2211} & M_{2222} & M_{2233} & M_{2223} + M_{2232} & M_{2213} + M_{2231} & M_{2212} + M_{2221} \\
M_{3311} & M_{3322} & M_{3333} & M_{3332} + M_{3332} & M_{3313} + M_{3331} & M_{3312} + M_{3321} \\
M_{2311} & M_{2322} & M_{2333} & M_{2323} + M_{2332} & M_{2313} + M_{2331} & M_{2312} + M_{2321} \\
M_{1311} & M_{1322} & M_{1333} & M_{1323} + M_{1332} & M_{1313} + M_{1331} & M_{1312} + M_{1321} \\
M_{1211} & M_{1222} & M_{1233} & M_{1223} + M_{1232} & M_{1213} + M_{1231} & M_{1212} + M_{1221}
\end{bmatrix} \quad (2.31)
\]

Using Eq.(2.24), the \( M \) matrix can also be written explicitly in terms of the direction cosines as

\[
M = \begin{bmatrix}
Q_{11}^2 & Q_{12}^2 & Q_{13}^2 \\
2Q_{11}Q_{12} & 2Q_{12}Q_{13} & 2Q_{11}Q_{13} \\
2Q_{11}Q_{21} & 2Q_{12}Q_{22} & 2Q_{13}Q_{23}
\end{bmatrix} \quad (2.32)
\]

The transformation of the stiffness matrix in the reduced form of Eq.(2.30) is due to BOND [5], and the \( M \) matrix is called BOND’s rotation matrix. Although this matrix is in a rather complex form, it can be written as a product of simpler matrices by noting that any general rotation of coordinates in three dimensions can be considered as the result of
1. a rotation through an angle $\xi$ about the original $x^m_3$-axis of an $(x^m_1, x^m_2, x^m_3)$ system to generate a $(x'_1, x'_2, x'_3)$ system,

2. a rotation through an angle $\eta$ about the $x'_2$-axis to generate a $(x''_1, x''_2, x''_3)$ system, and

3. a rotation through an angle $\xi'$ about the $x''_3$ axis to generate the final rotated coordinate system $(x_1, x_2, x_3)$.

In matrix form, we can write these successive rotations as

$$ M = M^\xi M^\eta M^\xi. $$

(2.33)

The matrices $M^\xi$ and $M^\eta$ are of the same form since both represent rotations about either the $x_3$ or $x'_3$-axis, respectively. Such a rotation generates a matrix of direction cosines (for a rotation angle given by

$$ Q^\xi = \begin{bmatrix}
\cos \xi & \sin \xi & 0 \\
-\sin \xi & \cos \xi & 0 \\
0 & 0 & 1
\end{bmatrix}. $$

(2.34)

which leads to the BOND rotation matrix

$$ M^\xi = \begin{bmatrix}
\cos^2(\xi) & \sin^2(\xi) & 0 & 0 & 0 & \sin(2\xi) \\
\sin^2(\xi) & \cos^2(\xi) & 0 & 0 & 0 & -\sin(2\xi) \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos(\xi) & -\sin(\xi) & 0 \\
0 & 0 & 0 & \sin(\xi) & \cos(\xi) & 0 \\
-\frac{1}{2} \sin(2\xi) & \frac{1}{2} \sin(2\xi) & 0 & 0 & 0 & \cos(2\xi)
\end{bmatrix}. $$

(2.35)

Similarly, the matrix of direction cosines for a rotation through an angle $\eta$ about the $x_2$-axis is given by

$$ Q^\eta = \begin{bmatrix}
\cos \eta & 0 & -\sin \eta \\
0 & 1 & 0 \\
\sin \eta & 0 & \cos \eta
\end{bmatrix}. $$

(2.36)
leading to a \textbf{BOND} rotation matrix given by

\[
M^\eta = \begin{bmatrix}
\cos^2(\eta) & 0 & \sin^2(\eta) & 0 & -\sin(2\eta) & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\sin^2(\eta) & 0 & \cos^2(\eta) & 0 & \sin(2\eta) & 0 \\
0 & 0 & 0 & \cos(\eta) & 0 & \sin(\eta) \\
\frac{1}{2}\sin(2\eta) & 0 & -\frac{1}{2}\sin(2\eta) & 0 & \cos(2\eta) & 0 \\
0 & 0 & 0 & -\sin(\eta) & 0 & \cos(\eta)
\end{bmatrix}.
\]  
(2.37)

Two \texttt{Matlab} subroutines (\texttt{rotCz.z}, \texttt{rotCy.m}) were written to perform the transformation. Both routines accept as an input the stiffness matrix in crystal coordinates and the angle of rotation $\xi, \eta$ about the $x_3, x_2$-axis. The stiffness matrix in the rotated coordinate system will be returned by the function as an output argument, as defined by Eq.(2.35). The \texttt{Matlab} code listed below shows the \texttt{rotCz.m} \texttt{matlab}-file.

\begin{verbatim}
function [Cp] = rotCz(C,xi)
% ROTCZ(C,phi)
% Rotation of the Stiffness matrix C of xi deg [radians] about the Z-axis
% Using Bonds rotation matrix (Auld vol1 p. 77 eq. 3.42)
M=[cos(xi)^2 sin(xi)^2 0 0 0 sin(2*xi) ;
    sin(xi)^2 cos(xi)^2 0 0 0 -sin(2*xi);
    0 0 1 0 0 0 ;
    0 0 0 cos(xi) -sin(xi) 0 ;
    0 0 0 sin(xi) cos(xi) 0 ;
    -0.5*sin(2*xi) 0.5*sin(2*xi) 0 0 0 cos(2*xi)];
Cp = M*C*M.';
\end{verbatim}

\textbf{Navier's Equation}

Combining \textbf{CAUCHY's} equation of motion, Eq.(2.9), with the general constitutive equation, Eq.(2.10), leads to

\[
\frac{\partial}{\partial x_j} C^\eta_{ijkl} \epsilon_{ki} = \rho \ddot{u}_i.
\]  
(2.38)
Substitution of the strain-displacement relation

\[ \varepsilon_{kl} = \frac{1}{2} \left( \frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l} \right), \]  

(2.39)
gives

\[ \frac{1}{2} \frac{\partial}{\partial x_j} C_{ijkl}^m \left( \frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l} \right) = \rho \ddot{u}_i. \]  

(2.40)

Using the symmetry conditions \( C_{ijkl}^m = C_{ijlk}^m \), we find

\[ C_{ijkl}^m \frac{\partial^2 u_l}{\partial x_k \partial x_j} = \rho \ddot{u}_i, \]  

(2.41)

which is NAVIER's equation for a homogeneous, non-dissipative general anisotropic solid.

In the case of isotropy, the elastic stiffness tensor \( C_{ijkl}^m \) reduces to

\[ C_{ijkl}^m = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \]  

(2.42)
in term of the LAMÉ constants, \( \lambda \), and \( \mu \), leaving NAVIER's equation in the form

\[ (\lambda + \mu) \frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \rho \ddot{u}_i. \]  

(2.43)

**Christoffel's Equation**

To find a solution to NAVIER's equations in terms of monochromatic plane waves we substitute

\[ u_i = Ud_i e^{i(k_i x_i - \omega t)} \]  

(2.44)

into Eq.(2.41) and obtain

\[ C_{ijkl}^m \delta_{jk} \delta_{dl} = \rho \omega^2 d_i, \]  

(2.45)

where \( \mathbf{d} = \partial u / \partial \mathbf{x} \) is the polarization vector and \( U \) is the displacement amplitude. In terms of the propagation direction \( \mathbf{l} = l_1 \hat{e}_1 + l_2 \hat{e}_2 + l_3 \hat{e}_3 \)

\[ \left( C_{ijkl}^m \delta_{jk} - \rho \omega^2 \delta_{il} \right) d_i = 0, \]  

(2.46)
where we used $k_i = k l_i$. The result is known as CHRISTOFFEL's equation. For simplicity it is possible to define the so-called CHRISTOFFEL tensor $\Gamma$ as

$$\Gamma_{ij} = C_{ijkl}^m l_j l_k,$$  \hspace{1cm} (2.47)

where, for the most general case (triclinic symmetry) the $\Gamma_{ij}$'s have the form

$$
\Gamma_{11} = C_{11}^{m1} l_1^2 + C_{66}^{m1} l_2^2 + C_{55}^{m1} l_3^2 + 2C_{56}^{m1} l_2 l_3 + 2C_{15}^{m1} l_3 l_1 + 2C_{16}^{m1} l_1 l_2,
$$

$$
\Gamma_{22} = C_{66}^{m2} l_1^2 + C_{22}^{m2} l_2^2 + C_{44}^{m2} l_3^2 + 2C_{24}^{m2} l_2 l_3 + 2C_{46}^{m2} l_3 l_1 + 2C_{46}^{m2} l_1 l_2,
$$

$$
\Gamma_{33} = C_{55}^{m3} l_1^2 + C_{44}^{m3} l_2^2 + C_{33}^{m3} l_3^2 + 2C_{34}^{m3} l_2 l_3 + 2C_{35}^{m3} l_3 l_1 + 2C_{45}^{m3} l_1 l_2,
$$

$$
\Gamma_{12} = C_{16}^{m2} l_1^2 + C_{26}^{m2} l_2^2 + C_{46}^{m2} l_3^2 + (C_{46}^{m} + C_{25}^{m}) l_2 l_3 + (C_{14}^{m} + C_{56}^{m}) l_3 l_1 + (C_{12}^{m} + C_{66}^{m}) l_1 l_2,
$$

$$
\Gamma_{13} = C_{15}^{m3} l_1^2 + C_{45}^{m3} l_2^2 + C_{35}^{m3} l_3^2 + (C_{45}^{m} + C_{36}^{m}) l_2 l_3 + (C_{13}^{m} + C_{55}^{m}) l_3 l_1 + (C_{14}^{m} + C_{56}^{m}) l_1 l_2,
$$

$$
\Gamma_{23} = C_{56}^{m3} l_1^2 + C_{24}^{m3} l_2^2 + C_{34}^{m3} l_3^2 + (C_{44}^{m} + C_{23}^{m}) l_2 l_3 + (C_{39}^{m} + C_{43}^{m}) l_3 l_1 + (C_{25}^{m} + C_{46}^{m}) l_1 l_2,
$$

$$
\Gamma_{21} = \Gamma_{12},
$$

$$
\Gamma_{31} = \Gamma_{13},
$$

$$
\Gamma_{32} = \Gamma_{23}. \hspace{1cm} (2.48)
$$

Using Eq.(2.47) we can write the CHRISTOFFEL equation as

$$
\left( \Gamma_{ij} - \rho v^2 \delta_{ij} \right) d_i = 0, \hspace{1cm} (2.49)
$$

or. in terms of $\Omega_{ij} = \Gamma_{ij} - \rho v^2 \delta_{ij}$,

$$
\Omega_{ij} d_j = 0. \hspace{1cm} (2.50)
$$

The latter equation defines a set of three homogeneous linear equations for the unit displacement components $(d_1, d_2, d_3)$. The phase velocity for a given direction $\hat{i}$ is unknown, but the existence of a non-trivial solution of the system of equations in Eq.(2.50) requires the determinant of the coefficients of that system to vanish. This constitutes an eigenvalue problem with the eigenvalues identified as $\rho v^2$. The eigenvector $(d_1, d_2, d_3)$ associated with each eigenvalue defines the polarization $\hat{d}$. Because of the symmetry of
The eigenvalues are real and the associated eigenvectors are mutually orthogonal to each other. The next chapter describes how to solve CHRISTOFFEL's equation and discusses aspects of the numerical algorithm. When dealing with reflection and refraction problems it is convenient to recast CHRISTOFFEL's equation in a different form. If we divide Eq.(2.46) by \( \nu^2 \) we have

\[
\left( C_{ijkl} s_j s_k - \rho \delta_{il} \right) d_l = 0, \quad (2.51)
\]

where \( s = 1/\nu \) defines the slowness vector (inverse velocity). We can write the latter equation in a short form as

\[
G_{ij} d_j = 0, \quad (2.52)
\]

with \( G_{ij} = C_{ijkl} s_l s_k - \rho \delta_{ij} \). Equation (2.51) is obviously an alternative expression to Eq.(2.46). In the next chapter Eq.(2.46) will be solved for \( \rho \nu^2 \) for a given propagation direction and that direction will then be varied to map out the complete slowness surface. In chapter four Eq.(2.51) will be solved instead for a particular component of the slowness vector which is needed as part of solving for reflection and transmission coefficients at a plane interface.
3 THE SLOWNESS SURFACE

The last chapter derived Christoffel's equation, Eq.(2.49), for an anisotropic solid. This equation describes the plane wave propagation in an anisotropic solid and is essential for the rest of the thesis. This chapter will show how to solve Eq.(2.49) numerically which leads to the so-called slowness-surfaces, defined by a solution of an eigenvalue problem, and the associated eigenvectors which give the polarization of the propagating waves. A detailed discussion of the numerical procedure, including *matlab*-examples, is given in a separate section. The important concept of the group-velocity will also be discussed and again related to the physical behavior of wave-propagation.

Finally, we will show how to obtain locally the slopes and curvatures of slowness-curves. These quantities are needed, as we will see later, as part of the solution process for obtaining the radiated wavefields of transducers in the paraxial approximation.

To solve Eq.(2.49) we first write down its matrix form

\[
\begin{pmatrix}
\Gamma_{11} - \rho v^2 & \Gamma_{12} & \Gamma_{13} \\
\Gamma_{12} & \Gamma_{22} - \rho v^2 & \Gamma_{13} \\
\Gamma_{13} & \Gamma_{23} & \Gamma_{33} - \rho v^2
\end{pmatrix}
\begin{pmatrix}
d_1 \\
d_2 \\
d_3
\end{pmatrix} = 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

(3.1)

The latter equation has a nontrivial solution if the characteristic determinant vanishes

\[
|\Omega_{ij}| = |\Gamma_{ij} - \rho v^2 \delta_{ij}| = 0.
\]

(3.2)

As we will see when we discuss the numerical solution of Eq.(3.2), it is convenient to assume \( \hat{1} \) lies in an \( x_1 - x_3 \) plane so that the \( l_2 \) component vanishes. Then Eq.(2.48)
reads as

$$\begin{align*}
\Gamma_{11} &= C_{11}^{\text{m}} t_1^2 + C_{55}^{\text{m}} t_3^2 + 2C_{13}^{\text{m}} t_3 t_1, \\
\Gamma_{22} &= C_{66}^{\text{m}} t_1^2 + C_{44}^{\text{m}} t_3^2 + 2C_{46}^{\text{m}} t_3 t_1, \\
\Gamma_{33} &= C_{55}^{\text{m}} t_1^2 + C_{33}^{\text{m}} t_3^2 + 2C_{35}^{\text{m}} t_3 t_1, \\
\Gamma_{12} &= C_{16}^{\text{m}} t_1 + C_{48}^{\text{m}} t_3 + (C_{14}^{\text{m}} + C_{56}^{\text{m}}) l_3 l_1, \\
\Gamma_{13} &= C_{15}^{\text{m}} t_1 + C_{35}^{\text{m}} t_3 + (C_{13}^{\text{m}} + C_{55}^{\text{m}}) l_3 l_1, \\
\Gamma_{23} &= C_{56}^{\text{m}} t_1 + C_{34}^{\text{m}} t_3 + (C_{36}^{\text{m}} + C_{45}^{\text{m}}) l_3 l_1, \\
\Gamma_{21} &= \Gamma_{12}, \\
\Gamma_{31} &= \Gamma_{13}, \\
\Gamma_{32} &= \Gamma_{23}. 
\end{align*}$$

(3.3)

**Determination of the Eigenvalues $\lambda$**

Using the the substitution $\lambda = \rho v^2$ we can write Eq.(3.2) as

$$\lambda^3 - \text{tr}\{\Gamma\} \lambda^2 + \text{tr}\{\text{adj}\{\Gamma\}\} \lambda - \det\{\Gamma\} = 0. \quad (3.4)$$

where $\text{tr}\{\cdot\}$ defines the trace, $\text{adj}\{\cdot\}$ the adjoint and $\det\{\cdot\}$ the determinant of a matrix.

The operation in Eq.(3.4) can be written explicitly in component form as

$$\begin{align*}
\text{tr}\{\Gamma\} &= \Gamma_{ii}, \\
\text{tr}\{\text{adj}\{\Gamma\}\} &= \frac{1}{2} (\epsilon_{11} \epsilon_{11} + \epsilon_{22} \epsilon_{22} + \epsilon_{33} \epsilon_{33} + \epsilon_{12} \epsilon_{12} + \epsilon_{13} \epsilon_{13} + \epsilon_{23} \epsilon_{23}), \\
\det\{\Gamma\} &= \epsilon_{ijk} \Gamma_{ij} \Gamma_{2j} \Gamma_{3k}. 
\end{align*}$$

(3.5) \hspace{1cm} (3.6) \hspace{1cm} (3.7)

Once the material properties and the propagation direction are chosen, the latter equation can be solved for $\lambda$. The roots of the cubic are the eigenvalues $\lambda$ of the Christoffel-matrix $\Gamma_{ij}$ from which the phase-velocities are then given by

$$v^\alpha = \sqrt{\frac{\lambda^\alpha}{\rho}}, \quad \alpha = 1, 2, 3 \quad (3.8)$$
For an arbitrary direction there exist – in general – three different phase-velocities. Two of those phase-velocities can be equal to each other in certain symmetry directions. The eigenvectors corresponding to these eigenvalues are the polarization directions. The polarizations will not be purely longitudinal (along \( \hat{1} \)) or transverse (perpendicular to \( \hat{1} \)), but there will usually be a wave with \textit{mostly} longitudinal polarization and two shear waves with \textit{mostly} transverse polarizations. The wave of \textit{mostly} longitudinal wave type will be referred to as quasi-pressure (qP) and the other two shear like waves will be called quasi-shear1,2 (qS1,qS2). We will subsequently refer to \( \alpha \) as the wave type where \( \alpha = qP,qS1,qS2 \). The deviation of the polarization direction of a wave of type \( \alpha \) from a purely longitudinal or transverse direction is commonly known as the deviation or skewing angle [39].

**Determination of the Eigenvectors**

As we will see in the next chapter a key problem in solving for the reflection and transmission coefficients is the computation of the displacement vectors (eigenvectors). However, general methods for obtaining eigenvectors in a manner that is convenient to implement numerically are seldom discussed. Direct methods, which involve the substitution of the eigenvalues back into Eq.(3.1) and solving a subset of those equations (see Nayfeh [39]) are not convenient because they fail under a variety of conditions. Here, we will describe a general procedure, based on Chen [7] for determining the eigenvectors which effectively handles all the exceptional cases.

We will have in general three eigenvalues for a given direction \( \hat{1} \). To determine the corresponding eigenvectors for these eigenvalues clearly depends on the matrix \( \Omega \), and one has to distinguish two cases:

1. The three eigenvalues are all different; In this case \( \det{\Omega} = 0 \) and \( \text{adj}{\Omega} \neq 0 \).
29

and $\Omega$ is said to be planar.

2. Two of the three eigenvalue coincide; In this case for the two coinciding eigenvalues $\det{\Omega} = 0$ and $\text{adj}\{\Omega\} = 0$ and $\Omega$ is said to be linear.

For the first case we consider the problem of finding solutions of the homogeneous equation

$$\Omega \cdot \dot{d} = 0, \quad (3.9)$$

with $\Omega$ being planar and singular (that is $\det{\Omega} = 0$). Clearly $\dot{d} = 0$ is a solution but we are interested in the non-trivial solution (that is $\dot{d} \neq 0$). Moreover if a nontrivial solution exists it will not be unique. To see this consider

$$\Omega \cdot \alpha \dot{d} = \alpha \Omega \cdot \dot{d} = 0 \quad (3.10)$$

where $\alpha$ is an arbitrary scalar. Eq.(3.10) shows that the homogeneous solution determines only the direction of the polarization vector not its magnitude. To overcome this problem we apply the constraint that the magnitude of the polarization vector has to equal unity. i.e.

$$|\dot{d}| = 1. \quad (3.11)$$

To obtain $\dot{d}$ in the first case we consider the identity

$$\Omega \cdot W = \det{\Omega} I \quad (3.12)$$

where $W = \text{adj}\{\Omega\}$ and its components are given by

$$W_{ij} = \frac{1}{2} \epsilon_{ikt} \epsilon_{jmn} \Omega_{km} \Omega_{ln}, \quad (3.13)$$

or written out explicitly

$$W_{11} = \Omega_{22} \Omega_{33} - \Omega_{32} \Omega_{23},$$

$$W_{22} = \Omega_{33} \Omega_{11} - \Omega_{13} \Omega_{31},$$
\[ W_{33} = \Omega_{11}\Omega_{22} - \Omega_{21}\Omega_{12}, \]
\[ W_{12} = \Omega_{31}\Omega_{23} - \Omega_{21}\Omega_{33}, \]
\[ W_{31} = \Omega_{12}\Omega_{33} - \Omega_{13}\Omega_{22}, \]
\[ W_{23} = \Omega_{31}\Omega_{12} - \Omega_{32}\Omega_{11}. \]  

(3.14)

Since \( \Omega \) is singular we have
\[ \Omega \cdot W = 0I. \]  

(3.15)

Dot multiplying this equation from the right by an arbitrary constant vector \( c \) we obtain
\[ \Omega \cdot (W \cdot c) = 0 \]  

(3.16)

Comparing Eq.(3.16) with Eq.(3.10) we find the nontrivial solution as
\[ \alpha \dot{d} = W \cdot c. \]  

(3.17)

If we define the constant \( \alpha \) as \( \alpha = |W \cdot c| \) we get
\[ \dot{d} = \frac{1}{|W \cdot c|} W \cdot c. \]  

(3.18)

Since \( c \) is arbitrary we can choose \( c = \hat{e}_\eta \) to pick out the column \( \eta \) of \( W \) which is non zero: i.e
\[ \dot{d} = \frac{1}{|W \cdot e_\eta|} W \cdot \hat{e}_\eta \quad (\text{no sum on } \eta ), \]  

(3.19)

or in index notation
\[ d_\eta = \frac{1}{\sqrt{W_{1\eta}^2 + W_{2\eta}^2 + W_{3\eta}^2}} W_{\eta} \quad (\text{no sum on } \eta ). \]  

(3.20)

For the second case, when two of the three eigenvectors coincide, the above method fails since the adjoint tensor \( W \) is zero for the coinciding eigenvalues. The matrix \( \Omega \) is said to be linear, and \( \Omega \) itself must be a single dyad [7], i.e \( \Omega = ab \). The homogeneous equation becomes
\[ \Omega \cdot \ddot{d} = ab \cdot \dot{d} = 0, \]  

(3.21)
where
\[ b_i = \frac{\Omega_{ij}}{\sqrt{\Omega_{i1}^2 + \Omega_{i2}^2 + \Omega_{i3}^2}} \]  \hspace{1cm} (no sum on \( \eta \)) \hspace{1cm} (3.22)

where \( \eta \) defines a non-zero row. Thus any vector \( \mathbf{d} \) perpendicular to vector \( \mathbf{b} \) is a solution of the homogeneous equation. If the vector \( \mathbf{b} \) is in the direction of propagation, \( \mathbf{i} \), we can take

\[ \mathbf{d}^{n2} = \mathbf{e}_2, \]  \hspace{1cm} (3.23)
\[ \mathbf{d}^{n1} = \mathbf{e}_2 \times \mathbf{i}. \]  \hspace{1cm} (3.24)

For the case that \( \mathbf{b} \) is not in the direction of propagation we will take the polarization of the two coinciding eigenvalues as

\[ \mathbf{d}^{q2} = \mathbf{b} \times \mathbf{i} \]  \hspace{1cm} (3.25)
\[ \mathbf{d}^{q1} = \mathbf{b} \times \mathbf{d}^{q2} \]  \hspace{1cm} (3.26)

### Numerical Solutions

To determine slownesses and polarizations numerically, consider the two coordinate systems shown in Fig. 3.1. The slowness curve in the \( x_1 - x_3 \) plane shown in Fig. 3.1 can be obtained by first rotating the stiffness tensor – which is originally defined in the material coordinate system \((x^m_1, x^m_2, x^m_3)\) – \( \xi \) degrees about the \( x^m_3 \) axis using the transformation rules Eq.(2.35). We then assume that the plane wave direction vector \( \mathbf{I} \) lies in that plane defined by \( x_1 \) and \( x_3 \). The propagation direction in this coordinate system can therefore be defined by the angle \( \theta \). The slowness curve is then constructed by incrementally changing \( \theta \) from \( 0^\circ \) to \( 360^\circ \). For each propagation direction the cubic equation, Eq.(3.4), is constructed and solved for its roots. The polarizations (eigenvectors) are determined by the scheme outlined in the previous section. Figure 3.2 shows the slowness curves for InAs for the plane defined by \( \xi = 0^\circ \). Computing similar
slowness curves in various planes by increasing the angle $\xi$ from $\xi = 0^\circ$ to $180^\circ$ we can construct an entire three dimensional slowness surface as shown in Fig. 3.3 for the qS2 wave of an austenitic steel.

After the determination of the three slowness values and polarizations one has to associate the appropriate mode with the slowness values. This association is needed since the slowness curves for the shear waves can touch (see Fig. 3.2) or cross, so that it is not possible to identify a particular shear mode by simply ordering the sizes of the slownesses. For the materials used in this thesis it is sufficient to associate the fastest mode with the qP-wave and distinguish the qS1-wave from the qS2-wave by a polarization argument.

Since all modes have to be mutually orthogonal it is possible to use the dot-product of the current eigenvectors with the eigenvectors for a previous angle as an sorting criteria. In this procedure, the polarizations of the previous angle are stored and the dot product is taken with polarizations of the current angle. To illustrate the sorting algorithm consider Fig. 3.4. This figure shows small sections of the qS2 and qS1 slowness curves.
Figure 3.2 Slowness curve: CSS $\xi = 0^\circ$; The $qP$-wave is displayed as a solid line, the $qS1$ as dotted and $qS2$ dashed

The $qP$-mode is not shown since, as mentioned earlier, this mode is identified as the fastest mode. For the first angle $\theta = \theta_1$ the quasi-shear eigenvalue with the largest $x_2$ polarization will be labeled $qS1$ consequently the other one $qS2$. The starting angle $\theta_1$ is a generic value (depending on the part of slowness section desired), but if two slowness curves touch at this point it is taken to be the original value plus one degree. If we now increase the angle to $\theta_1$, we form the dot product of the eigenvector at the point $P_1^1$ with the eigenvector at the point $P_2^1$. If the value of the dot product is close to unity the point $P_2^1$ will be labeled $qS1$, otherwise $qS2$. Doing so for all angles $\theta_1 \ldots \theta_n$ will sort the two shear modes. The \texttt{matlab}-code (\texttt{SortEV3d.m}) shows the sorting algorithm in vectorized form.

```matlab
idx = find(abs(dot(P_qS2(:,[1:end-1]),P_qS2(:,[2:end]))) < 0.5);
if ~isempty(idx)
    idx_1 = length(idx);
    if idx_1 == 1
```
Figure 3.3 Slowness surface of the qS2-wave traveling in austenitic steel

```matlab
idx(2) = theta_max;
end
for rr=1:2:(idx_l-1)
    if isempty(idx(rr+1)) == 1
        idx(rr+1) = theta_max;
    end
    idx_v = (idx(rr)+1):idx(rr+1);
    %... First save the information in a temporary variable
    tmp_s = s_qS1(idx_v);
    tmp_P = P_qS1(:,idx_v);
    %... Switch modes and polarization with index idx_v
    s_qS1(idx_v) = s_qS2(idx_v);
    s_qS2(idx_v) = tmp_s;
    P_qS1(:,idx_v) = P_qS2(:,idx_v);
    P_qS2(:,idx_v) = tmp_P;
end
end
```
Group Velocity

Up to now, we have considered the phase velocity or its inverse the slowness. But dealing with anisotropic media it is important to discuss another velocity called the group velocity. Originally the word group velocity has its origin in hydrodynamics [53]. For harmonic waves of frequency $\omega$ and a corresponding wave vector $k$ the group velocity is described mathematically by

$$g = \frac{\partial \omega}{\partial k}.$$  \hfill (3.27)

or in Cartesian coordinates

$$g = \frac{\partial \omega}{\partial k_i} \hat{e}_i.$$  \hfill (3.28)

With $k_i = k l_i$ we can also write the group velocity as

$$g = \frac{\partial v}{\partial l_i} \hat{e}_i.$$  \hfill (3.29)

Consider now Christoffel’s equation in the form

$$C_{ijkl} l_k l_j d_i = \rho v^2 d_i,$$  \hfill (3.30)
Multiplying both sides of Eq.(3.30) by \( d_i \) and taking the derivative \( \partial / \partial l_p \) we get

\[
C_{ijkl}d_i(l_k \frac{\partial}{\partial l_p}l_j + l_j \frac{\partial}{\partial l_p}l_k) = \rho \frac{\partial}{\partial l_p}v^2
\]

\[
C_{ijkl}d_i(l_k \delta_{jp} + l_j \delta_{kp}) = 2\rho v \frac{\partial}{\partial l_p}v
\]

\[
C_{ijkl}d_i l_k + C_{ijpj}dl_j = 2\rho v \frac{\partial}{\partial l_p}v
\]

\[
C_{ijkl}l_k dl = \rho v \frac{\partial}{\partial l_p}v. \tag{3.31}
\]

We can use the definition of the group velocity (Eq.(3.29)) to obtain

\[
g_j = \frac{1}{\rho v} C_{ijkl}l_k dl_i. \tag{3.32}
\]

In chapter 4 the group velocity will be used to determine the direction of the energy. In this chapter an explicit form of Eq.(3.32) will be given, involving the reduced stiffnesses. It can be shown [44] that the physical meaning of equation (3.32) is that the kinetic energy of a plane wave is equal to the stiffened strain energy. It can also be shown [39] that the direction of the group velocity coincides with that of the energy flow. This knowledge can be used to compute numerically the group velocity surfaces, also known as energy ray surfaces. The velocity of the energy transport in an anisotropic medium is the group velocity [48]. The group velocity has an interesting physical meaning. Consider two transducers \((T_A, T_B)\) located as depicted in Fig. 3.5. If a wave is radiated from transducer \( T_A \) into an anisotropic solid, the beam (highly idealized in Fig. 3.5) will travel along the trajectory shown, which is the group-velocity direction \( \hat{I}_{gr} \). Even so the phase-fronts travel along the phase-velocity direction \( \hat{I} \). As a consequence the receiving transducer \( T_B \) must be offset in order to receive the beam. The angle between the group velocity direction \( \hat{I}_{gr} \) and the phase-velocity direction \( \hat{I} \) is known as the power flow angle \( \psi \) and is given by

\[
\psi = \arccos(\hat{I} \cdot \hat{I}_{gr}), \tag{3.33}
\]

or in Cartesian coordinates

\[
\psi = \arccos[l_j(l_{gr})_j]. \tag{3.34}
\]
Figure 3.5  Beam trajectory in an anisotropic solid

Figure 3.6 shows the power flow angle on the slowness-surface and depicts that the group velocity vector $\mathbf{g}$ is normal to the slowness curve a result that is shown by Nayfeh [39]. One has to realize however that in general $\mathbf{g}$ does not necessarily lie in the $s_1 - s_3$ plane as depicted in Fig. 3.6 since the normal to the slowness surface may have an $s_2$ component.

In an isotropic medium the frequency is proportional to the wave number $k$. Therefore, the group velocity direction will be also the phase velocity direction, resulting to a zero power flow angle. Furthermore the magnitude of the phase velocity $v$ – in a lossless medium – is equal to the magnitude of the group velocity $v_{gr}$.

Figure 3.7 shows the group velocity curves for Graphite-Epoxy and Indium Arsenite.
Figure 3.7 Group velocity curves for Graphite-Epoxy (left) and InAs (right) for $\xi = 0^\circ$

Approximation to the Slowness Surface

The properties of the slowness surface play a key role in the beam models discussed in later chapters. The beam models which are based on the paraxial approximation use the local slopes and curvatures of the slowness curve in the vicinity of a particular direction of propagation (later called a fixed ray direction), as measured in a particular coordinate system. Specifically, in those paraxial models the $x_3$-axis is taken along the fixed ray direction and the $x_1 - x_3$ plane is taken as the plane of incidence (which will be defined in the next chapter). Mathematically, what is needed for those beam models are the values for the first and second order terms in the Taylor series expansion of the slowness $s^\alpha$ expressed in the form

$$S^\alpha(s_1^\alpha, s_2^\alpha, s_3^\alpha) = 0.$$  \hspace{1cm} (3.35)

about the point $(0, 0, s_3^\alpha)$. Since the slowness-surface is not given explicitly but is contained implicitly in the CHRISTOFFEL equation we expand the slowness surface in the form of a TAYLOR-series and determine unknown coefficients numerically from the eigen-
value problem. For example, consider the slowness-surface near the point $P$ as shown in Fig. 3.8. We can compute a **slowness patch** in the vicinity of $P$ – indicated by the mesh in Fig. 3.8– by solving the eigenvalue problem for the directions defined by the points on this mesh. We then transform those points in a coordinate system $(s_1^a, s_2^a, s_3^a)$ where the $s_3^a$-axis is aligned with the vector originating in the origin and pointing to $P$. Furthermore the plane defined by $s_1^a - s_3^a$ will be a plane of incidence. In this new

![Figure 3.8 Slowness patch of InAs with its approximation (white circles)](image)

coordinate system we can write the slowness surface as

$$ s^a = \tilde{s}^a(s_1^a, s_2^a) \approx \tilde{s}^a(0,0) + \frac{\partial \tilde{s}^a}{\partial s_1^a} \bigg|_{(0,0)} s_1^a + \frac{\partial \tilde{s}^a}{\partial s_2^a} \bigg|_{(0,0)} s_2^a + $$

$$ + \frac{1}{2} \left\{ \frac{\partial^2 \tilde{s}^a}{\partial (s_1^a)^2} \bigg|_{(0,0)} (s_1^a)^2 + 2 \frac{\partial^2 \tilde{s}^a}{\partial s_1^a \partial s_2^a} \bigg|_{(0,0)} s_1^a s_2^a + \frac{\partial^2 \tilde{s}^a}{\partial (s_2^a)^2} \bigg|_{(0,0)} (s_2^a)^2 \right\} $$

(3.36)

where we have truncated the Taylor-series after the second order terms. If we now define the parameters $s_0^a, A^a, B^a, C^a, D^a, E^a$ as

$$ s_0^a = \tilde{s}^a(0,0) $$

(3.37)

$$ A^a = \frac{\partial \tilde{s}^a}{\partial s_1^a} \bigg|_{(0,0)} $$

(3.38)
we can express the slowness as

\[ s^\alpha \approx s_0^\alpha + A^\alpha s_1^\alpha + B^\alpha s_2^\alpha + C^\alpha (s_1^\alpha)^2 + D^\alpha s_1^\alpha s_2^\alpha + E^\alpha (s_2^\alpha)^2. \] (3.43)

Here \( s_0^\alpha \) represents the slowness of a plane wave propagating in \( x_3 \) direction, the parameters \( A^\alpha \) and \( B^\alpha \) determine the change of the slowness curve with propagation direction, and hence determine the group velocity. To show how the group velocity is functionally dependent on the parameters \( A^\alpha \) and \( B^\alpha \) we consider

\[ g^\alpha = g^\alpha j^\alpha_{gr} \] (3.44)

with the energy propagation direction defined – in a local plane of incidence coordinate system with the \( x_3 \)-axis along the central ray – by

\[ j^\alpha_{gr} = \frac{\nabla S^\alpha}{|\nabla S^\alpha|}. \] (3.45)

where \( S^\alpha = s^\alpha - \hat{S}^\alpha(s_1^\alpha, s_2^\alpha) \). With the definition of the gradient

\[ \nabla S^\alpha = \frac{\partial S^\alpha}{\partial s_1^\alpha} \hat{e}_1 + \frac{\partial S^\alpha}{\partial s_2^\alpha} \hat{e}_2 + \frac{\partial S^\alpha}{\partial s_3^\alpha} \hat{e}_3 \] (3.46)

and Eqs.(3.38)(3.39) we get for the energy propagation direction

\[ j^\alpha_{gr} = \frac{-A^\alpha \hat{e}_1 - B^\alpha \hat{e}_2 + \hat{e}_3}{\sqrt{(A^\alpha)^2 + (B^\alpha)^2 + 1}} \] (3.47)

The phase velocity is related to the group velocity through the relation

\[ v^\alpha = g^\alpha \cos \psi^\alpha \] (3.48)
which can also be written as

\[ \nu^\alpha i^\alpha \cdot i^\alpha = g^\alpha i^\alpha \cdot i^\alpha = \frac{g^\alpha}{\sqrt{(A^\alpha)^2 + (B^\alpha)^2 + 1}}. \]  

(3.49)

Comparison of Eq.(3.48) with Eq.(3.49) yields for the cosine of the power flow angle

\[ \cos \psi^\alpha = \frac{1}{\sqrt{(A^\alpha)^2 + (B^\alpha)^2 + 1}}. \]  

(3.50)

Substitution of the power flow angle – as defined by Eq.(3.50) – into Eq.(3.48) defines the group velocity to be

\[ g^\alpha = \nu^\alpha \sqrt{(A^\alpha)^2 + (B^\alpha)^2 + 1}. \]  

(3.51)

It can be seen from Eq.(3.51) that the group velocity is \( \sqrt{A^\alpha + B^\alpha + 1} \) larger than the phase velocity.

