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Anindya Roy
Iowa State University

Barry Falk
Iowa State University

Wayne A. Fuller
Iowa State University, waf@iastate.edu

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Estimation of the Trend Model
with Autoregressive Errors

Staff Paper #327

by

Anindya Roy
Barry Falk
Wayne A. Fuller

Iowa State University
Ames, IA 50011

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Abstract

The variance of the feasible generalized least squares estimator of the trend coefficient is heavily dependent on the parameters of the autoregressive process. Estimators of the variance of the estimated trend coefficient are presented that perform much better than the direct feasible generalized least squares estimator. Limiting distributions are derived for the proposed test statistics.

1. Introduction

This paper is concerned with estimation and inference in a univariate p-th order autoregressive model with a time trend and, possibly, a unit root. Econometric interest in univariate autoregressions is partly due to the direct benefits that are attainable from a better understanding of the time series structures of individual economic variables. In addition, developments in the study of univariate time series typically lead the way to developments in the study of multivariate time series models.

There are two fundamental problems that complicate estimation and inference in autoregressive models with a possible unit root (i.e., AR/UR models). First, the ordinary least squares (OLS) estimator is biased and nonnormal in finite samples. Second, the asymptotic distribution of the OLS estimator is discontinuous at the boundary of the parameter space, being normal in the interior but nonstandard at the unit root endpoint.

The second problem would not be a serious concern if the unit root endpoint could safely be ignored. Following the work of Fuller (1976), Dickey and Fuller (1979) and Nelson and Plosser (1982), considerable attention has been directed toward development and application of tests for the presence of unit roots in economic time series. The results of these tests have been generally supportive of the unit root null hypothesis, leading to widespread theoretical and applied interest in what has become known as unit-root econometrics.

The rapid growth of unit-root econometrics has not occurred without some discomfort and skepticism. Unit root tests are known to have low power against most plausible alternatives, i.e., stationary or trend-stationary models with roots close to one. Therefore, the failure of unit root tests to reject the unit root null should not be interpreted as compelling evidence

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against stationarity. Furthermore, as Sims (1988) forcefully argued, economic theory almost never provides hypotheses whose validity rest upon unit root restrictions. These concerns have led to growing interest in the development of more impartial or objective approaches to estimation and inference in the autoregressive model when a unit root is acknowledged to be a distinct possibility, but it is not elevated to the status of null hypothesis.

One alternative approach relies on Bayesian procedures. illustrated, for example. in De-Jong and Whitman (1991). Although in many settings classical and Bayesian procedures lead to the same (or nearly the same) conclusions, the AR/UR model is a case in which important differences emerge. Despite some attractive features of this line of work, it highlights a central problem in Bayesian approach: the choice of prior. In particular, there appears to be strong disagreement in this literature regarding the type of prior that most appropriately reflects the econometrician's impartiality in the present setting. A 1991 special issue of the *Journal of Applied Econometrics* was devoted to Bayesian procedures for the autoregressive process.

More recently, Andrews (1993), Andrews and Chen (1994), and Fuller (1996) have developed median-unbiased or nearly unbiased estimators for the AR/UR model. The idea is to take advantage of the fact that, for example, in the AR(1) model fit with an intercept, the finite sample bias in the AR coefficient varies smoothly across the entire [-1,1] parameter space. In addition, although the finite-sample distribution of the *OLS* estimator of this coefficient is nonnormal, the distribution varies smoothly across the entire parameter space. The suggested estimators modify the ordinary (or weighted symmetric) least squares estimator. Andrews (1993) develops an exact median-unbiased estimator for the first-order AR/UR model, fit with or without a constant or a linear trend. The procedure is extended

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by Andrews and Chen (1994) to an approximately median-unbiased estimator for the p-th order case.

Fuller (1996) developed an approximately median-unbiased estimator for the AR(p) model fit with an intercept for autoregressive processes with a root close to or equal to one. Specifically, Fuller (1996, pp. 578-579) proposed the following procedure to estimate the AR(p) model when β is known to be zero. First, estimate (3) by weighted symmetric least squares and compute the statistic $\hat{\tau}_{WS,1}$,

$$\hat{\tau}_{WS,1} = (\hat{\alpha}_{WS} - 1)(\hat{\sigma}_{WS})^{-1}, \tag{1}$$

where $\hat{\alpha}_{WS}$ is the weighted symmetric least squares ($WSLS$) estimator of α and $\hat{\sigma}_{WS}$ is its estimated standard error. The weighted symmetric least squares estimator of the AR(p) model, described in Fuller (1996, pp. 413-419), appears to outperform OLS in finite samples, particularly when α is close to or equal to one. Second, the $WSLS$ estimator of α is modified according to

$$\tilde{\alpha}_{MU} = \hat{\alpha}_{WS} + \bar{C}(\hat{\tau}_{WS,1})\hat{\sigma}_{WS}, \tag{2}$$

where $C(\hat{\tau}_{WS,1})$ is a specified function of $\hat{\tau}_{WS,1}$. Fuller's procedure was recently extended by Roy and Fuller (1999) to allow for general deterministic regressors.

In certain applications the trend parameter β is the parameter of primary interest and testing and estimation of β have been studied by several authors. See Canjel and Watson (1997), Durlauf and Phillips (1988), Nelson and King (1983), Sun and Pantula (1998), Woodward and Gray (1995), and citations in those papers. We add to the literature concerning estimation of the trend coefficient in AR/UR models with trend. The variance of the feasible generalized least squares estimator for the trend coefficient when the largest AR

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root is close to or equal to one is heavily dependent on the parameters of the autoregressive process. This complicates the construction of a pivotal or nearly-pivotal statistic for use in inference with respect to the trend coefficient. We recommend a studentized statistic that uses a modification of the Roy-Fuller estimator of the autoregressive process along with a Gauss-Newton estimator of the variance of the estimator of the trend coefficient.

