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Efficiency Tradeoffs in Estimating the Trend and Error Structure of the Linear Model

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Abstract
Assume that the observed time series has been generated by the model Yt=a + bt + yt, t=l,...,T (1) yt = pyt-i+YiAyt-i+-.--h'p-i^yt-p+i+£t, st~i.i.d.(0,c^) (2) where A denotes the first difference operator and p e (-1,1] is the largest autoregressive root in the autoregressive representation of yt implied by (2). Thus, yt can be an 1(1) or an 1(0) process according to whether p = 1 or p e (-1,1), respectively. If p e (-1,1), the Grenander and Rosenblatt (1957) result implies that the ordinaiy least squares (OLS) estimator of (a,b) in (1) is asymptotically equivalent to the generalized least squares (GLS) estimator of (a,b) using (1) and (2). If p = 1, the parameter a is not identified and although the OLS estimator of b is consistent, it is not asymptotically efficient. In this case, the sample mean of Ayt is an asymptotically efficient estimator of b, being equivalent to the GLS estimator. We will refer to the f sample mean of Ayt as the first-difference estimatorof b. Of course, in practice we do not know a priori whether p is equal to or less than one.

Disciplines
Econometrics | Economic Theory | Statistical Models

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by

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Abstract: A number of median-unbiased estimators of the parameters of the error process in the deterministic trend model with stationary or unit root AR(p) errors have been developed recently. Although these estimators generally provide more precise estimates (than OLS estimates) of the error component of the model, we show that there is an important interval of values of the largest autoregressive root for which they provide less precise estimates of the trend coefficient.

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1. Introduction

Assume that the observed time series $Y_1,\ldots,Y_T$ has been generated by the model

$$Y_t = a + bt + y_t, \qquad t = 1,\ldots,T \tag{1}$$

$$y_t = \rho y_{t-1} + \gamma_1 \Delta y_{t-1} + \ldots + \gamma_{p-1}\Delta y_{t-p+1} + \varepsilon_t, \quad \varepsilon_t \sim \text{i.i.d.}(0,\sigma^2) \tag{2}$$

where Δ denotes the first difference operator and $\rho \in (-1,1]$ is the largest autoregressive root in the autoregressive representation of y_t implied by (2). Thus, y_t can be an I(1) or an I(0) process according to whether $\rho = 1$ or $\rho \in (-1,1)$, respectively.

If $\rho \in (-1,1)$, the Grenander and Rosenblatt (1957) result implies that the ordinary least squares (OLS) estimator of (a,b) in (1) is asymptotically equivalent to the generalized least squares (GLS) estimator of (a,b) using (1) and (2). If $\rho = 1$, the parameter a is not identified and although the OLS estimator of b is consistent, it is not asymptotically efficient. In this case, the sample mean of Δy_t is an asymptotically efficient estimator of b, being equivalent to the GLS estimator. We will refer to the sample mean of Δy_t as the first-difference estimator of b. Of course, in practice we do not know *a priori* whether ρ is equal to or less than one.

Canjels and Watson (1997) recently studied this problem. They used standard and local-to-unity asymptotic distribution theory, along with Monte Carlo simulations to evaluate the OLS estimator, the first-difference estimator, and several feasible GLS estimators of b when the researcher does not know ρ or commit *a priori* to either the I(0) or I(1) representation of y_t. They conclude that in this case the feasible Prais-Winsten (FPW) estimator of b, as described in Greene (1997) for example, is the preferred estimator.

2. OLS vs. Median-Unbiased Estimators of ρ

Although the OLS estimator of ρ is consistent for any ρ in the parameter space (and is super-consistent if $\rho = 1$), it is a downward-biased estimator of ρ with respect to both the mean and median. The bias is of the same order as the standard deviation of the estimator for values of ρ close to or equal to one.

Andrews (1993) proposed using a median-unbiased principle (i.e., choose an estimator such that the median of the distribution of the estimator is equal to ρ) to construct an improved estimator of ρ for model (1)-(2) when $k = 1$ and the errors are normally distributed. This estimator has better finite sample properties than the OLS estimator of ρ, especially for values of ρ close to or equal to one. The actual mechanics of the estimator construction depend upon which of the following three cases applies: 1) a and b are unrestricted; 2) b is known to be equal to 0 but a is unrestricted; or 3) a and b are known to be equal to 0. Andrews and Chen (1994) extended the estimator to obtain a nearly median-unbiased estimator when k is greater than one and the errors are not normally distributed. Estimators of ρ which are nearly median-unbiased have also been suggested by Rudebusch (1992), Fuller (1996), Roy and Fuller (1998) and Roy, Falk, and Fuller (1999).