The parameter \( C^\alpha, D^\alpha \) and \( E^\alpha \) define the curvatures of the slowness, and therefore determine the rate of divergence or convergence of the beam due to diffraction. Those slowness surfaces are widely used in the literature [37], [35], [41]. The local properties of the slowness surface \( A^\alpha, B^\alpha, C^\alpha, D^\alpha, E^\alpha \) near the fixed ray are then determined as follows. Near the fixed ray – specified by \( \xi_0^\alpha, \theta_0^\alpha \) – the slowness surface is sampled \( n \) times as shown in Fig. 3.9. The deviation angles \( \Delta \xi, \Delta \theta \) are taken to be 20 deg to get a good long range approximation as discussed by MINACHI [37]. The slowness values for all of these angles are computed. The slowness vectors are then transformed into ray-coordinates \( (s_1^\alpha, s_2^\alpha, s_0^\alpha) \) where an \( n \times 5 \) overdetermined system of equations is assembled

\[
\begin{align*}
S^{\gamma\alpha}(\xi_0^\alpha, \theta_0^\alpha) - s_0^{\gamma\alpha}(\xi_0^\alpha, \theta_0^\alpha) \\
S^{\gamma\alpha}(\xi_1^\alpha, \theta_1^\alpha) - s_0^{\gamma\alpha}(\xi_0^\alpha, \theta_0^\alpha) \\
\vdots \\
S^{\gamma\alpha}(\xi_n^\alpha, \theta_n^\alpha) - s_0^{\gamma\alpha}(\xi_0^\alpha, \theta_0^\alpha)
\end{align*}
\]
This overdetermined system, Eq. (3.52), is then solved in a least square sense, which minimizes the sum of the squares of the deviation of the computed slownesses from the fitted slownesses. The assembling of the matrix and the least squares solution is done by the routine mm2dpfit of the Mastering Matlab toolbox from Duane Hanselman [18]. Below are a couple of matlab lines out of the routine GetABCDE.m which determines those parameters:

```matlab
%... Collocate at n-points around cr.Xi
phi = rad(((Xi-dXi)+(j-1).*((Xi+dXi)-(Xi-dXi))./(n-1)));  

% Collocate at n-points around cr.T
theta = rad(((T-dT)+(j-1).*((T+dT)-(T-dT))./(n-1)));  

%... Get Slowness for the points , Returns Slowness for desired mode
[sgmt] = Slow3d(phi,theta);  
```
%... Transform Slowness-Segment into Ray-Coordinates
[sgmt] = crys2ray(sgmt);
%... Fit parabola to slowness
%... Compute Coefficients
p = mm2dpfit(sgmt.sx , sgmt.sy , sgmt.sz,2,2);
%... Pass back the coefficients
s0 = p(3); A = p(4); B = p(6); C = p(5)+1/(2*s0);
D = p(7); E = p(8)+1/(2*s0);

Figure 3.10 shows an example of an approximation to a slowness patch. The white circles represent the approximated slowness, using the best fit parameters on top of the solution of CHRIStOFFEL's equation. As we will see in chapter 5 the parameter $C^\alpha$ and $E^\alpha$ enter in the form $[C^\alpha - 1/(2s_0^2)]$ and $[E^\alpha - 1/(2s_0^2)]$ in paraxial beam models and are responsible for beam focusing effects in anisotropic materials. It is therefore interesting to see the angular dependence of those parameters. Since most of the important variations in these quantities are associated with the qS2 wave for the examples shown here, only the qS2 mode will be discussed. The slowness curves for both austenitic steel and graphite epoxy are shown in Fig. 3.11 and Fig. 3.12. In Fig. 3.13 the parameter
$C^{qS2}$ is displayed as a solid line and $1/(2s_0^{qS2})$ as a dashed line for austenitic steel. At an angle of $\theta = 15^\circ$ the quantity $[C^{qS2} - 1/(2s_0^{qS2})]$ changes its sign from positive to negative and at $75^\circ$ back to positive. The quantity $[E^\alpha - 1/(2s_0^\alpha)]$ only changes its sign once at an angle of $57^\circ$ as seen in Fig. 3.14. Similar plots for the qS2-wave are shown for Graphite-Epoxy in Fig. 3.15 and Fig. 3.16. At points where these curves cross we expect that beam models based on the paraxial approximation will fail so that it is important to know where these points occur on the slowness surface.

Figure 3.11 Right quadrant of the austenitic slowness curve ($\xi = 0^\circ, \eta = 0^\circ$); Displayed are the qP-wave as a solid line, the qS1-wave dotted and the qS2-wave dashed.
Figure 3.12 Right quadrant of the Graphite-Epoxy slowness curve \( (\xi = 0^\circ, \eta = 0^\circ) \); Displayed are the qP-wave as a solid line, the qS1-wave dotted and the qS2-wave dashed.

Figure 3.13 Parameter \( C^{qP} \) and \( 1/(2s_0^{qS2}) \) as a function of the angle \( \theta \) for austenitic steel.
Figure 3.14 Parameter $E^{qS2}$ and $1/(2s_0^{qS2})$ as a function of the angle $\theta$ for austenitic steel.

Figure 3.15 Parameter $C^{qS2}$ and $1/(2s_0^{qS2})$ as a function of the angle $\theta$ for graphite epoxy.
Figure 3.16 Parameter $E^{qS2}$ and $1/(2s_0^{qS2})$ as a function of the angle $\theta$ for graphite epoxy.
4 REFLECTION AND REFRACTION OF OBLIQUE INCIDENT PLANE WAVES ON A PLANE INTERFACE BETWEEN A FLUID HALFSPACE AND AN GENERAL ANISOTROPIC HALFSPACE

Plane wave reflection and refraction from an interface between two isotropic media has been studied intensively in the literature and can be found in numerous books such as those by Schmerr [50], Achenbach [1], Auld [3] and Graff [16]. If we allow one medium - or both media - to be anisotropic, we add further complications to the problem arising from the variation of the slowness with the direction of propagation. Also in the anisotropic medium, the incident, reflected and refracted plane waves do not have in general purely longitudinal or transverse polarizations. Even so the three polarizations are still mutually orthogonal to each other [38]; a fact that can often be used to help solve reflection and transmission problems. Although in later chapters we are interested in modeling the reflection and refraction of bounded beams from finite transducer sources, the plane wave results presented in this section are needed for those models. Reflection and refraction problems in anisotropic media have been discussed by several authors, including Musgrave [38], Fedorov [11], Henneke [19], Rokhlin [46] and Mandal [28]. The main ingredient in the studies of all these workers is the slowness curve, as discussed to some extent in the previous section. However, as we will see in this chapter when discussing reflection and refraction problems, it is more convenient to recast the solution of Christoffel's equation in a different form.
The reflection/refraction problem in anisotropic media we will consider here is for a plane boundary between a fluid and an anisotropic solid, as shown in Fig. 4.1, where the incident wave will be taken to be in the fluid. We let \( s' \) be the incident slowness vector, let \( s'' \) be the reflected slowness vector, and give the slowness for the three refracted waves as \( s^{\alpha P}, s^{\alpha S1}, s^{\alpha S2} \), respectively. The incident and reflected wave can be expressed as

\[
\mathbf{s}^{i,r} = \frac{k}{\omega} \mathbf{i}^{i,r}.
\]

where \( \mathbf{i}^{i,r} \) denotes the propagation direction of a incident or reflected plane wave in a fluid. We will assume that the incident propagation direction, as shown in Fig. 4.1, lies in the \( x_1 - x_3 \)-plane, which is the plane of incidence for this problem, where the plane of incidence is defined as the plane containing both the incident wave direction and the normal to the interface. Due to SNELL's law the other propagation directions \( \mathbf{i}^{r,qP,qS1,qS2} \) will also lie in that plane. If we take the origin of our coordinate system on the interface \( x_3 = 0 \), the intimate contact between the media requires continuity of the normal displacement

\[
u^i_3 + u^r_3 = \sum_{\alpha} u^\alpha_3 \quad : \quad \alpha = qP, qS1, qS2
\]

and the stress component

\[
\sigma^{i}_{33} + \sigma^{r}_{33} = \sum_{\alpha} \sigma^\alpha_{33} \quad ; \quad \alpha = qP, qS1, qS2.
\]

Also, the shear stresses must vanish for the solid at the interface, i.e.

\[
0 = \sum_{\alpha} \sigma^\alpha_{13} \quad ; \quad \alpha = qP, qS1, qS2.
\]

and

\[
0 = \sum_{\alpha} \sigma^\alpha_{23} \quad ; \quad \alpha = qP, qS1, qS2.
\]
Stress Components in the Fluid

The stress components in the fluid can be given by the stress-strain relation for a isotropic media

\[ \sigma_{ij}^{ir} = \lambda \epsilon_{kk}^{ir} a_{ij} + 2\mu \epsilon_{ij}^{(i.r)}, \]  

(4.6)

where \( \mu = 0 \) and \( \lambda = \rho_f v_f^2 \). This gives

\[ \sigma_{33}^{ir} = \rho_f v_f^2 (\epsilon_{11}^{ir} + \epsilon_{22}^{ir} + \epsilon_{33}^{ir}) \]  

(4.7)

As mentioned previously, due to our choice of the coordinate system the plane of incidence (POI) is the \( x_1 - x_3 \) plane. Furthermore, since only longitudinal waves - polarized in the POI - propagate in the fluid, the displacements of the incident and reflected waves
are given by
\[ u_{1}^{i,r} = U^{i,r} d_{1}^{i,r} e^{ik_{m}x_{m}}, \]
\[ u_{2}^{i,r} = 0, \]
\[ u_{3}^{i,r} = U^{i,r} d_{3}^{i,r} e^{ik_{m}x_{m}}, \] (4.8)

where \( k_{m} \) \((m = 1, 3)\) are the only non-zero components of the wavenumber vector in the fluid and throughout we will suppress the common time dependent factor \( \exp(-i\omega t) \).

Since the strains are given by
\[ \epsilon_{11}^{i,r} = \frac{\partial u_{1}^{i,r}}{\partial x_{1}}, \]
\[ \epsilon_{22}^{i,r} = \frac{\partial u_{2}^{i,r}}{\partial x_{2}} = 0 \]
\[ \epsilon_{33}^{i,r} = \frac{\partial u_{3}^{i,r}}{\partial x_{3}} \] (4.9)
we get for the stress in the fluid due to the incident and reflected waves
\[ \sigma_{33}^{i,r} = i\rho f v_{f}^{2} U^{i,r} (k_{1}^{i} d_{1}^{i,r} + k_{3}^{i} d_{3}^{i,r}) e^{ik_{m}x_{m}}. \] (4.10)

Using for the wave vector components
\[ k_{1}^{i} = k_{f} \sin \theta^{i}, \quad k_{1}^{r} = k_{f} \sin \theta^{r}, \]
\[ k_{3}^{i} = k_{f} \cos \theta^{i}, \quad k_{3}^{r} = -k_{f} \cos \theta^{r} \] (4.11)
and for the polarization components
\[ d_{1}^{i} = \sin \theta^{i}, \quad d_{1}^{r} = \sin \theta^{r}, \]
\[ d_{3}^{i} = \cos \theta^{i}, \quad d_{3}^{r} = -\cos \theta^{r} \] (4.12)
with SNELL’s law \( \theta^{i} = \theta^{r} \) we can write for the stress
\[ \sigma_{33}^{i,r} = i\rho f v_{f}^{2} k_{f} U^{i,r} e^{ik_{m}x_{m}}. \] (4.13)

Defining now
\[ D_{3}^{i,r} = \rho f v_{f} \] (4.14)
we arrive at
\[ \sigma_{33}^{i^r} = i\omega D_3^{i^r} U^{i^r} e^{ikmzm}. \] (4.15)
where we used \( k_f = \omega/v_f \).

**Stress Components in the Anisotropic Solid**

For the determination of the \( \sigma_{i3} \) components in the anisotropic media we first express the displacements as
\[ u_k^o = U^\alpha d_k^\alpha e^{ikmzm}. \] (4.16)
and then write down the stress-strain equation for the solid
\[ \sigma_{ij}^o = C_{ijkl} e_{kl}^o \] (4.17)
and the strain-displacement relation
\[ e_{kl}^o = \frac{1}{2} \left( \frac{\partial u_k^o}{\partial x_l} + \frac{\partial u_l^o}{\partial x_k} \right). \] (4.18)
Using the displacements then we can write the stress components as
\[ \sigma_{i3}^o = iC_{i3kl} k_k^o d_l^\alpha U^\alpha e^{ikmzm}. \] (4.19)
With \( k_k^o = \omega s_k^o \) and defining a quantity
\[ D_l^\alpha = C_{i3kl} s_k^o d_l^\alpha, \] (4.20)
we arrive at
\[ \sigma_{i3}^o = i\omega D_l^\alpha U^\alpha e^{ikmzm}. \] (4.21)
The \( D_l^{(\alpha)} \)'s can be given explicitly in terms of the reduced stiffness components as
\[
D_1^o = (s_1^o C_{15} + s_3^o C_{35})d_1^o + (s_1^o C_{36} + s_3^o C_{45})d_2^o + (s_1^o C_{55} + s_3^o C_{35})d_3^o
\]
\[
D_2^o = (s_1^o C_{14} + s_3^o C_{45})d_1^o + (s_1^o C_{46} + s_3^o C_{44})d_2^o + (s_1^o C_{15} + s_3^o C_{34})d_3^o
\]
\[
D_3^o = (s_1^o C_{13} + s_3^o C_{33})d_1^o + (s_1^o C_{36} + s_3^o C_{43})d_2^o + (s_1^o C_{35} + s_3^o C_{33})d_3^o
\] (4.22)
The phase matching at the interface leads to a generalized SNELL's law at the incident side

\[ \sin \theta^i = \sin \theta^r, \]  

(4.23)

and on the transmitted side

\[ s^i_1 = s^r_1. \]  

(4.24)

Substitution of Eq.(4.16) into Eq.(4.2) and Eq.(4.19) into Eq.(4.3), (4.4), (4.5) gives rise to a four by four system of equations.

\[
\begin{align*}
U^i d^i + U^r d^r &= \sum_\alpha U^\alpha d^\alpha \\
0 &= \sum_\alpha U^\alpha D^\alpha_1 \\
0 &= \sum_\alpha U^\alpha D^\alpha_2 \\
D^\alpha_3 &= \sum_\alpha U^\alpha D^\alpha_3.
\end{align*}
\]  

(4.25)

Definition of a reflection coefficient \( R = U^r / U^i \) and three transmission coefficients \( T^\alpha = U^\alpha / U^i \) then yields the following matrix equation

\[
\begin{bmatrix}
d^i_3 & d^s_1 & d^s_2 & d^r_3 \\
D^q_1 & D^q_1 & D^q_1 & 0 \\
D^q_2 & D^q_2 & D^q_2 & 0 \\
D^q_3 & D^q_3 & D^q_3 & D^r_3
\end{bmatrix}
\begin{bmatrix}
T^q_1 \\
T^q_2 \\
T^q_3 \\
R
\end{bmatrix}
= \begin{bmatrix}
d^i_1 \\
0 \\
0 \\
D^i_3
\end{bmatrix}.
\]

(4.26)

where recall, the incident \( x_3 \)-component is given by

\[ d^i_3 = \cos \theta^i, \]  

(4.27)

and for the reflected wave

\[ d^r_3 = -d^i_3. \]  

(4.28)

One can now solve Eq.(4.26) to obtain the reflection and transmission coefficients

\[ R = \frac{-\Delta_1 + \Delta_2}{\Delta_1 + \Delta_2} \]  

(4.29)
\[ T^{qP} = 2 \rho_f v_f \cos \theta^i G^{qP} / \Delta \] (4.30)
\[ T^{qS1} = 2 \rho_f v_f \cos \theta^i G^{qS1} / \Delta \] (4.31)
\[ T^{qS2} = 2 \rho_f v_f \cos \theta^i G^{qS2} / \Delta \] (4.32)

where

\[ \Delta = \Delta_1 + \Delta_2 \] (4.33)
\[ \Delta_1 = (G^{qP} d_3^{qP} + G^{qP} d_3^{qS2} + G^{qS2} d_3^{qS2}) \rho_f v_f \] (4.34)
\[ \Delta_2 = (G^{qP} D_3^{qP} + G^{qP} D_3^{qS2} + G^{qS2} D_3^{qS2}) \cos \theta^i \] (4.35)

and

\[ G^{qP} = D_1^{qS2} D_2^{qS1} - D_1^{qS1} D_2^{qS2} \] (4.36)
\[ G^{qS1} = D_1^{qP} D_2^{qS2} - D_1^{qS2} D_2^{qP} \] (4.37)
\[ G^{qS2} = D_1^{qS1} D_2^{qP} - D_1^{qP} D_2^{qS1} \] (4.38)

To evaluate Eq.(4.29) through Eq.(4.32) the polarizations \((d_1^o, d_2^o, d_3^o)\) and slowness vector values appearing in those equations have to be found. For that we go back to SNELL's law and some of the consequences arising from it. We mentioned earlier that because of SNELL's law all slowness vectors lie in the plane of incidence (POI). It is therefore convenient to select a coordinate system in such a way that the interface plane coincides with the \((x_1; x_2)\) coordinate plane and the POI coincides with the \((x_1; x_3)\) coordinate plane. Therefore, the stiffness tensor \(C_{ijkl}\) has to be transformed from the crystallographic coordinate system \((x_1^m, x_2^m, x_3^m)\) into the POI-coordinate system \((x_1, x_2, x_3)\), using the transformation rules established earlier by Eq.(2.22). In the POI coordinate system all slowness vectors have only two nonzero components \(s_1^o; s_2^o\). We also know that all \(s_1^o\)-components are equal to each other by SNELL's law and are known because \(s_1^i\) for the incident wave is a given input parameter. Thus, the only slowness components which are left to determine are the \(s_3^o\) \((\alpha = qP, qS1, qS2)\). Below we will outline a computational method for the determination of these \(s_3^o\) components. To find \(s_3^o\) it is convenient
to write an alternative expression for CHRISTOFFEL's equation Eq.(2.51), which is here repeated for convenience in the form

\[ G_{il} d_i = 0. \]  

(4.39)

where

\[ G_{il} = c_{ijkl} s_j s_k - \rho \delta_{il}. \]  

(4.40)

Again, for a solution of Eq.(4.40) we need to set \( \det \{ G \} = \det G = 0 \) which can be expanded out as a sixth order equation for \( s_3^\alpha \) since \( s_2^\alpha = 0 \) and \( s_1^\alpha \) is known to give

\[
\begin{align*}
|G| &= |A_1 A_2 A_3| + \left[ |A_1 A_2 B_3| + |A_1 B_2 A_3| + |B_1 A_2 A_3| \right] s_3^\alpha \\
&+ \left[ |A_1 A_2 D_3| + |A_1 B_2 B_3| + |A_1 D_2 A_3| + |B_1 A_2 B_3| + |B_1 B_2 A_3| + |D_1 A_2 A_3| \right] (s_3^\alpha)^2 \\
&+ \left[ |A_1 B_2 D_3| + |A_1 D_2 B_3| + |B_1 A_2 D_3| + |B_1 B_2 B_3| + |B_1 D_2 A_3| + |D_1 A_2 B_3| \right] (s_3^\alpha)^3 \\
&+ \left[ |B_1 D_2 D_3| + |D_1 A_2 D_3| + |D_1 D_2 B_3| + |D_1 D_2 A_3| \right] (s_3^\alpha)^4 \\
&+ \left[ |D_1 D_2 D_3| + |D_1 B_2 D_3| + |D_1 D_2 B_3| + |D_1 D_2 B_3| \right] (s_3^\alpha)^5 \\
&+ |D_1 D_2 D_3| (s_3^\alpha)^6. 
\end{align*}
\]  

(4.41)

where \( A_i, B_i, D_i \) denote the i-th columns of the matrices

\[
A = \begin{pmatrix}
C_{11} (s_1^\alpha)^2 - \rho & C_{16} (s_1^\alpha)^2 & C_{15} (s_1^\alpha)^2 \\
C_{16} (s_1^\alpha)^2 & C_{66} (s_1^\alpha)^2 & C_{56} (s_1^\alpha)^2 \\
C_{15} (s_1^\alpha)^2 & C_{65} (s_1^\alpha)^2 & C_{55} (s_1^\alpha)^2
\end{pmatrix},
\]  

(4.42)

\[
B = \begin{pmatrix}
2C_{15} & C_{16} + C_{56} & C_{13} + C_{55} \\
C_{14} + C_{56} & 2C_{46} & C_{36} + C_{45} \\
C_{13} + C_{55} & C_{36} + C_{45} & C_{35}
\end{pmatrix},
\]  

(4.43)

\[
D = \begin{pmatrix}
C_{55} & C_{45} & C_{35} \\
C_{45} & C_{44} & C_{34} \\
C_{35} & C_{34} & C_{33}
\end{pmatrix}.
\]  

(4.44)
The slowness-component \( s_z \) is an input parameter and is given by
\[
s_z = s_{1f} = \sin \frac{\theta_i}{\nu_f}.
\] (4.45)

For a given \( s_z \) six solutions will be found in the anisotropic medium as shown in Fig. 4.2. To determine which of these solutions represent acceptable solutions for the refracted waves, we must consider separately a number of different cases.

Figure 4.2 Six solutions of the polynomial (4.41) for a given incident angle \( \theta_i \)

If the angle of incidence corresponding to the \( s_z \)-component is smaller than the critical quasi longitudinal angle, \( \theta_{ci}^P \), all six roots will be real (see case 1 (C1) in Fig. 4.3). In this case three waves will be transmitted into the solid. To choose those three roots which lead to a transmitted propagating wave the energy of the associated wave has to have a component in the \(+x_3\) direction. In Fig. 4.2 the vectors correspond to the energy flow
Figure 4.3 Different roots for different incident angles (group velocity) direction $\mathbf{l}^g_{gr}$ of the different wave types are shown. One can see explicitly which roots have a positive $s_3$ components and, hence, lead to transmitted waves. It has to be noted that the sign of the $x_3$- component of the slowness is not sufficient enough for a selection criterion. One easily can find (especially in a material with cubic symmetry) examples where a root has a negative $s_3$-component yet still can be transmitted into the solid because it has a positive $g_3$-component for the group velocity (see Fig. 4.4). Figure 4.5 shows a schematic representation of the three transmitted waves of Fig. 4.2 propagating in the direction of the energy flow $\mathbf{l}^e_{gr}$ and the direction of the phase velocity $\mathbf{l}^e_{ph}$. This phenomena only occurs in the anisotropic case. For a isotropic material the wave fronts will be be always orthogonal to the propagation direction $\mathbf{l}^e_{gr}$, which implies
that the energy flow direction coincides with the phase velocity direction \( u_{gr,iso}^0 = l_{iso}^0 \).

If the incident angle is larger than the critical quasi-longitudinal angle, two of the six roots are complex, and the corresponding waves will be evanescent. To be physical the imaginary part of the complex solution must lead to an exponential decay of the amplitude with distance from the interface; case 2 (C2) in Fig. 4.3 shows such a case.

Increasing further the incident angle and passing the first critical quasi shear wave angle, four of the six roots will become generally complex (see case 3 (C3) in Fig. 4.3). The same criteria have to applied for choosing the three physical roots.

To compute the energy flow in \( x_3 \)-direction we consider Eq. (3.32) and write

\[
g_3^0 = \frac{1}{\rho u^0} C_{ijkl} l_k^0 d_l^0 d_i^0. \tag{4.46}
\]

Expanding the latter equations out and again adopting the contracted notation for the stiffnesses \( C_{ijkl} \), we can write the \( x_3 \)-component of the group velocity as

\[
g_3^0 = \frac{1}{\rho} \left( C_{15} d_1^0 + C_{55} d_2^0 + C_{35} d_3^0 \right) s_1^0 d_i^0 + \left( C_{14} d_1^0 + C_{46} d_2^0 + C_{45} d_3^0 \right) s_1^0 d_i^0 + \]

\[
\left( C_{13} d_1^0 + C_{36} d_2^0 + C_{35} d_3^0 \right) s_1^0 d_i^0 + \left( C_{55} d_1^0 + C_{15} d_2^0 + C_{35} d_3^0 \right) s_3^0 d_i^0 +
\]
Figure 4.5  Energy direction vectors (subscript $gr$) and the phase direction vectors (subscript $ph$) which are normal to the wavefronts corresponding to Fig. 4.2.

\[ (C_{45}d_1^0 + C_{44}d_2^0 + C_{34}d_3^0) s_3^0 d_2^0 + (C_{35}d_1^0 + C_{34}d_2^0 + C_{33}d_3^0) s_3^0 d_3^0 \]  

\[ (4.47) \]

**Numerical Solutions**

The numerical implementation is again done in `matlab`. The subroutine `CalTrCoeff.m` returns the reflection and transmission coefficients for all incident angles in question. The subroutine accepts the incident angles and the material properties — stored in a structure named `material` — as input. The computational procedure is summarized in the following steps:

1. For a given incident angle $\theta^i$ the $s_1$-component is determined by Eq.(4.45). Using this component the coefficients of the sixth-order polynomial are determined (Eq.(4.41)) and the roots of this polynomial, $s_3^0$, are found using the `matlab` func-
tion roots.

2. The wave polarizations corresponding to each root are computed using the procedure outlined in the last chapter. As mentioned there the case where the wave normal lies along an acoustic axis must be treated separately.

3. From the six roots obtained in step 1 only three are physical (either evanescent or propagating). The $x_3$-group velocity component is computed using Eq (4.47), based on the polarizations obtained in step 2, to select the propagating waves which are transmitted into the solid.

4. With the slowness components $(s_1^0, s_2^0)$ and the corresponding polarizations $(d_1^0, d_2^0, d_3^0)$ in the solid, the linear system of Eq.(4.26) is assembled according to (4.20), (4.14), (4.27), (4.28) and solved for the reflection and transmission coefficients.

5. Finally the transmission coefficients are then sorted using the polarization argument given in the previous chapter.

**Reflection and Transmission Coefficients at a Fluid/Transversely Isotropic or Cubic Solid Interface**

The formalism discussed above – valid for triclinic solids – also holds for higher symmetry classes such as monoclinic, orthotropic, transversely isotropic, cubic and isotropic materials. Traveling along symmetry direction of those materials will lead to a decoupling of the SH-wave motion and will further simplify the computation. Especially for transversely isotropic symmetry and higher it will be possible to write down explicitly the $s_3^0$ component since the sixth order polynomial (Eq.(4.41)) will reduce to a quadratic equation in $(s_3^0)^2$. For these cases the reflection and transmission coefficients stated in Eq.(4.29) through Eq.(4.32) can be written explicitly.
The stiffness constants for a transversely isotropic solid, with \( x_1 - x_2 \) plane being the plane of isotropy, are given by (see Appendix A)

\[
\begin{align*}
C_{11} & = C_{22} := C_{11} \\
C_{12} & := C_{12} \\
C_{13} & := C_{23} := C_{13} \\
C_{44} & := C_{55} := C_{44} \\
C_{66} & = (C_{11} - C_{12})/2
\end{align*}
\]

(4.48)

In those cases the plane of isotropy is the \( x_1 - x_2 \) plane and as a consequence the qS1-wave will have a pure horizontal polarization (qS1=SH), i.e

\[
\mathbf{d}^{qS1} = (0, d_2^{SH}, 0).
\]

(4.49)

The other two waves of the type \( \alpha = qP, qS2 \) will be polarized in the POI. i.e

\[
\begin{align*}
\mathbf{d}^{qP} & = (d_1^{qP}, 0, d_3^{qP}) \\
\mathbf{d}^{qS2} & = (d_1^{qS2}, 0, d_3^{qS2})
\end{align*}
\]

(4.50)

Using these polarization components and the stiffness constants for a transversely isotropic material we can write for the D’s

\[
\begin{align*}
D_1^\alpha & = C_{44} (s_1^\alpha d_1^\alpha + s_2^\alpha d_2^\alpha) \\
D_2^\alpha & = 0 \quad \alpha = qP, qS2 \\
D_3^\alpha & = s_1^\alpha C_{13} d_1^\alpha + s_3^\alpha C_{33} d_3^\alpha.
\end{align*}
\]

(4.51)

for the waves polarized in the plane of incidence, and

\[
\begin{align*}
D_1^{SH} & = 0 \\
D_2^{SH} & = s_3^2 C_{44} d_2^\alpha \\
D_3^{SH} & = 0
\end{align*}
\]

(4.52)
for the horizontally polarized wave. Substituting Eqs. (4.51) and (4.52) with the polarizations defined above into Eq. (4.26) we find

\[
\begin{bmatrix}
d_3^{QP} & 0 & d_3^{Q2} & \cos \theta^i \\
d_1^{QP} & 0 & d_1^{Q2} & 0 \\
0 & d_2^{SH} & 0 & 0 \\
d_3^{QP} & 0 & d_3^{Q2} & -\rho_l v_l
\end{bmatrix}
\begin{bmatrix}
T^{QP} \\
T^{SH} \\
T^{Q2} \\
R
\end{bmatrix}
= \begin{bmatrix}
\cos \theta^i \\
0 \\
0 \\
\rho_l v_l
\end{bmatrix}.
\] (4.53)

It can be seen that indeed the qS1=SH wave decouples from the other waves, i.e.

\[
\begin{bmatrix}
d_3^{QP} & d_3^{QS} & -\cos \theta^i \\
d_1^{QP} & d_1^{QS} & 0 \\
d_3^{QP} & d_3^{QS} & -\rho_l v_l
\end{bmatrix}
\begin{bmatrix}
T^{QP} \\
T^{QS} \\
R
\end{bmatrix}
= \begin{bmatrix}
\cos \theta^i \\
0 \\
\rho_l v_l
\end{bmatrix},
\] (4.54)

where we wrote qS=qS2 for simplicity. In fact, from Eq. (4.53) we see that \( T^{SH} = 0 \) so that the transmitted SH-wave is zero. This system of equations can be solved using Cramer's rule to give

\[
R = \frac{-\Delta_1 + \Delta_2}{\Delta_1 + \Delta_2} \quad (4.55)
\]

\[
T^{QP} = -2\rho_l v_l \cos \theta^i D_1^{QS} / \Delta \quad (4.56)
\]

\[
T^{QS} = 2\rho_l v_l \cos \theta^i D_1^{QP} / \Delta \quad (4.57)
\]

where

\[
\Delta = \Delta_1 + \Delta_2 \quad (4.58)
\]

\[
\Delta_1 = (D_1^{QP} d_3^{QS} - D_1^{Q2} d_3^{QP}) \rho_l v_l \quad (4.59)
\]

\[
\Delta_2 = (D_1^{QP} D_3^{QS} - D_1^{Q2} D_3^{QP}) \cos \theta^i \quad (4.60)
\]

To determine the \( s_3^o \) component we substitute the stiffnesses Eq. (4.48) into the sixth order polynomial Eq. (4.41). For the cases of symmetry considered here that sixth order polynomial reduces to

\[
|G| = (s_3^o)^4 + K_1(s_3^o)^2 - K_2K_3 \quad (4.61)
\]
where

\[ K_1 = - \left( \frac{C_{11}}{C_{44}} + \frac{C_{44}}{C_{33}} - \left( \frac{C_{13} + C_{44}}{C_{33}C_{44}} \right)^2 \right) s_1^3 + \frac{\rho_s}{C_{44}} + \frac{\rho_s}{C_{33}} \]  

\[ K_2 = \frac{C_{11}}{C_{33}} s_1^3 - \frac{\rho_s}{C_{33}} \]  

\[ K_3 = s_1^3 - \frac{\rho_s}{C_{44}} \]  

(4.62)  

(4.63)  

(4.64)

and \( \rho_s = \rho \) is the density of the solid. Eq.(4.61) is a quadratic equation in \( (s_3^a)^2 \) and can be solved to give

\[ s_3^{qP} = \pm \sqrt{\frac{1}{2} \left( K_1 - \sqrt{K_1^2 - 4K_2K_3} \right)} \]  

\[ s_3^{qS} = \pm \sqrt{\frac{1}{2} \left( K_1 + \sqrt{K_1^2 - 4K_2K_3} \right)} \]  

(4.65)  

(4.66)

Only two of the four solutions will represent physical waves. To decide which two to take, the same arguments apply as discussed previously for the triclinic case. For the real roots one has to choose those solutions with an energy flow direction pointing into the solid. Whereas for complex roots those having an imaginary part leading to exponentially decaying waves in the solid have to be selected. For the waves of type \( \alpha = qP,qS2 \).

CHRISTOFFEL's equation reduces for a transversely isotropic solid to

\[
\begin{bmatrix}
C_{11}(s_1^a)^2 + C_{11}(s_3^a)^2 - \rho_s & (C_{13} + C_{44})s_1^as_3^a \\
(C_{13} + C_{44})s_1^as_3^a & C_{44}(s_1^a)^2 + C_{33}(s_3^a)^2 - \rho_s
\end{bmatrix}
\begin{bmatrix}
d_1^a \\
d_3^a
\end{bmatrix} = 0.
\]

(4.67)

For given \( s_3^a \) the homogeneous system of equations can be solved with the requirement \(|d^a| = 1\) to give for the \( qP \)-polarization

\[ d_1^{qP} = \frac{C_{33}(s_3^{qP})^2 + C_{44}(s_1^{qP})^2 - \rho_s}{\sqrt{(C_{11}(s_1^{qP})^2 + C_{44}(s_3^{qP})^2 - \rho_s) + (C_{33}(s_3^{qP})^2 + C_{44}(s_1^{qP})^2 - \rho_s)}} \]  

(4.68)

\[ d_3^{qP} = \frac{C_{11}(s_3^{qP})^2 + C_{44}(s_1^{qP})^2 - \rho_s}{\sqrt{(C_{11}(s_1^{qP})^2 + C_{44}(s_3^{qP})^2 - \rho_s) + (C_{33}(s_3^{qP})^2 + C_{44}(s_1^{qP})^2 - \rho_s)}} \]  

(4.69)

and for the \( qS \)-wave

\[ d_1^{qS} = \frac{C_{33}(s_3^{qS})^2 + C_{44}s_1^{qS} - \rho_s}{\sqrt{(C_{11}(s_1^{qS})^2 + C_{44}(s_3^{qS})^2 - \rho_s) + (C_{33}(s_3^{qS})^2 + C_{44}(s_1^{qS})^2 - \rho_s)}} \]  

(4.70)
\[ d_{3}^{S} = -\sqrt{\frac{C_{11}(s_{1}^{S})^{2} + C_{44}(s_{3}^{S})^{2} - \rho_{s}}{(C_{11}(s_{1}^{S})^{2} + C_{44}(s_{3}^{S})^{2} - \rho_{s}) + (C_{33}(s_{3}^{S})^{2} + C_{44}(s_{3}^{S})^{2} - \rho_{s})}}. \] (4.71)

Those analytical expressions for the reflection and transmission coefficient will also hold for cubic symmetry.

For an fluid/isotropic solid interface the reflection and transmission coefficients are given by Eq.(4.55) through Eq.(4.57) are still valid. The slowness components for the transmitted P-wave are given by

\[ s_{i}^{P} = 1/v_{f} \sin \theta^{i} \]  
\[ s_{3}^{P} = 1/v_{P} \cos \theta^{P} \]  

with the displacements

\[ d_{1}^{P} = \sin \theta^{P} \]  
\[ d_{3}^{P} = \cos \theta^{P} \]  

since the P-wave will be polarized in the direction of propagation. The slowness and polarization components for the transmitted SV-wave (polarized in the POI) are given by

\[ s_{i}^{SV} = 1/v_{f} \sin \theta^{i} \]  
\[ s_{3}^{SV} = 1/v_{SV} \cos \theta^{SV} \]  

the displacements by

\[ d_{1}^{SV} = \cos \theta^{SV} \]  
\[ d_{3}^{SV} = -\sin \theta^{SV} \]  

and the refracted angles are determined by Snell's law

\[ \frac{\sin \theta^{i}}{v_{f}} = \frac{\sin \theta^{P}}{v_{P}} = \frac{\sin \theta^{SV}}{v_{SV}}. \]  

(4.80)
The D's are given in Eq. (4.51).

The following figures will show the reflection and transmission coefficients for a variety of materials. In all cases the P-wave transmission coefficient will only be plotted up until the first critical angle. Figure 4.6 shows the reflection behavior of a water/steel interface. Since the solid is isotropic only a pure P- and SV wave will propagate. The

![Figure 4.6](image)

Figure 4.6 Reflection coefficient for water/steel interface. Solid line is computed with the anisotropic scheme using isotropic stiffnesses. The analytical isotropic reflection coefficient is plotted as circles.

SH-wave motion is decoupled and its transmission coefficient is zero (see Fig. 4.7). The first critical angle is found at 15 degrees, beyond this angle the transmission coefficient for the SV-wave become complex (see Fig. 4.8) for the transmitted phases). Beyond the second critical angle (27.5 degrees) total reflection will occur as seen in Fig. 4.6. Figures 4.6-4.8 demonstrate that the general numerical scheme for isotropic solids as outlined in this chapter gives results identical to analytical solutions for the isotropic case.
Figures 4.7 Through Fig. 4.11 show a comparison of the general anisotropic scheme with the analytic solution found earlier by Eq. (4.57) for a transversely isotropic solid (centrifugally cast stainless steel (CSS)) having the axis of symmetry aligned with the $x_3$ axis. Again the agreement of the general scheme with the analytic solution is excellent. The shape of the curves are very similar to the isotropic case. However as it can be seen in Fig. 4.10 that a second shear wave will appear coming from the same branch of the $qS2$-mode [3]. This anomalous shear behavior occurs in the range 23° to 27° for the CSS. In this range two solutions of the sixth order polynomial will be real, resulting in two propagating waves.

Finally Fig. 4.12 through Fig. 4.17 show the reflection and transmission coefficients for a Graphite-Epoxy (GrEp) solid. For Figs. 4.12-4.14 the fibers of the unidirectional
GrEp are aligned with the $x_3$ axis which defines the $x_1 - x_3$ plane as the plane of isotropy. Again as in the case of the CSS the qSH-wave mode will decouple Fig. 4.13. The reflection (Fig. 4.12) and the transmission coefficients Fig. 4.13 show in principle the same behavior as the examples before. The dips near the critical angle are sharper which arises from the fact that the GrEp slowness deviates more from the isotropic one. If we now tilt the material $\eta = 90^\circ$ and define the plane of incidence to be $\zeta = 24^\circ$ a coupling between all three waves types occur. These results are shown in Fig. 4.15-4.16.