2. Model

Assume that the observed time series $Y_1, Y_2, \ldots, Y_T$ has been generated by a model of the form

$$
\begin{aligned}
Y_t &= \mu + \beta\, t + y_t \\
y_t &= \alpha y_{t-1} + \psi_1 \Delta y_{t-1} + \ldots + \psi_{p-1}\Delta y_{t-p+1} + u_t, \qquad t = 1, \ldots, T \\
u_t &\sim i.i..(0, \sigma^2),
\end{aligned}
\tag{3}
$$

where μ and β are the parameters of the trend function and Δ is the first difference operator, $\Delta x_t = x_t - x_{t-1}$. The parameter α lies in the half closed interval (-1,1]. If $\alpha \in (-1, 1)$, the parameters $\psi_1, \ldots, \psi_{p-1}$ are assumed to be such that the AR(p) model for y_t is stationary and the initial values $y_1, \ldots, \psi_{p-1}$ are drawn from the stationary distribution corresponding to that model. If $\alpha = 1$, the model reduces to

$$
\Delta Y_t = \beta + \Delta y_t
$$

$$
\Delta y_t = \sum_{j=2}^{p} \psi_{j-1}\Delta y_{t-j+1} + u_t,
\tag{4}
$$

where the parameters $\psi_1, \ldots, \psi_{p-1}$ are assumed to be such that the AR($p-1$) model for Δy_t is stationary, and y_1 can be any random variable.

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3. Estimator of the AR parameter for a model with Trend

Fuller (1996, pp 572) defines the weighted symmetric least squares regression for obtaining estimators of the coefficients of an autoregressive process. Let $\hat{\alpha}_{WS}$ denote the trend adjusted $WSLS$ estimator of α obtained by the weighted symmetric ($WSLS$) regression of $\hat{y}_t$ on $\hat{y}_{t-1}, \Delta \hat{y}_{t-1}, \ldots, \Delta \hat{y}_{t-p+1}$ where $\hat{y}_t$ is the de-trended series $Y_t - \hat{\mu}_{OLS} - \hat{\beta}_{OLS} t$ and $\hat{\mu}_{OLS}$ and $\hat{\beta}_{OLS}$ are the ordinary least squares estimators of μ and β. The approximately unbiased estimator of α defined by Roy and Fuller (1999) is the modified $WSLS$ estimator

$$\tilde{\alpha} = min(\ddot{\alpha}_{MU}, 1)$$

where

$$\ddot{\alpha}_{MU} = \hat{\alpha}_{WS} + C(\hat{\tau}_{WS,1}) \hat{\sigma}_{WS}, \tag{5}$$

$\hat{\sigma}_{WS}$ is the standard error of $\hat{\alpha}_{WS}$, $\hat{\tau}_{WS,1}$ is the statistic (1) calculated with trend-adjusted quantities and

$$
\begin{aligned}
C(\hat{\tau}_{WS,1}) &= \tau_{Med} + c_1(\hat{\tau}_{WS,1} - \tau_{Med}) & \hat{\tau}_{WS,1} &> \tau_{Med}, \\
&= I_p(T^{-1}\hat{\tau}_{WS,1}) - 3[\hat{\tau}_{WS,1} + k(\hat{\tau}_{WS,1} + 5)]^{-1} & -5 &< \hat{\tau}_{WS,1} \le \tau_{Med}, \\
&= I_p(T^{-1}\hat{\tau}_{WS,1}) - 3[\hat{\tau}_{WS,1}]^{-1} & -(3T)^{1/2} &< \hat{\tau}_{WS,1} \le -5, \\
&= 0 & \hat{\tau}_{WS,1} &\le -(3T)^{1/2}.
\end{aligned}
$$

I_p is the integer part of $2^{-1}(p+1)$, τ_{Med} is the median of the limiting distribution of $\hat{\tau}_{WS,1}$ when $\alpha = 1$,

$$k = [3T - \tau_{Med}^2(I_p + T)][\tau_{Med}(5 + \tau_{Med})(I_p + T)]^{-1}$$

$$c_1 = (1.12 - 1.5T^{-1})(1.65)^{-1}.$$

where the value c_1 is chosen to give a continuous function with $C(\tau_{0.975}) = 2 - \tau_{0.975}$ and $\tau_{0.975}$ is the 97.5% point of the limiting distribution of $\hat{\tau}_{WS,1}$ when $\alpha = 1$. We will use this estimator and similar estimators to construct feasible generalized least squares (FGLS) estimates for the trend coefficient.

4. The AR(1) Model: FGLS Estimators of β

We define the FGLS estimator of μ and β for an estimator $\hat{\alpha}$ of α as

$$
\begin{aligned}
(\hat{\mu}, \hat{\beta})' &= (X'\Sigma^{-1}(\hat{\alpha})X)^{-1}X'\Sigma^{-1}(\hat{\alpha})Y \quad & |\hat{\alpha}| < 1 \\
\hat{\beta} &= (T-1)^{-1}\sum_2^T \Delta Y_{t-1} \quad & \hat{\alpha} = 1
\end{aligned}
\tag{6}
$$

where $X = (x_1', \ldots, x_T')'$, $x_t = (1, t)$, $\Sigma(\hat{\alpha})$ is the variance-covariance matrix of the first T observations of a first order autoregressive process with autoregressive coefficient $\hat{\alpha}$. $Y = (Y_1, \ldots, Y_T)'$ and $\Delta Y_t = Y_t - Y_{t-1}$. The FGLS t-statistic for β is defined as