These (approximately) median-unbiased estimators of ρ improve upon the OLS estimator of ρ for values of ρ close to or equal to one by reducing the (mean and median) bias and the mean-squared error of the estimator.[1/] Even for values of ρ substantially less than one, where the bias in the OLS estimator is quite small, these alternative estimators seem to perform at least as well as the OLS estimator.

3. Implications for Estimation of b

In light of these results, it would seem that the FPW estimator of the trend coefficient b in (2) could be improved by using one of these median-unbiased estimators of ρ in place of the OLS estimator in the final step of the FPW procedure. This view seems to be implicit in Andrews and Chen (1994). Although they do not report a direct comparison between their estimator of b and the estimator of b based upon the OLS estimator of ρ, they do report simulation comparisons of estimators of α and β in the following model that is an alternative representation of (1)-(2):

$$Y_t = \alpha + \beta t + \rho Y_{t-1} + \gamma_1 \Delta Y_{t-1} + \ldots + \gamma_{k-1} \Delta Y_{t-k+1} + \varepsilon_t \qquad (3)$$

where the parameters α and β in (3) are related to the parameters in (1)-(2) according to $\alpha = a(1-\rho) + b(\rho-\gamma_1-\ldots-\gamma_{k-1})$ and $\beta = b(1-\rho)$. They conclude that the OLS estimator of β conditioned upon the estimates of $\rho, \gamma_1 \ldots, \gamma_{k-1}$ derived from their median-unbiased estimation procedure, is more accurate (in the mean-square sense) than the unconditional OLS estimator of β from (3).

We conducted a simulation experiment to investigate the performance of the FPW estimator of the trend parameter b using alternative estimators of ρ. Five thousand realizations of $y_1, \ldots, y_{100}$ were generated according to (1)-(2) where a=b=0 (but are unrestricted in the estimation process), k = 1, and the ε_t's are independently drawn from the standard normal distribution. The initial value ε_0 is set equal to zero when $\rho = 1$. Otherwise, ε_0 is determined by drawing from its stationary distribution and allowing for a burn-in period.

For each simulated realization of the y's, the following estimators of b were constructed:

1. Apply the OLS estimator to (1) to obtain b_{OLS}.
2. Apply the exact GLS estimator to (1)-(2) to obtain b_{GLS}.
3. Apply the FPW estimator to (1)-(2) using the OLS estimator of ρ to obtain $b_{FPW}(\rho_{OLS})$.
4. Apply the FPW estimator to (1)-(2) using Andrews's exact median-unbiased estimator of ρ to obtain $b_{FPW}(\rho_{MU})$.

The results are presented in Table 1.

Notice that for each value of ρ considered, the MSE of b_{OLS} is greater than the MSE of b_{GLS}. The MSE's of the two feasible Prais-Winsten estimators, $b_{FPW}(\rho_{OLS})$ and $b_{FPW}(\rho_{MU})$ always fall between MSE(b_{OLS}) and MSE(b_{GLS}). For the values of ρ equal to 1, .99, and .975, the MSE of $b_{FPW}(\rho_{OLS})$ is greater than the MSE of $b_{FPW}(\rho_{MU})$. The reverse is true for the values of ρ equal to .95, .90, .85, and .80. For smaller values of ρ the MSE's for all four estimators are about equal.

Therefore, with respect to the precision of estimating the trend parameter b in model (1)-(2), we conclude although the feasible Prais-Winsten estimator outperforms the OLS estimator (as is well known), using a more precise estimator of ρ in constructing this estimator is not necessarily desirable. In particular the FPW estimator of b based on the OLS estimator of ρ outperforms the FPW estimator of b based on Andrews's median-unbiased estimator of ρ except for values of ρ that are equal or sufficiently close to one. This occurs even though the latter is a better estimator of ρ and a better estimator of the parameter β in (3).

4. Discussion

The intuition underlying the conclusion that the OLS estimator of ρ can lead to an FPW estimator superior to the FPW constructed from the median unbiased estimator of ρ is straightforward. It is that the variance of the FGLS estimator of b is an increasing function of ρ and this function increases at an increasing rate. Consequently, the penalty for overestimating ρ by a given amount is greater than the penalty for underestimating ρ by that same amount. This is illustrated in Figure 1, which was derived using the same simulation structure as used above with T = 100 and using the OLS estimator of ρ to construct the FGLS estimator of b.