Figure 4.8 Transmission coefficients (phase) for water/steel interface. Solid line is computed with the anisotropic scheme using isotropic stiffnesses. The analytical isotropic reflection coefficient is plotted as circles.
Figure 4.9  Reflection coefficient for a water/CSS ($\xi = 0^\circ; \eta = 0^\circ$) interface. Solid line is computed with the anisotropic scheme. The analytical transversely isotropic reflection coefficient are plotted as circles and squares for the different modes.
Figure 4.10 Transmission coefficients (absolute value) for a water/CSS interface ($\xi = 0^\circ; \eta = 0^\circ$). Solid line is computed with the anisotropic scheme. The analytical transversely isotropic transmission coefficients is plotted as circles. The transmission coefficient for the qS1-wave is zero.
Figure 4.11 Transmission coefficients (phase) for a water/CSS interface ($\xi = 0^\circ; \eta = 0^\circ$). Solid line is computed with the anisotropic scheme. The analytical transversely isotropic transmission coefficients are plotted as circles and squares for the different modes. The transmission coefficient for the qS1-wave is zero.
Figure 4.12 Reflection coefficient for water/GrEp ($\xi = 0^\circ; \eta = 0^\circ$) interface.

Figure 4.13 Transmission coefficients (absolute value) for water/GrEp interface ($\xi = 0^\circ; \eta = 0^\circ$)
Figure 4.14 Transmission coefficients (phase) for water/GrEp interface
($\xi = 0^\circ; \eta = 0^\circ$)

Figure 4.15 Reflection coefficient for water/GrEp ($\xi = 24^\circ; \eta = 90^\circ$)
Figure 4.16 Transmission coefficients (absolute value) for water/GrEp interface ($\xi = 24^\circ; \eta = 90^\circ$)
Figure 4.17 Transmission coefficients (phase) for water/GrEp interface ($\xi = 24^\circ; \eta = 90^\circ$)
5 BOUNDARY DIFFRACTION WAVE MODEL

Brief Overview

The primary purpose of this thesis is to develop models of the sound fields generated in immersion NDE inspections, where a transducer is located in a fluid and radiates through an interface into an anisotropic solid. Since it is difficult to have a single, general model to handle all possible cases, three different models will be developed, each of which is appropriate for certain applications.

In this chapter, we will develop a boundary diffraction wave model which is a generalization of similar models developed for isotropic materials by Schmerr [50] for the case where the interface is planar and the solid is a one of general anisotropy. This boundary diffraction wave model represents the radiated field as a superposition of direct and edge wave terms and relies on the paraxial approximation to obtain expressions for the radiated field in terms of simple line integrals that can easily be evaluated numerically. This model can handle planar transducers of various shapes and can serve as the basis of fast simulations.

Like all models based on the paraxial approximation, this boundary diffraction wave model can however fail under certain conditions such as in the very near field. Our model for an anisotropic material may also fail at points on the slowness surface where the curvatures are large or for points in the solid where a focal point is developed (due to material focusing effects). Although these cases where the boundary diffraction wave model breaks down are rather special and are not likely to occur often in practice, they
must be identified if the model is to be used reliably.

In the following chapter, a multiple Gaussian beam model will be developed which removes some of the limitations of the boundary diffraction wave model. This model also is based on the paraxial approximation and represents the fields as a superposition of a small number of Gaussian beams. Unlike the boundary diffraction wave model, the multiple Gaussian model is designed to handle both general curved interfaces as well as a general anisotropic solid. The multiple Gaussian model can fail for points on the interface where the surface curvature may be rapidly varying and for points on the slowness surface where the curvature is rapidly changing. But the multiple Gaussian model does not break down at any points in the solid due to focusing effects caused either by the curved interfaces or the material properties. The multiple Gaussian beam model can easily accommodate non-planar (focused) transducers but it is restricted at present to transducers of circular cross-section. This multiple Gaussian model is even more numerically efficient than the boundary diffraction wave model since it only requires a small number of function evaluations rather than a numerical integration.

Finally, in chapter 7, we will consider a more exact model based on an angular plane wave spectrum approach. This model will treat the radiation of a planar transducer through a planar interface into a general anisotropic solid. This model will not rely on the paraxial approximation but will use a two-dimensional numerical integration of an angular plane wave spectrum representation of the fields in the solid. Thus, this model can be used to test the paraxial models and study where the paraxial approximation may fail. This more exact model, however, requires one to perform a two-dimensional numerical integration so that it is considerably slower than the paraxial models.
Boundary Diffraction Wave Model

Figure 5.1 shows the general setup that will be simulated with the boundary diffraction wave model. A plane transducer is located in a fluid and is oriented at an angle to a plane interface between the fluid and a general anisotropic solid. In analyzing this problem, it will be convenient to use a number of different coordinate systems, as shown in Fig. 5.1. One "transducer" coordinate system, which will be identified by a 't' superscript, will be located at a fixed point on the transducer face and will have one of its axes oriented normal to the planar transducer and the other two coordinates in the plane of the transducer face. The normal to the transducer and the normal to the interface together define a plane (which contains both normals) that we will call the plane of incidence. If a point \( x \) in either the fluid or the solid is measured in the 't' coordinate system (i.e. \( x = (x^1, x^2, x^3) \)) then the \( x^3 \) coordinate is measured in the direction normal to the transducer, the \( x^1 \) coordinate is in the plane of incidence and the \( x^2 \) coordinate is in a direction normal to the plane of incidence. Similarly, for a point \( y = (y^1, y^2, y^3) \),

![Figure 5.1](image-url)  
Figure 5.1  Schematic of a piston transducer radiating into a fluid
on the transducer face the $y_1^i$ and $y_2^i$ coordinates are taken to be in and normal to the plane of incidence, respectively. A second "interface" coordinate system will be identified by an 'i' superscript. The origin of this coordinate system will be located at a point on the interface where an incident ray extending normal to the transducer from the origin of the transducer coordinates intersects the interface. In this interface coordinate system the ($x_1^i$, $x_3^i$) coordinates will lie in the plane of incidence with the $x_3^i$ axis taken normal to the interface as shown in Fig. 5.1, and the $x_2^i$ coordinate will be normal to that plane. Finally a third coordinate system will be a refracted 'ray' coordinate system denoted by a 'r' superscript. The origin of this coordinate system will be the same as for the interface coordinates with the ($x_1^r$, $x_2^r$) coordinates again lying in the plane of incidence, but where now the $x_3^r$ axes is oriented along a refracted ray whose direction is obtained from the incident ray direction and SNELL's law. Note that these refracted ray coordinates therefore depend on the type of transmitted wave being considered by the model. Again, the $x_2^r$ coordinate is normal to the plane of incidence. The distance from the transducer to the interface is given by the water path $D_r$. The angle of incidence $\theta^i$ is given by the angle formed by the transducer normal $n^i$ and the $x_3^i$-axis. The refracted angle $\theta^o$ on the other hand is given by the angle between the refracted base ray and the $x_3^i$-axis.

The pressure field of an piston transducer, at a location $x$ in the fluid can be represented as a superposition of spherical waves over the entire aperture of the transducer, and is given in the form of the RAYLEIGH-SOMMERFELD integral [50]

$$p(x, \omega) = \frac{-i\omega \rho v_0}{2\pi} \int \int \frac{e^{ik|x-y|}}{|x-y|} dS_r. \quad (5.1)$$

Since we are later interested in predicting the wave fields on the other side of a planar interface it is useful to represent the spherical waves in terms of an angular spectrum of
plane waves by employing WEYL's [65] formula

\[
e^{i k_f |x-y|} \frac{e^{i k_f |x-y|}}{|x-y|} = \frac{i}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i k_f (x-y)}}{k_{3f}^2} \, dk_{1f} \, dk_{2f}.
\]  

(5.2)

where \( k_f = \{ k_{1f}, k_{2f}, k_{3f} \} \) in the 'transducer' coordinates (see Fig. 5.1). The derivation of WEYL’s integral is straightforward using spatial Fourier transforms [50]. In Eq.(5.2) \( k_{3f} \) is given by

\[
k_{3f} = \begin{cases} 
\sqrt{\frac{1}{2} (k_{1f}^2 + (k_{2f}^2 - k_f^2) \quad : \quad k_f^2 < (k_{1f}^2)^2 + (k_{2f}^2)^2} \\
\sqrt{\frac{1}{2} (k_{1f}^2 - (k_{2f}^2)^2 - (k_{1f}^2)^2) \quad : \quad k_f^2 > (k_{1f}^2)^2 + (k_{2f}^2)^2}
\end{cases}
\]

(5.3)

It can be seen from Eq.(5.3) that for some parts of the \( (k_{1f}, k_{2f}) \) integration the plane waves are inhomogeneous. This occurs for \( k_f^2 < (k_{1f}^2)^2 + (k_{2f}^2)^2 \) so that \( k_{3f} \) becomes positive and imaginary; which physically means that the corresponding inhomogeneous waves propagate parallel to the \( x_1 x_2 \) - plane and decrease exponentially in the \( x_3 \) direction, i.e. they become evanescent. Substituting Eq.(5.2) into RAYLEIGH-SOMMERFELD integral Eq.(5.1) then gives

\[
p(\mathbf{x}, \omega) = \frac{\omega \rho_f v_f}{(2\pi)^2} \int \int \int_{-\infty}^{\infty} \left\{ \int e^{i k_f (x-y)} dk_{1f} \, dk_{2f} \right\} dS_f.
\]

(5.4)

Eq.(5.4) is in a form that makes it easy to propagate the transducer wavefields across the interface. To see this, consider an incident plane wave in the form

\[
p = P e^{i k_f (x-y)}.
\]

(5.5)

The displacement vector that this plane wave transmits through the interface into an anisotropic solid is given by [50]

\[
u^\alpha = U^\alpha e^{i (k_{1f}^\alpha (x-y) + (k_{3f}^\alpha - k_{3f}^{\alpha_0}) D_f \cos \theta)},
\]

(5.6)

where the displacement amplitude \( U^\alpha \) can be written in terms of the transmission coefficient \( T^\alpha \) as

\[
U^\alpha = -\frac{T^\alpha P}{i \omega \rho_f v_f} \mathbf{d}^\alpha.
\]

(5.7)
The transmission coefficient here is defined as the ratio of the incident displacement to the transmitted displacement, which is consistent with the coefficients obtained in chapter 4. Since Eq. (5.4) is a superposition of waves similar to that in Eq. (5.5), it follows by superposition that the displacement in the solid is given by

\[ u^a(x, \omega) = \frac{i}{(2\pi)^2} \frac{v_0}{v_f} \tilde{d}^a \int \int_{-\infty}^{\infty} T^a(k_{1f}, k_{2f}, k_{3f}) \frac{e^{i(k^a \cdot (x-y) + (k_{3f}^{k_\alpha} - k_{3f}^{k_\alpha}) D_f \cos \theta)}}{k_{3f}^{k_\alpha}} \, dk_{1f} \, dk_{2f} \, dS_f. \]  

(5.8)

The above equation is an exact expression for the displacement in the solid. However this expression involves a surface and two spectral integrals, and is therefore - in its current form - not practical to evaluate numerically. The next section shows how a combination of the paraxial approximation and the method of stationary phase will allow us to reduce the above integrations to a single line integral that can easily be evaluated numerically. A key point in this reduction is the evaluation of phase of the integral appearing in Eq. (5.8), a topic which will be considered in the next section.

Phase Evaluation at Stationary Phase Point

Consider the phase in Eq. (5.8) and introducing a point on the interface \( x_f \) (see Fig. 5.1), we can rewrite the phase as

\[ \phi^a = \mathbf{k}^a \cdot (x_f - y) + \mathbf{k}^a \cdot (x - x_f) + (k_{3f}^{k_\alpha} - k_{3f}^{k_\alpha}) D_f \cos \theta. \]  

(5.9)

Using Snell’s law we may express the first term in Eq. (5.9) as

\[ \mathbf{k}^a \cdot (x_f - y) = \mathbf{k}_f \cdot (x_f - y) - (k_{3f}^{k_\alpha} - k_{3f}^{k_\alpha}) D_f \cos \theta. \]  

(5.10)

Then we have for the phase

\[ \phi^a = \mathbf{k}^a \cdot (x - x_f) + \mathbf{k}_f \cdot (x_f - y). \]  

(5.11)
Expressing the phase in the \((x_1, x_2, x_3)\) coordinate system (see Fig. 5.1) and defining the vector \(\mathbf{x}_i = D_t \cos \theta^i \mathbf{e}_3\) we can write out the phase in a more explicit form

\[
\phi^\alpha = k_1^\alpha x_1 + k_2^\alpha x_2 + (x_3 - D_f \cos \theta^i)k_3^\alpha + k_3^i D_f \cos \theta^i - k_f \cdot y. \tag{5.12}
\]

As shown in the last chapter \(k_3^\alpha\) is determined by the shape of the slowness surface, which is the solution of an eigenvalue problem, so it is not given in an explicit form for a anisotropic medium. To overcome this problem we expand the slowness surface in a Taylor series about the origin of a coordinate system where the \(x_3\) axis is aligned with the refracted central ray (see Fig. 5.1). As mentioned previously, this is the ray coordinate system \((x_1^r, x_2^r, x_3^r)\). To proceed, we first transform the phase of Eq. (5.12) into the interface coordinate system with the transformation rules for the wave vectors

\[
k_1^\alpha = k_1^{i\alpha},
\]

\[
k_2^\alpha = k_2^{i\alpha},
\]

\[
k_3^\alpha = k_3^{i\alpha},
\]

\[
k_3^r = k_3^{i\alpha}.
\tag{5.13}
\]

and for the spatial coordinates

\[
x_1 = x_1^i + D_t \sin \theta^i.
\]

\[
x_2 = x_2^i,
\]

\[
x_3 = x_3^i + D_t \cos \theta^i. \tag{5.14}
\]

Using those transformations and defining the \(x_1^i - x_3^i\)-plane as the plane of incidence we get for the phase

\[
\phi^\alpha = k_1^{i\alpha} (x_1^i + D_f \sin \theta^i) + k_3^{i\alpha} x_3^i + k_3^i D_f \cos \theta^i - k_f \cdot y. \tag{5.15}
\]

We now transform the wave vector and the spatial components into the ray coordinates system by a rotation of \(\theta^\alpha\) degrees about the \(x_2^i\)-axis. The wave vector components then
transform as

\[ k_1^{i\alpha} = k_1^{r\alpha} \cos \theta^\alpha + k_3^{r\alpha} \sin \theta^\alpha, \]
\[ k_3^{i\alpha} = -k_1^{r\alpha} \sin \theta^\alpha + k_3^{r\alpha} \cos \theta^\alpha, \]  \hfill (5.16)

and the coordinates transform as

\[ x_1^i = x_1^i \cos \theta^\alpha + x_3^i \sin \theta^\alpha, \]
\[ x_3^i = -x_1^i \sin \theta^\alpha + x_3^i \cos \theta^\alpha, \]  \hfill (5.17)

(see Fig. 5.1). After some minor algebra we get for the phase

\[ \phi = k_1^{r\alpha} x_1^i + k_3^{r\alpha} x_3^i + k_1^{r\alpha} D_f \sin \theta^i \cos \theta^\alpha + k_3^{r\alpha} D_f \cos \theta^i \cos \theta^\alpha + k_3^{r\alpha} D_f \sin \theta^i \sin \theta^\alpha - k_f \cdot y. \]  \hfill (5.18)

In Eq. (5.18) the only term not explicitly expressed in ray coordinates is \( k_{3f}^{i\alpha} \). This term can be written – using SNELL's law – as

\[ k_{3f}^{i\alpha} = \sqrt{k_f^2 - (k_1^{i\alpha})^2 - (k_2^{i\alpha})^2}. \]  \hfill (5.19)

Using again the transformation rule, Eq. (5.16), and \( k_2^{i\alpha} = k_2^{r\alpha} \) we can also write

\[ k_{3f}^{i\alpha} = \sqrt{k_f^2 - (k_1^{r\alpha} \cos \theta^\alpha + k_3^{r\alpha} \sin \theta^\alpha)^2 - (k_2^{r\alpha})^2}. \]  \hfill (5.20)

When Eq. (5.20) is substituted into Eq. (5.18) the phase is expressed entirely in terms of wave vector components and coordinates in the ray coordinate system. However the component \( k_3^{r\alpha} \), appearing in Eq. (5.20), is still unknown since it is given implicitly from the CHRISTOFFEL equation. To obtain this form we expand the slowness surface – as discussed in chapter 3 – locally about the refracted ray in a TAYLOR series

\[ s^\alpha = \tilde{S}^\alpha(x_1^{r\alpha}, x_2^{r\alpha}) \approx s_0^\alpha + A^\alpha s_1^{r\alpha} + B^\alpha s_2^{r\alpha} + C^\alpha(s_1^{r\alpha})^2 + D^\alpha s_1^{r\alpha} s_2^{r\alpha} + E^\alpha(s_2^{r\alpha})^2, \]  \hfill (5.21)

and express the slowness surface as

\[ (s^\alpha)^2 = (s_1^{r\alpha})^2 + (s_2^{r\alpha})^2 + (s_3^{r\alpha})^2 \]  \hfill (5.22)
or

\[ s_3^{(a)} = \sqrt{(s^a)^2 - (s_1^{(a)})^2 - (s_2^{(a)})^2}. \]  

(5.23)

If we assume \( s_1^{(a)}, s_2^{(a)} \) are small and expand the square root in a power series, we find

\[ s_3^{(a)} = s_0^{(a)} + s_1^{(a)} s_2^{(a)} \left( 1 - \frac{(s_1^{(a)})^2 + (s_2^{(a)})^2}{2(s_0^{a})^2} + \mathcal{O} \right), \]

(5.24)

and substitute Eq.(5.21) into Eq.(5.24) we find

\[
\begin{align*}
\frac{1}{s_0^{(a)}} & = \frac{1}{s_0^{(a)}} + \frac{1}{s_0^{(a)}} \left[ C^{(a)} - \frac{1}{(2s_0^{a})} \right] (s_1^{(a)})^2 + D^{(a)} s_1^{(a)} D_2^{(a)} + \left[ E^{(a)} - \frac{1}{(2s_0^{a})} \right] (s_2^{(a)})^2 \right] + \mathcal{O}. \quad (5.25)
\end{align*}
\]

If we now neglect the higher order terms and let \( s_m^{(a)} = k_m^{(a)}/\omega \) (m=1,2,3) we obtain

\[
\begin{align*}
\frac{1}{s_3^{(a)}} & = \frac{1}{s_0^{(a)}} + C^{(a)} + D^{(a)} s_1^{(a)} D_2^{(a)} + \left[ E^{(a)} - \frac{1}{(2s_0^{a})} \right] (k_2^{(a)})^2 \right] + \mathcal{O}. \quad (5.26)
\end{align*}
\]

Substituting Eq.(5.26) into Eq.(5.20), expanding the squares and grouping the wave vector components yields

\[
\begin{align*}
k_{3f}^{(i)} & = \{ k_f^{(i)} - \omega^2 (s_0^{a})^2 \sin^2 \theta^a - k_1^{(i) a} [2 \omega s_0^{a} A^{a} \sin^2 \theta^a + \omega s_0^{a} \sin(2\theta^a)] \} \\
& - 2k_2^{(i) a} \omega^2 B^a \sin^2 \theta^a \\
& - (k_1^{(i) a})^2 [ (A^{a})^2 \sin^2 \theta^a + (2C^{a} s_0^{a} - 1) \sin^2 \theta^a + A^{a} \sin(2\theta^a) + \cos^2 \theta^a] \\
& - (k_2^{(i) a})^2 [1 + (B^{a})^2 \sin^2 \theta^a + (2E^{a} s_0^{a} - 1) \sin^2 \theta^a] \\
& - k_1^{(i) a} k_2^{(i) a} [2D^{a} s_0^{a} \sin^2 \theta^a + B^a \sin(2\theta^a) + 2A^a B^a \sin^2 \theta^a] + \cdots \frac{1}{2} \] 
\end{align*}
\]

(5.27)

Expanding now the square root of the latter equation in an power series

\[
\begin{align*}
k_{3f}^{(i)} & = \sqrt{k_f^{(i)} - \omega^2 (s_0^{a})^2 \sin^2 \theta^a \{ 1 - (k_1^{(i) a})^2 [2 \omega s_0^{a} A^{a} \sin^2 \theta^a + \omega s_0^{a} \sin(2\theta^a)] \} \\
& + 2k_2^{(i) a} \omega^2 B^a \sin^2 \theta^a \\
& + (k_1^{(i) a})^2 ((A^{a})^2 \sin^2 \theta^a + (2C^{a} s_0^{a} - 1) \sin^2 \theta^a + A^{a} \sin(2\theta^a) + \cos^2 \theta^a] \\
& + (k_2^{(i) a})^2 [1 + (B^{a})^2 \sin^2 \theta^a + (2E^{a} s_0^{a} - 1) \sin^2 \theta^a] \\
& - k_1^{(i) a} k_2^{(i) a} [2D^{a} s_0^{a} \sin^2 \theta^a + B^a \sin(2\theta^a) + 2A^a B^a \sin^2 \theta^a] + \cdots \frac{1}{2} \] 
\end{align*}
\]
+ \left( k_1^{ra} k_2^{ra} \right)^2 [1 + (B^a)^2 \sin^2 \theta^a + (2E^a s_0^a - 1) \sin^2 \theta^a] \\
+ k_1^{ra} k_2^{ra} [2D^a s_0^a \sin^2 \theta^a + B^a \sin(2\theta^a) + 2A^a B^a \sin^2 \theta^a] / 2(k_f^2 - \omega^2(s_0^a)^2 \sin^2 \theta^a) \\
- \left( k_1^{ra} \right)^2 [2\omega s_0^a A^a \sin^2 \theta^a + \omega s_0^a \sin(2\theta^a)]^2 + 4\omega^2(k_2^{ra})^2(s_0^a)^2(B^a)^2 \sin^4 \theta^a \\
+ 4k_1^{ra} k_2^{ra} [2\omega s_0^a A^a \sin^2 \theta^a + \omega s_0^a \sin(2\theta^a)] \omega s_0^a B^a \sin^2 \theta^a + \cdots \times \\
\frac{1}{8(k_1^2 - \omega^2(s_0^a)^2 \sin^2 \theta^a)^2 + \cdots}, \quad (5.28)

and substituting that result and Eq.(5.26) into Eq.(5.18) we obtain after considerable algebra

\begin{align*}
\phi^a &= k_f D_f + \omega s_0^a x_0^r - k_f \cdot y + (x_1^r + A^a x_3^a) k_1^{ra} + (x_2^r + B^a x_3^a) k_2^{ra} \\
&+ \frac{1}{\omega} \left( C^a - \frac{1}{2s_0^a} \right) x_3^r - \frac{D_f}{2k_f} \left( A^a \sin \theta^a + \cos \theta^a \right)^2 \\
&+ \frac{1}{\omega} \left( E^a - \frac{1}{2s_0^a} \right) x_3^r - \frac{D_f}{2k_f} \left( 1 + (B^a)^2 \sin^2 \theta^a \right) \left( k_2^{ra} \right)^2 \\
&+ \frac{D^a}{\omega} x_3^r - \frac{D_f}{2k_f} 2A^a B^a \sin^2 \theta^a + \omega \left( B^a \sin(2\theta^a) \right) \omega s_0^a B^a \sin^2 \theta^a + \cdots \right) \left( k_1^{ra} k_2^{ra} \right). \quad (5.29)
\end{align*}

Equation (5.29) expresses the phase term of Eq.(5.8) in an explicit form which was obtained by expanding the exact phase expression to second order about a fixed refracted ray direction in the solid. This expression is the paraxial approximation of the phase, which physically is a statement that all the waves in the solid travel approximately normal from the transducer face along a ray path into the solid which satisfies Snell's law. Consistent with that approximation we keep the amplitude terms in Eq.(5.8) only to first order, i.e. we set $T^a(k_{1f}^a, k_{2f}^a, k_{3f}^a) = T^a(0,0,k_{3f}^a)$ and $k_{3f}^a = k_f$ to give

$$u^a(x,\omega) = \frac{i}{\omega} T_0^a d^a \int \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\theta^a} d^a k_{1f}^a \, d^a k_{2f}^a \right\} dS_T. \quad (5.30)$$

where $T_0^a = T^a(0,0,k_{3f}^a)$ is the transmission coefficient along the fixed ray direction. In the next section we will evaluate Eq.(5.30) by the method of stationary phase. That stationary phase evaluation will require that we obtain the value of the phase term at the stationary phase point. In preparation for the full evaluation of Eq.(5.30) we will
obtain here the stationary phase evaluation of $\phi^\alpha$. Since the integration in Eq.(5.30) is with respect to 'transducer' coordinates it is convenient to transform the wave vector components back into transducer coordinates. For that we express

$$k_{1f} = k_{1}^{l\alpha}$$

(5.31)

with

$$k_{1f} = k_{1f}^{l} \cos \theta^l + k_{3f}^{l} \sin \theta^l$$

$$k_{1}^{l\alpha} = k_{1}^{r\alpha} \cos \theta^\alpha + k_{3}^{r\alpha} \sin \theta^\alpha$$

(5.32)

to get

$$k_{1f}^{l} \cos \theta^l + k_{3f}^{l} \sin \theta^l = k_{1}^{r\alpha} \cos \theta^\alpha + k_{3}^{r\alpha} \sin \theta^\alpha.$$

(5.33)

Substituting now Eq.(5.26) into Eq.(5.33) and considering only linear terms, we obtain

$$k_{1f}^{l} \cos \theta^l + k_{3f}^{l} \sin \theta^l = k_{1}^{r\alpha} \cos \theta^\alpha + k_{3}^{r\alpha} (\omega s_0^\alpha + A^\alpha k_{1}^{r\alpha} + B^\alpha k_{2}^{r\alpha} + H.O.T) \sin \theta^\alpha.$$ 

(5.34)

If we now approximate $k_{3f}^{l} = k_{f}$ and substitute back into Eq.(5.34) we yield after neglecting the higher order terms

$$k_{1f}^{l} \cos \theta^l + k_{f} \sin \theta^l = k_{1}^{r\alpha} \cos \theta^\alpha + k_{3}^{r\alpha} (\omega s_0^\alpha + A^\alpha k_{1}^{r\alpha} + B^\alpha k_{2}^{r\alpha}) \sin \theta^\alpha.$$ 

(5.35)

Applying Snell's law in the form

$$k_{f} \sin \theta^l = \omega s_0^\alpha \sin \theta^\alpha.$$ 

(5.36)

yields after rearranging

$$k_{1}^{r\alpha} = \frac{k_{1f}^{l} \cos \theta^l - k_{2f}^{l} B^\alpha \sin \theta^\alpha}{A^\alpha \sin \theta^\alpha + \cos \theta^\alpha}.$$ 

(5.37)

Similarly we get for

$$k_{2}^{r\alpha} = k_{2f}^{r}.$$ 

(5.38)
Substituting the latter relations (Eq. (5.38), Eq. (5.37)) back into Eq. (5.41) yield finally - after some minor algebra - for the phase

$$\phi^\alpha = k_1 D_f + \omega s_0^\alpha x_3^r - [y_1^r - (x_1^r + A^\alpha_1 x_3^r) \cos \theta^i] k_1^0$$

(5.39)

$$+ \left[ x_2^r - x_1^r \frac{B^\alpha \sin \theta^\alpha}{F^\alpha} + x_3^r \frac{B^\alpha \cos \theta^\alpha}{F^\alpha} \right] k_2^r$$

(5.40)

$$+ \left[ \frac{1}{\omega} \left( C^\alpha - \frac{1}{2 s_0^\alpha} \right) \left( \frac{\cos \theta^i}{F^\alpha} \right)^2 x_3^r - \frac{D_f}{2 k_f} \right] (k_1^r)^2$$

$$+ \left[ \frac{1}{\omega} \left( C^\alpha - \frac{1}{2 s_0^\alpha} \right) \frac{x_3^r}{x_3^r} \frac{(\sin \theta^i)^2}{F^\alpha} + \frac{1}{\omega} \left( E^\alpha - \frac{1}{2 s_0^\alpha} \right) x_3^r - \frac{D^\alpha}{\omega} \frac{x_3^r}{x_3^r} \frac{B^\alpha \sin \theta^\alpha}{F^\alpha} - \frac{D_f}{2 k_f} \right] (k_2^r)^2$$

(5.41)

where $F^\alpha = A^\alpha \sin \theta^\alpha + \cos \theta^\alpha$. To compute the stationary phase point we write the phase in the more compact form

$$\phi^\alpha = k_1 D_f + \omega s_0^\alpha x_3^r - c_1 k_1^0 - c_2 k_2^0 + \frac{a_{11}^\alpha}{2} (k_1^r)^2 + \frac{a_{22}^\alpha}{2} (k_2^r)^2 + a_{12}^\alpha k_1^r k_2^r$$

(5.42)

with

$$a_{11}^\alpha = 2 \left[ \frac{1}{\omega} \left( C^\alpha - \frac{1}{2 s_0^\alpha} \right) \left( \frac{\cos \theta^i}{F^\alpha} \right)^2 x_3^r - \frac{D_f}{2 k_f} \right]$$

$$a_{12}^\alpha = \frac{D^\alpha \cos \theta^i}{F^\alpha} x_3^r - \frac{2}{\omega} \left( C^\alpha - \frac{1}{2 s_0^\alpha} \right) x_3^r \frac{B^\alpha \sin \theta^\alpha \cos \theta^i}{F^\alpha}$$

$$a_{22}^\alpha = 2 \left[ \frac{1}{\omega} \left( C^\alpha - \frac{1}{2 s_0^\alpha} \right) x_3^r \frac{(\sin \theta^i)^2}{F^\alpha} + \frac{1}{\omega} \left( E^\alpha - \frac{1}{2 s_0^\alpha} \right) x_3^r - \frac{D^\alpha}{\omega} \frac{x_3^r}{x_3^r} \frac{B^\alpha \sin \theta^\alpha}{F^\alpha} - \frac{D_f}{2 k_f} \right]$$

$$c_1^\alpha = y_1^r - (x_1^r + A^\alpha x_3^r) \frac{\cos \theta^i}{F^\alpha}$$

$$c_2^\alpha = - \left( x_2^r - y_2^r - x_1^r \frac{B^\alpha \sin \theta^\alpha}{F^\alpha} + x_3^r \frac{B^\alpha \cos \theta^\alpha}{F^\alpha} \right)$$

(5.43)

At the stationary phase points we must set the derivatives of the phase with respect to the wave vectors to be zero, i.e.

$$\frac{\partial}{\partial k_1^r} \phi^\alpha = 0,$$

(5.44)

$$\frac{\partial}{\partial k_2^r} \phi^\alpha = 0.$$
Those conditions lead to the stationary phase point \((k_{1f}^{t^s}, k_{2f}^{t^s})\) defined by

\[
k_{1f}^{t^s} = \frac{a_{12}^0 a_{11}^0 - a_{12}^0 a_{22}^0}{\Delta^\alpha}, \tag{5.46}
\]

\[
k_{2f}^{t^s} = \frac{a_{11}^0 a_{22}^0 - a_{12}^0 a_{12}^0}{\Delta^\alpha}, \tag{5.47}
\]

where

\[
\Delta^\alpha = a_{11}^0 a_{22}^0 - a_{12}^0 a_{12}^0, \tag{5.48}
\]

and the 's' superscript is used here to indicate explicitly the components here are evaluated at the stationary phase point. Substituting the stationary phase point back into the phase of Eq.(5.42) we get after simplification

\[
\phi^\alpha(k_{1f}^{t^s}, k_{2f}^{t^s}) = k_I D_I + \omega s_0 x_I^2 - \frac{(c_1^\alpha)^2 a_{22}^0}{2\Delta^\alpha} + \frac{c_1^\alpha c_2^\alpha a_{12}^0}{\Delta^\alpha} - \frac{(c_2^\alpha)^2 a_{11}^0}{2\Delta^\alpha}. \tag{5.49}
\]

From Eq.(5.43) one can see that the quantities \(a_{11}^0, a_{12}^0, a_{22}^0\) depend on the properties of the slowness surface along the fixed ray and the geometrical quantities (distances, angles) also associated with that ray. The quantities \(c_1^\alpha\) and \(c_2^\alpha\) are both of the form

\[
c_1^\alpha = y_1^i - y_{10}, \tag{5.50}
\]

\[
c_2^\alpha = y_2^i - y_{20} \tag{5.51}
\]

where \((y_1^i, y_2^i)\) are general points on the transducer and

\[
y_{10} = (x_1^i + A^\alpha x_3^i) \frac{\cos \theta^i}{F^\alpha}, \tag{5.52}
\]

\[
y_{20} = x_2^i - x_1^i \frac{B^\alpha \sin \theta^\alpha}{F^\alpha} + x_3^i \frac{B^\alpha \cos \theta^\alpha}{F^\alpha}. \tag{5.53}
\]

can be represented as some fixed point \(y_0 = (y_{10}, y_{20}, 0)\) on the plane of the transducer face (Fig.5.2). If instead of using the coordinates \((y_1^i, y_2^i)\) to represent the location of the general point \(y\), we introduce polar coordinates \((\rho, \varphi)\) centered at \(y_0\), we have

\[
c_1^\alpha = - \left[ (x_1^i + A^\alpha x_3^i) \frac{\cos \theta^i}{F^\alpha} - y_1^i \right] = \rho \cos \varphi \tag{5.54}
\]

\[
c_2^\alpha = - \left[ x_2^i - x_1^i \frac{B^\alpha \sin \theta^\alpha}{F^\alpha} + x_3^i \frac{B^\alpha \cos \theta^\alpha}{F^\alpha} - y_2^i \right] = \rho \sin \varphi \tag{5.55}
\]
With the relations (5.54) and (5.55) the phase then can be written as

$$\phi^\alpha(k_{1f}^\alpha, k_{2f}^\alpha) = k_1 D_t + \omega s_0^2 x_3^f + (K_1^\alpha \cos^2 \varphi + K_2^\alpha \sin \varphi \cos \varphi + K_3^\alpha \sin^2 \varphi) \rho^2 / \Delta^\alpha$$  (5.56)

with

$$K_1^\alpha = -\frac{1}{2} a_{22}^\alpha$$
$$K_2^\alpha = a_{12}^\alpha$$
$$K_3^\alpha = -\frac{1}{2} a_{11}^\alpha$$  (5.57)

For the case of propagation in plane of symmetry $B^\alpha = D^\alpha = 0$ and the fixed point $y_0 = (y_{10}, y_{20}, 0)$ can be given an explicit geometrical interpretation. In this case we have

$$y_{10} = (x_1^i + A^\alpha x_3^i) \frac{\cos \theta^i}{F^\alpha}$$  (5.58)
$$y_{20} = x_2^i.$$  (5.59)

But, as shown later in chapter 6 (see Eqs.(6.213) and (6.214)) for this case we also have $A^\alpha = -\tan \psi^\alpha$ and $F^\alpha = \cos(\theta^\alpha + \psi^\alpha) / \cos \psi^\alpha$ where $\psi^\alpha$ is the angle the group velocity
direction makes with respect to the refracted ray (Fig. 5.3). Thus, Eq. (5.59) gives

\[ y_{10} = (\cos \psi^\alpha x_1^i - \sin \psi^\alpha x_3^i) \frac{\cos \theta^i}{\cos(\theta^\alpha + \psi^\alpha)} \]

\[ = x_1^g \frac{\cos \theta^i}{\cos(\theta^\alpha + \psi^\alpha)} \]

\[ y_{20} = x_2^g \]

where \((x_1^g, x_2^g, x_3^g)\) are coordinates in a coordinate system where the \(x_3^g\) axis is along the group velocity direction. Eq. (5.61) shows that the points \((y_{10}, y_{20})\) are obtained by following a ray from point \(x\) along the group velocity direction in the solid and parallel to the incident ray direction in the fluid (see Fig. 5.3) where only \(y_{10}\) is shown since \(y_{20} = x_2^g\)

Figure 5.3 Geometrical interpretation for the case of propagation in a plane of symmetry
Stationary Phase Evaluation

In the last section we derived the phase at the stationary phase point. Having this phase expression now at our disposal we can continue with the complete stationary phase evaluation of the spectral integrals. For that we return to Eq.(5.30). Since the phase term in the integrand of Eq.(5.30) varies rapidly only certain regions contribute significantly to the integral. The assumption of high frequencies is not very critical in most NDE problems, since the characteristic lengths involved with most transducer applications are many wavelengths long. Assuming now high frequencies we can apply the stationary phase method [50] to approximate the spectral integrals appearing in Eq.(5.30). If we now write Eq.(5.8) in the form

\[ u^\alpha(x, \omega) = \frac{i}{(2\pi)^2} \int_0 T_0^\alpha \oint I \, dS_T. \]  

(5.61)

with

\[ I = \int \int_{-\infty}^{\infty} e^{i\omega \alpha} \, dk_1^1 \, dk_2^1 \]  

(5.62)

and \( \phi^\alpha(k_1^1, k_f) = i k_1^1 \cdot (x - y) + (k_3^1 - k_3^1) D_f \cos \theta, \) we can apply the stationary phase method to the integral, Eq.(5.62) to give

\[ I \approx \frac{2\pi}{\sqrt{|H^\alpha|}} e^{i\phi^\alpha(k_1^1, k_2^1)} e^{i\psi^\alpha}. \]  

(5.63)

Where \( H^\alpha \) is the Hessian of the phase and \( \psi^\alpha \) an additional phase term. \( \phi^\alpha(k_1^1, k_2^1) \) is the phase at the stationary phase point as determined in the previous section and given by Eq.(5.56), but we still need to be determine the Hessian of the phase, \( H^\alpha \), and the additional phase function \( \psi^\alpha \). The Hessian of the phase is defined by

\[ H^\alpha = \frac{\partial^2 \phi^\alpha}{\partial (k_1^1)^2} \frac{\partial^2 \phi^\alpha}{\partial (k_2^1)^2} - \left( \frac{\partial^2 \phi^\alpha}{\partial k_1^1 k_2^1} \right)^2. \]  

(5.64)

With the phase given by Eq.(5.42) and the definition of the determinant, Eq.(5.48), the Hessian can be written as

\[ H^\alpha = a_{11}^\alpha a_{22}^\alpha - (a_{12}^\alpha)^2 = \Delta^\alpha. \]  

(5.65)
The phase $\exp(i\psi^\alpha)$, appearing in the stationary phase approximation, is defined by [57]

$$\psi^\alpha = \frac{\pi}{4} \sigma. \quad (5.66)$$

The function $\sigma$ can be computed, from Eq.(5.42), with

$$\sigma = \text{sgn}\{\sigma_1\} + \text{sgn}\{\sigma_2\} \quad (5.67)$$

$$\sigma_1 = \frac{1}{2} \frac{\partial^2 \phi^\alpha}{\partial (k_{2f})^2} = \frac{1}{2} a_{22}^a = -K_1^a \quad (5.68)$$

$$\sigma_2 = \frac{1}{2} \frac{\partial^2 \phi^\alpha}{\partial (k_{1f})^2} \left( \frac{1}{2} \frac{\partial^2 \phi^\alpha}{\partial (k_{2f})^2} \right)^2 \frac{\partial^2 \phi^\alpha}{\partial (k_{2f})^2} = \left( \frac{1}{2} a_{11}^a - \left( \frac{1}{2} a_{12}^a \right)^2 \right)$$

$$= -K_3^a + \frac{(K_2^a)^2}{8K_1^a}, \quad (5.69)$$

where $\text{sgn}\{\cdot\}$ is the signum function, to give

$$\psi^\alpha = -\frac{\pi}{4} \left( \text{sgn}\{K_1^a\} + \text{sgn}\left\{ K_3^a - \frac{(K_2^a)^2}{8K_1^a} \right\} \right). \quad (5.70)$$

Going now back to Eq.(5.63) and substituting Eq.(5.63) into Eq.(5.61) we obtain

$$u^\alpha(x, \omega) = \frac{i}{2\pi \omega} \frac{v_0}{\sqrt{|\Delta^\alpha|}} T_0 \hat{d}^\alpha e^{i\omega^\alpha} \int_S e^{i\omega^\alpha(k_{1f} - k_{2f})} dS(y) \quad (5.71)$$

To further evaluate the integral in Eq.(5.71) we set up cylindrical coordinates $(\rho, \varphi)$ in the transducer plane about the point $y_0$ as shown in Fig.5.2 and consider three cases separately: (1) Point $x$ in the main beam of the transducer, (2) point $x$ outside this main beam, and (3) point $x$ on the edge of the main beam.