$$t(\hat{\beta}) = (\hat{\beta} - \beta)se(\hat{\beta})^{-1} \tag{7}$$

$se(\hat{\beta})$ is the square root of the $(2,2)$th element of the matrix $\hat{\sigma}^2(X'\Sigma^{-1}(\hat{\alpha})X)^{-1}$ if $\hat{\alpha} = 1$ or is equal to $T^{1/2}\hat{\sigma}$ if $\hat{\alpha} = 1$, where

$$
\begin{aligned}
\hat{\sigma}^2 &= (T-3)^{-1}\sum_2^T \hat{u}_t^2 \quad & |\hat{\alpha}| < 1 \\
&= (T-1)^{-1}\sum_2^T \Delta Y_t^2 \quad & \hat{\alpha} = 1
\end{aligned}
\tag{8}
$$

and $\hat{u}_t$ are the residuals from the regression of $\hat{y}_t$ on $\hat{y}_{t-1}$.

To illustrate the properties of FGLS estimators of β for the AR(1) model. we constructed FGLS estimators with three different estimators of α. Table 1 contains the result where $\hat{\beta}_{OLS}(\hat{\beta}_{WS}, \tilde{\beta})$ refers to the feasible generalized least squares estimators of β when $\hat{\alpha}_{OLS}(\hat{\alpha}_{WS}, \tilde{\alpha})$ is used to estimate α. The feasible generalized least squares estimator of β is unbiased for β when u_t's of (3) are symmetrically distributed because the error in the estimator is an odd function of the u_t's.

The MSE's of these estimators are similar for all values of α, and are nearly the same when α is less than or equal to 0.70. When α is close to or equal to one (i.e., $\alpha = 1, 0.98. 0.97$). the MSE of $\tilde{\beta}$ is slightly smaller than that of $\hat{\beta}_{OLS}$ and $\hat{\beta}_{WS}$. When $\alpha = 0.95. 0.90$ or 0.80. the MSEs of $\hat{\beta}_{OLS}$ and $\hat{\beta}_{WS}$ are smaller than the MSE of $\tilde{\beta}$, and the difference is grater than 10% when $\alpha = 0.90$. The result that over a range of α. the more precise estimators of α lead to less precise $FGLS$ estimators has been previously noted by Falk and Roy (1998).

The fact that ordinary test statistics for β have much higher significance levels that the nominal levels has been noted by a number of authors. See Canjel and Watson (1997). The 97.5% point of the simulated distributions of the studentized statistics are given in Table 1. Because the error in the estimator is an odd function of the u_t, the 2.5% and 97.5% points have the same absolute value. The first three statistics are t-statistics constructed from the second stage regression of the feasible generalized least squares procedure using $\hat{\alpha}_{OLS}, \hat{\alpha}_{WS}$ and $\tilde{\alpha}$ respectively. as estimates of α and using the estimated generalized least squares standard error.

The 97.5% point converges to 2.0 as α decreases toward zero for all three statistics. However, when α is close to one. none of the t-statistics has a 97.5% point that is close enough to the Student's t percentile to suggest its use as an approximate pivotal statistic

over the entire range of α values. Simulations run with sample sizes (50. 200. 500) produced similar results.

These t-statistics have large variances because the variance of the trend coefficient is a highly nonlinear function of α. For a sample of size 100, the $FGLS$ variance at $\alpha = 1$ is about 1,000 times that of the $FGLS$ variance at $\alpha = 0$ and about ten times the $FGLS$ variance at $\alpha = 0.90$. This is because the variance is proportional to $\sum_{t=2}^{T} = O(T^3)$ for $\alpha = 0$ and proportional to T for $\alpha = 1$. Consequently, because of the convexity of the variance function, the $FGLS$ estimator of the variance of $\hat{\beta}$ grossly underestimates the true variance of the estimator in the neighborhood of $\alpha = 1$. The maximum variance occurs at $\alpha = 1$, the boundary of the parameter space which automatically leads to a negative bias in the ariance estimation. Even a small negative bias in the estimation of α gets heavily magnified through the estimated variance function, which has a very steep slope when α is near one.

5. Gauss Newton Estimation

To create a test statistic with distribution closer to that of Student's t. we suggest a Gauss-Newton estimator of the variance of $\hat{\beta}$. The Gauss Newton estimator for the first order process obtained by expanding the representation

$$Y_1 = \mu + \beta + y_1$$
$$Y_t = \mu(1 - \rho) + \beta[t - \rho(t - 1)] + \rho[Y_{t-1} - \mu - \beta(t - 1)] + e_t$$

in a first order Taylor expansion about an initial estimator $(\ddot{\mu}, \ddot{\beta}, \ddot{\alpha})$. Then the Gauss-Newton

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regression equations, with parameters $\Delta\ddot{\mu}, \Delta\ddot{\beta}$ and $\Delta\ddot{\alpha}$ are

$$\tilde{e}_1 = \Delta\ddot{\mu}[(1-\ddot{\alpha}^2)^{1/2}] + \Delta\ddot{\beta}[(1-\ddot{\alpha}^2)^{1/2}] + \nu_1, \quad t=1 \quad \text{and} \quad \ddot{\alpha}^2 < 1$$

$$= \Delta\ddot{\mu} + \Delta\ddot{\beta} + \nu_1, \qquad\qquad t=1 \quad \text{and} \quad \ddot{\alpha} = 1$$

and

$$\tilde{e}_t = \Delta\ddot{\mu}(1-\ddot{\alpha}) + \Delta\ddot{\beta}[t-\ddot{\alpha}(t-1)] + \ddot{\alpha}\ddot{y}_{t-1} + \nu_t, \qquad t=2,\ldots,T, \tag{9}$$

where

$$\ddot{y}_t = Y_t - \ddot{\mu} - \ddot{\beta}t, \quad t = 1,\ldots,T$$

$$\tilde{e}_1 = (1-\ddot{\alpha}^2)^{-1/2}\ddot{y}_1 \quad t = 1 \text{ and } \ddot{\alpha}^2 < 1$$

$$= \ddot{y}_1 \qquad\qquad t = 1 \text{ and } \ddot{\alpha} = 1$$

and

$$\tilde{e}_t = \ddot{y}_t - \ddot{\alpha}\ddot{y}_{t-1}, \qquad t = 2,\ldots,T.$$