When ρ is sufficiently close to or equal to one, the bound on the size of the overestimation combined with the size of the downward median-bias in the OLS estimator of ρ make $b_{FPW}(\rho_{MU})$ a better estimator of b than $b_{FPW}(\rho_{OLS})$. When ρ is sufficiently small, the median-unbiased estimator of ρ and the OLS estimator of ρ are nearly equivalent and so are the corresponding FPW estimators of b. However, there will be an intermediate range of values for ρ for which the median-unbiased approach to estimation of ρ will yield less precise estimates of the trend parameter b than the (downward-biased) OLS approach.

NOTES

1. Our focus will be on the effects of reducing the median-bias, the mean-bias, and the mean squared error an estimator of ρ on the distribution of the estimator of the trend parameter b in I(0)/I(1) environments. MacKinnon and Smith's (1998) study of bias-variance tradeoffs in estimators of AR(1) models focuses on the effects of reducing the mean-bias in an estimator of ρ on the mean squared error of that estimator of ρ, primarily in explosive autoregressive settings.

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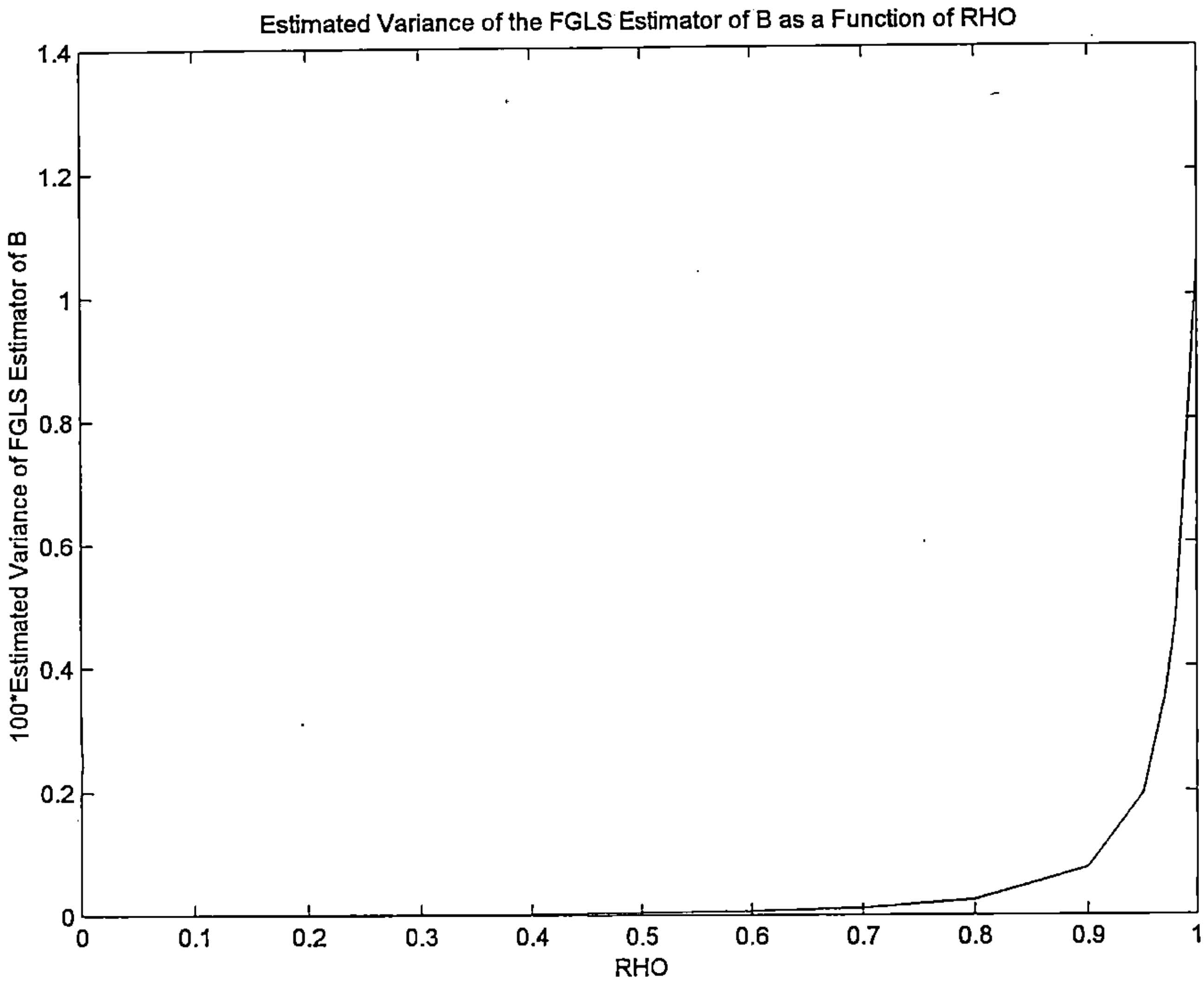
TABLE 1: Estimators of b (= 0)