**Case 1: Point $y_0$ on the transducer face.** In this case (see Fig.5.4) $\rho$ varies from zero to the edge value of $\rho_{e}(\varphi))$. Thus we can write Eq.(5.71) in cylindrical coordinates as

$$u^\alpha(x, \omega) = \frac{i}{2\pi \omega} \frac{v_0}{\sqrt{|\Delta^\alpha|}} T_0 \hat{d}^\alpha e^{i\omega^\alpha} e^{i(k_{1f}D_{\rho} + \omega_{\rho}^\rho \rho_{\rho})}$$

$$\times \int_0^{2\pi} \rho e^{iK_1^0 \cos^2 \varphi + K_2^0 \sin \varphi \cos \varphi + K_3^0 \sin^2 \varphi \rho^2 / \Delta^\alpha} d\varphi d\rho \quad (5.72)$$

To evaluate the integral with respect to $\rho$ the quantity
Figure 5.4  Geometry for \( y_0 \) on the transducer face

\[
\phi^a = (K_1^a \cos^2 \varphi + K_2^a \sin \varphi \cos \varphi + K_3^a \sin^2 \varphi)
\]  

is defined so that Eq.(5.72) is simplified to the form

\[
u^a(x, \omega) = \frac{i}{2\pi} \frac{v_0}{\omega \sqrt{|\Delta^a|}} T_0^a d^a e^{i\omega \alpha} e^{i(k_1 D_l + \omega_s \phi^a \phi^a)} \int_0^{2\pi} \rho^a e^{i\omega \phi^a} d\phi^a d\rho.
\]

The \( \rho \) integration can be done exactly to give

\[
u^a(x, \omega) = \frac{i}{2\pi} \frac{v_0}{\omega \sqrt{|\Delta^a|}} T_0^a d^a e^{i\omega \alpha} e^{i(k_1 D_l + \omega_s \phi^a \phi^a)} \int_0^{2\pi} \left( \frac{\Delta^a}{2i\omega \phi^a} - \frac{\Delta^a}{2i\omega \phi^a} e^{i\omega \phi^a \phi^a} \right) d\phi^a
\]

or finally

\[
u^a(x, \omega) = \frac{1}{2\pi} \frac{v_0}{\omega \sqrt{|\Delta^a|}} T_0^a d^a e^{i\omega \alpha} e^{i(k_1 D_l + \omega_s \phi^a \phi^a)} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\phi^a} \left( 1 - e^{i\omega \phi^a \phi^a} \right) d\phi^a
\]

**Case 2:** Point \( y_0 \) outside the transducer face

In this case \( \rho \) is not a single valued function of \( \varphi \) (Fig.5.5). To account for this we divide the edge of the transducer into two parts \( C_+ \) and \( C_- \), where the values of \( \rho \) are single valued functions \( \rho^+ (\varphi) \) and \( \rho^- (\varphi) \), respectively. With this decomposition we can write
the integration in Eq.(5.71) as

$$u^{a}(x, \omega) = \frac{i v_0}{2\pi \omega \sqrt{|A|}} T_0^{a} \dot{d}^{a} e^{i\omega''t} e^{i(k_1 D_t + \omega s^3 \xi')} \int_{\varphi'}^{+\gamma} \int_{\rho^+} \rho \frac{e^{i\omega''t}}{3\sigma^2} d\varphi d\rho.$$  (5.77)

The $\rho$ integration can be performed to give

$$u^{a}(x, \omega) = \frac{i v_0}{2\pi \omega \sqrt{|A|}} T_0^{a} \dot{d}^{a} e^{i\omega''t} e^{i(k_1 D_t + \omega s^3 \xi')} \int_{\varphi'}^{+\gamma} \frac{\Delta^a}{2i\omega} \left( e^{i\omega''t} - e^{i\omega''t} \right) d\varphi.$$  (5.78)

As $\varphi$ increases from $\varphi$ to $\varphi + \gamma$, $C_+$ is traversed counterclockwise while $C_-$ is traversed in a clockwise fashion. If we now reverse the direction of integration on $C_-$ also to counterclockwise, and let $\rho_+ = \rho_+^+$ on $C_+$ and $\rho_+ = \rho_+^-$ on $C_-$ we can write Eq.(5.78) compactly as

$$u^{a}(x, \omega) = -\frac{1}{2} \frac{v_0}{\omega \sqrt{|A|}} T_0^{a} \dot{d}^{a} e^{i\omega''t} e^{i(k_1 D_t + \omega s^3 \xi')} \frac{1}{2\pi} \int_{\varphi} \frac{1}{\Delta^a} \left( -e^{i\omega''t} \frac{\Delta^a}{2\sigma^2} \int_{\varphi} \right) d\varphi$$  (5.79)

where $C = C_+ + C_-$ is the entire edge traversed counterclockwise.

Case 3: Point $y_0$ on the edge of the transducer face (Fig.5.6)
To compute the displacement for this case we consider the limit of \( y_0 \) moving from outside the face of the transducer to the edge. Since the displacement is continuous as we go across the beam we could obtain the same limit coming from the inside. Going back to Eq. (5.79)

\[
\mathbf{u}^a(x, \omega) = -\frac{1}{2} \frac{\Delta^a}{\omega} T_0^a \frac{d^a e^{i\omega \phi^a} e^{i(k_D + \omega \rho^2 z_f)}}{2\pi} \frac{1}{\phi^a} \left( \frac{1}{\phi^a} \right) \exp \left( \frac{-e^{i\phi^a \rho^2}}{2\pi} \right) d\phi
\]

(5.80)

and letting \( y_0 \) be near the transducer edge, we see that on \( C_- = C_\epsilon \) we have \( \rho^- \approx 0 \) and therefore \( \exp(i\rho^-)^2/\Delta^\phi \approx 1 \) so that with \( \rho^+ = \rho \), we get

\[
\mathbf{u}^a(x, \omega) = -\frac{1}{2} \frac{\Delta^a}{\omega} T_0^a \frac{d^a e^{i\omega \phi^a} e^{i(k_D + \omega \rho^2 z_f)}}{2\pi} \frac{1}{\phi^a} \lim_{\epsilon \to 0} \left( -\int_{C_-} \frac{1}{\phi^a} e^{i\phi^a \rho^2} d\phi - \int_{C_\epsilon} \frac{1}{\phi^a} d\phi \right)
\]

(5.81)

Since the angle varies from \( \phi^- + \pi \) to \( \phi^- \) the \( C_\epsilon \) integral can be written as

\[
\int_{C_\epsilon} \frac{1}{\phi^a} d\phi = \int_{\phi^- + \pi}^{\phi^-} \frac{1}{\phi^a(\phi)} d\phi
\]

\[
= -\int_{\phi^-}^{\phi^- + \pi} \frac{1}{\phi^a(\phi)} d\phi
\]
\[
\begin{align*}
&= - \left\{ \int_0^\pi \frac{1}{\phi^\alpha (\varphi)} \, d\varphi + \int_{\pi}^{\varphi^- + \pi} \frac{1}{\phi^\alpha (\varphi)} \, d\varphi - \int_0^{\varphi^-} \frac{1}{\phi^\alpha (\varphi)} \, d\varphi \right\}.
\end{align*}
\]  
(5.82)

Since \(\phi^\alpha (\pi) = \phi^\alpha (0)\) and \(\phi^\alpha (\varphi^- + \pi) = \phi^\alpha (\varphi^-)\) the latter two integrals cancel out and we get

\[
\int \frac{1}{\phi^\alpha} \, d\varphi = - \int_0^\pi \frac{1}{\phi^\alpha (\varphi)} \, d\varphi
\]

\[
= - \frac{1}{2} \int_0^{2\pi} \frac{1}{\phi^\alpha (\varphi)} \, d\varphi,
\]  
(5.83)

and substituting Eq. (5.83) into Eq. (5.81) gives

\[
\begin{align*}
\mathbf{u}^\alpha (x, \omega) &= - \frac{1}{2} \frac{\Delta^\alpha}{\omega} T_0 \mathbf{a} e^{i\omega^0} e^{i(k_x D_t + \omega x_3)} \frac{1}{\phi^\alpha} \left( \frac{1}{2} - e^{i\phi^\alpha (\varphi)} \right) \times 2 \pi \, d\varphi
\end{align*}
\]  
(5.84)

Comparison of Eqs. (5.76), (5.79) and (5.84) suggests that we can combine all three cases by defining the quantity

\[
\Theta = \begin{cases} 
1 & : y_0 \text{ inside the transducer face} \\
0 & : y_0 \text{ outside of the transducer face} \\
\frac{1}{2} & : y_0 \text{ on the edge of the transducer face}
\end{cases}
\]  
(5.85)

leading to the compact form

\[
\begin{align*}
\mathbf{u}^\alpha (x, \omega) &= - \frac{1}{2} \frac{\Delta^\alpha}{\omega} T_0 \mathbf{a} e^{i\omega^0} e^{i(k_x D_t + \omega x_3)} \\
&\times \frac{1}{2\pi} \int_0^{2\pi} \Theta - e^{i\omega^0} \left( C_1^\alpha \cos^2 \varphi + C_2^\alpha \sin \varphi \cos \varphi + C_3^\alpha \sin^2 \varphi \right) d\varphi
\end{align*}
\]  
(5.86)

where

\[
\begin{align*}
C_1^\alpha &= - \left[ \left( C^\alpha - \frac{1}{2s_0^\alpha} \right) x_3 \left( \frac{B^\alpha \sin \theta^\alpha}{F^\alpha} \right)^2 + \left( E^\alpha - \frac{1}{2s_0^\alpha} \right) x_3 - D^\alpha x_3 \frac{B^\alpha \sin \theta^\alpha}{F^\alpha} - \frac{D_\text{tr}_{\text{tr}}}{2} \right] \\
C_2^\alpha &= D^\alpha \frac{\cos \theta^\alpha}{F^\alpha} x_3 - 2 \left( C^\alpha - \frac{1}{2s_0^\alpha} \right) x_3 \frac{B^\alpha \sin \theta^\alpha \cos \theta^\alpha}{F^\alpha} \\
C_3^\alpha &= - \left[ \left( C^\alpha - \frac{1}{2s_0^\alpha} \right) \left( \frac{\cos \theta^\alpha}{F^\alpha} \right)^2 x_3 - \frac{D_\text{tr}_{\text{tr}}}{2} \right] \\
F^\alpha &= A^\alpha \sin \theta^\alpha + \cos \theta^\alpha
\end{align*}
\]  
(5.87)
with
\[ \delta^\alpha = 4C_1^\alpha C_3^\alpha - (C_2^\alpha)^2. \]  
(5.88)

and
\[ \psi^\alpha = -\frac{\pi}{4} \left( \text{sgn}\{C_1^\alpha\} + \text{sgn}\left\{C_3^\alpha - \frac{(C_2^\alpha)^2}{8C_1^\alpha}\right\} \right). \]  
(5.89)

**Reduction to Transversely Isotropic Symmetry**

The expression above is valid for all symmetry classes. If we consider a transversely isotropic solid the anisotropic parameters \( B^\alpha \) and \( D^\alpha \) vanish, i.e.
\[ B^\alpha = D^\alpha = 0. \]  
(5.90)

As a consequence the quantities \( C_1^\alpha, C_2^\alpha, C_3^\alpha \) reduce to
\[ C_1^\alpha = -\left[ \left(E^\alpha - \frac{1}{2\eta^0}\right)x_3^\alpha - \frac{D_1^\alpha v_t}{2}\right] \]
\[ C_2^\alpha = 0 \]
\[ C_3^\alpha = -\left[ \left(\frac{C^\alpha}{2\eta^0}\right)^2 \left(\cos\theta^\alpha\right)^2 x_3^\alpha - \frac{D_1^\alpha v_t}{2}\right] \]  
(5.91)

which can be rewritten as
\[ C_1^\alpha = \frac{1}{2\eta^0} \left[ (-2E^\alpha s_0^\alpha + 1)x_3^\alpha + D_1^\alpha v_t^\alpha \right] \]
\[ C_2^\alpha = 0 \]
\[ C_3^\alpha = \frac{1}{2\eta^0} \left[ (-2C^\alpha s_0^\alpha + 1) \left(\frac{\cos\theta^\alpha}{F^\alpha}\right)^2 x_3^\alpha + D_1^\alpha v_t^\alpha \right] \]  
(5.92)

or
\[ C_1^\alpha = \frac{1}{2} v_t \left[ (1 - 2E^\alpha s_0^\alpha) \frac{v_t^\alpha}{v_t} x_3^\alpha + D_1^\alpha \right] = \frac{1}{2} v_t \Delta_{x}^{\alpha} \]
\[ C_2^\alpha = 0 \]
\[ C_3^\alpha = \frac{1}{2} v_t \left[ (1 - 2C^\alpha s_0^\alpha) \left(\frac{\cos\theta^\alpha}{F^\alpha}\right) x_3^\alpha + D_1^\alpha \right] = \frac{1}{2} v_t \Delta_{x}^{\alpha} \]  
(5.93)
and using Eqs. (5.93) we get
\[ \delta^\alpha = v^2 \Delta_1 \Delta_2. \] (5.94)

The superscript ‘ti’ here indicates the transversely isotropic symmetry. Substitution of
Eq. (5.93) and Eq. (5.94) into Eq. (5.86) gives for the displacement
\[
\mathbf{u}^\alpha(x, \omega) = -\frac{v_0}{\omega} \frac{1}{\sqrt{\Delta_1 \Delta_2}} T_0 \mathbf{d}^\alpha e^{i\omega \mathbf{e} \cdot \mathbf{e}(k_f D_1 + \omega s_2 x_2^2)} \frac{1}{2\pi} \int_0^{2\pi} \Theta - e^{i\frac{1}{2} k_f \left( \frac{\cos^2 \phi}{\Delta_1} + \frac{\sin^2 \phi}{\Delta_2} \right) \rho^2} \sin \phi \sin \theta \sin \phi \sin \theta \int_0^\pi d\phi.
\] (5.95)

where
\[
\Delta_1 = (1 - 2C_2^\alpha s_3^\alpha) \left( \frac{\cos \theta^i}{F^i} \right)^2 \frac{v_0}{v_f} x_3 + D_f \] (5.96)
\[
\Delta_2 = (1 - 2E_2^\alpha s_3^\alpha) \frac{v_0}{v_f} x_3 + D_f \] (5.97)
\[
\psi^\alpha = -\frac{\pi}{4} \left( \text{sgn}\{\Delta_1^\alpha\} + \text{sgn}\{\Delta_2^\alpha\} \right). \] (5.98)

Reduction to Isotropy

If we further increase the level of symmetry, we have for the isotropic case the same
structure for the displacement as in the transversely isotropic case, i.e
\[
\mathbf{u}^\alpha(x, \omega) = -\frac{v_0}{\omega} \frac{1}{\sqrt{\Delta_1 \Delta_2}} T_0 \mathbf{d}^\alpha e^{i\omega \mathbf{e} \cdot \mathbf{e}(k_f D_1 + \omega s_2 x_2^2)} \frac{1}{2\pi} \int_0^{2\pi} \Theta - e^{i\frac{1}{2} k_f \left( \frac{\cos^2 \phi}{\Delta_1^\alpha} + \frac{\sin^2 \phi}{\Delta_2^\alpha} \right) \rho^2} \sin \phi \sin \theta \sin \phi \sin \theta \int_0^\pi d\phi.
\] (5.99)

is still valid. But since we have \( A^\alpha = B^\alpha = C^\alpha = D^\alpha = E^\alpha = 0 \) the isotopic parameters
\( \Delta_1 \) reduce to
\[
\Delta_1^\text{iso} = \left( \frac{\cos \theta^i}{\cos \theta^\alpha} \right)^2 \frac{v_0}{v_f} x_3 + D_f
\]
\[
\Delta_2^\text{iso} = \frac{v_0}{v_f} x_3 + D_f \] (5.100)
since \( F^\alpha = \cos \theta^\alpha \). The additional phase term, \( \psi^\alpha \), reduces to

\[
\psi^\alpha = -\frac{\pi}{4} \left( \text{sgn} \{ \Delta^\text{iso}_x \} + \text{sgn} \{ \Delta^\text{iso}_y \} \right).
\]  

(5.101)

Since \( \Delta^\text{iso}_x \) and \( \Delta^\text{iso}_y \) are always positive for the isotropic case we get

\[
\exp \left( -\frac{i \pi}{2} \right) = -i;
\]  

(5.102)

and we can write the displacement, Eq.(5.99), as

\[
\mathbf{u}^\alpha(x, \omega) = -\frac{\sqrt{\chi}}{i \omega} \frac{1}{\sqrt{\Delta^\text{ii}_x \Delta^\text{ii}_y}} \mathbf{T}_0^\alpha \mathbf{d}^\alpha e^{i(k_l \rho_l + \omega s^\alpha_0 z^\alpha_0)} \frac{1}{2\pi} \int_0^{2\pi} \Theta - e \frac{1}{2} k_l \left( \frac{\cos^2 \phi + \sin^2 \phi}{\Delta^\text{iso}_x} + \frac{\sin^2 \phi}{\Delta^\text{iso}_y} \right) i r^2 d\phi. 
\]  

(5.103)

Eq.(5.103) coincides with the result obtained previously by SCHMERR [50] (Eq.(8.227) in that reference) if we take into account that SCHMERR uses a transmission coefficient, \( T_{12}^\alpha \), based on a stress/pressure ratio instead of the transmission coefficient used here, \( T_0^\alpha \), based on displacement/displacement ratios. Thus, setting

\[
T_0^\alpha = -\frac{\rho TV_l}{\rho_s v_l} T_{12}^\alpha
\]  

(5.104)

we find Eq.(5.103) is identical with Eq.(8.227) of [50].

**Comparison of some Isotropic Results**

In a later chapter the anisotropic BDW model is compared with a variety of models for many different cases. Here we will evaluate the anisotropic BDW model for the isotropic problem considered previously by LERCH [21]. All the examples here model a lucite/steel interface where the shear strength of the Lucite is neglected, allowing the use of a fluid-solid model. The transducer is taken as a 5MHz, 1/2 inch diameter, unfocused probe in all cases. The incident angle was adjusted to produce refracted angles of 45° and 60° for both P-P and P-SV waves. The distance \( D_l \) in the lucite wedge was 1.8 cm. Fig.5.7 and Fig.5.8 show central axis scans at a refracted angle of 60°. Both figures
give responses which are identical in form to those shown by LERCH in his thesis, but since LERCH did not give the normalization factor used in his plots different amplitudes appear in Fig. 5.7 and Fig. 5.8. One should also note that the x-axis in our plots displays the actual distance on the central ray axis $x'_3$ whereas the x-axis used in LERCH [21] is taken along the $x'_3$-axis. Figure 5.9 shows the P-SV wave displacement in a cross axis profile taken 1 cm deep into the elastic solid. The transducer is oriented so its P-SV central axis is refracted at 45 degrees, and Figure 5.9 shows for the same setup a cross axis scan 2 cm deep into the solid. Again both figures show excellent qualitative agreement with the results obtained by LERCH.

![Graph](image)

**Figure 5.7** P-P on-axis scan; 60° into steel
Figure 5.8  P-SV on-axis scan: 60° into steel

Figure 5.9  P-SV cross axis 1 cm deep into steel (θ^SV = 45°)
Figure 5.10  P-SV cross axis 2 cm deep into steel ($\theta_{SV} = 45^\circ$)
6 GAUSSIAN BEAM MODEL

This chapter describes the development of a Gaussian beam model. Throughout the derivation the validity of the Fresnel or paraxial approximation is assumed. Due to this assumption the beam profile radiated by a circular transducer remains Gaussian in all cross-sectional planes. This is in general true for the case of propagation through a curved interface between a fluid and a anisotropic solid. Typical NDE transducers do not produce Gaussian beams but the next chapter will show how to model wave fields of a circular piston transducer by simply superimposing ten Gaussian beam solutions of the type discussed in this chapter. The beam model resulting from this superposition will be referred as the multi-Gaussian beam model (MGB).

The setup for the derivation of the Gaussian beam propagating through a curved fluid-solid interface is given in Fig. 6.1. For the development of Gaussian propagation laws, it is assumed that the beam has in the $x_3 = 0$ plane the cross sectional variation

$$p(x_1, x_2, 0) = p_0 \exp \left[ i \frac{k_f (x_1^2 + x_2^2)}{2q_0} \right] \quad : \quad \frac{1}{q_0} = \frac{1}{R_0} + i \frac{2}{k_f w_0^2},$$

(6.1)

where $w_0$ is the half width of the Gaussian, and $R_0$ is the radius of curvature of the wavefronts. It has to be pointed out that due to the sign convention the beam converges for $R_0 < 0$ and diverges for $R_0 > 0$. see Fig. 6.2.

THOMPSON and LOPES [59] developed a simple propagation rule to propagate such a Gaussian beam through a curved interface. Later THOMPSON and NEWBERRY [41] extended that procedure to the case where the materials on either side of the interface could be anisotropic, and replaced the Gaussians with a set of Gauss-Hermite basis
Figure 6.1 Schematic of a piston transducer radiating into a fluid

Figure 6.2 Schematic illustration of a Gaussian beam
functions. Although the formulation considered in THOMPSON [59] and NEWBERRY [41] were quite general, explicit results of the propagation rule were only developed for special cases (plane of incidence aligned with the principle axes of the surface, propagation along a material plane of symmetry, etc.). Here, we will describe a new approach, based on the use of an angular eikonal, that produces simple, explicit expressions for a Gaussian that propagates across a general curved interface into a general anisotropic solid. Since many of the details for the anisotropic case are nearly identical to that for an isotropic solid, we will first present the Gaussian beam model for that simpler case below.

**Gaussian Beam Incident on an Isotropic Solid Interface**

To consider how a Gaussian beam propagates through an interface, consider a Gaussian pressure profile on the plane \( x_3 = 0 \) given by Eq.(6.1), and compute the 2-D spatial Fourier transformation of that pressure given by

\[
p(k_{1f}^i, k_{2f}^i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x_1^i, x_2^i, 0) e^{-i(k_{1f}^i x_1^i + k_{2f}^i x_2^i)} \, dx_1^i \, dx_2^i.
\]  

(6.2)

Using the known integral formula \[14\]

\[
\int e^{-ax^2 + bx} \, dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}; \quad \Re\{a\} > 0.
\]  

(6.3)

(where \( \Re\{\cdot\} \) indicates "real part of ") and completing the square in the exponent, the 2-D spectrum of the incident pressure can be written as

\[
p(k_{1f}^i, k_{2f}^i) = \frac{2\pi \rho_0 q_0}{k_f} \exp \left[ -i \frac{q_0}{2k_f} ((k_{1f}^i)^2 + (k_{2f}^i)^2) \right].
\]  

(6.4)

Given the spectrum of the pressure distribution at the transducer surface, the spectrum of the pressure at a distance \( D_f \) in the fluid is (see. Fig.6.1)

\[
p^-(k_{1f}^i, k_{2f}^i) = p(k_{1f}^i, k_{2f}^i) e^{ik_{2f}^i D_f},
\]  

(6.5)
so that the spectrum of the pressure in the fluid right in front of the interface, $p^-(k_{1f}^1, k_{2f}^2)$, is explicitly given by

$$p^-(k_{1f}^1, k_{2f}^2) = \frac{2\pi p_0q_0}{k_f} \exp \left[-i \frac{q_0}{2k_f} (i(k_{1f}^1)^2 + (k_{2f}^2)^2)\right] \exp \left[ik_{1f}^1D_f\right]. \tag{6.6}$$

Using the paraxial approximation for $k_{3f}^3$, i.e.

$$k_{3f}^3 \approx k_f \left(1 - \frac{(k_{1f}^1)^2 + (k_{2f}^2)^2}{2(k_f)^2}\right), \tag{6.7}$$

and substituting Eq.(6.7) into Eq.(6.6) gives

$$p^-(k_{1f}^1, k_{2f}^2) = \frac{2\pi p_0q_0}{k_f} \exp \left[-i \frac{q(D_f)}{2k_f} (i(k_{1f}^1)^2 + (k_{2f}^2)^2)\right] \exp \left[ik_fD_f\right]. \tag{6.8}$$

with $q(D_f) = q_0 + D_f$. Since the displacement in the solid at the interface depends on the pressure distribution in the fluid at the interface it is reasonable to assume, following the approach of WALTHER [62], that a linear integral transformation connects the spectrum of the displacement $u^0(k_{1f}^1, k_{2f}^2)$ with the pressure spectrum given by Eq.(6.8), i.e.

$$u^0(k_{1f}^1, k_{2f}^2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K^\alpha(k_{1f}^1, k_{2f}^2, k_{1r}^1, k_{2r}^2) p^-(k_{1f}^1, k_{2f}^2) \, dk_{1r}^1 \, dk_{2r}^2. \tag{6.9}$$

The kernel in the integral transformation can be split up into an amplitude function $A(k_{1f}^1, k_{2r}^2, k_{2r}^2, k_{2r}^2)$ and an path function $\tilde{W}(k_{1f}^1, k_{2r}^2, k_{2r}^2)$

$$K^\alpha(k_{1f}^1, k_{2r}^2, k_{2r}^2, k_{2r}^2) = A(k_{1f}^1, k_{2r}^2, k_{2r}^2, k_{2r}^2) \exp \left[i\tilde{W}(k_{1f}^1, k_{2r}^2, k_{2r}^2, k_{2r}^2)\right]. \tag{6.10}$$

It is seen that to determine the displacement field spectrum in the solid we have to first find the kernel $K^\alpha(k_{1f}^1, k_{2r}^2, k_{2r}^2, k_{2r}^2)$. In the paraxial approximation the contribution to $u^\alpha$ of the amplitude part of $K^\alpha(k_{1f}^1, k_{2r}^2, k_{2r}^2, k_{2r}^2)$ can, as we will see, be determined by simply considering the transmission of a plane wave across a planar interface. However, the path function $\tilde{W}(k_{1f}^1, k_{2r}^2, k_{2r}^2, k_{2r}^2)$ is more complex. To see what requirements $\tilde{W}(k_{1f}^1, k_{2r}^2, k_{2r}^2, k_{2r}^2)$ must satisfy, we connect the pressure in the fluid at a
point $x^i = (x_1^i, x_2^i, x_3^i)$ and the displacement at $x^f = (x_1^f, x_2^f, x_3^f)$ with a similar integral relation

$$u^a(x_1^i, x_2^i, x_3^i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_2^a \left( x_1^i, x_2^i, x_3^i, x_1^f, x_2^f, x_3^f \right) p^-(x_1^i, x_2^i, x_3^i) \, dx_1^i \, dx_2^i. \quad (6.11)$$

This relation can be obtained by substituting Eq.(6.9) into

$$u^a(x_1^i, x_2^i, x_3^i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u^a(k_1^{1\alpha}, k_2^{1\alpha}) e^{i(k_1^{1\alpha} x_1^i + k_2^{1\alpha} x_2^i + k_3^{1\alpha} x_3^i)} \, dk_1^{1\alpha} \, dk_2^{1\alpha}$$

and calculating $p^-(k_1^{1f}, k_2^{1f})$ by means of an inverse Fourier transform. In Eq.(6.11) the $K_2^a$ kernel can then be defined, leading to

$$K_2^a \left( x_1^i, x_2^i, x_3^i, x_1^f, x_2^f, x_3^f \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K^a \left( k_1^{1f}, k_2^{1f}, k_1^{1\alpha}, k_2^{1\alpha} \right)$$

$$\times \exp \left[ i \left( k_1^{1\alpha} x_1^i + k_2^{1\alpha} x_2^i + k_3^{1\alpha} x_3^i \right) - \left( k_1^{1f} x_1^i + k_2^{1f} x_2^i + k_3^{1f} x_3^i \right) \right] \, dk_1^{1\alpha} \, dk_2^{1\alpha} \, dk_1^{1f} \, dk_2^{1f}. \quad (6.13)$$

To see what relations are satisfied by $K_2^a$, we evaluate the quadruple integral by the method of stationary phase. Substituting Eq.(6.10) into Eq.(6.13) gives

$$K_2^a \left( x_1^i, x_2^i, x_3^i, x_2^f \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A^a \left( k_1^{1f}, k_2^{1f}, k_1^{1\alpha}, k_2^{1\alpha} \right)$$

$$\times \exp \left[ i \left( W^a(k_1^{1f}, k_2^{1f}, k_1^{1\alpha}, k_2^{1\alpha}) + \left( k_1^{1\alpha} x_1^i + k_2^{1\alpha} x_2^i + k_3^{1\alpha} x_3^i \right) \right) \right] \, dk_1^{1\alpha} \, dk_2^{1\alpha} \, dk_1^{1f} \, dk_2^{1f}. \quad (6.14)$$

To apply the stationary phase technique to the integration over $k_1^{1\alpha}, k_2^{1\alpha}, k_1^{1f}, k_2^{1f}$ we set the derivative of the phase function with respect to the wave vector components to zero, i.e.

$$\frac{\partial W^a}{\partial k_1^{1f}} - x_1^i - \frac{\partial k_1^{1f}}{\partial x_1^i} x_3^i = 0,$$

$$\frac{\partial W^a}{\partial k_2^{1f}} - x_2^i - \frac{\partial k_2^{1f}}{\partial x_2^i} x_3^i = 0,$$

$$\frac{\partial W^a}{\partial k_1^{1\alpha}} + x_1^i + \frac{\partial k_1^{1\alpha}}{\partial x_1^i} x_3^i = 0,$$

$$\frac{\partial W^a}{\partial k_2^{1\alpha}} + x_2^i + \frac{\partial k_2^{1\alpha}}{\partial x_2^i} x_3^i = 0. \quad (6.15)$$
Those conditions can be rewritten in the form

\[
\begin{align*}
      x_1' &= \frac{\partial \tilde{V}}{\partial k_{1f}} - \frac{\partial k_{3f}^i}{\partial k_{1f}} x_3', \\
      x_2' &= \frac{\partial \tilde{V}}{\partial k_{2f}} - \frac{\partial k_{3f}^i}{\partial k_{2f}} x_3', \\
      x_1^r &= -\frac{\partial \tilde{V}}{\partial k_{1a}^i} - \frac{\partial k_{3a}^i}{\partial k_{1a}^i} x_3^r, \\
      x_2^r &= -\frac{\partial \tilde{V}}{\partial k_{2a}^i} - \frac{\partial k_{3a}^i}{\partial k_{2a}^i} x_3^r. 
\end{align*}
\]  

(6.16)

For the case where the \( x^f \) point is taken to be in the \( x_3 = 0 \) plane and the point \( x^{r,a} \) lies in the \( x^{r,a}_3 = 0 \) plane the stationary conditions become

\[
\begin{align*}
x_1' &= \frac{\partial \tilde{V}}{\partial k_{1f}}, \\
x_2' &= \frac{\partial \tilde{V}}{\partial k_{2f}}, \\
x_1^r &= -\frac{\partial \tilde{V}}{\partial k_{1a}^i}, \\
x_2^r &= -\frac{\partial \tilde{V}}{\partial k_{2a}^i} .
\end{align*}
\] 

(6.17)

To determine the function \( \tilde{V}(k_{1f}, k_{2f}, k_{1a}^{r,a}, k_{2a}^{r,a}) \) explicitly, we need to consider in detail some of the properties of propagation of waves through an interface. Let point \( y \) be a point in the fluid, point \( x \) a point in the solid, and point \( x_i \) a point on the interface. If \( x_0 \) is a fixed point on the interface, then the total change of phase for a wave traveling from \( y \) to \( x \), \( \phi^a \), can be written as (see Fig. 6.3)

\[
\phi^a = k_f \sqrt{(x_s + r_{10}) \cdot (x_s + r_{10})} + k_a^{r,a} \sqrt{(r_{20} - x_s) \cdot (r_{20} - x_s)} 
\] 

(6.18)

where

\[
\begin{align*}
r_{10} &= x_0 - y, \\
r_{20} &= x - x_0.
\end{align*}
\] 

(6.19)
and $x_s$ is the position vector of $x_i$ as measured from $x_0$. The quantities $k_f, k^\alpha$ are the wave numbers for the fluid and a wave of type $\alpha$ in the solid, respectively.

If we fix points $y$ and $x$ the phase is a function of the point $x_s$ only, which can be parameterized as $x_s = x_s(s_1, s_2)$, where $s_\alpha (\alpha = 1, 2)$ are the surface arc length coordinates of $x_s$. Thus, we also have $\phi^\alpha = \phi^\alpha(s_1, s_2)$. If we assume point $x_0$ is a stationary phase point on the interface, then

$$
\frac{\partial \phi^\alpha}{\partial s_\alpha} \bigg|_{x_0} = k_f \frac{\partial x_s}{\partial s_1} \cdot \hat{e}_{r_1} - k^\alpha \frac{\partial x_s}{\partial s_2} \cdot \hat{e}_{r_2} \bigg|_{x_0} = 0
$$

which is just a statement of Snell’s law.