The Gauss Newton estimator is

$$(\hat{\mu}_{GN}, \hat{\beta}_{GN}, \hat{\alpha}_{GN}) = (\ddot{\mu}, \ddot{\beta}, \ddot{\alpha}) + (\Delta\hat{\mu}, \Delta\hat{\beta}, \Delta\hat{\alpha}) \tag{10}$$

where $(\Delta\hat{\mu}, \Delta\hat{\beta}, \Delta\hat{\alpha})$ are the least squares coefficients for the regression associated with (9). The estimated standard error of the one-step Gauss Newton estimator of β is the square root of the second diagonal element of the inverse of the sums of squares and products matrix of the explanatory variables of the regression associated with (9) multiplied by the residual mean square.

To offset the effects of the steepness of the variance function when α is near one, we construct the one-step Gauss Newton estimator of β using an initial estimator of α that intentionally tends to overestimate α when α is near one. The estimator is similar to the

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estimator $\tilde{\alpha}$ given by (5), but is constructed with a positive median-bias, estimating α to be equal to one about 80% of the time when α is equal to one. Specifically, we propose the estimator

$$\ddot{\alpha} = \hat{\alpha}_{WS} + \ddot{C}(\hat{\tau}_{WS,1})$$ (11)

where

$$
\begin{aligned}
\ddot{C}(\hat{\tau}_{WS,1}) &= -\hat{\tau}_{WS,1} & \hat{\tau}_{WS,1} &> -2.85, \\
&= T^{-1}\hat{\tau}_{WS,1} - 3[\hat{\tau}_{WS,1} + k(\hat{\tau}_{WS,1} + 5)]^{-1} & -5 &< \hat{\tau}_{WS,1} \leq -2.85, \\
&= T^{-1}\hat{\tau}_{WS,1} - 3[\hat{\tau}_{WS,1}]^{-1} & -(3T)^{1/2} &< \hat{\tau}_{WS,1} \leq -5, \\
&= 0 & \hat{\tau}_{WS,1} &\leq -(3T)^{1/2}.
\end{aligned}
$$

(12)

where k is chosen such that $\ddot{C}(\hat{\tau}_{WS,1}) = 2.85$ for $\hat{\tau}_{WS,1} = -2.85$. Sun and Puntula (1998) used a similar of α in constructing an estimator of the variance of the estimated β Their adjustment would set the estimator of α to one, whenever the unit root test is bigger than -3.5. This corresponds to setting the estimator equal to one about 97.5% times when α = 1. The Sun-Pantula procedure improves the estimated variance of the $FGLS$ slightly when $\alpha = 1$. However, when α is close to but not equal to one, the variance is severely overestimated resulting in significance levels that are much smaller than the nominal levels.

Let $\ddot{\mu}$ and $\ddot{\beta}$ be feasible generalized least squares estimators of μ and β, respectively, constructed using $\ddot{\alpha}$.

We use the estimated standard error from the one-step Gauss Newton procedure, which

we denote by $se(\tilde{\beta}_{GN})$, to define the test statistic as

$$t_{GN}(\tilde{\beta}) = (\tilde{\beta} - \beta)se(\tilde{\beta}_{GN})^{-1} \tag{13}$$

Before investigating the finite sample properties of the suggested test statistics, we will study their limiting properties. The following theorem derives the limiting distribution of the test statistics for the first order processes. The result for higher order processes can be derived along the same lines.

Theorem 1   Let y_t satisfy model (3). Let $\tilde{\beta}$ and $\hat{\beta}_{WS}$ be the feasible generalized least squares estimates of β using $\tilde{\alpha}$ and $\hat{\alpha}_{WS}$, respectively. Let $se(\tilde{\beta}_{GN})$ denote the standard error of the estimate of β obtained from the Gauss-Newton procedure using $\tilde{\alpha}$ as the initial estimate of α and let $se(\hat{\beta}_{GN})$ be that using $\hat{\alpha}_{WS}$ as the initial estimate of α. Let $t_{GN}(\tilde{\beta})$ be as defined in (13), $t(\hat{\beta}_{WS})$ be the feasible generalized t-statistic for β defined in (7) and let

$$t_{GN}(\hat{\beta}_{WS}) = (\hat{\beta}_{WS} - \beta)se(\hat{\beta}_{GN})^{-1}$$