	Median	Mean	TxMSE
		$\rho = 1$:	
$\hat{b}_{OLS}$	.0006	.0009	1.175
$\hat{b}_{FPW}(\hat{\rho}_{OLS})$	.0007	.0009	1.044
$\hat{b}_{FPW}(\hat{\rho}_{MU})$	.0001	.0009	1.009
$\hat{b}_{GLS}$	-.0009	.0009	0.995
		$\rho = .99$:	
$\hat{b}_{OLS}$	-.0016	.0015	.8673
$\hat{b}_{FPW}(\hat{\rho}_{OLS})$	-.0006	.0018	.7409
$\hat{b}_{FPW}(\hat{\rho}_{MU})$	.0014	.0018	.7120
$\hat{b}_{GLS}$	.0016	.0022	.6956
		$\rho = .975$:	
$\hat{b}_{OLS}$	.0001	-.0027	.5004
$\hat{b}_{FPW}(\hat{\rho}_{OLS})$	.0002	-.0012	.4173
$\hat{b}_{FPW}(\hat{\rho}_{MU})$	-.0007	.0000	.4027
$\hat{b}_{GLS}$	-.0011	.0003	.3864
		$\rho = .95$:	
$\hat{b}_{OLS}$	-.0021	-.0012	.2497
$\hat{b}_{FPW}(\hat{\rho}_{OLS})$	-.0014	-.0010	.2097
$\hat{b}_{FPW}(\hat{\rho}_{MU})$	-.0007	-.0007	.2139
$\hat{b}_{GLS}$	-.0009	-.0009	.2004
		$\rho = .90$	
$\hat{b}_{OLS}$	-.0004	.0003	.0870
$\hat{b}_{FPW}(\hat{\rho}_{OLS})$	.0001	.0004	.0749
$\hat{b}_{FPW}(\hat{\rho}_{MU})$	.0005	.0004	.0805
$\hat{b}_{GLS}$	.0001	.0005	.0732

TABLE 1 (Continued)

	Median	Mean	TxMSE
		$\rho = .85$	
$\hat{b}_{OLS}$	.0002	.0003	.0436
$\hat{b}_{FPW}(\hat{\rho}_{OLS})$	.0002	.0005	.0393
$\hat{b}_{FPW}(\hat{\rho}_{MU})$	.0005	.0005	.0405
$\hat{b}_{GLS}$	.0006	.0005	.0386
		$\rho = .80$	
$\hat{b}_{OLS}$	.0003	.0002	.0254
$\hat{b}_{FPW}(\hat{\rho}_{OLS})$	.0001	.0002	.0232
$\hat{b}_{FPW}(\hat{\rho}_{MU})$	.0001	.0002	.0235
$\hat{b}_{GLS}$	.0001	.0002	.0229
		$\rho = .70$	
$\hat{b}_{OLS}$	.0000	-.0000	.0118
$\hat{b}_{FPW}(\hat{\rho}_{OLS})$	-.0002	-.0000	.0111
$\hat{b}_{FPW}(\hat{\rho}_{MU})$	-.0003	-.0000	.0112
$\hat{b}_{GLS}$	-.0001	-.0000	.0111
		$\rho = .50$	
$\hat{b}_{OLS}$	.0001	.0002	.0045
$\hat{b}_{FPW}(\hat{\rho}_{OLS})$	.0001	.0000	.0044
$\hat{b}_{FPW}(\hat{\rho}_{MU})$	.0001	.0000	.0044
$\hat{b}_{GLS}$	.0001	.0000	.0044
		$\rho = .30$	
$\hat{b}_{OLS}$	.0001	.0000	.0023
$\hat{b}_{FPW}(\hat{\rho}_{OLS})$	.0001	.0000	.0023
$\hat{b}_{FPW}(\hat{\rho}_{MU})$	.0001	.0000	.0023
$\hat{b}_{GLS}$	.0001	.0000	.0023

FIGURE 1



Note: The estimates of the variance of the FGLS estimator of b were constructed from simulations using the OLS estimator of rho and sample size 100.