Now, consider the function

$$
W^\alpha = \phi^\alpha + k_f \cdot y - k^\alpha \cdot x
$$

where

$$
k_f = k_f \hat{e}_{r_1},
$$

$$
k^\alpha = k^\alpha \hat{e}_{r_2}.
$$
Geometrically, $W^\alpha$ is the total phase change experienced in going along the path $OO'$ shown in Fig.6.4. Since $k_{3f} = k_{3f}(k_{1f}, k_{2f})$, $k_3^\alpha = k_3^\alpha(k_1^\alpha, k_2^\alpha)$, and for $y$ and $x$ fixed, $W^\alpha = W^\alpha(y, x, k_{1f}, k_{2f}, k_1^\alpha, k_2^\alpha, s_1, s_2)$ in general. However, at a stationary phase point Eq.(6.21) can be considered to be a relationship specifying $s^\alpha = s^\alpha(k_{1f}, k_{2f}, k_1^\alpha, k_2^\alpha)$, so that when the stationary phase condition is satisfied we could write

$$W^\alpha = W^\alpha(k_{1f}, k_{2f}, k_1^\alpha, k_2^\alpha). \quad (6.25)$$

It follows from Eq.(6.22) that at the stationary phase point we have (using Eq.(6.21) to eliminate the terms depending on $\phi^\alpha$)

$$\frac{\partial W^\alpha}{\partial k_{1f}} = y_1 + \frac{\partial k_{3f}}{\partial k_{1f}} y_3$$
$$\frac{\partial W^\alpha}{\partial k_{2f}} = y_2 + \frac{\partial k_{3f}}{\partial k_{2f}} y_3$$
$$\frac{\partial W^\alpha}{\partial k_1^\alpha} = -x_1 - \frac{\partial k_3^\alpha}{\partial k_1^\alpha} x_3$$
$$\frac{\partial W^\alpha}{\partial k_2^\alpha} = -x_2 - \frac{\partial k_3^\alpha}{\partial k_2^\alpha} x_3 \quad (6.26)$$

and if we choose coordinate systems at $y$ and $x$ such that $y = (y_1, y_2, 0)$ and $x = (x_1, x_2, 0)$. as shown in Fig.6.5, Eq.(6.26) reduces to

$$\frac{\partial W^\alpha}{\partial k_{1f}} = y_1$$
$$\frac{\partial W^\alpha}{\partial k_{2f}} = y_2$$
$$\frac{\partial W^\alpha}{\partial k_1^\alpha} = -x_1$$
$$\frac{\partial W^\alpha}{\partial k_2^\alpha} = -x_2. \quad (6.27)$$

In optics, a phase function defined by Eq.(6.22) and satisfying Eq.(6.27) is called an angular eikonal [62]. These are also the same equations satisfied by $\tilde{W}^\alpha$ (see Eqs.(6.17)) so it is reasonable to assume $\tilde{W}^\alpha$ is the angular eikonal (i.e.$\tilde{W}^\alpha = W^\alpha$), a result we will assume henceforth.
Figure 6.4 Geometrical interpretation of $W^\alpha$

Figure 6.5 Location of particular coordinate systems
Since we can write the phase $\phi^\alpha$ in the alternative form

$$\phi^\alpha = k_f \cdot (x_s - y) + k^\alpha \cdot (x - x_s)$$  \hspace{1cm} (6.28)$$

it follows that

$$W^\alpha = (k_f - k^\alpha) \cdot x_s$$  \hspace{1cm} (6.29)$$
is also an expression for the angular eikonal provided the expression in Eq.(6.29) is made stationary with respect to $s_\alpha (\alpha = 1, 2)$. In Appendix C, these stationary conditions are applied to Eq.(6.29) for the case of refraction through a general curved fluid-solid interface for a general anisotropic solid to obtain an explicit expression for $W^\alpha$ in the paraxial approximation (i.e. to second order in $k_{1f}^i, k_{2f}^i, k_{1^\alpha}^i, k_{2^\alpha}^i$). For the isotropic solid considered here, the result can be simply written in terms of 2x2 matrices, $P^\alpha, Q^\alpha$ and $R^\alpha$ in the form

$$W^\alpha(k_{1f}^i, k_{2f}^i, k_{1^\alpha}^i, k_{2^\alpha}^i) = \frac{1}{2} (k_{1f}^i, k_{2f}^i)P^\alpha(k_{1f}^i, k_{2f}^i)^T + (k_{1f}^i, k_{2f}^i)Q^\alpha(k_{1^\alpha}^i, k_{2^\alpha}^i)^T + \frac{1}{2} (k_{1^\alpha}^i, k_{2^\alpha}^i)R^\alpha(k_{1^\alpha}^i, k_{2^\alpha}^i)^T$$  \hspace{1cm} (6.30)$$

with

$$P^\alpha = \frac{1}{\Gamma^\alpha} \begin{bmatrix} h_{22} \cos^2 \theta^i & -h_{12} \cos \theta^i \\ -h_{12} \cos \theta^i & h_{11} \end{bmatrix}$$

$$Q^\alpha = -\frac{1}{\Gamma^\alpha} \begin{bmatrix} h_{22} \cos \theta^i \cos \theta^\alpha & -h_{12} \cos \theta^i \\ -h_{12} \cos \theta^\alpha & h_{11} \end{bmatrix}$$  \hspace{1cm} (6.31)$$

$$R^\alpha = \frac{1}{\Gamma^\alpha} \begin{bmatrix} h_{22} \cos^2 \theta^\alpha & -h_{12} \cos \theta^\alpha \\ -h_{12} \cos \theta^\alpha & h_{11} \end{bmatrix}$$

$$\Gamma^\alpha = (k^\alpha \cos \theta^\alpha - k_f \cos \theta^i)(h_{11}h_{22} - h_{12}^2).$$

**Determination of the Displacement $u^\alpha$ in the Solid at the Interface**

In the last section we derived the angular eikonal as it appears in the integral transformation (6.9). We also determined the pressure spectral distribution at the incident
side of the interface, Eq. (6.8), repeated here for convenience in the form

\[ p^-(k_{1f}, k_{2f}) = \frac{2\pi p_0 q_0}{k_f} \exp[i k_f D_f] \exp \left[ -i \frac{1}{2} (a_t^T G_f a_t) \right] \]

(6.32)

where

\[ a_t = \begin{bmatrix} k_{1f}^r \\ k_{2f}^r \end{bmatrix}, \quad \text{and} \quad G_f = \frac{1}{k_f} \begin{bmatrix} q(D_f) & 0 \\ 0 & q(D_f) \end{bmatrix}. \]

(6.33)

and the \( T \) superscript in Eq. (6.32) denotes the transpose of the vector. To determine the displacement on the refracted side of the interface we perform the integration as required by the integral transformation (6.9). For that we substitute the pressure spectrum Eq. (6.32), the angular eikonal, Eq. (6.30), and the integral kernel, Eq. (6.10), into Eq. (6.9). Defining the vector

\[ a_r = \begin{bmatrix} k_{1f}^{r\alpha} \\ k_{2f}^{r\alpha} \end{bmatrix}, \]

(6.34)

the spectrum of the displacement at the interface is given by

\[ u^\alpha(k_{1f}^{r\alpha}, k_{2f}^{r\alpha}) = \frac{2\pi p_0 q_0}{k_f} \exp[i k_f D_f] \int \int_{-\infty}^{\infty} A(k_{1f}^{l\beta}, k_{2f}^{l\beta}, k_{1f}^{r\alpha}, k_{2f}^{r\alpha}) \]

\[ \times \exp \left[ i \frac{1}{2} \left( (a_t^r)^T (P_f^\alpha - G_f) a_t + 2 a_t^T Q^\alpha a_r + (a_r^T) R^\alpha a_r \right) \right] \] \( \text{d}k_{1f}^{l\beta} \text{d}k_{2f}^{l\beta}. \) (6.35)

However,

\[ a_t^T Q^\alpha a_r = (Q^\alpha a_r)^T a_t \]

(6.36)

so

\[ u^\alpha(k_{1f}^{r\alpha}, k_{2f}^{r\alpha}) = \frac{2\pi p_0 q_0}{k_f} \exp[i k_f D_f] \int \int_{-\infty}^{\infty} A(k_{1f}^{l\beta}, k_{2f}^{l\beta}, k_{1f}^{r\alpha}, k_{2f}^{r\alpha}) \]

\[ \times \exp \left[ i \frac{1}{2} \left( (a_t^r)^T (P_f^\alpha - G_f) a_t + 2 (Q^\alpha a_r)^T a_t + (a_r^T) R^\alpha a_r \right) \right] \] \( \text{d}k_{1f}^{l\beta} \text{d}k_{2f}^{l\beta}. \) (6.37)

If we let \( \tilde{P}_f^\alpha = (P_f^\alpha - G_f) \) and define a vector \( q^\alpha \) by

\[ q^\alpha = a_t + (\tilde{P}_f^\alpha)^{-1} Q^\alpha a_r, \]

(6.38)
the terms in the exponent linear in \( a_r \) can be removed, leaving Eq.(6.37) as

\[
\begin{align*}
\mathbf{u}^\alpha(k_1^{r\alpha}, k_2^{r\alpha}) &= i \frac{2\pi \rho_0 q_0}{k_f} e^{ik_t D_t} \exp \left[ i \frac{1}{2} \left( (a_r)^T \mathbf{R}^\alpha a_r - (a_r)^T (Q^\alpha)^T (\tilde{P}^\alpha)^{-1} Q^\alpha a_r \right) \right] \\
\times &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(k_{1f}^{t\alpha}, k_{2f}^{t\alpha}, k_1^{r\alpha}, k_2^{r\alpha}) \exp \left[ i \frac{1}{2} \left( (q^\alpha)^T \tilde{P}^\alpha q^\alpha \right) \right] \, dk_{1f}^{t\alpha} \, dk_{2f}^{t\alpha}.
\end{align*}
\]  

(6.39)

Letting

\[
\mathcal{B}(k_1^{r\alpha}, k_2^{r\alpha}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(k_{1f}^{t\alpha}, k_{2f}^{t\alpha}, k_1^{r\alpha}, k_2^{r\alpha}) \exp \left[ i \frac{1}{2} \left( (q^\alpha)^T \tilde{P}^\alpha q^\alpha \right) \right] \, dk_{1f}^{t\alpha} \, dk_{2f}^{t\alpha} 
\]

(6.40)

the displacement spectrum is then given by

\[
\begin{align*}
\mathbf{u}^\alpha(k_2^{r\alpha}, k_2^{r\alpha}) &= i \frac{2\pi \rho_0 q_0}{k_f} \exp \left[ ik_t D_t \right] \exp \left[ -i \frac{1}{2} \left( (a_r)^T \mathbf{G}^\alpha a_r \right) \right] \mathcal{B}(k_1^{r\alpha}, k_2^{r\alpha}),
\end{align*}
\]

(6.41)

where we defined

\[
\mathbf{G}^\alpha = (Q^\alpha)^T (P^\alpha - G_f)^{-1} Q^\alpha - R^\alpha.
\]

(6.42)

We have now derived an expression for the displacement spectrum at the interface in the solid, but the function \( \mathcal{B} \) is still unknown. In order to determine this function we need to transform the spectrum on both sides of the interface back into spatial coordinates. To accomplish these transformations as a first step the displacement in the solid is written as the two dimensional \textsc{fourier} transformation

\[
\begin{align*}
\mathbf{u}^\alpha(x_1^r, x_2^r, 0) &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{u}^\alpha(k_1^{r\alpha}, k_2^{r\alpha}) e^{i(k_1^{r\alpha} x_1^r + k_2^{r\alpha} x_2^r)} \, dk_1^{r\alpha} \, dk_2^{r\alpha}.
\end{align*}
\]

(6.43)

If we now approximate \( \mathcal{B}(k_1^{r\alpha}, k_2^{r\alpha}) \) to first order\(^1\), i.e.

\[
\mathcal{B}(k_1^{r\alpha}, k_2^{r\alpha}) \approx \mathcal{B}(0, 0) := \mathcal{B}_0.
\]

(6.44)

and substituting Eq.(6.41) into the Eq.(6.43) we get

\[
\mathbf{u}^\alpha(x_1^r, x_2^r, 0) = i \frac{\rho_0 q_0}{2\pi k_f} \mathcal{B}_0 e^{ik_t D_t I}.
\]

(6.45)

\(^1\)A justification will be given later (see discussion on Eq.(6.60))
Then the only remaining integral to calculate can be written as
\[ I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\frac{1}{2}(a_r^T G^\alpha a_r)} e^{i(k_1^{\alpha r} x_1^r + k_2^{\alpha r} x_2^r)} \, dk_1^{\alpha r} \, dk_2^{\alpha r} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\phi_1} e^{\phi_2} \, dk_1^{\alpha r} \, dk_2^{\alpha r}. \] (6.46)

with
\[ \phi_1 = -i \frac{1}{2} (a_r^T G^\alpha a_r) \]
\[ \phi_2 = i (k_1^{\alpha r} x_1^r + k_2^{\alpha r} x_2^r). \] (6.47)

To evaluate the spectral integrals we expand the phase \( \phi_1 \) as
\[ \phi_1 = -i \frac{1}{2} (a_r^T G^\alpha a_r) = -i \frac{1}{2} (G^\alpha_{11} (k_1^{\alpha r})^2 + 2G^\alpha_{12} k_1^{\alpha r} k_2^{\alpha r} + G^\alpha_{22} (k_2^{\alpha r})^2) \] (6.48)
and
\[ \phi = \phi_1 + \phi_2 = -i \frac{1}{2} G^\alpha_{11} (k_1^{\alpha r})^2 - i G^\alpha_{12} k_1^{\alpha r} k_2^{\alpha r} - i \frac{1}{2} G^\alpha_{22} (k_2^{\alpha r})^2 + i x_1^r k_1^{\alpha r} + i x_2^r k_2^{\alpha r}. \] (6.49)

Hence we can write the spectral integrals, Eq.(6.46), as
\[ I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[-i \frac{1}{2} G^\alpha_{11} (k_1^{\alpha r})^2 - i G^\alpha_{12} k_1^{\alpha r} k_2^{\alpha r} - i \frac{1}{2} G^\alpha_{22} (k_2^{\alpha r})^2 + i x_1^r k_1^{\alpha r} + i x_2^r k_2^{\alpha r} \right] \, dk_1^{\alpha r} \, dk_2^{\alpha r}. \] (6.50)
and evaluate them using Eq.(6.3) to obtain
\[ I = \sqrt{\frac{2\pi}{i G^\alpha_{11}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i \frac{1}{2} G^\alpha_{11} (k_1^{\alpha r})^2 + (i x_1^r - i G^\alpha_{12} k_2^{\alpha r}) k_2^{\alpha r}} \, dk_2^{\alpha r} \]
\[ = \sqrt{\frac{2\pi}{i G^\alpha_{11}}} \sqrt{\frac{2\pi G^\alpha_{11}}{i (G^\alpha_{11} G_{22} - (G^\alpha_{12})^2)}} \exp \left[i \frac{(G^\alpha_{12})^2 (x_1^r)^2 + 2G^\alpha_{11} G^\alpha_{12} x_1^r x_2^r - (G^\alpha_{11})^2 x_2^r}{2G^\alpha_{11} (G^\alpha_{11} G_{22} - (G^\alpha_{12})^2)} \right] \] (6.51)

Note that special attention has to be given to the evaluation of the square root terms because of the conditions imposed on the phase terms in Eq.(6.50) by Eq.(6.3). The
phase term of the final expression given in Eq.(6.51) can be conveniently expressed in
matrix notation, by defining
\[
x = \begin{pmatrix} x_1^r \\ x_2^r 
\end{pmatrix},
\] (6.52)
to give for the displacement in the solid at the interface
\[
u^a(x_1^r, x_2^r, 0) = i \frac{p_0 q_0}{k_f} \sqrt{\frac{1}{i G_{11}^a}} \sqrt{\frac{G_{11}^a}{i \left( G_{11}^a G_{22}^a - (G_{12}^a)^2 \right)}} B_0 e^{ik_f D_f e^{i \frac{1}{2} (x^T (G^a)^{-1} x)}},
\] (6.53)
which can be written alternatively as
\[
u^a(x_1^r, x_2^r, 0) = U^a e^{ik_f D_f} \exp \left[ i \frac{1}{2} (x^T (G^a)^{-1} x) \right];
\] (6.54)
with the displacement amplitude given by
\[
U^a = i \frac{p_0 q_0}{k_f} \sqrt{\frac{1}{i G_{11}^a}} \sqrt{\frac{G_{11}^a}{i \left( G_{11}^a G_{22}^a - (G_{12}^a)^2 \right)}} B_0.
\] (6.55)
Similarly, the pressure spectrum on the fluid side at the interface can be transformed
back into space by the means of the 2D Fourier transformation. We have
\[
p^-(x_1^r, x_2^r, x_3^r) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{p}^-(k_1^r, k_2^r) e^{i(k_1^r x_1^r + k_2^r x_2^r)} dk_1^r dk_2^r.
\] (6.56)
Using the pressure spectrum at the interface defined by Eq.(6.8) and applying Eq.(6.3)
the Fourier transformation of Eq.(6.56) can be evaluated as
\[
p^-(x_1^r, x_2^r, x_3^r) = p_0 \frac{q_0}{q(D_f)} \exp [ik_f D_f] \exp \left[ i \frac{k_f ((x_1^2 + (x_2^r)^2)}{2q(D_f)} \right],
\] (6.57)
\[
= P \exp [ik_f D_f] \exp \left[ i \frac{k_f ((x_1^2 + (x_2^r)^2)}{2q(D_f)} \right],
\] (6.58)
where the incident pressure amplitude is defined as
\[
P = p_0 \frac{q_0}{q(D_f)}.
\] (6.59)
and \(q(D_f) = q_0 + D_f\). If we examine Eq.(6.58), we see that to first order (in phase) it
represents merely an incident plane wave on the interface. Thus, to first order \(U^a\) is just
the displacement amplitude of a plane wave transmitted across a plane interface given by

\[ U^\alpha = \frac{T^\alpha P}{i\omega \rho_l v_l} \hat{d}^\alpha; \quad \alpha = P, S \]  \hspace{1cm} (6.60)

where \( \hat{d}^\alpha \) is the polarization of the wave of type \( \alpha \). \( T^\alpha \) denotes the transmission coefficient (based on velocity or displacement ratios) going from medium 1 (fluid) to medium 2 (solid), and \( v_l, \rho_l \) represents the velocity and density in the fluid. This approximation is consistent with the paraxial assumption where we keep amplitude terms to only first order about a fixed ray but retain phase terms to second order.

The coefficient \( B_0 \) can now be found by equating Eq. (6.60) and Eq. (6.55) to obtain

\[ B_0 = -\frac{T^\alpha k_l}{\omega \rho_l v_l q(D_l)} \left[ \frac{1}{iG_{11}^\alpha} \right]^{-\frac{1}{2}} \left[ \frac{G_{11}^\alpha}{i(G_{11}^\alpha G_{22}^\alpha - (G_{12}^\alpha)^2)} \right]^{-\frac{1}{2}} \hat{d}. \]  \hspace{1cm} (6.61)

Substituting Eq. (6.61) into Eq. (6.41), the displacement spectrum in the solid at the interface can be written finally as

\[ u^\alpha(k_1^{\alpha}, k_2^{\alpha}) = -i \frac{2\pi p_0 q_0 T^\alpha}{\omega \rho_l v_l q(D_l)} \left[ \frac{1}{iG_{11}^\alpha} \right]^{-\frac{1}{2}} \left[ \frac{G_{11}^\alpha}{i(G_{11}^\alpha G_{22}^\alpha - (G_{12}^\alpha)^2)} \right]^{-\frac{1}{2}} \hat{d}^\alpha \times \exp\left[ik_rD_r\right] \exp \left[-i\frac{1}{2}a^r G^\alpha a^r\right]. \]  \hspace{1cm} (6.62)

**Determination of the Displacement \( u^\alpha \) in the Solid**

Equation (6.62) gives the displacement spectrum at the interface. To obtain the displacement field in the elastic solid at a distance \( x_3 \) from the interface we again consider

\[ u^\alpha(x_1^r, x_2^r, x_3^r) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u^\alpha(k_1^{\alpha}, k_2^{\alpha}) e^{i(k_1^{\alpha}x_1^r + k_2^{\alpha}x_2^r + k_3^{\alpha}x_3)} dk_1^{\alpha} dk_2^{\alpha}. \]  \hspace{1cm} (6.63)

Substituting Eq. (6.62) into Eq. (6.63) gives for the displacement field

\[ u^\alpha(x_1^r, x_2^r, x_3^r) = -i \frac{p_0 q_0 T^\alpha}{2\pi \omega \rho_l v_l q(D_l)} \left[ \frac{1}{iG_{11}^\alpha} \right]^{-\frac{1}{2}} \left[ \frac{G_{11}^\alpha}{i(G_{11}^\alpha G_{22}^\alpha - (G_{12}^\alpha)^2)} \right]^{-\frac{1}{2}} \hat{d}^\alpha e^{ik_rD_rI} \]  \hspace{1cm} (6.64)

with

\[ I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\phi_1} e^{\phi_2} dk_1^{\alpha} dk_2^{\alpha}. \]  \hspace{1cm} (6.65)
where
\begin{align*}
\phi_1 &= -\frac{i}{2} a_r^r G^\alpha a_r \\
\phi_2 &= i(k_1^r + k_2^r + k_3^r)
\end{align*}
\tag{6.66}
and \( k_3^r = \sqrt{(k^\alpha)^2 - (k_1^r)^2 - (k_2^r)^2} \) with \( k^\alpha \) being the wave vector in the solid. If we now expand this square root in a power series and keeping only terms up to second order, we can write for the wave vector component
\begin{equation}
k_3^r \approx k^\alpha \left( 1 - \frac{(k_1^r)^2 + (k_2^r)^2}{2(k^\alpha)^2} \right).
\tag{6.67}
\end{equation}
Substituting Eq.(6.67) into the phase expression \( \phi_2 \) of Eq.(6.65) we get for the sum of the phase expressions \( \phi_1 \) and \( \phi_2 \)
\begin{equation}
\phi = \phi_1 + \phi_2 = -\frac{i}{2} \left( G_{11}^\alpha + \frac{x_3^r}{k} \right) (k_1^r)^2 - iG_{12}^\alpha k_1^r k_2^r - \frac{i}{2} \left( G_{22}^\alpha + \frac{x_3^r}{k} \right) (k_2^r)^2 + x_1^r k_1^r + x_2^r k_2^r.
\tag{6.68}
\end{equation}
Defining
\begin{align*}
G_{11}^{z,\alpha} &= G_{11}^\alpha + \frac{x_3^r}{k} \\
G_{22}^{z,\alpha} &= G_{22}^\alpha + \frac{x_3^r}{k} \\
G_{12}^{z,\alpha} &= G_{12}^\alpha
\tag{6.69}
\end{align*}
we can finally express the phase in a form similar to Eq.(6.49), namely
\begin{equation}
\phi = -\frac{i}{2} G_{11}^{z,\alpha} (k_1^r)^2 - iG_{12}^{z,\alpha} k_1^r k_2^r - \frac{i}{2} G_{22}^{z,\alpha} (k_2^r)^2 + ix_1^r k_1^r + ix_2^r k_2^r.
\tag{6.72}
\end{equation}
The integrals in Eq.(6.65) evaluate in exactly the same way as previously shown for Eq.(6.50), with the phase now defined by Eq.(6.72). We find
\begin{equation}
I = \sqrt{\frac{2\pi}{iG_{11}^{z,\alpha}}} \exp \left[ \frac{1}{2} (x^T(G^{z,\alpha})^{-1}x) \right].
\tag{6.73}
\end{equation}
where the vector, \( x \), is defined as.
\begin{equation}
x = \begin{pmatrix} x_1^r \\ x_2^r \end{pmatrix}.
\tag{6.74}
\end{equation}
Again, in evaluating these square roots, one has to ensure that the condition present in Eq.(6.3) is satisfied. If we now substitute Eq.(6.73) into Eq.(6.64) we get for the displacement in the solid at a location $x_3^f$

$$\mathbf{u}^\alpha(x_1^f, x_2^f, x_3^f) = -i \frac{p_0 T^\alpha}{\omega \rho \nu r} \frac{q_0}{q(D_t)} \mathbf{S} \mathbf{d}^\alpha \exp \left[ i(k \nu r + k^\alpha x_3^f) \right] \left[ \frac{1}{2} (\mathbf{x}^T (\mathbf{G}^{\alpha}_1)^{-1} \mathbf{x}) \right]. \quad (6.75)$$

with

$$S = \sqrt{\frac{i G_{11}^{\alpha}}{\sqrt{G_{11}^{\alpha} G_{22}^{\alpha} - (G_{12}^{\alpha})^2}}} \sqrt{\frac{i G_{11}^{\alpha} G_{22}^{\alpha} - (G_{12}^{\alpha})^2}{\sqrt{G_{11}^{\alpha} G_{22}^{\alpha} - (G_{12}^{\alpha})^2}}} \quad (6.76)$$

and using the definitions Eq.(6.69) through (6.71) we can write in matrix notation

$$\mathbf{G}^{\alpha}_1 = \mathbf{G}^\alpha + \frac{x_3^f}{k^\alpha} \mathbf{I} \quad (6.77)$$

where recall

$$\mathbf{G}^\alpha = (\mathbf{Q}^\alpha)^T (\mathbf{P}^\alpha - \mathbf{G}_f)^{-1} \mathbf{Q}^\alpha - \mathbf{R}^\alpha. \quad (6.78)$$

Unfortunately, $\mathbf{G}^\alpha$ expressed in terms of the $\mathbf{P}^\alpha, \mathbf{Q}^\alpha$ and $\mathbf{R}^\alpha$ matrices is not a useful form since it becomes indefinite in the limit as the interface becomes planar ($h_{11} = h_{12} = h_{22} = 0$). This problem, however, can be avoided by simply rewriting the form of $\mathbf{G}^\alpha$ in terms of new matrices $\mathbf{A}^\alpha, \mathbf{B}^\alpha, \mathbf{C}^\alpha$ and $\mathbf{D}^\alpha$ defined by

$$\mathbf{A}^\alpha = - (\mathbf{Q}^\alpha)^{-1} \quad (6.79)$$

$$\mathbf{B}^\alpha = - (\mathbf{Q}^\alpha)^{-1} \mathbf{P}^\alpha \quad (6.80)$$

$$\mathbf{C}^\alpha = - \mathbf{R}^\alpha (\mathbf{Q}^\alpha)^{-1} \quad (6.81)$$

$$\mathbf{D}^\alpha = (\mathbf{Q}^\alpha)^T - \mathbf{R}^\alpha (\mathbf{Q}^\alpha)^{-1} \mathbf{P}^\alpha \quad (6.82)$$

then $\mathbf{G}^\alpha$ can be expressed in a completely non-degenerate form as (see Appendix D)

$$\mathbf{G}^\alpha = (\mathbf{C}^\alpha \mathbf{G}_f - \mathbf{D}^\alpha) (\mathbf{B}^\alpha - \mathbf{A}^\alpha \mathbf{G}_f)^{-1}. \quad (6.83)$$
Substitution of Eq. (6.31) in Eq. (6.79) though Eq. (6.82) yields after some algebra, which is conveniently done in Mathematica,

\[ A^{\alpha} = \frac{k^{\alpha} \cos \theta^{\alpha} - k_{t} \cos \theta^{t}}{\cos \theta^{i} \cos \theta^{\alpha}} \begin{bmatrix} h_{11} & h_{12} \cos \theta^{i} \\ h_{12} \cos \theta^{\alpha} & h_{22} \cos \theta^{i} \cos \theta^{\alpha} \end{bmatrix} \]  

(6.84)

\[ B^{\alpha} = \begin{bmatrix} \cos \theta^{i} \\ \cos \theta^{\alpha} \\ 0 \end{bmatrix} \begin{bmatrix} \cos \theta^{i} \\ \cos \theta^{\alpha} \\ 0 \end{bmatrix} \]  

(6.85)

\[ C^{\alpha} = \begin{bmatrix} \cos \theta^{i} \\ \cos \theta^{\alpha} \\ 0 \end{bmatrix} \begin{bmatrix} \cos \theta^{i} \\ \cos \theta^{\alpha} \\ 0 \end{bmatrix} \]  

(6.86)

\[ D^{\alpha} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]  

(6.87)

and

\[ G_{r} = \frac{1}{k_{f}} \begin{bmatrix} q(D_{t}) & 0 \\ 0 & q(D_{t}) \end{bmatrix} \]  

(6.88)

**Planar Interface**

If we consider the special case of a planar interface the curvatures will all be zero. i.e.

\[ h_{11} = 0 \]

\[ h_{22} = 0 \]

\[ h_{12} = 0. \]  

(6.89)

so that \( A^{\alpha} = 0 \) and the matrix multiplications in Eq. (6.83) yield (with the aid of Mathematica)

\[ G_{z^{\alpha}} = \begin{bmatrix} \frac{q(D_{t}) \cos^{2} \theta^{\alpha}}{k_{f} \cos^{2} \theta^{i}} + \frac{x_{3}^{f}}{k^{\alpha}} & 0 \\ 0 & \frac{q(D_{t})}{k_{f}} + \frac{x_{3}^{f}}{k^{\alpha}} \end{bmatrix} \]  

(6.90)
Similarly the inverse is given by

$$(G^{z,0})^{-1} = k^a \begin{bmatrix} \frac{1}{q(D_t) \frac{\partial}{\partial x^{0}} \left( \frac{\cos^2 \theta_0}{\cos^2 \theta} \right)^2 + x^3} & 0 \\ 0 & \frac{1}{x^3 + \frac{\omega}{v} q(D_t)} \end{bmatrix} ,$$  

so we can write for the displacement, finally

$$u^a(x^i, x^2, x^3) = -i \frac{p^a_0 T^a}{\omega \rho_0 v_t} q_0 \hat{A}^a \exp \left[ i (k^a D_t + k^b x^b) \right] \sqrt{\frac{1}{iC_{11}} \frac{1}{iC_{22}}} \sqrt{\frac{1}{iC_{11}} \frac{1}{iC_{22}}} \exp \left[ i \frac{1}{2} k^a \left( \frac{(x^i_1)^2}{q(D_t) \frac{\partial}{\partial x^{0}} \left( \frac{\cos^2 \theta_0}{\cos^2 \theta} \right)^2 + x^3} + \frac{(x^3_1)^2}{q(D_t) \frac{\partial}{\partial x^{0}} + x^3} \right) \right]$$

**Gaussian Beam Incident on an Anisotropic Solid Interface**

The derivation for the propagation rules for an anisotropic solid follows closely the derivation of the isotropic case. We again invoke the paraxial expression, where we expand the slowness curve to second order close to the base ray, to account for the anisotropic effects. The geometry and the coordinate systems are shown in Fig. 6.1.

**The Angular Eikonal; Anisotropic Case**

As in the isotropic case the kernel of the integral transformation, Eq.(6.9), has to be determined to find the displacement spectrum at the interface in the solid. The derivation of the angle eikonal follows closely the isotropic case. The starting point is again the angle eikonal at the stationary point, as derived in Appendix C, and repeated here for convenience

$$W^a = -\frac{1}{2(h_{11} h_{22} - h_{12}^2) \mathcal{N}} \left[ h_{22} (\mathcal{L}^a)^2 - 2 h_{12} \mathcal{L}^a \mathcal{M}^a + h_{11} (\mathcal{M}^a)^2 \right] ,$$  

with

$$\mathcal{L}^a = k^a_{1} - k^a_{1} ,$$

(6.94)
\[ M^\alpha = k_{2f}^i - k_{2i}^i, \]  \hspace{1cm} (6.95)

\[ N^\alpha = k_{3f}^i - k_{3i}^i. \]  \hspace{1cm} (6.96)

The integral transformation, Eq.(6.9), requires the wave vector components of the fluid in transducer coordinates, and the solid wave vector components in ray coordinates. Since \( L^\alpha, M^\alpha, N^\alpha \) are expressed in interface coordinates a transformation has to be performed to get the components in the desired coordinate system. The transformation of the wave vector components in the fluid is identical to the one performed in the isotropic case. The \( x_1^i, x_2^i \) components of the wave vector in the fluid are given by

\[ k_{1f}^i = k_{1f}^i \cos \theta^i + k_{1f}^i \sin \theta^i \]  \hspace{1cm} (6.97)

\[ k_{2f}^i = k_{2f}^i. \]  \hspace{1cm} (6.98)

(for detailed discussion see Appendix C). Since the transformation is about the \( x_2^i \) axis we can write in the solid for a ray close to the central ray

\[ k_{2i}^{i\alpha} = k_{2}^{r\alpha}. \]  \hspace{1cm} (6.99)

and

\[ k_{1i}^{\alpha} = \cos(\theta^\alpha + d\theta^\alpha)k_{1}^{r\alpha} + \sin(\theta^\alpha + d\theta^\alpha)k_{2}^{r\alpha}. \]  \hspace{1cm} (6.100)

However, in contrast to the isotropic media the \( k_{3}^{r\alpha} \) component now is not given by a simple algebraic expression. In general \( k_{3}^{r\alpha} \) is found by the solving a sixth order polynomial as discussed in chapter 4. To overcome this problem we again use the procedure discussed in chapter 3 and chapter 5, where we expressed the slowness surface in a TAYLOR series in the ray coordinate system. For a detailed discussion see chapter 3 and chapter 5. Here we just quote the result from Eq.(5.26) as

\[ k_{3}^{r\alpha} = \omega s_\theta^\alpha + A^\alpha k_{1}^{r\alpha} + B^\alpha k_{2}^{r\alpha} \]

\[ + \frac{1}{\omega} \left\{ \left[ C^\alpha - \frac{1}{2s_\theta^\alpha} \right] (k_{1}^{r\alpha})^2 + D^\alpha k_{1}^{r\alpha}k_{2}^{r\alpha} + \left[ E^\alpha - \frac{1}{2s_\theta^\alpha} \right] (k_{2}^{r\alpha})^2 \right\}, \]  \hspace{1cm} (6.101)
where we only retain terms up to second order. Substitution of Eq.(6.101) into Eq.(6.100) gives

$$k_i^{\alpha} = \cos(\theta^\alpha + d\theta^\alpha)k_i^{\alpha} + \sin(\theta^\alpha + d\theta^\alpha)\left\{\omega s_0^\alpha + A^\alpha k_1^{\alpha} + B^\alpha k_2^{\alpha}\right\}$$

$$+ \frac{1}{\omega}\left\{\left[C^\alpha - \frac{1}{(2s_0^\alpha)}\right](k_1^{\alpha})^2 + D^\alpha k_1^{\alpha}k_2^{\alpha} + \left[E^\alpha - \frac{1}{(2s_0^\alpha)}\right](k_2^{\alpha})^2\right\}, \quad (6.102)$$

or

$$k_i^{\alpha} = (\cos \theta^\alpha \cos d\theta^\alpha - \sin \theta^\alpha \sin d\theta^\alpha)k_i^{\alpha} + (\sin \theta^\alpha \cos d\theta^\alpha - \cos \theta^\alpha \sin d\theta^\alpha)$$

$$\times \left\{\omega s_0^\alpha + A^\alpha k_1^{\alpha} + B^\alpha k_2^{\alpha} + \mathcal{H.O.T}\right\} \quad (6.103)$$

If we again assume that $d\theta^\alpha \approx 0$ we get

$$k_i^{\alpha} = \cos \theta^\alpha k_i^{\alpha} + \sin \theta^\alpha (\omega s_0^\alpha + A^\alpha k_1^{\alpha} + B^\alpha k_2^{\alpha} + \mathcal{H.O.T}) \quad (6.104)$$

As in the isotropic case we take for $\mathcal{N}^\alpha$ the central ray value

$$\mathcal{N}^\alpha = k_f \cos \theta^i - k^\alpha \cos \theta^\alpha \quad (6.105)$$

The wave vector in the solid for the central ray is given by

$$k^\alpha = \omega s_0^\alpha \quad (6.106)$$

Substituting Eqs.(6.97), (6.104) into Eq.(6.94), Eqs.(6.98), (6.99) into Eq.(6.95) and Eq. (6.106) into Eq.(6.105), and applying SNELL’s law $k_f \sin \theta^i = \omega s_0^\alpha \sin \theta^\alpha$. gives

$$\mathcal{L}^\alpha = k_1^\alpha \cos \theta^i + k_f \sin \theta^i - (\cos \theta^\alpha + A^\alpha \sin \theta^\alpha)k_1^{\alpha} - B^\alpha \sin \theta^\alpha k_2^{\alpha} + \mathcal{H.O.T} \quad (6.107)$$

$$\mathcal{M}^\alpha = k_2^\alpha - k_2^{\alpha} \quad (6.108)$$

$$\mathcal{N}^\alpha = k_f \cos \theta^i - \omega s_0^\alpha \cos \theta^\alpha \quad (6.109)$$

Substituting Eqs.(6.107), (6.108) and (6.109) into Eq.(6.93) and keeping at most quadratic terms in $k_1^{\alpha}, k_2^{\alpha}$ gives

$$W^\alpha(k_1^i, k_2^f, k_2^{\alpha}, k_2^{\alpha}) = -\frac{1}{2(h_{11}h_{22} - h_{12}h_{21})(k_f \cos \theta^i - \omega s_0^\alpha \cos \theta^\alpha)}$$
\[
\times \frac{1}{2} \left\{ h_{22} \left( k_{1f}^i \cos \theta^i + k_{1f} \sin \theta^i - (\cos \theta^a + A^a \sin \theta^a) k_{1f}^{r,a} - B^a \sin \theta^a k_{2f}^{r,a} \right)^2 \\
- 2h_{12} \left( k_{1f}^i \cos \theta^i + k_{1f} \sin \theta^i - (\cos \theta^a + A^a \sin \theta^a) k_{1f}^{r,a} - B^a \sin \theta^a k_{2f}^{r,a} \right) \left( k_{2f}^i - k_{2f}^{r,a} \right) \\
+ h_{11} \left( k_{2f}^i - k_{2f}^{r,a} \right)^2 \right\},
\]

which can be simplified and rewritten in matrix form as

\[
W^\alpha(k_{1f}, k_{2f}^i, k_{2f}^{r,a}, k_{2f}^{s,a}) = \frac{1}{\Gamma^\alpha}(W_{11}^\alpha + W_{12}^\alpha + W_{22}^\alpha)
\]

with

\[
W_{11}^\alpha = \frac{1}{2} \{ k_{1f}^i, k_{2f}^i \} \left[ \begin{array}{cc}
 h_{22} \cos^2 \theta^i & -h_{12} \cos \theta^i \\
 -h_{12} \cos \theta^i & h_{11}
\end{array} \right] \{ k_{1f}^i \}
\]

and

\[
W_{12}^\alpha = -\{ k_{1f}^i, k_{2f}^i \} \left[ \begin{array}{cc}
 (W_{12}^{\alpha})_{11} & (W_{12}^{\alpha})_{12} \\
 (W_{12}^{\alpha})_{21} & (W_{12}^{\alpha})_{22}
\end{array} \right] \{ k_{1f}^{\alpha} \}
\]

where the components of \( W_{12}^\alpha \) are given by

\[
(W_{12}^{\alpha})_{11} = h_{22} \cos \theta^i (\cos \theta^a + A^a \sin \theta^a)
\]

\[
(W_{12}^{\alpha})_{12} = (-h_{12} + h_{22} B^a \sin \theta^a) \cos \theta^i
\]

\[
(W_{12}^{\alpha})_{21} = -h_{12} (\cos \theta^a + A^a \sin \theta^a)
\]

\[
(W_{12}^{\alpha})_{22} = h_{11} - h_{12} B^a \sin \theta^a.
\]

Similarly,

\[
W_{22}^\alpha = \frac{1}{2} \{ k_{1f}^{\alpha}, k_{2f}^{\alpha} \} \left[ \begin{array}{cc}
 (W_{22}^{\alpha})_{11} & (W_{22}^{\alpha})_{12} \\
 (W_{22}^{\alpha})_{21} & (W_{22}^{\alpha})_{22}
\end{array} \right] \{ k_{1f}^{\alpha} \}
\]

with its components

\[
(W_{22}^{\alpha})_{11} = h_{22} (\cos \theta^a + A^a \sin \theta^a)^2
\]

\[
(W_{22}^{\alpha})_{12} = (-h_{12} + h_{22} B^a \sin \theta^a)(\cos \theta^a + A^a \sin \theta^a)
\]

\[
(W_{22}^{\alpha})_{21} = (-h_{12} + h_{22} B^a \sin \theta^a)(\cos \theta^a + A^a \sin \theta^a)
\]

\[
(W_{22}^{\alpha})_{22} = h_{11} - 2h_{12} B^a \sin \theta^a + h_{22} (B^a \sin \theta^a)
\]
\[ \Gamma^\alpha = -(k_t \cos \theta^\alpha - \omega s_0^\alpha \cos \theta)(h_{11} h_{22} - h_{12}^2). \]  \hspace{1cm} (6.123)