Then

1. if $|\alpha| < 1$

$$t_{GN}(\tilde{\beta}) \xrightarrow{\mathcal{L}} N(0,1)$$

$$t_{GN}(\hat{\beta}_{WS}) \xrightarrow{\mathcal{L}} N(0,1)$$

$$t(\hat{\beta}_{WS}) \xrightarrow{\mathcal{L}} N(0,1).$$

2. If $\alpha = 1$

$$t_{GN}(\tilde{\beta}) \xrightarrow{\mathcal{L}} [\eta_{33}\eta_{22} - \eta_{32}^2]^{1/2}\eta_{33}^{-1/2}\xi_0$$

$$t_{GN}(\hat{\beta}_{WS}) \xrightarrow{\mathcal{L}} [\zeta_{33}\zeta_{22} - \zeta_{32}^2]^{1/2}\zeta_{33}^{-1/2}\xi_0$$

$$t(\hat{\beta}_{WS}) \xrightarrow{\mathcal{L}} \xi_0[X^{-1}I(X > 0) + I(X \le 0)].$$

where $X = 0.5Z^{-1}\{[(W(1) - 2H)(W(1) - 6K) - 1] + (H - 3K)^2 - Z\}$ $Y = Z^{-1/2}X$

$Z = (G - H^2 - 3K^2)$, $G = 2\int_0^1 W^2(t)dt$; $H = \int_0^1 W(t)dt$ $K = 2\int_0^1 tW(t)dt - H$.

$$\eta_{22} = 0.5[X + \ddot{C}(Y)Z^{1/2}] + 1$$

$$\eta_{32} = [X + \ddot{C}(Y)Z^{1/2}][\int_0^1 tW(t)dt - 0.5\int_0^1 W(t)dt - 3^{-1}\xi_1]$$

$$\eta_{33} = [\int_0^1 W^2(t)dt - (\int_0^1 W(t)dt)^2 - 2\xi_1\int_0^1 tW(t)dt + \xi_1(\int_0^1 W(t)dt) + 3^1\xi_1^2]$$

$$\zeta_{22} = 0.5X + 1$$

$$\zeta_{32} = X[\int_0^1 tW(t)dt - 0.5\int_0^1 W(t)dt - 3^{-1}\xi_1]$$

$$\zeta_{33} = \eta_{33}$$

ξ_1, ξ_0 are standard normal variables, $W(t)$ is the standard Brownian Motion.

The Gauss-Newton standard error is always larger than the FGLS standard error. See the comment at the end of the proof of Theorem (1). This provides theoretical justification for using the Gauss-Newton procedure to reduce the FGLS underestimation of the standard error of the estimator of β. The simulation results indicate that the Gauss-Newton standard error provides much better normalizer for the test statistic than the FGLS standard error. Also the Gauss-Newton standard error is an increasing function of the initial estimate of α. Because the modified estimator $\ddot{\alpha}$ is bigger than the weighted symmetric estimator $\hat{\alpha}_{WS}$, we get a substantial improvement in the estiamtes standard error using the modified estimator of α.

An interesting fact that comes out of the proof of Theorem 1 is that the limiting distribution of the Gauss-Newton test statistic depends on the error in estimation of α when $\alpha = 1$. Also the Gauss Newton standard error is a decreasing function of the error in estimation of

α when $\alpha = 1$. These results suggest that an initial estimator biased towards one will give less biased estimator of the standard error of the estimator of β.

The 97.5% points of t-statistic defined in (13) is reported in the last column of Table 2 as $t_{.975}(\tilde{\beta}_{GN})$. The values in the table are the signed average of the 0.025 and 0.975 values. These values are much closer to the percentiles of the Student's t than those for the simple FGLS. Similar results were obtained for sample sizes 50 and 500. Thus $t(\tilde{\beta}_{GN})$ can be used as an approximate pivotal statistic for inference with regard to the trend coefficient β in model (3), recognizing that the percentiles for $\alpha = 1$ and α near one deviate from the percentiles of Student's t. Because the variance function changes so rapidly near one, we are unable to define a statistic that 2.5% points close to 2.0 for all values of α.

5..1 The AR(p) Model with Trend

The Roy-Fuller estimator is based upon adjusting the $WSLS$ estimator of α using the test for unit root. Therefore, it is nearly as easy to implement in the general AR(p) case as in the AR(1) case. In the AR(p) case, the de-trended data, $\hat{y}_t$, are regressed on $\hat{y}_{t-1}, \Delta\hat{y}_{t-1}, \ldots, \Delta\hat{y}_{t-p+1}$ by $WSLS$. Then the $WSLS$ estimator of α is adjusted according to (5) to obtain the modified estimator of α. Finally, $\psi_1, \ldots, \psi_{p-1}$ are re-estimated by applying OLS to estimate the regression of $\hat{y}_t - \tilde{\alpha}\hat{y}_{t-1}$ on $\Delta\hat{y}_{t-1}, \ldots, \Delta\hat{y}_{t-p+1}$.

To illustrate the performance of the modified estimator in the second order AR case, we simulated 10,000 time series according to (3) with $(T, p) = (100, 2)$ for several values of α and ψ_1.

Table 3 gives a comparison of properties of the distribution of $\hat{\beta}_{WS}$ to the corresponding

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properties of the distribution of $\tilde{\beta}$. A description of the Gauss-Newton procedure for the higher order autoregressive process is given in the appendix. In computing the variance estimator, we included only the term $1, t$ and y_{t-1} in the Gauss-Newton equations. Similar to the first order case, the MSEs of $\hat{\beta}_{WS}$ and $\tilde{\beta}$ are nearly the same for all values of α and ψ_1 investigated. When ψ_1 is a large positive value, $\tilde{\beta}$ is about 3.5% less efficient than $\hat{\beta}_{WS}$ for values of α close to one. However for negative values of ψ_1, $\tilde{\beta}$ has smaller MSE than $\hat{\beta}_{WS}$ for all values of α investigated. When $\psi_1 = -0.80$, $\tilde{\beta}$ has an MSE that is about 10% smaller than that of $\hat{\beta}_{WS}$.