If we now define \( P^\alpha \), \( Q^\alpha \) and \( R^\alpha \) matrices, as done in the isotropic case, we finally find the angle eikonal in the form

\[ W^\alpha(k_1^t, k_2^t, k_1^{r_{10}}, k_2^{r_{10}}) = \frac{1}{2}\{k_1^t, k_2^t\} P^\alpha \begin{bmatrix} k_1^t \\ k_2^t \end{bmatrix} + \frac{1}{2}\{k_1^{r_{10}}, k_2^{r_{10}}\} Q^\alpha \begin{bmatrix} k_1^{r_{10}} \\ k_2^{r_{10}} \end{bmatrix} + \frac{1}{2}\{k_1^{r_{10}}, k_2^{r_{10}}\} R^\alpha \begin{bmatrix} k_1^{r_{10}} \\ k_2^{r_{10}} \end{bmatrix} \hspace{1cm} (6.124) \]

with

\[ P^\alpha = \frac{1}{\Gamma^\alpha} \begin{bmatrix} h_{22} \cos^2 \theta^i - h_{12} \cos \theta^i \\ -h_{12} \cos \theta^i & h_{11} \end{bmatrix} \hspace{1cm} (6.125) \]

\[ Q^\alpha = -\frac{1}{\Gamma^\alpha} \begin{bmatrix} h_{22} \cos \theta^i (\cos \theta^\alpha + A^\alpha \sin \theta^\alpha) & (-h_{12} + h_{22} B^\alpha \sin \theta^\alpha) \cos \theta^i \\ -h_{12} (\cos \theta^\alpha + A^\alpha \sin \theta^\alpha) & h_{11} - h_{12} B^\alpha \sin \theta^\alpha \end{bmatrix} \hspace{1cm} (6.126) \]

and

\[ R^\alpha = \frac{1}{\Gamma^\alpha} \begin{bmatrix} R_{11}^\alpha & R_{12}^\alpha \\ R_{21}^\alpha & R_{22}^\alpha \end{bmatrix} \hspace{1cm} (6.127) \]

where

\[ R_{11}^\alpha = h_{22}(\cos \theta^\alpha + A^\alpha \sin \theta^\alpha)^2 \hspace{1cm} (6.128) \]

\[ R_{12}^\alpha = (-h_{12} + h_{22} B^\alpha \sin \theta^\alpha)(\cos \theta^\alpha + A^\alpha \sin \theta^\alpha) \hspace{1cm} (6.129) \]

\[ R_{21}^\alpha = (-h_{12} + h_{22} B^\alpha \sin \theta^\alpha)(\cos \theta^\alpha + A^\alpha \sin \theta^\alpha) \hspace{1cm} (6.130) \]

\[ R_{22}^\alpha = h_{11} - 2h_{12} B^\alpha \sin \theta^\alpha + h_{22}(B^\alpha \sin \theta^\alpha) \hspace{1cm} (6.131) \]

**Displacement in the Anisotropic Solid**

For the derivation of the displacement field in the anisotropic solid we can use some of the results already obtained in the isotropic case. In that case we found the spectrum of
the displacement by means of a integral transformation, Eq.(6.9). The incident pressure spectrum was given by Eq.(6.8). The unknown amplitude term was determined by comparison of the pressure distribution in the fluid with the displacement field in the solid at the interface. In the derivation the $G^0$ played an important role and was given by

$$G^0 = (Q^0)^{(P^0 - G^T)}^{-1}R^0.$$  \hspace{1cm} (6.132)

The result obtained for the displacement spectrum in the isotropic solid is also valid for the anisotropic case, except that we now have to use definitions of the $Q^a, P^a, R^a$ matrices given by Eqs.(6.125), (6.126) and (6.127). Then the displacement spectrum at the interface in the anisotropic solid is given by

$$u^a(k^{r,a}_1, k^{r,a}_2) = -i \frac{2\pi p_0^0 T^a}{\omega \rho_v u q(D_f)} \left[ \frac{1}{iG^0_{11}} \right]^{-\frac{1}{2}} \left[ \frac{G^0_{11}}{i[G^0_{11}G^0_{22} - (G^0_{12})^2]} \right]^{-\frac{1}{2}} \delta^a e^{ik_f D_f} e^{-i\frac{1}{2}(a_r^T G^a a_r)}.$$  \hspace{1cm} (6.133)

with

$$a_t = \left\{ \begin{array}{c} k^t_1 \\ k^t_2 \end{array} \right\}, \quad a_r = \left\{ \begin{array}{c} k^{r,a}_1 \\ k^{r,a}_2 \end{array} \right\}.$$  \hspace{1cm} (6.134)

The displacement field in the elastic anisotropic solid at a distance $x^*_i$ can be obtained

$$u^a(x_1^r, x_2^r, x_3^r) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u^a(k^{r,a}_1, k^{r,a}_2)e^{i(k^{r,a}_1 x_1^r + k^{r,a}_2 x_2^r + k^{r,a}_3 x_3^r)} dk^{r,a}_1 dk^{r,a}_2.$$  \hspace{1cm} (6.135)

Substitution of the displacement spectrum, Eq(6.133), into Eq.(6.135) gives

$$u^a(x_1^r, x_2^r, x_3^r) = -i \frac{2\pi p_0^0 T^a}{\omega \rho_v u q(D_f)} \left[ \frac{1}{iG^0_{11}} \right]^{-\frac{1}{2}} \left[ \frac{G^0_{11}}{i[G^0_{11}G^0_{22} - (G^0_{12})^2]} \right]^{-\frac{1}{2}} \delta^a e^{ik_f D_f} I,$$  \hspace{1cm} (6.136)

with

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[ -i \frac{1}{2}(a_r^T G^a a_r) \right] \exp \left[ i(k^{r,a}_1 x_1^r + k^{r,a}_2 x_2^r + k^{r,a}_3 x_3^r) \right] dk^{r,a}_1 dk^{r,a}_2.$$  \hspace{1cm} (6.137)
As in the derivation of the angle eikonal we expand $k_3^{ra}$ about the central ray in a TAYLOR series Eq.(6.101) which is rewritten here for convenience as

$$k_3^{ra} \approx \omega s_0^a + A^a k_1^{ra} + B^a k_2^{ra} + \left[ \frac{C^a}{\omega - 2 \omega s_0^a} \right] (k_1^{ra})^2 + \frac{D^a}{\omega} k_1^{ra} k_2^{ra} + \left[ \frac{E^a}{\omega - \frac{1}{2 \omega s_0^a}} \right] (k_2^{ra})^2. \quad (6.139)$$

If we now substitute Eq.(6.139) into the phase $\phi_2$ of Eq.(6.137) we get for the sum of the phases $\phi_1$ and $\phi_2$

$$\phi = \phi_1 + \phi_2 = \frac{i}{2} \left[ G_{11}^a - \left( \frac{2C^a}{\omega - 1 \omega s_0^a} \right) x_3^r \right] (k_1^{ra})^2 - i \left( G_{12}^a - \frac{D^a}{\omega} x_3^r \right) k_1^{ra} k_2^{ra}$$

$$- \frac{i}{2} \left[ G_{22}^a - \left( \frac{2E^a}{\omega - 1 \omega s_0^a} \right) x_3^r \right] (k_2^{ra})^2 + i (x_3^r + A^a x_3^r) k_1^{ra}$$

$$+ i (x_2^r + B^a x_3^r) k_2^{ra}. \quad (6.140)$$

Then, by defining the following quantities

$$G_{11}^{x_2 a} = G_{11}^a - \left( \frac{2C^a}{\omega - 1 \omega s_0^a} \right) x_3^r \quad (6.141)$$

$$G_{22}^{x_2 a} = G_{22}^a - \left( \frac{2E^a}{\omega - 1 \omega s_0^a} \right) x_3^r \quad (6.142)$$

$$G_{12}^{x_2 a} = G_{12}^a - \frac{D^a}{\omega} x_3^r \quad (6.143)$$

$$\tilde{x}_1 = x_1^r + A^a x_3^r \quad (6.144)$$

$$\tilde{x}_2 = x_2^r + B^a x_3^r \quad (6.145)$$

we can write the phase term in the form

$$\phi = -\frac{i}{2} G_{11}^{x_2 a} (k_1^{ra})^2 - i G_{12}^{x_2 a} k_1^{ra} k_2^{ra} - \frac{i}{2} G_{22}^{x_2 a} (k_2^{ra})^2 + \tilde{x}_1 k_1^{ra} + \tilde{x}_2 k_2^{ra}. \quad (6.146)$$
This phase term is in the same form as Eq.(6.49) and we can use the results of the integration of Eq.(6.50) to obtain

\[ I = \sqrt{\frac{2\pi}{iG_{11}^{\alpha\alpha}}} \sqrt{\frac{2\pi G_{11}^{\alpha\alpha}}{i[G_{11}^{\alpha\alpha}G_{22}^{\alpha\alpha} - (G_{12}^{\alpha\alpha})^2]}} \exp \left[ i \frac{1}{2} (\tilde{x}^T (G^{\alpha\alpha})^{-1} \tilde{x}) \right] . \]  

(6.147)

where the vector, \( \tilde{x} \), is given by

\[
\tilde{x} = \begin{cases} 
\tilde{x}_1 \\
\tilde{x}_2
\end{cases} = \begin{cases} 
x_1^r + A^\alpha x_3^r \\
x_2^r + B^\alpha x_3^r
\end{cases} .
\]  

(6.148)

If we now substitute Eq.(6.147) into Eq.(6.136) we obtain the displacement in the solid at a location \( x_3^r \) as

\[
u^\alpha(x_1^r, x_2^r, x_3^r) = -i \frac{p_0 T^\alpha}{\omega \rho_f \gamma_f q(D_f)} S^\alpha \exp \left[ i(k_r D_f + k_\alpha x_3^r) \right] \exp \left[ i \frac{1}{2} (\tilde{x}^T (G^{\alpha\alpha})^{-1} \tilde{x}) \right] .
\]  

(6.149)

where

\[ S = \frac{\sqrt{\frac{1}{iG_{11}^{\alpha\alpha}}} \sqrt{\frac{G_{11}^{\alpha\alpha}}{i[G_{11}^{\alpha\alpha}G_{22}^{\alpha\alpha} - (G_{12}^{\alpha\alpha})^2]}}}{\sqrt{\frac{1}{iG_{11}^{\alpha\alpha}}} \sqrt{\frac{G_{11}^{\alpha\alpha}}{i[G_{11}^{\alpha\alpha}G_{22}^{\alpha\alpha} - (G_{12}^{\alpha\alpha})^2]}}} .
\]  

(6.150)

Using the definitions Eq.(6.141) through (6.143) we can write in matrix notation

\[
G^\alpha_r = G^\alpha + \frac{2C^\alpha}{\omega} - \frac{1}{\omega \delta^\alpha} \frac{D^\alpha}{\omega} \begin{bmatrix} x_1^r \\
x_2^r \\
C^\alpha x_3^r
\end{bmatrix} .
\]  

(6.151)

and again we can rewrite \( G^\alpha \) in terms of the refraction matrices \( A^\alpha, B^\alpha, C^\alpha \) and \( D^\alpha \); (see Appendix D for the derivation) in the form

\[
G^\alpha = (C^\alpha G_r - D^\alpha)(B^\alpha - A^\alpha G_r)^{-1}.
\]  

(6.152)

where, recall

\[
A^\alpha = -(Q^\alpha)^{-1}
\]  

(6.153)

\[
B^\alpha = -(Q^\alpha)^{-1} P^\alpha
\]  

(6.154)

\[
C^\alpha = -R^\alpha (Q^\alpha)^{-1}
\]  

(6.155)

\[
D^\alpha = (Q^\alpha)^T - R^\alpha (Q^\alpha)^{-1} P^\alpha.
\]  

(6.156)
Substitution of Eqs. (6.125), (6.126) and (6.127) into Eqs. (6.153) though Eq. (6.156) yields (after some manipulation which is conveniently done in Mathematica) the refraction matrices explicitly as

\[
A = \frac{\omega_0^2 \cos \theta^i - k f \cos \theta^i}{\cos \theta^i (\cos \theta^a + A^\alpha \sin \theta^a)} \times \begin{bmatrix}
h_{11} - h_{12} B^\alpha \sin \theta^a & (h_{12} - h_{22} B^\alpha \sin \theta^a) \cos \theta^i \\
h_{12} (\cos \theta^a + A^\alpha \sin \theta^a) & h_{22} \cos \theta^i (\cos \theta^a + A^\alpha \sin \theta^a)
\end{bmatrix}
\]  

\[\text{Eq. (6.157)}\]

\[
B = \frac{1}{\cos \theta^a + A^\alpha \sin \theta^a} \begin{bmatrix}
\cos \theta^a & -B^\alpha \sin \theta^a \\
0 & \cos \theta^a + A^\alpha \sin \theta^a
\end{bmatrix}
\]  

\[\text{Eq. (6.158)}\]

\[
C = \frac{1}{\cos \theta^i} \begin{bmatrix}
\cos \theta^a + A^\alpha \sin \theta^a & 0 \\
B^\alpha \sin \theta^a & \cos \theta^i
\end{bmatrix}
\]  

\[\text{Eq. (6.159)}\]

\[
D = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]  

\[\text{Eq. (6.160)}\]

and as before

\[
G_f = \frac{1}{k_1} \begin{bmatrix}
q(D_f) & 0 \\
0 & q(D_f)
\end{bmatrix}
\]  

\[\text{Eq. (6.161)}\]

**Planar Interface** \( h_{11} = 0, h_{12} = 0, h_{22} = 0 \)

If we now consider the case of a planar fluid/anisotropic solid interface the curvatures are zero, so

\[
h_{11} = 0.
\]

\[
h_{22} = 0,
\]

\[
h_{12} = 0.
\]  

\[\text{Eq. (6.162)}\]
As seen from Eq. (6.157) the refraction matrix $A^\alpha$ will vanish; i.e. $A^\alpha = 0$. Performing now the matrix multiplications, as required by Eq. (6.152), with Mathematica we obtain

$$G^\alpha = \begin{bmatrix} G_{11}^\alpha & G_{12}^\alpha \\ G_{12}^\alpha & G_{22}^\alpha \end{bmatrix}$$  \hspace{1cm} (6.163)

with

$$G_{11}^\alpha = \frac{q(D_f)(\cos \theta^\alpha + A^\alpha \sin \theta^\alpha)^2}{k_f \cos^2 \theta^i}$$  \hspace{1cm} (6.164)

$$G_{12}^\alpha = \frac{q(D_f)(\cos \theta^\alpha + A^\alpha \sin \theta^\alpha)B^\alpha \sin \theta^\alpha}{k_f \cos^2 \theta^i}$$  \hspace{1cm} (6.165)

$$G_{22}^\alpha = \frac{q(D_f)[\cos^2 \theta^i + (B^\alpha \sin \theta^\alpha)^2]}{k_f \cos^2 \theta^i}$$  \hspace{1cm} (6.166)

Thus, the matrix $G^{z_\alpha}$ given by Eq. (6.151) is now

$$G_{11}^{z_\alpha} = \frac{q(D_f)(\cos \theta^\alpha + A^\alpha \sin \theta^\alpha)^2}{k_f \cos^2 \theta^i} + \left( \frac{1}{\omega s_0^\alpha} - \frac{2C^\alpha}{\omega} \right) x_3^i$$

$$G_{12}^{z_\alpha} = \frac{q(D_f)(\cos \theta^\alpha + A^\alpha \sin \theta^\alpha)B^\alpha \sin \theta^\alpha}{k_f \cos^2 \theta^i} - \frac{D^\alpha}{\omega} x_3^i$$

$$G_{22}^{z_\alpha} = \frac{q(D_f)[\cos^2 \theta^i + (B^\alpha \sin \theta^\alpha)^2]}{k_f \cos^2 \theta^i} + \left( \frac{1}{\omega s_0^\alpha} - \frac{2E^\alpha}{\omega} \right) x_3^i$$  \hspace{1cm} (6.168)

As required in the phase function the inverse can be computed as

$$(G^{z_\alpha})^{-1} = \begin{bmatrix} \frac{G_{11}^{z_\alpha}}{G_{11}^{z_\alpha}G_{22}^{z_\alpha} - (G_{12}^{z_\alpha})^2} & \frac{-G_{12}^{z_\alpha}}{G_{11}^{z_\alpha}G_{22}^{z_\alpha} - (G_{12}^{z_\alpha})^2} \\ \frac{G_{12}^{z_\alpha}}{G_{11}^{z_\alpha}G_{22}^{z_\alpha} - (G_{12}^{z_\alpha})^2} & \frac{G_{11}^{z_\alpha}}{G_{11}^{z_\alpha}G_{22}^{z_\alpha} - (G_{12}^{z_\alpha})^2} \end{bmatrix}$$  \hspace{1cm} (6.169)

Substitution of Eq. (6.168) into Eq. (6.149) then yields for the displacement

$$u^\alpha(x_1^i, x_2^i, x_3^i) = -i \frac{p_0 T^\alpha}{\omega \rho v_f} \frac{q_0}{q(D_f)} S d^\alpha \exp[i k_f D_f + k^\alpha x_3^i] \exp \left[ \frac{i}{2} \left( \frac{G_{22}^{z_\alpha}(x_1^i + A^\alpha x_3^i)^2}{G_{11}^{z_\alpha}G_{22}^{z_\alpha} - (G_{12}^{z_\alpha})^2} + 2 \frac{-G_{12}^{z_\alpha}(x_1^i + A^\alpha x_3^i)(x_2^i + B^\alpha x_3^i)}{G_{11}^{z_\alpha}G_{22}^{z_\alpha} - (G_{12}^{z_\alpha})^2} + \frac{G_{11}^{z_\alpha}(x_2^i + B^\alpha x_3^i)^2}{G_{11}^{z_\alpha}G_{22}^{z_\alpha} - (G_{12}^{z_\alpha})^2} \right) \right]$$  \hspace{1cm} (6.170)

with

$$S = \sqrt{\frac{i G_{11}^{z_\alpha}}{G_{11}^{z_\alpha}}} \sqrt{\frac{G_{11}^{z_\alpha}G_{22}^{z_\alpha} - (G_{12}^{z_\alpha})^2}{G_{11}^{z_\alpha}}}$$  \hspace{1cm} (6.171)

and the $G_{ij}^{z_\alpha}$ defined by Eqs. (6.168).
Propagation along Axis of Symmetry \( h_{12} = 0, B^\alpha = 0, D^\alpha = 0 \)

As in the BDW chapter we consider now the special case of propagating along an axis of symmetry. In this case the anisotropic parameters \( B^\alpha, D^\alpha \) vanish, i.e \( B^\alpha = D^\alpha = 0 \). If we also propagate along a principal axis, \( h_{12} = 0 \) and the refraction matrices reduce to

\[
A^\alpha = \frac{\omega s^\alpha_0 \cos \theta^\alpha - k_1 \cos \theta^i}{\cos \theta^i(\cos \theta^\alpha + A^\alpha \sin \theta^\alpha)} \begin{bmatrix} h_{11} & 0 \\ 0 & h_{22} \cos \theta^i(\cos \theta^\alpha + A^\alpha \sin \theta^\alpha) \end{bmatrix}
\]

(6.172)

\[
B^\alpha = \frac{1}{\cos \theta^\alpha + A^\alpha \sin \theta^\alpha} \begin{bmatrix} \cos \theta^\alpha & 0 \\ 0 & \cos \theta^\alpha + A^\alpha \sin \theta^\alpha \end{bmatrix}
\]

(6.173)

\[
C^\alpha = \frac{1}{\cos \theta^i} \begin{bmatrix} \cos \theta^\alpha + A^\alpha \sin \theta^\alpha & 0 \\ 0 & \cos \theta^i \end{bmatrix}
\]

(6.174)

\[
D^\alpha = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

(6.175)

The transformation of the \( G_f \) matrix through the interface was defined by

\[
G^\alpha = (C^\alpha G_f - D^\alpha)(B^\alpha - A^\alpha G_f)^{-1}
\]

(6.176)

so with \( A^\alpha, B^\alpha, C^\alpha, D^\alpha \) given by Eq.(6.175), we get for \( G^\alpha \)

\[
G^\alpha = \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22}^\alpha \end{bmatrix}
\]

(6.177)

with

\[
G_{11}^\alpha = \frac{q(D_f)(\cos \theta^\alpha + A^\alpha \sin \theta^\alpha)^2}{k_f \cos^2 \theta^i + q(D_f)h_{11}(k_f \cos \theta^i - \omega s^\alpha_0 \cos \theta^\alpha)}
\]

(6.178)

\[
G_{22}^\alpha = \frac{q(D_f)}{k_f + q(D_f)h_{22}(k_f \cos \theta^i - \omega s^\alpha_0 \cos \theta^\alpha)}
\]

(6.179)

The determination of the \( G^{z:a} \) matrix is straightforward and can be written as

\[
G^{z:a} = \begin{bmatrix} G_{11}^{z:a} & 0 \\ 0 & G_{22}^{z:a} \end{bmatrix}
\]

(6.180)
with

\[
G_{11}^{z;\alpha} = \frac{q(D_{f})(\cos \theta^{\alpha} + A^{\alpha} \sin \theta^{\alpha})^2}{k_{f}\cos^2 \theta^{i} + q(D_{f}) h_{11}(k_{f}\cos \theta^{i} - \omega s_0^\alpha \cos \theta^{\alpha})} + x_3^{f} \left( \frac{1}{\omega s_0^\alpha} - \frac{2C^{\alpha}}{\omega} \right) (6.181)
\]

\[
G_{22}^{z;\alpha} = \frac{q(D_{f})}{k_{f} + q(D_{f}) h_{22}(k_{f}\cos \theta^{i} - \omega s_0^\alpha \cos \theta^{\alpha})} + x_3^{f} \left( \frac{1}{\omega s_0^\alpha} - \frac{2E^{\alpha}}{\omega} \right) (6.182)
\]

Since $G^{z;\alpha}$ is a diagonal matrix its inverse is simply given by

\[
(G^{z;\alpha})^{-1} = \begin{bmatrix} \frac{1}{G_{11}^{z;\alpha}} & 0 \\ 0 & \frac{1}{G_{22}^{z;\alpha}} \end{bmatrix}, \quad (6.183)
\]

and we can write finally for the displacement field

\[
u^{\alpha}(x_1^f, x_2^f, x_3^f) = -i \frac{p_0 T^\alpha}{\omega \rho_{r_{11}} q(D_{f})} \frac{q_0}{\sqrt{i G_{11}^{z;\alpha}}} \sqrt{i G_{22}^{z;\alpha}} \frac{1}{\sqrt{i G_{11}^{z;\alpha}}} \frac{1}{\sqrt{i G_{22}^{z;\alpha}}}
\]

\[
\times \exp \left[ -i \frac{1}{2} k^{\alpha} \left( \frac{(x_1^f + A^{\alpha} x_3^f)^2}{G_{11}^{z;\alpha}} + \frac{(x_2^f)^2}{G_{22}^{z;\alpha}} \right) \right] (6.184)
\]

with

\[
G_{11}^{\alpha} = \frac{q(D_{f})(\cos \theta^{\alpha} + A^{\alpha} \sin \theta^{\alpha})^2}{k_{f}\cos^2 \theta^{i} + q(D_{f}) h_{11}(k_{f}\cos \theta^{i} - \omega s_0^\alpha \cos \theta^{\alpha})} (6.185)
\]

\[
G_{22}^{\alpha} = \frac{q(D_{f})}{k_{f} + q(D_{f}) h_{22}(k_{f}\cos \theta^{i} - \omega s_0^\alpha \cos \theta^{\alpha})} (6.186)
\]

\[
G_{11}^{z;\alpha} = \frac{q(D_{f})(\cos \theta^{\alpha} + A^{\alpha} \sin \theta^{\alpha})^2}{k_{f}\cos^2 \theta^{i} + q(D_{f}) h_{11}(k_{f}\cos \theta^{i} - \omega s_0^\alpha \cos \theta^{\alpha})} + x_3^{f} \left( \frac{1}{\omega s_0^\alpha} - \frac{2C^{\alpha}}{\omega} \right) (6.187)
\]

\[
G_{22}^{z;\alpha} = \frac{q(D_{f})}{k_{f} + q(D_{f}) h_{22}(k_{f}\cos \theta^{i} - \omega s_0^\alpha \cos \theta^{\alpha})} + x_3^{f} \left( \frac{1}{\omega s_0^\alpha} - \frac{2E^{\alpha}}{\omega} \right) (6.188)
\]

**Reduction to Isotropy** $A^{\alpha} = 0, B^{\alpha} = 0, C^{\alpha} = 0, D^{\alpha} = 0, E^{\alpha} = 0$

Substituting the isotropy conditions, $A^{\alpha} = 0, B^{\alpha} = 0, C^{\alpha} = 0, D^{\alpha} = 0, E^{\alpha} = 0$, into the refraction matrices defined by Eqs.(6.160) gives

\[
A = \frac{k^{\alpha} \cos \theta^{\alpha} - k_{f} \cos \theta^{i}}{\cos \theta^{i} \cos \theta^{\alpha}} \begin{bmatrix} h_{11} & h_{12} \cos \theta^{i} \\ h_{12} \cos \theta^{\alpha} & h_{22} \cos \theta^{i} \cos \theta^{\alpha} \end{bmatrix} (6.189)
\]

\[
B = \begin{bmatrix} \cos \theta^{i} & 0 \\ \cos \theta^{\alpha} & 0 \end{bmatrix} (6.190)
\]
\[ C = \begin{bmatrix} \cos \theta^0 & 0 \\ \cos \theta^i & 1 \end{bmatrix} \]  
\[ D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \]  
which agree with the previously determined refraction matrices from the isotropic case (see Eqs.(6.84)-(6.87)). Furthermore the vector \( \tilde{x} \) reduces to
\[ \tilde{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \]  
and the displacement is given by
\[ u^a(x_1^f, x_2^f, x_3^f) = -i \frac{D_0 T^a}{\omega \rho_T \nu_T q(D_f)} S^a \exp \left[ i(k_D + k^a x_3) \right] \exp \left[ \frac{i}{2} (x^T (G^{a^{\alpha}})^{-1} x) \right] \]  
agreeing with the isotropic case (see Eq.(6.92)).

**Transformation of the Beam-Waist and Curvature Parameters through an Interface: A Comparison**

In NEWBERRY [40] a transformation is given for relating the parameters of a Gaussian beam on both sides of an interface between two anisotropic solids. The q-parameters of the beam on the incident side were given by NEWBERRY as \( q_x \) and \( q_y \), describing the radii of curvature and width of the beam in the \( x \) and \( y \) directions. They were related to their counterparts (see Eq.(63a) and Eq.(63b) of NEWBERRY [40]) on the refracted side by
\[ \left( \frac{1}{q_x} \right)_T^{NB} = \left( \frac{s_a \cos^2 \psi^a \cos^2 \phi^i}{s_b \cos^2 \psi^i \cos^2 \phi^a} \right) \left( \frac{1}{q_z} \right)_I^{NB} - \left( \frac{\cos^2 \psi^a (s_a/s_b \cos \theta^i - \cos \theta^a)}{B_x \cos^2 \phi^a} \right) \]  
and
\[ \left( \frac{1}{q_y} \right)_T^{NB} = \frac{s_a}{s_b} \left( \frac{1}{q_y} \right)_I^{NB} - \frac{s_a/s_b - 1}{B_y} \]  
(6.196)
where \( B_x \) and \( B_y \) are the radii of curvature defined as positive for a concave surface; i.e. \( B_x = -1/h_{11}, B_y = -1/h_{22} \). Eqs.(6.195), (6.196) were derived for the case that the incident plane is a plane of symmetry, therefore the results presented in the previous section 'Propagation along axis of symmetry' \( h_{12} = 0, B^o = 0, D^o = 0 \) will be applicable. Since in our case medium 1 is an isotropic fluid in Eqs.(6.195) and (6.196) we have

\[
\psi^i = 0, \\
\phi^i = \theta^i, \\
s_a = s_f. 
\] (6.197)

In Newberry the angle between the group velocity direction and the \( x_3 \) coordinate is denoted \( \phi^o \), whereas the beam skew angle is given by \( \psi^o \) (see Fig. 6.6). If we also consider the case of an initially symmetric Gaussian, i.e. \( 1/q = 1/q_x = 1/q_y \), we can write Eq.(6.195) and Eq.(6.196) with the definitions given by Eqs.(6.197) as

\[
\left( \frac{1}{q_x} \right)^N_{B} = \left( \frac{s_f \cos^2 \psi^o \cos^2 \theta^i}{s_0^2 \cos^2 \phi^o} \right) \left( \frac{1}{q} \right)_I^{N^B} + h_{11} \left( \frac{\cos^2 \psi^o (s_f/s_0^2 \cos \theta^i - \cos \theta^o)}{\cos^2 \phi^o} \right) 
\] (6.198)

and

\[
\left( \frac{1}{q_y} \right)^N_{B} = \frac{s_f}{s_0^2} \left( \frac{1}{q} \right)^N_{B} + h_{22} \left( \frac{s_f}{s_0^2} - 1 \right). 
\] (6.199)

where \( s_b = s_0^{a} \). To compare Newberry's results with ours, we will extract Newberry's parameters out of our scheme. The equivalent parameters for the beam in the fluid are given by

\[
G_{t}^{-1} = k_{t} \begin{bmatrix} \frac{1}{q(D_t)} & 0 \\ 0 & \frac{1}{q(D_t)} \end{bmatrix} 
\] (6.200)

where

\[
\frac{1}{q(D_t)} \equiv \left( \frac{1}{q} \right)_I^{N^B}. 
\] (6.201)
Figure 6.6 Location of the group velocity and phase velocity with angles

For the solid we can use some of the results already established in a previous section. The \((G^\alpha)^{-1}\) matrix was given by Eq.(6.177) and is repeated here in the form

\[
(G^\alpha)^{-1} = \begin{bmatrix} 1/G_{11}^\alpha & 0 \\ 0 & 1/G_{22}^\alpha \end{bmatrix}
\] (6.202)

with

\[
(G^\alpha)^{-1}_{11} = \frac{1}{(F^\alpha)^2} \left( \frac{k_I \cos^2 \theta^I}{q(D_I)} - h_{11}(\omega s_0^\alpha \cos \theta^\alpha - k_I \cos \theta^I) \right)
\] (6.203)

\[
(G^\alpha)^{-1}_{22} = \frac{k_I}{q(D_I)} - h_{22}(\omega s_0^\alpha \cos \theta^\alpha - k_I \cos \theta^I)
\] (6.204)

where \(F^\alpha = \cos \theta^\alpha + \lambda^\alpha \sin \theta^\alpha\) and \(h_{12} = 0\). For comparison with NEWBERRY’s results we set

\[
(G^\alpha)_{11}^{-1} = \frac{k_I}{q_{11}^\alpha}, \quad (G^\alpha)_{22}^{-1} = \frac{k_I}{q_{22}^\alpha}
\] (6.205)

and we want to check if

\[
\frac{1}{q_{11}^\alpha} = \left( \frac{1}{q_z} \right)_T^{NB}, \quad \frac{1}{q_{22}^\alpha} = \left( \frac{1}{q_y} \right)_T^{NB}
\] (6.206)
To express the \( A^\alpha \) parameter in terms of the power flow angle \( \psi^\alpha \) we refer to Fig. 6.6. The power flow angle is the deviation angle of the group velocity from the phase velocity direction and its relation to the \( A \) parameter was already determined by Eq.(3.50), repeated here for the case \( B^\alpha = 0 \) as

\[
\cos \psi^\alpha = \frac{1}{\sqrt{(A^\alpha)^2 + 1}}. \tag{6.207}
\]

This relationship can also be written as

\[
A^\alpha = \pm \sqrt{\frac{1}{\cos^2 \psi^\alpha} - 1} = \pm \tan \psi^\alpha. \tag{6.208}
\]

To determine the appropriate sign in Eq.(6.208) we consider

\[
\mathbf{v}^\alpha \times \mathbf{g}^\alpha = |\mathbf{v}^\alpha||\mathbf{g}^\alpha| \sin \psi^\alpha \mathbf{\hat{e}}_2
\]

(6.209)

or

\[
\sin \psi^\alpha \mathbf{\hat{e}}_2^r = \frac{\mathbf{v}^\alpha \times \mathbf{g}^\alpha}{|\mathbf{v}^\alpha||\mathbf{g}^\alpha|}. \tag{6.210}
\]

With \( \mathbf{v}^\alpha = 1/s^\alpha \mathbf{\hat{e}}_3^r \) and \( \mathbf{g}^\alpha = 1/s^\alpha (-A\mathbf{\hat{e}}_1 + \mathbf{\hat{e}}_3) \) we find

\[
\sin \psi^\alpha \mathbf{\hat{e}}_2^r = -\frac{A^\alpha}{\sqrt{(A^\alpha)^2 + 1}} \mathbf{\hat{e}}_2^r, \tag{6.211}
\]

so the sine of the power flow angle is given by

\[
\sin \psi^\alpha = -\frac{A^\alpha}{\sqrt{(A^\alpha)^2 + 1}}. \tag{6.212}
\]

Dividing Eq.(6.212) by Eq.(6.207) gives

\[
\tan \psi^\alpha = \frac{\sin \psi^\alpha}{\cos \psi^\alpha} = -A^\alpha, \tag{6.213}
\]

which determines the appropriate sign to be minus. Then we can rewrite \( F^\alpha \) in terms of the angles \( \psi^\alpha \) and \( \phi^\alpha \) (see Fig. 6.6) as

\[
F = \cos \theta^\alpha + A \sin \theta^\alpha
= \cos \theta^\alpha - \tan \psi^\alpha \sin \theta^\alpha
= \frac{\cos \theta^\alpha \cos \psi^\alpha - \sin \psi^\alpha \sin \theta^\alpha}{\cos \psi^\alpha}
= \frac{\cos (\theta^\alpha + \psi^\alpha)}{\cos \psi^\alpha} = \frac{\cos \phi^\alpha}{\cos \psi^\alpha}. \tag{6.214}
\]
If we now return to Eq.(6.205) and substitute Eq.(6.214) into Eq.(6.205) we find, with the definitions of the \((G^a)^{-1}\) matrix given by Eq.(6.203) and Eq.(6.204),

\[
\frac{k^a}{q_{11}^a} = \frac{\cos^2 \psi^a}{\cos^2 \phi^a} \left( \frac{k_f \cos^2 \theta^i}{q(D_f)} - h_{11}(\omega s_0^a \cos \theta^a - k_f \cos \theta^i) \right).
\] (6.215)

Dividing both sides by \(k^a = \omega/s_0^a\) then yields

\[
\frac{1}{q_{11}^a} = \frac{s_f \cos^2 \psi^a}{s_0^a \cos^2 \phi^a} \frac{1}{q(D_f)} + h_{11} \frac{\cos^2 \psi^a (s_f/s_0^a \cos \theta^i - \cos \theta^a)}{\cos^2 \phi^a}
\] (6.216)

Comparing Eq.(6.216) to NEWBERRY's result, Eq.(6.198), we find complete agreement.

Finally, substituting Eq.(6.204) into Eq.(6.205) gives

\[
\frac{k^a}{q_{22}^a} = \frac{k_f}{q(D_f)} + h_{22}(\omega s_0^a \cos \theta^a - k_f \cos \theta^i)
\] (6.217)

or

\[
\frac{1}{q_{22}^a} = \frac{s_f}{s_0^a} \frac{1}{q(D_f)} - h_{22} \left( \frac{s_f}{s_0^a} \cos \theta^i - \cos \theta^a \right)
\] (6.218)

Comparison of Eq.(6.218) with Eq.(6.199) reveals that in NEWBERRY's result the cosine functions are missing. Hence, Eq.(63b) given in NEWBERRY [40] is incorrect.
7 MULTI-GAUSSIAN BEAM MODEL

In the last chapter we described a Gaussian beam passing through an interface. Using a series of such Gaussian functions it is possible to express the wave field of a circular piston transducer as a superposition of these basis functions. In the superposition the beam waist \( w_0 \) and the focal length \( R_0 \) are varied for each Gaussian in the sum. Since this Gaussian expansion breaks down only in the extreme nearfield (0.1×nearfield distance) it is a very attractive alternative to other modeling schemes. Another advantage of the multi Gaussian model is that propagation of a Gaussian beam though a interface can be completely described by simple analytic functions. This is one reason why the MGB model is currently of significant interest to the NDE community [30], [56].

The procedure for obtaining the wave field of a circular piston transducer from a superposition of Gaussian beams is described in Wen and Breazeale [64] so we will only quote their end results here. They show that the velocity potential of a circular piston transducer radiating into a fluid can be approximated as

\[
\phi(\xi, \sigma) = \frac{i}{k_f} \sum_{n=1}^{10} \frac{A_n}{1 + iB_n\sigma} \exp \left( -\frac{B_n\xi^2}{1 + iB_n\sigma} + i k_f z_R \sigma \right)
\]

with the Rayleigh distance \( z_R = \frac{1}{2} k_f a^2 \) and

\[
\xi = \rho/a, \quad \sigma = x_3/z_r,
\]

where \( \rho = (x_1^2 + x_2^2) \). Wen and Breazeale found this solution by expressing the boundary conditions in terms of a nonorthogonal basis set and requiring that the field

\[\text{Martin Spies calls the multi-Gaussian beam model Gaussian superposition}\]
solution satisfies those boundary conditions. The beam waist parameters $B_n$ and the amplitude factors $A_n$ are then found by a numerical minimization procedure and are listed in Tab. 7. Figure 7.1 compares the surface velocity distribution obtained from Eq.(7.1) with the corresponding one of a rigid piston. In order to use the multi-

![Figure 7.1 Velocity distribution on the transducer surface. Solid line real part of velocity distribution, dashed line imaginary part. Dotted line represents ideal velocity distribution of rigid piston transducer.](image)

Gaussian beam model with the previously derived Gaussian beam model we simply have to relate the parameters of that model to the $A_n$ and $B_n$. Recall that the incident pressure was defined as

$$p(x_1, x_2, 0) = p_0 \exp \left[ i \frac{k_f((x_1^1)^2 + (x_2^1)^2)}{2q_0} \right] ; \quad q_0 = \frac{1}{R_0} + i \frac{2}{k_f w_0^2}.$$  \hspace{1cm} (7.3)

Comparing the exponent of Eq.(7.3) with Eq.(7.1)

$$\frac{k_f((x_1^1)^2 + (x_2^1)^2)}{2q_0} = \frac{B_n \xi^2}{1 + iB_n \sigma} + ik_f z_R \sigma,$$  \hspace{1cm} (7.4)
and using furthermore the definitions for $\rho, \sigma, z_r$ we get for $q(x_3^1)$

$$q(x_3^1) = x_3^1 - i\frac{k_f a^2}{2B_n}.$$  \hspace{1cm} (7.5)

Since $q(x_3^1) = x_3^1 + q_0$, $q_0$ is given by

$$q_0 = -i\frac{k_f a^2}{2B_n}.$$  \hspace{1cm} (7.6)

To obtain the amplitude factor we have to convert the velocity potential expression of WEN and BREAZEALE [64] to pressure with $p(x_1^1, x_3^1, 0) = -i\omega \rho \phi$. Comparison of the amplitudes of Eq.(7.3) with Eq.(7.1) gives

$$\left. \frac{A_n}{1 + iB_n \sigma} \right|_{x_3=0} = p_0,$$  \hspace{1cm} (7.7)

which gives

$$p_0 = A_n.$$  \hspace{1cm} (7.8)

It is also possible to express the parameters $R_0$ and $w_0$ in terms of $B_n$. To do that, we consider

$$\frac{1}{q_0} = \frac{1}{R_0} + \frac{2i}{k_f w_0^3}.$$  \hspace{1cm} (7.9)

From Eq.(7.6) we have

$$\frac{1}{q_0} = \frac{2iB_n}{k_f a^2}.$$  \hspace{1cm} (7.10)

To determine $R_0$ and $w_0$ as a function of $B_n$ the real and imaginary part of Eq.(7.9) and Eq.(7.10) have to be equal. This condition leads to

$$\frac{1}{R_0} = -\frac{2\Im\{B_n\}}{k_f a^2},$$

$$R_0 = -\frac{k_f a^2}{2\Im\{B_n\}}$$  \hspace{1cm} (7.11)

and

$$\frac{2}{k_f w_0^3} = \frac{2\Re\{B_n\}}{k_f a^2}$$

$$w_0 = \frac{a}{\sqrt{\Re\{B_n\}}}$$  \hspace{1cm} (7.12)
To show some typical values for $R_0$ and $w_0$ in [mm], the last two columns in Tab. 7 list these parameters for a circular half inch diameter transducer radiating at 5MHz into water.

### Numerical Examples

The numerical examples considered in this section all employ the Gaussian solutions just developed, with the multi-Gaussian approach described in this chapter. The problem which is considered is a circular non-focused piston transducer radiating through cylindrical fluid-solid interface into an isotropic elastic solid. A schematic representation of the problem is depicted in Fig. 7.2. Two sets of comparisons will be made. First, on-axis P-wave displacement profiles will be computed with the multi-Gaussian and Gauss-Hermite beam model for concave and convex interfaces, where the radius of curvature is taken to be 50.8mm. The paraxial models will be qualitatively compared to the edge element results given in LERCH [23]. The second set compares the refracted SV-wave on-axis- and cross-axis scans of the multiple Gaussian beam model, as presented in this section, with the Gauss-Hermite beam model and a multiple Gaussian

### Table 7.1 Coefficients used in Gaussian beam model after Wen and Breazeale

<table>
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<tr>
<th>$n$</th>
<th>$A_n$</th>
<th>$B_n$</th>
<th>$R_0$</th>
<th>$w_0$</th>
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<td>4.0697</td>
<td>-1883.14</td>
<td>3.14769</td>
</tr>
<tr>
<td>2</td>
<td>0.06002</td>
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<td>20.4443</td>
<td>5.91344</td>
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<td>4.4608</td>
<td>-83.4755</td>
<td>3.0065</td>
</tr>
<tr>
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<td>4.3521</td>
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<td>3.0438</td>
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<tr>
<td>5</td>
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<td>4.5443</td>
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<td>2.9787</td>
</tr>
<tr>
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<td>3.23718</td>
</tr>
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<td>89.1439</td>
<td>3.4851</td>
</tr>
<tr>
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<td>27.0519</td>
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</tr>
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<td>3.91260</td>
</tr>
</tbody>
</table>
beam model independently derived by MARGETAN [29]. The two media modeled are water \( (v_f=1480\text{m/s}) \) and steel \( (v_p=5900\text{m/s}; v^{SV}=3200\text{m/s}) \). The transducer is modeled as a 1/2 inch diameter, 5MHz, unfocused probe. The water path is taken as \( D_f=2\text{cm} \). Figure 7.3, 7.4 show the on axis displacement for convex and concave water/steel interfaces for the normal incidence case \( (\theta^i = 0) \). It can be seen - as expected - the defocusing effect of the convex interface (the case shown in Fig. 7.3) reduces the predicted amplitudes significantly below that for the planar interface amplitude (shown in Fig. 7.5). The opposite effect is produced by the concave interface (see Fig. 7.4). The multiple Gaussian beam model shows excellent agreement with the Gauss Hermite predictions. Furthermore both models show also excellent qualitative agreement with the results given in LERCH [23] (comparisons not shown here).

The second set of comparisons considers the setup shown in Fig. 7.2 with \( \alpha = 0 \), but examines the refracted SV-wave behavior in the solid. Figures 7.6, 7.7 and 7.8 show the on axis and cross axis scans of the transmitted SV-wave in the near and far field at \( x_3 = 15\text{mm} \) and \( x_3 = 80\text{mm} \) respectively. The transducer is adjusted \( (\theta^i = 19.089^\circ) \) to generate a 45° refracted shear wave (SV). All three models show excellent agreement in the near and far field. In Fig. 7.9, 7.10 and 7.11, \( \alpha = 45^\circ \) is taken so the plane of
incidence and the principal planes of curvature do not coincide. The incident angle was kept at $\theta^i = 19.089^\circ$. Only a comparison of the two multiple Gaussian beam models are shown since the GH beam model is not capable of handling a general alignment of this type. The two multiple Gaussian predictions show no difference in the displacement amplitude. In the last set of figures, Figs. 7.12, 7.13 and 7.14, the plane of incidence was adjusted ($\alpha = 90^\circ$) to coincide with the principal plane with radius of curvature $r_2 = 50.8\text{mm}$. Again, all models show excellent agreement. Since the angle $\alpha$ has a significant effect on the response, Fig. 7.15 shows the changes of the on axis behaviour as $\alpha$ goes from $0^\circ$ to $90^\circ$. Note that this type of parametric study can not be done by the Gauss-Hermite because it cannot handle an arbitrary transducer alignment. The multi-Gaussian beam model has no such restrictions.
Figure 7.4 On-axis P wave displacement profiles for a concave cylindrical water/steel interface
Figure 7.5  On-axis P wave displacement profiles for a planar water/steel interface
Figure 7.6 On-axis SV wave displacement profiles for a cylindrical water/steel interface; $\theta^{SV} = 45^\circ$, $\alpha = 0^\circ$.

Figure 7.7 P-SV $x_2^f$-cross axis scan at $x_3^f = 15\text{mm}$ for a cylindrical water/steel interface $\theta^{SV} = 45^\circ$, $\alpha = 0^\circ$. 
Figure 7.8  P-SV $x_2$ -cross axis scan at $x_3=80$mm for a cylindrical water/steel interface $\theta^{SV} = 45^\circ$, $\alpha = 0^\circ$

Figure 7.9  P-SV on-axis scan; $\theta^{SV} = 45^\circ$, $\alpha = 45^\circ$
Figure 7.10  P-SV $x^f_2$-cross axis scan at $x^f_3=15\text{mm}$: $\theta^{SV} = 45^\circ$, $\alpha = 45^\circ$

Figure 7.11  P-SV $x^f_2$-cross axis scan at $x^f_3=80\text{mm}$: $\theta^{SV} = 45^\circ$, $\alpha = 45^\circ$
Figure 7.12  P-SV on-axis scan; $\theta_{SV} = 45^\circ$, $\alpha = 90^\circ$

Figure 7.13  P-SV $x_2$ -cross axis scan at $x_3=15\text{mm}$; $\theta_{SV} = 45^\circ$, $\alpha = 90^\circ$
Figure 7.14  P-SV $x_2^f$-cross axis scan at $x_3^f=80\text{mm}$; $\theta^{SV} = 45^\circ$, $\alpha = 90^\circ$

Figure 7.15  Comparison of the on-axis-scans; Collection of the $45^\circ$ refracted shear wave on axis plot; Case1 : $\alpha = 0^\circ$; Case2 : $\alpha = 45^\circ$; Case3: $\alpha = 90^\circ$
8 ANGULAR PLANE WAVE MODEL

In chapter 5 we developed a boundary diffraction wave model for the case where the interface is planar and the solid is a one of general anisotropy. In chapter 6 a multiple Gaussian beam model was introduced which could also handle generally curved interfaces. Both of these models are based on the paraxial approximation and fail under certain conditions as pointed out in the introduction of chapter 5.

In this chapter we consider a more 'exact' model based on an angular plane wave spectrum approach. This model will not rely on the paraxial approximation, but will use HOPKINS' method [20] for the two dimensional integration of an angular plane wave spectrum representation of the fields in the solid. The model will treat the radiation of a planar transducer through a planar interface into a general anisotropic solid. This model will be used to verify the paraxial models and can serve as an alternative beam model when needed.

The displacement in a general anisotropic solid expressed in terms of a superposition of plane waves is given by Eq.(5.8) and is repeated here for convenience in the form

$$u^\alpha(x, \omega) = \frac{i}{(2\pi)^2} \frac{V_0}{v_t} \hat{d}^\alpha \int \int \int_{-\infty}^{\infty} T^\alpha(k_{1f}^t, k_{2f}^t, k_{3f}^t) \frac{e^{i\phi^\alpha}}{k_{1f}^t} dk_{1f}^t \ dS_T. \quad (8.1)$$

where the phase is given by Eq.(5.12)

$$\phi^\alpha = k_1^\alpha x_1 + k_2^\alpha x_2 + (x_3 - D_f \cos \theta^i)k_3^\alpha + k_3fD_f \cos \theta^i - \mathbf{k}_r \cdot \mathbf{y} \quad (8.2)$$

For the location of the coordinate system see Fig. 5.1. Interchanging the surface and
the spectral integrations of Eq.(8.1) gives

$$u^\alpha(x, \omega) = \frac{i}{(2\pi)^2} \frac{v_0}{v_f} \hat{a}^\alpha \int_{-\infty}^{\infty} I(k_{1f}^i, k_{2f}^i) T^\alpha(k_{1f}^i, k_{2f}^i, k_{3f}^i) \frac{e^{i\phi^\alpha}}{k_{3f}^i} \, dk_{1f}^i \, dk_{2f}^i. \quad (8.3)$$

where we defined the phase

$$\phi^\alpha = k_1^0 x_1 + k_2^0 x_2 + (x_3 - D_f \cos \theta^i) k_3^0 + k_{3f} D_f \cos \theta^i \quad (8.4)$$

and the quantity

$$I(k_{1f}^i, k_{2f}^i) = \int e^{-ik_f \cdot y} \, dS(y). \quad (8.5)$$

The double integral over the surface of the transducer appearing in Eq.(8.5), can be evaluated analytically for the case of a circular transducer with radius $a$. The surface integral evaluates to (see Appendix E)

$$I(k_{1f}^i, k_{2f}^i) = \int e^{-ik_f \cdot y} \, dS(y) = 2\pi a \frac{J_1(k_i^0 a)}{k_i^0} \quad (8.6)$$

with $k_i = \sqrt{k_{1f}^2 + k_{2f}^2}$. In Eq.(8.6) $J_1$ is the Bessel function of the first kind of integral order one. Substituting Eq.(8.6) into Eq.(8.3) gives for the displacement field in the solid

$$u^\alpha(x, \omega) = \frac{ia}{2\pi} \frac{v_0}{v_f} \hat{a}^\alpha \int_{-\infty}^{\infty} J_1(k_i^0 a) T^\alpha(k_{1f}^i, k_{2f}^i, k_{3f}^i) \frac{e^{i\phi^\alpha}}{k_{3f}^i k_i^0} \, dk_{1f}^i \, dk_{2f}^i. \quad (8.7)$$

The integration in (8.7) over the components $k_{1f}^i$ and $k_{2f}^i$ of the wave vector $k_f$ can be replaced by integration over the angles $\Theta$ and $\Phi$, which characterize the direction of propagation of each plane wave (see Fig. 8.1). From Fig. 8.1 we have

$$k_{1f}^i = k_f \sin \Theta \cos \Phi,$$

$$k_{2f}^i = k_f \sin \Theta \sin \Phi,$$

$$k_{3f}^i = k_f \cos \Theta. \quad (8.8)$$

The area element in polar coordinates

$$dk_{1f}^i \, dk_{2f}^i = k_p^i \, dk_p^i \, d\Phi. \quad (8.9)$$
can be written, using \( k^l_p = k_f \sin \Theta \), as

\[
dk_1^l \, dk_2^l = k_f^2 \sin \Theta \cos \Theta \, d\Theta \, d\Phi. \tag{8.10}
\]

Substituting Eq.(8.10) and Eqs.(8.8) into Eq.(8.7) gives

\[
u^\alpha(x, \omega) = \frac{ia \, v_0}{2\pi \, v_f} \int_{0}^{2\pi} \int_{0}^{\pi/2} J_1(k_f a \sin \Theta) T(\Theta, \Phi) e^{i\phi(\Theta, \Phi)} \, d\Theta \, d\Phi. \tag{8.11}
\]

where

\[
\tilde{\phi}^\alpha(\Theta, \Phi) = [k_1^\alpha x_i + (k_{3f}^\alpha - k_3^\alpha) D_f \cos \theta]^\|_{\Theta, \Phi}. \tag{8.12}
\]

The subscripts \( \Theta, \Phi \) in Eq.(8.12) denote that all the wave vector components are expressed in terms of \( \Theta, \Phi \) according to Eqs.(8.8). It should be noted that strictly speaking the limit of the \( \Theta \) is \( \pi/2 - i\infty \), the imaginary part accounting for the evanescent waves. However since the evanescent waves contribute only in the region very close to the transducer face the integration can be truncated to \( \pi/2 \). Actually, as will be shown...
shortly, even this entire range on $\Theta$ is not needed since some waves will also be cut off by the interface.

Since we need to write the phase explicitly in terms of the wave vector components of the fluid, we rewrite the phase as

$$\tilde{\phi}^\alpha(\Theta, \Phi) = \left[ k_1^\alpha x_1 + k_2^\alpha x_2 + (x_3 - D_f \cos \theta^i) k_3^\alpha + k_{3f} D_f \cos \theta^i \right]_{\Theta, \Phi}. \quad (8.13)$$

To denote that for a general anisotropic solid the wave vector component $k_3^\alpha$ is given implicitly by Christoffel's equation we set $k_3^\alpha = k_3^\alpha(k_1^\alpha, k_2^\alpha)$, hence

$$\tilde{\phi}^\alpha(\Theta, \Phi) = \left[ k_1^\alpha x_1 + k_2^\alpha x_2 + (x_3 - D_f \cos \theta^i) k_3^\alpha(k_1^\alpha, k_2^\alpha) + k_{3f} D_f \cos \theta^i \right]_{\Theta, \Phi}. \quad (8.14)$$

The projections of the wave vector in the fluid and solid onto the interface are equal to each other so we can write

$$\tilde{\phi}^\alpha(\Theta, \Phi) = \left[ k_{1f} x_1 + k_{2f} x_2 + (x_3 - D_f \cos \theta^i) k_{3f}^\alpha(k_{1f}, k_{2f}) + k_{3f} D_f \cos \theta^i \right]_{\Theta, \Phi}. \quad (8.15)$$

Since the integration is conveniently done in transducer coordinates, we transform Eq.(8.15) into this coordinate system by applying the transformation (see Fig. 8.2)

$$k_{1f} = k_{1f}^1 \cos \theta^i + k_{1f}^3 \sin \theta^i, \quad k_{2f} = k_{2f}^1, \quad k_{3f} = -k_{1f}^3 \sin \theta^i + k_{3f}^1 \cos \theta^i. \quad (8.16)$$

If we further express the wave vector components in spherical coordinates we get

$$k_{1f} = k_f \left( \sin \Theta \cos \Phi \cos \theta^i + \cos \Theta \sin \theta^i \right), \quad k_{2f} = k_f \sin \Theta \sin \Phi, \quad k_{3f} = k_f \left( -\sin \Theta \cos \Phi \sin \theta^i + \cos \Theta \cos \theta^i \right). \quad (8.17)$$

Substituting Eqs.(8.17) into Eq.(8.15) finally gives

$$\tilde{\phi}^\alpha(\Theta, \Phi) = k_f \left[ \left( \sin \Theta \cos \Phi \cos \theta^i + \cos \Theta \sin \theta^i \right) x_1 + \sin \Theta \sin \Phi x_2 \right]$$

$$+ (x_3 - D_f \cos \theta^i) k_{3f}^\alpha(\Theta, \Phi) + k_f \left( \cos \Theta \cos \theta^i - \sin \Theta \cos \Phi \sin \theta^i \right) D_f \cos \theta^i. \quad (8.18)$$
The wave vector component \( k_3^f(\Theta, \Phi) \) has to be computed numerically. For a given \((\Theta, \Phi)\) the wave-vector components in the interface plane \((k_{1f}, k_{2f})\) are computed with Eqs.(8.17). Then the sixth order polynomial, Eq.(4.41), is solved to obtain \( k_3^f(\Theta, \Phi) \). In preparation for the numerical treatment of the spectral integrals we have to consider, that the Bessel function appearing in Eq.(8.7) is also oscillatory. This poses a problem with some numerical schemes because there are two competing highly oscillatory functions, \( J_1 \) and \( \exp[i\bar{\omega}a] \) that must be integrated. This problem, however, can be avoided entirely by noting that at sufficiently high frequencies of oscillation the Bessel function can be written in terms of its asymptotic expansion whose oscillatory behavior can be combined with the \( \exp[i\bar{\omega}a] \) term. We have [6]

\[
J_1(x) \approx \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{3}{4} \pi \right). 
\]

(8.19)

Figure (8.3) shows the Bessel function (dashed) indeed agrees well with its asymptotic expansion for all \( x > 5 \). Using the asymptotic form of the Bessel function we can then write Eq.(8.7) as the sum of two integrals, i.e.

\[
\mathbf{u}^a(\mathbf{x}, \omega) = \frac{ia}{2\pi} \frac{v_2}{v_f} \mathbf{d}^a \left\{ \int_0^{2\pi} J_1(k_f a \sin \Theta) T(\Theta, \Phi) e^{i\omega(\Theta, \Phi)} \ d\Theta \ d\Phi \right\}
\]
Figure 8.3 Approximation of the $J_1$ Bessel function. Shown are the Bessel function (solid line) and its approximation (dashed line)

\[
J_1(x) = \int_0^{2\pi} \int_0^\infty \frac{\sin \Theta}{\pi a \sin \Theta} \cos \left( k r a \sin \Theta - \frac{3}{4} \pi \right) T(\Theta, \Phi) e^{i\phi(\Theta, \Phi)} d\Theta d\Phi
\] (8.20)

If we now express the cosine function in exponential form

\[
\cos x = \frac{1}{2} \left( e^{ix} + e^{-ix} \right)
\] (8.21)

we get

\[
u(\mathbf{x}, \omega) = \frac{ia v_0}{2\pi \nu_t} \int_0^{2\pi} \int_0^\infty \frac{2}{\pi k r a \sin \Theta} \cos \left( k r a \sin \Theta - \frac{3}{4} \pi \right) T(\Theta, \Phi) e^{i\phi(\Theta, \Phi)} d\Theta d\Phi
\]

\[
+ \int_0^{2\pi} \int_0^\infty \frac{2}{\pi k r a \sin \Theta} T(\Theta, \Phi) \left[ e^{i\phi(\Theta, \Phi) + (k r a \sin \Theta - \frac{3}{4} \pi)} + e^{i\phi(\Theta, \Phi) - (k r a \sin \Theta - \frac{3}{4} \pi)} \right] d\Theta d\Phi
\]

(8.22)

where the phase is given by Eq.(8.18) and repeated here for convenience

\[
\tilde{\phi}(\Theta, \Phi) = k_t \left[ (\sin \Theta \cos \Phi \cos \theta + \cos \Theta \sin \theta) x_1 + \sin \Theta \sin \Phi x_2 \right]
\]

\[+(x_3 - D_f \cos \theta) k_1^0(\Theta, \Phi) + k_t \left( \cos \Theta \cos \theta - \sin \Theta \cos \Phi \sin \theta \right) D_f \cos \theta.
\] (8.23)
The integration limit $\Theta_t$ is determined by

$$k_f a \sin \Theta_t = 5 \quad \Rightarrow \quad \Theta_t = \arcsin \left( \frac{5}{k_f a} \right). \quad (8.24)$$

**Numerical Integration of Angular Spectrum Integrals**

The numerical evaluation of angular spectrum integrals falls in the category of evaluating integrals with rapid, irregular, oscillating integrands. For this class of integrals employing a general integration scheme, like **GAUSSIAN**- quadrature, **NEWTON-Cotes** etc. will prove inadequate. The reason is that due to the oscillating nature of the integrand general integration schemes need many function evaluations to achieve the desired accuracy and hence are computationally very demanding.

For many problems like the one being considered here, oscillatory integrals have, in general, the form

$$I = \int_a^b \int_c^d A(\Theta, \Phi) e^{iP(\Theta, \Phi)} \, d\Theta \, d\Phi \quad (8.25)$$

where the oscillatory part $\exp[iP(\Theta, \Phi)]$ undergoes normally a large number of oscillations over the aperture domain. This integral has attracted a large number of approaches in the past. One of the earliest methods was suggested in 1928 by **FILON** [12]. **FILON**'s method is equivalent to **SIMPSON**'s rule but for an oscillatory weight. Later, **CLENDENIN** [8] formulated a method equivalent to the trapezium rule and **FLINN** [13] developed a rule with polynomials of degree five. All these methods rely on the basic idea of fitting the integrand to a polynomial.

There are also other ways to approach finite range oscillatory integrals with non-trigonometric oscillations that do not fit the integrand to a polynomial and are therefore appropriate for treating non-trigonometric oscillations. **EHRENMARK** [10], for example, formulated a simple three point formula for the evaluation of general oscillatory integrals. The main advantage of **EHRENMARKS** formula lies in the simpleness of the formula itself, but the method has low accuracy. **LEVIN** [24] chose a different approach, in the spirit of
FILON's method, and reduced the problem of integration to solving linear systems. This method is well suited for multidimensional integration of rapidly oscillatory functions. Our method of choice, however, is HOPKINS' [20] algorithm since this method was found to be very stable and easy to implement, in contrast to LEVIN's method, which we initially considered as a candidate numerical scheme, where the resulting linear system of equations were badly conditioned.

Hopkins' Algorithm

In HOPKINS' method the integration domain is divided into $N \times M$ rectangular subdomains (see Fig. 8.4). Each of the rectangles have the sides of length $\delta_\theta$ in $\Theta$-direction and $\delta_\phi$ in $\Phi$ direction. In each subdomain the phase function is expanded in a Taylor series about the midpoint, discarding second and higher terms in the Taylor series. i.e.

$$P(\Theta, \Phi) = P(\Theta_m, \Phi_n) + P_\Theta(\Theta_m, \Phi_n)(\Theta - \Theta_m) + P_\Phi(\Theta_m, \Phi_n)(\Phi - \Phi_n). \quad (8.26)$$

where $P_\Theta$ represents the derivative of $P$ in respect to $\Theta$. If we further approximate the amplitude function $A(\Theta, \Phi)$ with the value of the function at the midpoint $A(\Theta_m, \Phi_n)$ we can express the integral over each subdomain in terms of a simple formula, and find the value of the integral by adding the contributions from all subdomains. Thus we have for one subdomain

$$I_{mn} = A(\Theta_m, \Phi_n) \int_{\Theta_m - \frac{1}{2} \delta_\theta}^{\Theta_m + \frac{1}{2} \delta_\theta} \int_{\Phi_n - \frac{1}{2} \delta_\phi}^{\Phi_n + \frac{1}{2} \delta_\phi} e^{i(P(\Theta_m, \Phi_n) + P_\Theta(\Theta_m, \Phi_n)(\Theta - \Theta_m) + P_\Phi(\Theta_m, \Phi_n)(\Phi - \Phi_n))} \, d\Theta \, d\Phi \quad (8.27)$$

This integration can be done analytically to yield

$$I_{mn} = \delta_\theta \delta_\phi A(\Theta_m, \Phi_n) e^{iP(\Theta_m, \Phi_n)} \text{sinc} \left[ \frac{1}{2} \delta_\theta P_\Theta(\Theta_m, \Phi_n) \right] \text{sinc} \left[ \frac{1}{2} \delta_\phi P_\Phi(\Theta_m, \Phi_n) \right]. \quad (8.28)$$
where \( \text{sinc}(x) = \frac{\sin(x)}{x} \). The whole integral is then found by summing all contributions

\[
I = \sum_{m=1}^{M} \sum_{n=1}^{N} \delta_\Theta \delta_\Phi \cdot \Gamma(\Theta_m, \Phi_n) e^{iP(\theta_m, \phi_n)} \text{sinc} \left[ \frac{1}{2} \delta_\Theta P_\Theta(\Theta_m, \Phi_n) \right] \text{sinc} \left[ \frac{1}{2} \delta_\Phi P_\Phi(\Theta_m, \Phi_n) \right]
\]

Both spectral integrals appearing in Eq.(8.22) will be evaluated with Hopkins' method. For the first spectral integral containing the Bessel function explicitly the \( \Theta - \Phi \)-domain is discretized using a \( 128 \times 128 \) fine grid. For the second integral containing the asymptotic form of \( J_1 \) the \( \Theta - \Phi \)-domain is discretized by a \( 128 \times 64 \) mesh. For the \( \Theta \)-integral special attention has to be paid for the case where \( k^2_3(\Theta, \Phi) \) becomes complex. For this case the wave vector components \( k^2_3(\Theta, \Phi) \) are evanescent in the solid so that they do not contribute to the bulk refracted wave and are not taken into account. The computational procedure for finding the \( k^2_3(\Theta, \Phi_m) \) and \( T^\alpha(\Theta_n, \Phi_m) \) values is described below.

A schematic representation of the \( (\Theta - \Phi) \)-domain is given in Fig. 8.5. The double
line in this figure represents the transition of \( s_3^\Theta \) from a real quantity to complex. Hence, to the right of that line the waves are evanescent and this region does not contribute to the response we are calculating. To accurately determine this transition line the following procedure is employed. The whole domain is divided into three regions (see Fig. 8.5). Region 1 is a rectangle taken from \( \Theta = 0^\circ \) to a \( \Theta \)-value where \( s_3^\Theta \) first becomes complex. In this first zone the \( s_3^\Theta(\Theta_m, \Phi_n) \) and the \( T^\Theta(\Theta_m, \Phi_n) \) are evaluated with the procedures described in chapter 4 for a coarse \( 16 \times 16 \) mesh. Since the numerical integration scheme requires a finer mesh for accuracy in evaluating the phase term, those additional points needed are found by means of interpolation. In Region 2 (which normally is very small in comparison to Region 1) the values \( s_3^\Theta(\Theta_n, \Phi_m) \) and \( T^\Theta(\Theta_m, \Phi_n) \) are found directly on a fairly fine mesh (\( 16 \times 16 \)). With this fine mesh it is possible to then determine the transition line and carry out the integrations over Region 1 and Region 2.

![Figure 8.5 Schematic representation of the computational region, separated into three zones](image-url)
9 RESULTS

Isotropic Cases

In this chapter the beam models presented in chapter 5 through 8 will be evaluated and compared to each other. For the first cases we consider a 1/2 inch 5 MHz piston probe mounted on a lucite wedge radiating a 45° refracted P-wave into steel where the shear strength of the lucite is neglected so we can use our fluid solid model. The distance $D_f$ in the lucite wedge was 1.8 cm. Fig. 9.1 shows a central axis scan of the P-wave at a refracted angle of 45°. Both paraxial models are compared (multi-Gaussian (solid), BDW (circles)) along with the angular plane wave spectrum model (dashed) and the surface integral model of Lerch (cross) [21]. Good agreement of all four models is found in the far field as expected. In the near field the predictions of paraxial models differ somewhat from both of the more exact models. To examine their differences in the very near field more closely we present a scan along the $x^j$-axis (parallel to the interface) 4.5 cm below the interface (Fig. 9.2). It can be seen, that the general shape of the curve is predicted well by both paraxial models. Overall, the paraxial models locate the main null adequately, although their profiles are slightly shifted. This shifting is likely the cause of the on-axis disagreement in the near field. The detailed off-axis structure is not as well predicted, as expected. Fig. 9.3 shows a cross-axis scan along the $x^i$-direction, also in the near field but not as close to the interface. Excellent agreement is found between the paraxial models and the APW model. As shown in Fig. 9.4 the paraxial models have no problem in predicting the far field cross axis response. Figures
9.5 through 9.7 show the same kinds of plots for a $45^\circ$ refracted SV-wave. It is interesting to notice that the paraxial models seem to predict the near field in the shear case better than in the P-wave case (compare Fig. 9.1 with Fig. 9.5). There are no significant difference between the exact and the paraxial models in the cross-axis scans (Fig. 9.6 and Fig. 9.7).

![Diagram showing P-P on-axis scan; $\theta^P = 45^\circ$](image)

Figure 9.1 P-P on-axis scan; $\theta^P = 45^\circ$
Figure 9.2  P-P cross axis scan at $x_1 = 4.5$ mm parallel to the interface; $\theta^P = 45^\circ$.

Figure 9.3  P-P cross axis scan along the $x_1$ axis at $x_3 = 15$ mm; $\theta^P = 45^\circ$. 
Figure 9.4  P-P cross axis scan along the $x_1^f$ axis at $x_3^f=80$mm: $\theta^P = 45^\circ$

Figure 9.5  P-SV on-axis scan: $\theta^{SV} = 45^\circ$
Figure 9.6 P-SV cross axis scan along the $x_i^r$ axis at $x_3^r=15\text{mm}$: $\theta^\text{SV} = 45^\circ$

Figure 9.7 P-SV cross axis scan along the $x_i^r$ axis at $x_3^r=80\text{mm}$: $\theta^\text{SV} = 45^\circ$
Anisotropic Cases

The anisotropic examples we will discuss are divided into twelve different cases. The first six cases deal with planar interfaces to demonstrate the effects of the anisotropy upon the transmitted beam, whereas cases six through twelve examine the additional effects arising from a curved interface. Tab. 9.1 summerizes the parameters used for the different cases.

Table 9.1 Different cases for anisotropic examples: The (*) in the \( \theta \) column indicates that the incident angle is adjusted to produce a 45° refracted wave.

<table>
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<tr>
<th>Case</th>
<th>Interf.</th>
<th>( R_x )</th>
<th>( R_y )</th>
<th>( \alpha )</th>
<th>( \theta )</th>
<th>Material</th>
<th>( \xi )</th>
<th>( \eta )</th>
<th>Wave Type</th>
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<td>( \infty )</td>
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<td>( \infty )</td>
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<td>45 (*)</td>
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<td>0</td>
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<td>( \infty )</td>
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<td>20</td>
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<td>CSS</td>
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<td>20</td>
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<td>qP</td>
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The first five cases model a 1/2 inch diameter, 5MHz transducer. For case one, the transducer is radiating at normal incidence into austenitic steel (CSS). The coupling media is again lucite \((D_f=1.8\text{cm})\) and the grains of the CSS are aligned with the \( x_3 \)-axis \((\eta = 0^\circ)\). The slowness curve for this case is shown in Fig. 9.9. As indicated by the arrow in the slowness curve the group velocity direction is along the \( x_3 \)-axis. Figure 9.10 shows the beam profile for this case as predicted from the multiple Gaussian beam model. A comparison of the on-axis responses as predicted from all three beam models is shown in Fig. 9.11 for the on-axis displacement. Very good agreement is achieved in the
far field and the near field by the paraxial models. Furthermore, good agreement is seen as well in the cross axis scans (taken in the plane of incidence normal to the refracted ray), shown in Figs. 9.12-9.15 at the distances $x_1^f = 5\text{mm}$, $x_2^f = 1\text{cm}$, $x_3^f = 2.5\text{cm}$ and $x_4^f = 5\text{cm}$ below the interface respectively, except near the central axis in the very near field (Fig. 9.12). As we have seen in the isotropic case the paraxial models predict the general shape of all the curves very well. Differences normally are seen only in the detailed side lobe structure as expected.

For case 2, we now rotate the austenitic steel so that the grains of the steel form a $10^\circ$ angle with the $x_3$-axis. As it can be seen from Fig. 9.16 the group velocity does not coincide with the phase velocity in this case. As a consequence the beam will shift to the group velocity direction as seen in the beam profile (Fig. 9.17). The cross-axis profiles taken at $x_1^f = 5\text{mm}$, $x_2^f = 10\text{mm}$ and $x_3^f = 50\text{mm}$ again show good agreement between all the beam models (Fig. 9.18-9.21). It is also interesting to note that the paraxial models predict a symmetric beam profile (see Fig. 9.21) but it is really somewhat unsymmetric.

In the third case of comparisons we adjust the transducer to an incident angle of $10^\circ$. radiating into austenitic steel ($\eta = 0^\circ$). The transmitted qP beam profile for this case is shown in Fig. 9.22. Figs. 9.23-9.26 show the P-qP wave displacements in cross-axis profiles taken $5\text{mm}$, $10\text{mm}$, $25\text{mm}$ and $50\text{mm}$ deep into the steel, respectively. Again very good agreement is seen between the paraxial models and the APW model. Also the near field structure is predicted very well by the paraxial models (compare Fig. 9.2 of the isotropic case).

In case 4 we consider the focusing/defocusing effects caused by the properties of the qSV-slowness surface. Fig. 9.27 shows the beam profile for case 4 where we have a $45^\circ$ refracted qSV-wave into austenitic steel ($\eta = 45^\circ$). In this case the phase velocity direction coincides with the group velocity direction and the slowness surface is concave near the refracted ray. The slowness surface together with the slowness patch is depicted in Fig. 9.28. To see the goodness of the fit for the $A^\alpha$, $B^\alpha$, $D^\alpha$, $E^\alpha$ parameters, Fig.
9.29 shows the slowness patch near the refracted ray as a mesh and the approximation plotted as stars.

In case 5 we adjust the transducer to produce a 45° refracted qSV-wave, but this time the grains are in $x_3^*$-direction and the slowness surface is convex near the refracted ray. The corresponding beam profile is shown in Fig. 9.30, the slowness surface with patch in Fig. 9.31 and the patch with its approximation in Fig. 9.32. Comparing Figs. 9.27 and Fig. 9.30 one can see there are indeed significant changes in the beam patterns caused by the difference in the slowness surface properties.

In case 6 we model a 2.25MHz, 1/2 inch diameter transducer radiating into a graphite-epoxy solid ($\xi = 5^\circ, \eta = 20^\circ$). In this case all three wave types (qP, qS1, qS2) will be generated. Fig. 9.33 through Fig. 9.39 show the beam profiles of the different type of waves. It can be seen from the beam profiles of the qS1 and qS2-wave (compare Fig. 9.34 with Fig. 9.35), that the beam steering is to the left of the phase velocity for the qS1-wave and to the right for the qS2-wave. The corresponding slowness surface and their approximations for this case are shown for the qP and qS1 wave in Fig. 9.36 through Fig. 9.39.

In the remaining cases to be discussed we consider the case where now the interface is curved and the solid anisotropic (see Fig. 9.8). In all these cases we return to a 5MHz, 1/2 inch diameter transducer. First, in case 7 the transducer is adjusted at normal incidence to a cylindrical interface having a radius of curvature of -50.8mm (setup 2). The grains of the austenitic steel are at 20° to the $x^*_3$-axis. Fig. 9.40 shows the beam profile of the qP wave. Two effects can be seen in this figure, the focusing arising from the curvature of the interface and the beam steering due to the anisotropy. A cross axis profile, at a distance of $x^*_3=10$mm below the interface, is shown in Fig. 9.41. Similarly, a beam profile and both on-axis and cross axis plots are shown for an incident angle of $\theta^i = 10^\circ$ in Figs. 9.43-9.45. Figs. 9.46-9.49 show the beam profile and both on- and cross axis scans for case 9 with a transducer radiates at normal incidence into a concave
curvature ($R_x = 50.8$mm, setup 1), and in Fig. 9.50-9.53 for case 10 where $\theta^i = 10^\circ$. Again beam steering, arising from the anisotropy, and now defocusing effects from the curvature can be seen.

In Case 11 the plane of incidence and the principal plane of curvature does not coincide anymore ($\alpha = 45^\circ$). The incident angle was kept at $10^\circ$. Fig. 9.54 shows the beam profile of the qP-wave in the $x_1^i - x_3^i$-plane whereas Fig. 9.55 shows the beam profile in the $x_2^i - x_3^i$ plane. Fig. 9.56 through Fig. 9.58 show on axis and cross axis scans at different locations.