As in the first order case, the $FGLS$ estimator of variance underestimates the variance of the $FGLS$ estimator for values of α near one. The amount of underestimation is also a function of ψ_1. Consequently, the ordinary $FGLS$ test statistics constructed from the second stage regression of the feasible generalized least squares procedure have much higher significance levels than the nominal levels, with 97.5% points as large as 10.

The Gauss-Newton estimator of variance of $\hat{\beta}$ is much superior to the $FGLS$ estimator. As in the first order case, we constructed the estimator of β, denote by $\ddot{\beta}$, using the adjustment function (12). For values of α near one, the estimator constructed using the C-function in (12) has similar behavior for all values of ψ_1.

The average of the 2.5% and the 97.5% points of the t-statistic constructed using the Gauss-Newton variance is given in Table 3 as $t_{.975}$ under the column of $\tilde{\beta}$. The percentiles of the Gauss-Newton t-statistic for β are much superior to those of the $FGLS$ t-statistic, but are larger than 2.00 for $\alpha = 1$ and smaller than 2.00 for α close to, but less than one. The Gauss-Newton percentiles deviate more from two than those in the first order case. However, they are much more stable across the range of values of α, compared to the percentiles of

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the $FGLS$ t-statistics. The ratio of the 97.5 percentile of the test on β when $\alpha = 1$ to the percentile when $\alpha = 0.98$ is about 3 for the $FGLS$ t statistic when $\psi_1 = 0.8$, whereas, the 97.5 percentile of the Gauss-Newton t-statistic for β when $\alpha = 1$ and $\psi_1 = 0.8$ is about 2 times the 97.5 percentile of the Gauss-Newton t-statistic for β when $\alpha = 0.98$ and $\psi_1 = 0.8$. Thus, while the percentiles of the Gauss-Newton leave much to be desired they are much superior to those of $FGLS$. The behavior of the percentiles is slightly better for negative values of ψ_1 than for positive values.

Examples

Example 1

In this example, we illustrate the computations for the proposed statistics using the annual real per capita U.S. gross national product in 1992 constant dollars. The data for 1909 through 1994 are an extended version of the series analyzed by Nelson and Plosser (1982) and later by Schotman and van Dijk (1991). Schotman and van Dijk used the series from 1909-1988 given in 1987 constant dollars. Our primary data source is the U.S. Department of Commerce, Bureau of Economic Analysis web page. The website contains data from 1929-1994. We used the simple linear regression of Schotman-van Dijk data on the web page data to create a 1992 level series for the years 1909-1928. Following the previous authors, we analyze the natural logarithm of the original series and assume the series to be a second order autoregressive process where the mean function is a linear trend. Thus, we assume model (3) with $p = 2$. Some authors analyzing the series have postulated a second order process with the mean function being a linear trend with a breakpoint. See Vogelsang (1998) and the references there in. Since we are focusing on the computational aspect, we do not include the breakpoint.

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The ordinary least squares estimator of trend is

$$\hat{Y}_t = 8.346 + 0.02158\, t.$$

The fitted weighted symmetric autoregressive equation is

$$\hat{u}_t = \underset{(0.047)}{0.850}\ \hat{u}_{t-1} + \underset{(0.100)}{0.417}\ \Delta\hat{u}_{t-1}$$

and the residual mean square is 0.00282. The value of the test statistic for testing for a unit root is -3.21. A formal test fails to reject the hypothesis of a unit root at the 5% level of significance because the 5% value of the unit root distribution obtained from tables in Fuller (1996) for a sample of size 86 is about -3.25.

The modified estimator for the autoregressive process defined by (5) is

$$\hat{u}_t = \underset{(0.057)}{0.896}\ \hat{u}_{t-1} + \underset{(0.097)}{0.391}\ \Delta\hat{u}_{t-1}.$$

The standard error of 0.057 was constructed using the h function defined in Roy and Fuller (1999). An approximate 95% confidence interval for α constructed using the modified estimator and the modified standard error is $((0.782, 1.000]$. The value one is included in the set because the confidence interval is symmetric about the estimate and a 0.025 level test would accept a unit root.

The $FGLS$ estimator of the trend based on the modified weighted symmetric estimated coefficients is

$$\hat{Y}_t = \underset{(0.028)}{8.374} + \underset{(0.0014)}{0.0210\, t}$$

where the standard errors in parantheses are those computed from the estimated generalized least squares treating the estimated autoregressive coefficients as they are the true values.

The t-statistic for testing the significance of the trend coefficients constructed using this standard error is 15.10. The test statistic constructed using the standard error from Gauss-Newton procedure described in the appendix is 7.81. The biased estimator of α constructed using the C-function (12) was used as an initial estimator in the Gauss-Newton procedure. Both tests reject the hypothesis of zero trend at all reasonable levels, but the standard error from the Gauss-Newton procedure is about twice that obtained from the estimated generalized least squares procedure, which is consistent with the Monte Carlo results of Table 4.

Example 2

In this example we use the interest rate series analyzed by Schotman and van Dijk (1991) which is an extended version of the Bond Yield series analyzed by Nelson and Plosser (1982). There are 89 observations in the series. The ordinary least squares estimator of trend is

$$\hat{Y}_t = 2.129 + 0.062\,t$$

Following Schotman and van Dijk, we fit a third order autoregressive model. The weighted symmetric fitted equation is

$$\hat{u}_t = \underset{(0.033)}{0.946}\ \hat{u}_{t-1} + \underset{(0.106)}{0.220}\ \Delta\hat{u}_{t-1} - \underset{(0.106)}{0.152}\ \Delta\hat{u}_{t-2}$$

and the residual mean square is 0.347. The value of the unit root test statistic is -1.63.