The last set of figures (Case 12, setup 3) considers the case where the curvature in $x$-direction is positive $R_x = 50.8$mm and negative in $y$-direction ($R_y = -50.8$mm). Again Fig. 9.59 displays the beam profile in the $x_1^i - x_3^i$-plane whereas Fig. 9.60 the beam profile in the $x_2^i - x_3^i$ plane shows. One can see from Fig. 9.59 and Fig. 9.60 that we have definite focusing effects in one plane and defocusing effects in the other. Fig. 9.61 shows the on-axis scan, which displays a combination of these focusing/defocusing effects (compare Fig. 9.43 and Fig. 9.51).
Figure 9.9 Case 1: qP-slowness curve for austenitic steel
Figure 9.10  Case 1: Beam Profile austenitic steel

Figure 9.11  Case 1: P-qP on-axis scan
Figure 9.12  Case 1: P-qP cross axis scan along the $x_1^r$ axis at $x_5^r=5\text{mm}$
Figure 9.13  Case 1: P-qP cross axis scan along the \( x_1 \)-axis at \( x_3 = 10 \text{mm} \)

Figure 9.14  Case 1: P-qP cross axis scan along the \( x_1 \)-axis at \( x_3 = 25 \text{mm} \)
Figure 9.15  Case 1: P-qP cross axis scan along the $x^*_1$ axis at $x^*_3=50\text{mm}$

Figure 9.16  Case 2: qP slowness curve for austenitic steel
Figure 9.17 Case 2: Beam profile for the qP-wave in austenitic steel

Figure 9.18 Case 2: P-qP cross axis scan along the $x_i^1$ axis at $x_5^i$=5mm
Figure 9.19  Case 2: P-qP cross axis scan along the $x_1$ axis at $x_3 = 10 \text{mm}$

Figure 9.20  Case 2: P-qP cross axis scan along the $x_1$ axis at $x_3 = 25 \text{mm}$
Figure 9.21  Case 2: P-qP cross axis scan along the $x_1$ axis at $x_1=50\text{mm}$

Figure 9.22  Case 3: Beam profile of a qP-wave traveling in austenitic steel
Figure 9.23  Case 3: P-qP cross axis scan along the $x_1^i$ axis at $x_3^f$=5mm

Figure 9.24  Case 3: P-qP cross axis scan along the $x_1^i$ axis at $x_3^f$=10mm
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Figure 9.37  Case 6: Slowness patch of Fig.(9.36) with its approximation
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Figure 9.39  Case 6: Slowness patch of Fig.(9.38) with its approximation
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Figure 9.42 Case 8: Beam profile of a qP wave in an anisotropic cylinder: $R_x = -50.8$mm, $R_y = 0$
Figure 9.43  Case 8: qP on-axis scan: $R_x = -50.8\text{mm}, R_y = 0$

Figure 9.44  Case 8: P-qP $x_1^i$-cross axis scan at $x_3^i=15\text{mm}$ below the interface; $R_x = -50.8\text{mm}, R_y = 0$
Figure 9.45  Case 8: P-qP $x_1^i$-cross axis scan at $x_3^i=40$mm below the interface: $R_x = -50.8$mm, $R_y = 0$
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Figure 9.47  Case 9: qP on-axis scan:  $R_x = 50.8\text{mm}, R_y = 0$

Figure 9.48  Case 9: P-qP $x_1^j$-cross axis scan at $x_3^j=15\text{mm}$ below the interface;  $R_x = 50.8\text{mm}, R_y = 0$
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\( R_x = 50.8\text{mm}, R_y = 0 \)
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Figure 9.52 Case 10: \( P-qP \) \( x_3^i \)-cross axis scan at \( x_3^i = 15 \text{mm} \) below the interface; \( R_x = 50.8 \text{mm}, R_y = 0 \)
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$R_x = 50.8\text{mm}, R_y = 0$

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$R_x = 50.8\text{mm}, R_y = 0$
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Figure 9.61 Case 12: qP on-axis scan in an anisotropic cylinder: $R_x = 50.8\,\text{mm}, R_y = -50.8$
10 CONCLUSIONS AND DISCUSSION

We have developed in detail three beam models for an immersion testing setup where the waves must travel from a transducer located in a fluid across a fluid-solid interface into an anisotropic medium. Two of these models, the Boundary diffraction wave (BDW) model and the multi-Gaussian beam model (MG), represent significant modeling advances. The third model, the angular spectrum of plane waves (APW) model, while not new, nevertheless is an important 'exact' model that can be used to evaluate the limitations other two models and to supply an alternative model when the others may fail.

The BDW model as presented can model the propagation of a sound beam from a planar transducer through a plane interface into a general anisotropic medium. One important property of the BDW model is that it represents the solution in terms of direct and edge wave contributions that have an explicit physical meaning. In addition, the model can treat planar transducers of arbitrary shape, a feature that is not shared by many other models. There are however, two limitations of the BDW model. First, because the model is based on the paraxial approximation, it inherently can fail under special conditions such as in the very near field, near critical angles in the solid where the transmission coefficient can be rapidly varying, at high refracted angles, and in cases where the material properties of the anisotropic solid may be rapidly varying over the beam of sound cross-section. Second, because the model is also based on a ray theory approach, it can fail at locations in the anisotropic solid where the anisotropy produces a focal point. This latter limitation occurs only for the shear waves in an anisotropic
medium so it does not arise for inspections that use quasi-P waves. Because many of these conditions do not ordinarily arise in many NDE tests, however, the BDW model can be a very useful model in practice. Since the model only involves a single integration, it is numerically very efficient and can be used for real time simulations.

The multi-Gaussian model presented here is a major advance in that it is a model that can handle both general curved interfaces and general anisotropic materials and does so in a manner that only requires a small number of analytical evaluations. Although multi-Gaussian modeling approaches are not new, our formulation of the MGB model using the angular eikonal approach leads to simple analytical expressions that are explicit, general and numerically efficient. Like the BDW model the multi-Gaussian approach relies on the paraxial approximation, so it also can fail under those special conditions where that approximation breaks down, but the MG model does not have difficulty with material focusing effects as found in the BDW model. Unlike the BDW model. However, the MGB model is currently limited to modeling only circular transducers.

The angular spectrum of plane waves model is based on numerically integrating a spectrum of plane waves to simulate a beam response using Hopkins' method. Since the APW does not rely on the paraxial approximation, it is a more 'exact' model, but also one that is not computationally very efficient. As implemented, the model is for the radiation of circular piston transducers through a plane interface into a general anisotropic solid, so it cannot deal with the complexity of a general curved interface.

Nevertheless, it can serve to validate the paraxial models and to replace them in special situations where the paraxial models fail.

As can be seen from the many results discussed in this thesis and from the comparisons made between models, the paraxial models provide in many cases powerful modeling tools that are both fast and accurate. This is remarkable given that the MG model, in particular, can handle both complicated geometries and material behavior.

There are, of course, a number of places where these models could be extended
or improved. The BDW model, for example, could be readily used to model planar transducers of shapes other than the circular ones considered here. The MG model, on the other hand, can be readily extended to treat the reflection and transmission of beams at multiple curved interfaces. Also, the MG model can be extended to handle focused probes. In fact, to model a spherically focused probe of focal length $F$ and radius $a$, all that is needed is to modify the Wen and Breazeale $B_n$ coefficients, e.g.

$$B_n \rightarrow B_n + \frac{ik_1a^2}{2F}$$

(10.1)

where $k_1$ is the wave number for the fluid. It may be possible to extend the MG model to model more general transducer shapes and output velocity profiles, but those extensions will likely not be easy to make. One modification of the MG model that would be easy is to use average interface surface curvatures in the model rather than the local values along the central fixed ray, as currently implemented. This would be in the same spirit as done with the slowness surface curvatures and might help improve the accuracy of the MGB model in cases where the curvatures vary over the beam "footprint" on the interface. The APW model also could be generalized in several ways. In particular, the spectrum kernel could be changed to handle other transducer shapes, such as elliptical or rectangular, without significantly affecting the numerical procedures. Also, other, more efficient numerical methods could certainly be used to replace Hopkins' method, which is based on a rather low order approximation. A more significant extension would be to find an 'exact' method that could efficiently handle the complexity of a curved interface as well as the material anisotropy. A boundary element approach that uses wavelets as a set of basis functions is one possibility that might be able to deal with the highly oscillatory fields found in such transducer models.

Perhaps the most significant next step with all these models, however, is in the area of experimental validation. Experimental tests that define the range of applicability of the models in real testing situations are essential. Such experiments could also help in
obtaining more general acceptance by the NDE community of such modeling approaches.

Another area that is a natural follow-on to the models presented in this thesis is the incorporation of these models into the general measurement model approach described in chapter one. To completely model ultrasonic responses in anisotropic media, however, effective models of flaw scattering responses in anisotropic media need also to be developed.

Finally, one long term generalization of these beam models that could also be considered is their extension to inhomogeneous, anisotropic media. Such extensions are important because anisotropy is often accompanied with material texture, particularly in weld testing problems, for example. Although direct numerical techniques such as finite elements could be used for such problems, more explicit models of the type described here could be invaluable.
APPENDIX A  SYMMETRY CLASSES AND STIFFNESS MATRICES FOR DIFFERENT MATERIALS

This appendix lists explicitly the stiffness matrices for certain symmetry classes

Matrix forms of stiffness

Triclinic

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{pmatrix} =
\begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\
C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\
C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}
\end{pmatrix}
\begin{pmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
\gamma_{23} \\
\gamma_{13} \\
\gamma_{12}
\end{pmatrix} \quad (A.1)
\]
Monoclinic

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{pmatrix} =
\begin{pmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\
C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\
C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\
0 & 0 & 0 & C_{44} & C_{45} & 0 \\
0 & 0 & 0 & C_{45} & C_{55} & 0 \\
C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66}
\end{pmatrix}
\begin{pmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
\gamma_{23} \\
\gamma_{13} \\
\gamma_{12}
\end{pmatrix}.
\]

Orthotropic

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{pmatrix} =
\begin{pmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{pmatrix}
\begin{pmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
\gamma_{23} \\
\gamma_{13} \\
\gamma_{12}
\end{pmatrix}.
\]

Transversely Isotropic

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{pmatrix} =
\begin{pmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\
C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{pmatrix}
\begin{pmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
\gamma_{23} \\
\gamma_{13} \\
\gamma_{12}
\end{pmatrix}.
\]
Cubic

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{pmatrix} = 
\begin{pmatrix}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\
C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{66} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{66} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{pmatrix} 
\begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{23} \\
\gamma_{13} \\
\gamma_{12}
\end{pmatrix}
\tag{A.5}
\]

Isotropic

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{pmatrix} = 
\begin{pmatrix}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\
C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{44}
\end{pmatrix} 
\begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{23} \\
\gamma_{13} \\
\gamma_{12}
\end{pmatrix}
\tag{A.6}
\]

The constants $C_{11}$ and $C_{44}$ are defined, by the longitudinal $v_l$ and shear velocity $v_t$, as

\[
C_{12} = C_{11} - 2C_{44} \tag{A.7}
\]

\[
C_{11} = \rho v_l^2. \tag{A.8}
\]

\[
C_{44} = \rho v_t^2. \tag{A.9}
\]

**Stiffness matrices for various materials**

The units for

- Listed stiffnesses $C_{ij}$ are $10^9 \text{N/m}^2$
- Listed material densities $\rho$ are $10^3 \text{kg/m}^3$
Graphite-Epoxy (Transversely Isotropic)

\[
C_{ij}^{GrEp} = \begin{bmatrix}
145.8 & 3.72 & 3.72 & 0 & 0 & 0 \\
3.72 & 16.34 & 4.96 & 0 & 0 & 0 \\
3.72 & 4.96 & 16.34 & 0 & 0 & 0 \\
0 & 0 & 0 & 3.37 & 0 & 0 \\
0 & 0 & 0 & 0 & 7.48 & 0 \\
0 & 0 & 0 & 0 & 0 & 7.48
\end{bmatrix}
\]

\[\rho^{GrEp} = 1.6\] (A.10)

Graphite-Epoxy [03/90] (orthotropic)

\[
C_{ij}^{GrEp} = \begin{bmatrix}
94 & 7.4 & 8.2 & 0 & 0 & 0 \\
7.4 & 13 & 9.1 & 0 & 0 & 0 \\
8.2 & 9.1 & 34 & 0 & 0 & 0 \\
0 & 0 & 0 & 3.6 & 0 & 0 \\
0 & 0 & 0 & 0 & 7.2 & 0 \\
0 & 0 & 0 & 0 & 0 & 4.2
\end{bmatrix}
\]

\[\rho^{GrEp} = 1.6\] (A.12)
Graphite-Epoxy $[00/90]$ (orthotropic)

\[
C_{ij}^{\text{GrEp}} = \begin{pmatrix}
70 & 6.9 & 7.0 & 0 & 0 & 0 \\
6.9 & 13 & 7.8 & 0 & 0 & 0 \\
7.0 & 7.8 & 69 & 0 & 0 & 0 \\
0 & 0 & 0 & 3.8 & 0 & 0 \\
0 & 0 & 0 & 0 & 5.6 & 0 \\
0 & 0 & 0 & 0 & 0 & 3.5
\end{pmatrix}
\]  
(A.14)

\[\rho^{\text{GrEp}} = 1.6 \]  
(A.15)

Graphite-Epoxy $[00/45/90/-45]$ (orthotropic)

\[
C_{ij}^{\text{GrEp}} = \begin{pmatrix}
50 & 6.8 & 19.0 & 0 & 0 & 0 \\
6.8 & 13 & 8.2 & 0 & 0 & 0 \\
19.0 & 8.2 & 57 & 0 & 0 & 0 \\
0 & 0 & 0 & 3.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 21.0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3.3
\end{pmatrix}
\]  
(A.16)

\[\rho^{\text{GrEp}} = 1.6 \]  
(A.17)
Austenitic Steel (Transversely Isotropic)

\[
C_{\text{Austenit}}^{ij} = \begin{pmatrix}
2.4110 & 0.9692 & 1.3803 & 0 & 0 & 0 \\
0.9692 & 2.4110 & 1.3803 & 0 & 0 & 0 \\
1.3803 & 1.3803 & 2.4012 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.1229 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.1229 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.7209
\end{pmatrix}
\]  
(A.18)

\[
\rho_{\text{Austenit}} = 7.82
\]  
(A.19)

InAs (Cubic)

\[
C_{\text{InAs}}^{ij} = \begin{pmatrix}
83.29 & 45.26 & 45.26 & 0 & 0 & 0 \\
45.26 & 83.29 & 45.26 & 0 & 0 & 0 \\
45.26 & 45.26 & 83.29 & 0 & 0 & 0 \\
0 & 0 & 0 & 39.59 & 0 & 0 \\
0 & 0 & 0 & 0 & 39.59 & 0 \\
0 & 0 & 0 & 0 & 0 & 39.59
\end{pmatrix}
\]  
(A.20)

\[
\rho_{\text{InAs}} = 5.67
\]  
(A.21)
GaAs (Cubic)

\[
C_{ij}^{GaAs} = \begin{pmatrix}
83.29 & 45.26 & 45.26 & 0 & 0 & 0 \\
45.26 & 83.29 & 45.26 & 0 & 0 & 0 \\
45.26 & 45.26 & 83.29 & 0 & 0 & 0 \\
0 & 0 & 0 & 39.59 & 0 & 0 \\
0 & 0 & 0 & 0 & 39.59 & 0 \\
0 & 0 & 0 & 0 & 0 & 39.59
\end{pmatrix} \tag{A.22}
\]

\[
\rho^{GaAs} = 5.67 \tag{A.23}
\]
This appendix lists the three dimensional slowness surfaces for different materials for the \( qP, qS1 \) and \( qS2 \) waves.

Figure B.1  Slowness surface of the \( qP \)-wave traveling in graphite-epoxy for \( \eta = 90^\circ \)
Figure B.2  Slowness surface of the qS1-wave traveling in graphite-epoxy for \( \eta = 90^\circ \)

Figure B.3  Slowness surface of the qS2-wave traveling in graphite-epoxy for \( \eta = 0^\circ \)
Figure B.4  Slowness surface of the qP-wave traveling in austenitic steel for $\eta = 0^\circ$

Figure B.5  Slowness surface of the qS1-wave traveling in austenitic steel for $\eta = 0^\circ$
Figure B.6 Slowness surface of the qS2-wave traveling in austenitic steel for $\eta = 0^\circ$
APPENDIX C  THE ANGLE EIKONAL $W(k_{1f}^i, k_{2f}^i, k_2^{r\alpha}, k_2^{r\alpha})$

As shown in chapter 6 the angle eikonal can be written as

$$W^\alpha(k_{1f}^i, k_{2f}^i, k_1^{i\alpha}, k_2^{i\alpha}) = \mathcal{L}^{i\alpha}x_1^i + \mathcal{M}^{i\alpha}x_2^i + \mathcal{N}^{i\alpha}x_3^i$$

(C.1)

where

$$\mathcal{L}^{i\alpha} = k_{1f}^{i} - k_1^{i\alpha} ; \quad \mathcal{M}^{i\alpha} = k_{2f}^{i} - k_2^{i\alpha} ; \quad \mathcal{N}^{i\alpha} = k_{3f}^{i} - k_3^{i\alpha}.$$  

(C.2)

provided that Eq.(C.1) is made stationary with respect to the surface coordinates $x_1^i, x_2^i$ (see Fig.(C.1)), i.e.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{angle_eikonal.png}
\caption{Determination of the angle eikonal}
\end{figure}
\[
\begin{align*}
\frac{\partial W^\alpha}{\partial x_1} &= 0 \\
\frac{\partial W^\alpha}{\partial x_2} &= 0.
\end{align*}
\] (C.3) (C.4)

Since we are only interested in the close vicinity of the central ray we can express the local shape of the surface as

\[
x^j_3 = \frac{1}{2} h_{11} x^1_1 x^2_2 + \frac{1}{2} (h_{12} + h_{21}) x^1_1 x^3_3 + \frac{1}{2} h_{22} x^2_2 x^3_3, \quad h_{12} = h_{21},
\] (C.5)

where \( h_{ij} \) is the curvature tensor. For cases where the plane of incidence is aligned with the principal directions of the surface the \( h_{ij} \)'s are related to the principal radii of curvature \( r_1, r_2 \) by

\[
\begin{align*}
h_{11} &= \frac{1}{r_1} \\
h_{22} &= \frac{1}{r_2} \\
h_{12} &= 0.
\end{align*}
\] (C.6)

Requiring that the angle eikonal be stationary, the derivatives of the eikonal with respect to the interface coordinates have to vanish, i.e.

\[
\begin{align*}
\frac{\partial W^\alpha}{\partial x_1} &= L^{i\alpha} + N^{i\alpha} (h_{11} x^1_1 + h_{12} x^3_3) = 0 \quad \text{(C.7)} \\
\frac{\partial W^\alpha}{\partial x_2} &= M^{i\alpha} + N^{i\alpha} (h_{12} x^1_1 + h_{22} x^3_3) = 0. \quad \text{(C.8)}
\end{align*}
\]

Solving the latter system of equations for the stationary angular eikonal point then gives

\[
\begin{align*}
x^1_i &= -\frac{M^{i\alpha} h_{12} - h_{22} L^{i\alpha}}{(h_{11} h_{22} - h_{12}^2) N^{i\alpha}}, \quad \text{(C.9)} \\
x^3_i &= -\frac{L^{i\alpha} h_{12} - h_{11} M^{i\alpha}}{(h_{11} h_{22} - h_{12}^2) N^{i\alpha}}. \quad \text{(C.10)}
\end{align*}
\]

so that substituting the stationary angular eikonal point back into the angular eikonal yields

\[
W^\alpha = -\frac{1}{2 (h_{11} h_{22} - h_{12}^2) N} \left[ h_{22} (L^\alpha)^2 - 2 h_{12} L^\alpha M^\alpha + h_{11} (M^\alpha)^2 \right], \quad \text{(C.11)}
\]
with $L^{i\alpha}, M^{i\alpha}$ and $N^{i\alpha}$ given by Eq.(C.2). This form of the angular eikonal involves wave vector components measured in interface coordinates. As required in the integral transformation (6.9) the wave vector components in the fluid have to be expressed in transducer coordinates and the wave vector components in the solid in ray coordinates. Consistent with the paraxial approximation, in both the solid and the fluid we will only keep wave vector components to second order. For performing that expansion we define the origins of the transducer and the ray coordinate systems to be at the point of intersection of the base ray with the surface. The $x_1^i$ and $x_3^i$ axes lie in the plane established by the base ray and the surface normal. For the central ray itself

$$k_{1f}^i = k_{1f}^{i\alpha} \quad \Rightarrow \quad L = 0$$

(C.12)
due to Snell’s law. Since $x_1^i, x_3^i$ lies in the plane of incidence

$$k_{2f}^i = k_{2f}^{i\alpha} = 0 \quad \Rightarrow \quad M = 0.$$  \hspace{1cm} (C.13)

and

$$N = k_{3f}^i - k_{3f}^{i\alpha} = k_f \cos \theta^i - k^\alpha \cos \theta^\alpha.$$  \hspace{1cm} (C.14)

If we now consider an ray close to the central ray (see. Fig.(C.2)) we can transform the wave vectors as measured in the interface coordinate system into wave vectors measured in transducer coordinates via

$$k_{1f}^i = \cos(\theta^i + d\theta^i)k_{1f}^i + \sin(\theta^i + d\theta^i)k_{3f}^i$$  \hspace{1cm} (C.15)

$$k_{2f}^i = k_{2f}^i$$ \hspace{1cm} (C.16)

The $k_{1f}^i$ component can be approximated by first expanding Eq.(C.15) using trigonometric relations, i.e.

$$k_{1f}^i = k_{1f}^i (\cos \theta^i \cos d\theta^i - \sin \theta^i \sin d\theta^i) + k_{3f}^i (\sin \theta^i \cos d\theta^i - \cos \theta^i \sin d\theta^i).$$  \hspace{1cm} (C.17)
Since $d\theta^i$ is very small, $\cos d\theta^i \approx 1$, $\sin d\theta^i \approx 0$ and if we further expand the square root into a power series we find
\[
k_{1f}^i = k_{1f}^i \cos \theta^i + k_f \left( 1 - \frac{(k_{1f}^i)^2 + (k_{2f}^i)^2}{k_f^2} \right) \sin \theta^i.
\] (C.18)

Now $k_{1f}^i, k_{2f}^i$ are small for rays close to the base ray, so we may use the approximation
\[
k_{1f}^i \approx k_{1f}^i \cos \theta^i + k_f \sin \theta^i.
\] (C.19)

For the wave vectors components in the solid we get, using the same arguments as in the fluid case
\[
k_{1i}^{\alpha} \approx k_{1i}^{\alpha \alpha} \cos \theta^\alpha + k^\alpha \sin \theta^\alpha
\]
\[
k_{2i}^{\alpha} = k_{2i}^{\alpha \alpha}
\] (C.20)

With those transformations then we find
\[
\mathcal{L}^\alpha = k_{1f}^i \cos \theta^i - k_{1i}^{\alpha \alpha} \cos \theta^\alpha
\]
\[
\mathcal{M}^\alpha = k_{2f}^i - k_{2i}^{\alpha \alpha}
\] (C.21)

Consistent with the second order approximation of $\mathcal{L}$ and $\mathcal{M}$ we need only the base ray value defined by Eq.(C.14) for $\mathcal{N}$. Substituting Eqs.(C.21) and Eq.(C.14) into Eq.(C.11)
we get after some algebra

\[ W^\alpha(k_{1f}^i, k_{2f}^i, k_{2}^{r\alpha}, k_{2}^{r\alpha}) = \frac{1}{\Gamma^\alpha} (W^\alpha_{11} + W^\alpha_{12} + W^\alpha_{22}) \]  

\[ W^\alpha_{11} = \frac{1}{2} \begin{pmatrix} k_{1f}^i \\ k_{2f}^i \end{pmatrix} \begin{bmatrix} h_{22} \cos^2 \theta^i & -h_{12} \cos \theta^i \\ -h_{12} \cos \theta^i & h_{11} \end{bmatrix} \begin{pmatrix} k_{1f}^i \\ k_{2f}^i \end{pmatrix} \]  

\[ W^\alpha_{12} = -(k_{1f}^i, k_{2f}^i) \begin{bmatrix} h_{22} \cos \theta^i \cos \theta^\alpha & -h_{12} \cos \theta^i \\ -h_{12} \cos \theta^\alpha & h_{11} \end{bmatrix} \begin{pmatrix} k_{1}^{r\alpha} \\ k_{2}^{r\alpha} \end{pmatrix} \]  

\[ W^\alpha_{22} = \frac{1}{2} \begin{pmatrix} k_{1}^{r\alpha} \\ k_{2}^{r\alpha} \end{pmatrix} \begin{bmatrix} h_{22} \cos^2 \theta^\alpha & -h_{12} \cos \theta^\alpha \\ -h_{12} \cos \theta^\alpha & h_{11} \end{bmatrix} \begin{pmatrix} k_{1}^{r\alpha} \\ k_{2}^{r\alpha} \end{pmatrix} \]  

\[ \Gamma^\alpha = (k_{2}^{r\alpha} \cos \theta^\alpha - k_{1} \cos \theta^i)(h_{11} h_{22} - h_{12}^2) \]  

which can be rewritten in terms of \( P^\alpha \), \( Q^\alpha \) and \( R^\alpha \) matrices

\[ W^\alpha(k_{1f}^i, k_{2f}^i, k_{2}^{r\alpha}, k_{2}^{r\alpha}) = \frac{1}{2} (k_{1f}^i, k_{2f}^i) P^\alpha(k_{1f}^i, k_{2f}^i)^T + (k_{1f}^i, k_{2f}^i) Q^\alpha(k_{1}^{r\alpha}, k_{2}^{r\alpha})^T + \frac{1}{2} (k_{1}^{r\alpha}, k_{2}^{r\alpha}) R^\alpha(k_{1}^{r\alpha}, k_{2}^{r\alpha})^T \]  

\[ P^\alpha = \frac{1}{\Gamma^\alpha} \begin{bmatrix} h_{22} \cos^2 \theta^i & -h_{12} \cos \theta^i \\ -h_{12} \cos \theta^i & h_{11} \end{bmatrix} \]  

\[ Q^\alpha = \frac{-1}{\Gamma^\alpha} \begin{bmatrix} h_{22} \cos \theta^i \cos \theta^\alpha & -h_{12} \cos \theta^i \\ -h_{12} \cos \theta^\alpha & h_{11} \end{bmatrix} \]  

\[ R^\alpha = \frac{1}{\Gamma^\alpha} \begin{bmatrix} h_{22} \cos^2 \theta^\alpha & -h_{12} \cos \theta^\alpha \\ -h_{12} \cos \theta^\alpha & h_{11} \end{bmatrix} \]  

\[ \Gamma^\alpha = (k_{2}^{r\alpha} \cos \theta^\alpha - k_{1} \cos \theta^i)(h_{11} h_{22} - h_{12}^2). \]
APPENDIX D  DERIVATION OF THE REFRACTION MATRICES

Defining the vectors

\[ a_t = \begin{pmatrix} k_{1t}^\alpha \\ k_{2t}^\alpha \end{pmatrix}, \quad a_r = \begin{pmatrix} k_{1r}^\alpha \\ k_{2r}^\alpha \end{pmatrix} \]

the angle eikonal at the stationary point, Eq.(6.30), is given by

\[
W^\alpha(k_{1f}^t, k_{2f}^t, k_{1r}^r, k_{2r}^r) = \frac{1}{2} a_t^T P^\alpha a_t + a_t^T Q^\alpha a_r + \frac{1}{2} a_r^T R^\alpha a_r
\]

which can also be written as

\[
W^\alpha(k_{1f}^t, k_{2f}^t, k_{1r}^r, k_{2r}^r) = \frac{1}{2} \{a_t, a_r\}^T \begin{bmatrix} P^\alpha & Q^\alpha \\ (Q^\alpha)^T & R^\alpha \end{bmatrix} \begin{bmatrix} a_t \\ a_r \end{bmatrix}.
\]

Differentiation with respect to the wave vector components yields

\[
\frac{\partial W^\alpha}{\partial k_{1f}^t} = \{P_1^\alpha, Q_1^\alpha\} \begin{bmatrix} a_t \\ a_r \end{bmatrix},
\]

\[
\frac{\partial W^\alpha}{\partial k_{2f}^t} = \{P_2^\alpha, Q_2^\alpha\} \begin{bmatrix} a_t \\ a_r \end{bmatrix},
\]

\[
\frac{\partial W^\alpha}{\partial k_{1r}^r} = \{(Q^\alpha)_1^T, R_1^\alpha\} \begin{bmatrix} a_t \\ a_r \end{bmatrix},
\]

\[
\frac{\partial W^\alpha}{\partial k_{2r}^r} = \{(Q^\alpha)_2^T, R_2^\alpha\} \begin{bmatrix} a_t \\ a_r \end{bmatrix},
\]
where $M_i$ represents the $i$'th row of the matrix $M$. We also know that from the definition of the angle eikonal [62]

$$\frac{\partial W^a}{\partial k^\alpha} = x^i_1, \quad \frac{\partial W^a}{\partial k^\alpha} = x^i_2$$

Thus, defining the vectors

$$x_t = \begin{pmatrix} x^1_1 \\ x^1_2 \end{pmatrix}, \quad x_r = \begin{pmatrix} x^r_1 \\ x^r_2 \end{pmatrix}. \quad \text{(D.9)}$$

we can write

$$\begin{pmatrix} x_t \\ -x_r \end{pmatrix} = \begin{bmatrix} P^\alpha & Q^\alpha \\ (Q^\alpha)^T & R^\alpha \end{bmatrix} \begin{pmatrix} a_t \\ a_r \end{pmatrix}. \quad \text{(D.10)}$$

If we rearrange Eq.(D.10) to bring the variables in transducer coordinates on one side and the ray based variables on the other, the resulting form in terms of refraction matrices $A, B, C, D$ is

$$\begin{pmatrix} a_r \\ x_r \end{pmatrix} = \begin{bmatrix} B^\alpha & -A^\alpha \\ -D^\alpha & C^\alpha \end{bmatrix} \begin{pmatrix} a_t \\ x_t \end{pmatrix}. \quad \text{(D.11)}$$

To find these refraction matrices in terms of the $P^\alpha, Q^\alpha, R^\alpha$ matrices we write the first row from Eq.(D.10) as

$$x_t = P^\alpha a_t + Q^\alpha a_r$$

$$(Q^\alpha)^{-1} x_t = (Q^\alpha)^{-1} P^\alpha a_t + a_r$$

which gives

$$a_r = (Q^\alpha)^{-1} x_t - (Q^\alpha)^{-1} P^\alpha a_t. \quad \text{(D.12)}$$

And the second equation of system (D.10) yields

$$-x_r = (Q^\alpha)^T a_t + R^\alpha a_r, \quad \text{(D.13)}$$
substitution of Eq.(D.12) into Eq.(D.13) gives

\[-x_r = (Q^\alpha)^T a_t + R^\alpha \left(-\left(Q^\alpha\right)^{-1} P^\alpha a_t + (Q^\alpha)^{-1} x_t \right),\]

\[x_r = \left(R^\alpha (Q^\alpha)^{-1} P^\alpha - (Q^\alpha)^T\right) a_t - R^\alpha (Q^\alpha)^{-1} x_t.\]  \hspace{1cm} \text{(D.14)}

and we get from E.(D.14) and Eq.(D.12)

\[
\begin{pmatrix}
a_r \\
x_r
\end{pmatrix} = \begin{bmatrix}
-(Q^\alpha)^{-1} P^\alpha & (Q^\alpha)^{-1} \\
(R^\alpha (Q^\alpha)^{-1} P^\alpha - (Q^\alpha)^T) & -R(Q^\alpha)^{-1}
\end{bmatrix}
\begin{pmatrix}
a_t \\
x_t
\end{pmatrix}.\]  \hspace{1cm} \text{(D.15)}

Comparison of Eq.(D.15) with Eq.(D.11) gives

\[A^\alpha = -(Q^\alpha)^{-1}\]  \hspace{1cm} \text{(D.16)}

\[B^\alpha = -(Q^\alpha)^{-1} P^\alpha\]  \hspace{1cm} \text{(D.17)}

\[C^\alpha = -R^\alpha (Q^\alpha)^{-1}\]  \hspace{1cm} \text{(D.18)}

\[D^\alpha = (Q^\alpha)^T - R^\alpha (Q^\alpha)^{-1} P^\alpha\]  \hspace{1cm} \text{(D.19)}

or

\[Q^\alpha = -(A^\alpha)^{-1}\]  \hspace{1cm} \text{(D.20)}

\[P^\alpha = (A^\alpha)^{-1} B^\alpha\]  \hspace{1cm} \text{(D.21)}

\[R^\alpha = C^\alpha (A^\alpha)^{-1}\]  \hspace{1cm} \text{(D.22)}

\[G\] in terms of the refraction matrices

In chapter 6 the matrix \(G^\alpha\) was derived as

\[G^\alpha = (Q^\alpha)^T \left(P^\alpha - G_f\right)^{-1} - R.\]  \hspace{1cm} \text{(D.23)}

To express \(G^\alpha\) as a function of the refraction matrices we first consider

\[P^\alpha - G_f = -(A^\alpha)^{-1} B^\alpha - G_f\]

\[A^\alpha (P^\alpha - G_f) = B^\alpha - A^\alpha G_f\]
\[(A^\alpha (P^\alpha - G_f))^{-1} = (B^\alpha - A^\alpha G_f)^{-1}\]
\[(P^\alpha - G_f)^{-1} (A^\alpha)^{-1} = (B^\alpha - A^\alpha G_f)^{-1}\]
\[(P^\alpha - G_f)^{-1} = (B^\alpha - A^\alpha G_f)^{-1} A^\alpha \quad (D.24)\]

now we get

\[G^\alpha = (Q^\alpha)^T (P^\alpha - G_f)^{-1} Q^\alpha - R, \]
\[G^\alpha = ((A^\alpha)^{-1})^T (B^\alpha - A^\alpha G_f)^{-1} A^\alpha (A^\alpha)^{-1} - C^\alpha (A^\alpha)^{-1}, \]
\[G^\alpha = \left[ ((A^\alpha)^{-1})^T - C^\alpha (A^\alpha)^{-1} (B^\alpha - A^\alpha G_f) \right] (B^\alpha - A^\alpha G_f)^{-1}, \]
\[G^\alpha = \left[ ((A^\alpha)^{-1})^T - C^\alpha (A^\alpha)^{-1} B^\alpha - C^\alpha G_f \right] (B^\alpha - A^\alpha G_f)^{-1}. \quad (D.25)\]

From the definition of \(D^\alpha\), Eq.(D.19), we have

\[D^\alpha = (Q^\alpha)^T - R^\alpha (Q^\alpha)^{-1} P^\alpha \]
\[D^\alpha = -((A^\alpha)^{-1})^T + C^\alpha (A^\alpha)^{-1} A^\alpha (A^\alpha)^{-1} B^\alpha \]
\[D^\alpha = -((A^\alpha)^{-1})^T + C^\alpha (A^\alpha)^{-1} B^\alpha \quad (D.26)\]

and substitution of Eq.(D.26) into Eq.(D.25) leaves

\[G^\alpha = (C^\alpha G_f - D^\alpha) (B^\alpha - A^\alpha G_f)^{-1} \quad (D.27)\]
In chapter 8, we needed to evaluate the surface integral

$$I(k_1, k_2) = \iint e^{-ik \cdot y} \, dS(y).$$  \hspace{1cm} (E.1)

If we assume a circular shape for the transmitter the integrations can be easily performed in polar coordinates. We change the integration variables from $(x_1, x_2) \rightarrow (r', \phi)$ via

$$x_1' = r' \cos \phi : \quad r' = \sqrt{x_1'^2 + x_2'^2},$$  \hspace{1cm} (E.2)
$$x_2' = r' \sin \phi.$$  \hspace{1cm} (E.3)

and introduce $(\tilde{k}; \psi)$ as the polar coordinates related to $(k_1; k_2)$

$$k_1' = \tilde{k} \cos \psi ; \quad \tilde{k} = \sqrt{k_1'^2 + k_2'^2}$$  \hspace{1cm} (E.4)
$$k_2' = \tilde{k} \sin \psi.$$  \hspace{1cm} (E.5)

If we now use $dS(R) = r' dr' d\phi$ we obtain

$$I(k_1; k_2) = \int_0^{2\pi} \int_0^{2\pi} e^{-jkr' \cos \phi \cos \psi + \sin \phi \sin \psi} \, dr' \, d\phi.$$  \hspace{1cm} (E.6)

Using $\cos \phi \cos \psi + \sin \phi \sin \psi = \cos(\phi + \psi)$

$$I(k_1; k_2) = \int_0^{2\pi} \int_0^{2\pi} e^{-jkr' \cos(\phi + \psi)} \, dr' \, d\phi$$  \hspace{1cm} (E.7)

and introducing a change of variables $\xi = \phi + \psi \Rightarrow d\xi = d\phi$ yields

$$I(k_1; k_2) = \int_0^{2\pi} \int_0^{2\pi} e^{-jkr' \cos \xi} \, dr' \, d\xi$$  \hspace{1cm} (E.8)
which can be evaluated using the known integral relation

\[ 2\pi J_0(\alpha) = \int_0^{2\pi} e^{i\alpha \cos \xi} d\xi, \quad (E.9) \]

to give

\[ I(k_1; k_2) = 2\pi \int_0^{a} J_0(-k r') dr', \quad (E.10) \]

where \( J_0(\alpha) \) is the zeros-order Bessel function. Using the definite integral relationship

\[ \int_0^{a} J_0(\alpha r') dr' = \frac{2\pi a}{\alpha} J_1(\alpha a) \quad (E.11) \]

we obtain finally

\[ I(k_1; k_2) = 2\pi a^2 \frac{J_1(\tilde{k}(k_1; k_2)a)}{\tilde{k}(k_1; k_2)} \quad (E.12) \]
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