The modified estimator of the autoregressive process is

$$\hat{u}_t = \underset{(0.048)}{1.000}\ \hat{u}_{t-1} + \underset{(0.107)}{0.204}\ \Delta\hat{u}_{t-1} - \underset{(0.108)}{0.184}\ \Delta\hat{u}_{t-2}$$

where $\hat{\alpha} = 1.000$ because the test statistic (-1.63) is greater than the median of the unit root null distribution (-1.96). To provide a confidence interval for α, we use the modified

estimator of α that is not restricted to the interval $(-1,1]$. The estimated value is 1.016. Then

an approximate 95% confidence interval is $(0.920,1.000]$ where the interval is the intersection

of a usual 95% interval and the interval $(-1,1]$. Based on the weighted symmetric estimator

of α, ψ_1 and ψ_2, the $FGLS$ estimator of trend is

$$\hat{Y}_t = \underset{(0.131)}{2.335} + \underset{(0.028)}{0.066\,t}$$

where the standard errors are those from $FGLS$ treating the estimated autoregressive

coefficients as true values. The value of the $FGLS$ t-statistic is 2.35. The estimated $FGLS$

trend equation using the modified estimator of the autoregressive coefficients is

$$\Delta\hat{Y}_t = \underset{(0.063)}{0.068}$$

where the estimator is the mean of the first differences. The standard error of 0.063 was con-

structed using the Gauss-Newton procedure described in appendix and the biased estimator

of α constructed using the C-function in (12). Because the t-statistic is 1.07 the hypothesis

if zero trend is easily accepted. As in the preceding example and consistent with the Monte

Carlo results of Table 4, the Gauss-Newton standard error is about twice that of $FGLS$.

6.   Summary and Conclusions

Estimators of the autoregressive parameters were used to construct feasible generalized least

squares estimators of the trend coefficient in the trend plus AR error model. The variances

of the feasible generalized least squares estimators of the trend coefficient are heavily depen-
dent on the parameters of the autoregressive process. Consequently. studentized statistics
constructed from the estimators using the feasible generalized least squares estimated vari-
ances have variances much greater than one. We propose a studentized statistic constructed
using the Gauss-Newton estimator of the variance of the feasible generalized least squares
estimator of the trend coefficient. Simulation results show that the 2.5% and 97.5% points
of this statistic are much closer to the percentiles used in practice than those constructed
with estimated generalized least squares variances. However, there remains a large difference
between the behavior of the test statistic for trend for $\alpha = 1$ and that for α close to but less
than one.

Acknowledgments

APPENDIX

Proof of Theorem 1

Case: $|\alpha| < 1$

Since $\tilde{\alpha}$ and $\hat{\alpha}_{WS}$ are $T^{1/2}$ consistent estimators of α and $\tilde{\beta}$ and $\hat{\beta}_{WS}$ are the feasible genralized least squares estimators of β based on $\tilde{\alpha}$ and $\hat{\alpha}_{WS}$, respectively, we have

$$T^{3/2}(\tilde{\beta} - \beta) \xrightarrow{\mathcal{L}} N(0, \sigma_\beta^2)$$

$$T^{3/2}(\hat{\beta}_{WS} - \beta) \xrightarrow{\mathcal{L}} N(0, \sigma_\beta^2)$$

where $\sigma_\beta^2 = 12(1-\alpha)^{-2}$. See Canjel and Watson (1997) or Grenander and Rosenblatt (1958). Now we need to get the probability limit of $se(\tilde{\beta}_{GN})$. Let us denote the (ij)-th element of the sums of squares and products matrix from the Gauss-Newton regression by v_{ij}. Then

$$
\begin{aligned}
v_{11} &= (T-1)(1-\ddot{\alpha})^2 + [(1-\ddot{\alpha})^{1/2}I(\ddot{\alpha} < 1) + I(\ddot{\alpha} = 1)]^2 \\
v_{12} &= \sum_{t=2}^{T}(1-\ddot{\alpha})[t(1-\ddot{\alpha}) + \ddot{\alpha}] + [(1-\ddot{\alpha})^{1/2}I(\ddot{\alpha} < 1) + I(\ddot{\alpha} = 1)]^2 \\
v_{13} &= \sum_{t=2}^{T}(1-\ddot{\alpha})\ddot{y}_{t-1} \\
v_{22} &= \sum_{t=2}^{T}[t(1-\ddot{\alpha}) + \ddot{\alpha}]^2 + [(1-\ddot{\alpha})^{1/2}I(\ddot{\alpha} < 1) + I(\ddot{\alpha} = 1)]^2 \\
v_{23} &= \sum_{t=2}^{T}(1-\ddot{\alpha})[t(1-\ddot{\alpha}) + \ddot{\alpha}]\ddot{y}_{t-1} \\
v_{33} &= \sum_{t=2}^{T}\ddot{y}_{t-1}^2
\end{aligned}
$$

where $I(A)$ denotes the indicator function of the event A.

Because for $|\alpha| < 1$, $\sum_{t=2}^{T} u_{t-1} = O_p(T^{1/2})$, $\sum_{t=2}^{T} u_{t-1}^2 = O_p(T)$, $\sum_{t=2}^{T} t\, u_{t-1} = O_p(T^{3/2})$.

we have

$$V = \begin{pmatrix} O_p(T) & O_p(T^2) & O_p(T^{1/2}) \\ O_p(T^2) & O_p(T^3) & O_p(T^{3/2}) \\ O_p(T^{1/2}) & O_p(T^{3/2}) & O_p(T) \end{pmatrix}.$$

where V is the sums of squares and products matrix from the Gauss-Newton regression. Then

$$se(\tilde{\beta}_{GN}) \approx [v_{11}v_{22} - v_{12}^2]^{-1/2} v_{11}^{1/2}$$

where $a \approx b$ means that $p\lim_{T\to\infty} ab^{-1} = 1$. Now $[v_{11}v_{22} - v_{12}^2]^{-1} v_{11}$ is the same as the leading term in the variance of the feasible generalized least squares estimator of β Hence the result.

Case: $\alpha = 1$

When $\alpha = 1$, we have

$$T^{1/2}(\tilde{\beta} - 1) \xrightarrow{\mathcal{L}} N(0,1)$$

$$T^{1/2}(\hat{\beta}_{WS} - 1) \xrightarrow{\mathcal{L}} N(0,1)$$

See Canjel and Watson (1997). Also

$$T(\hat{\alpha}_{WS}) \xrightarrow{\mathcal{L}} X$$

See Fuller (1996). Because $(\ddot{\alpha} - 1) = O_p(T^{-1})$, $\sum_{t=2}^{T} u_{t-1} = O_p(T^{3/2})$, $\sum_{t=2}^{T} u_{t-1}^2 = O_p(T^2)$, $\sum_{t=2}^{T} t\, u_{t-1} = O_p(T^{5/2})$, we have

$$V = \begin{pmatrix} O_p(1) & O_p(1) & O_p(T^{1/2}) \\ O_p(1) & O_p(T) & O_p(T^{3/2}) \\ O_p(T^{1/2}) & O_p(T^{3/2}) & O_p(T^2) \end{pmatrix}.$$

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Therefore

$$se(\tilde{\beta}_{GN}) \approx [v_{33}v_{22} - v_{32}^2]^{-1/2}v_{33}^{1/2}$$

or

$$T^{1/2}se(\tilde{\beta}_{GN}) \approx [(T^{-2}v_{33})(T^{-1}v_{22}) - (T^{-3/2}v_{32})^2]^{-1/2}(T^{-2}v_{33})^{1/2}$$

Noting that $\ddot{C}(.)$ is a continuous function we have

$$T(\ddot{\alpha} - 1) \xrightarrow{\mathcal{L}} X + \ddot{C}(Y)Z^{1/2}$$

Now because

$$T^{-3/2}\sum_{t=2}^{T} u_{t-1} \xrightarrow{\mathcal{L}} \int_0^1 W(t)dt,$$

$$T^{-5/2}\sum_{t=2}^{T} t\, u_{t-1} \xrightarrow{\mathcal{L}} \int_0^1 tW(t)dt,$$

$$T^{-2}\sum_{t=2}^{T} u_{t-1}^2 \xrightarrow{\mathcal{L}} \int_0^1 W^2(t)dt,$$

and $n^{1/2}(\hat{\beta} - \beta)$ converges in distribution to a standard normal variable, we have $T^{-1}v_{22} \xrightarrow{\mathcal{L}}$ η_{22}. $T^{-3/2}v_{32} \xrightarrow{\mathcal{L}} \eta_{32}$, and $T^{-2}v_{33} \xrightarrow{\mathcal{L}} \eta_{33}$ Therefore

$$t_{GN}(\tilde{\beta}) \xrightarrow{\mathcal{L}} [\eta_{33}\eta_{22} - \eta_{32}^2]^{1/2}\eta_{33}^{-1/2}\xi_0$$

The limiting distribution of $t_{GN}(\hat{\beta}_{WS})$ is derived similarly and hence omitted. The leading term in the variance of the FGLS estimator of β is $(1 - \hat{\alpha}_{WS})^2T^3I(\hat{\alpha}_{WS} < 1) + I(\hat{\alpha}_{WS} \geq 1)T$. Because

$$T(\hat{\alpha}_{WS} - 1) \xrightarrow{\mathcal{L}} X$$

we have the result. ▲

Note: The expression for the standard error from the FGLS procedure is $w^{(22)}$ where $w^{(ij)}$ are the elements of $\boldsymbol{W}^{-1}$ and $\boldsymbol{W}$ is the upper left 1×1 block of $\boldsymbol{V}$ if $\hat{\alpha}_{WS} = 1$ or the

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upper left 2×2 block of V if $\hat{\alpha}_{WS} < 1$. Thus the standard error from the FGLS procedure is smaller than the $v^{(22)}$ where $v^{(ij)}$ are the elements of V^{-1}. Also the expression for the leading term in the Gauss-Newton variance expression

$$[(\sum_2^T \ddot{y}_{t-1}^2)\{\sum_1^T (\Delta(t+1)+1)^2\} - (\sum_1^T \Delta(\Delta(t+1)+1)\ddot{y}_{t-1})^2]^{-1}$$

is an increasing function of the error in estimation, $\Delta = (\ddot{\alpha} - 1)$.

We provide here the Gauss-Newton equations used for estimating the variance of the estimated generalized least squares estimator of the trend coefficient for p-th order autoregressive process.

Let $\mathbf{F}^{*}$ be a $T \times 3$ matrix defined by

$$(F_{11}^{*}, \dots, F_{p1}^{*})' = (1, 1, \dots, 1)' \quad \ddot{\alpha} < 1$$
$$= (1, 0, \dots, 0)' \quad \ddot{\alpha} = 1$$

$$(F_{12}^{*}, \dots, F_{p2}^{*})' = (1, 2, \dots, p)' \quad \ddot{\alpha} < 1$$
$$= (1, 1, \dots, 1)' \quad \ddot{\alpha} = 1$$

$$(F_{13}^{*}, \dots, F_{p3}^{*})' = (0, 0, \dots, 0)'$$

and for $t = p + 1, \dots, T$,

$$F_{t1}^{*} = (1 - \ddot{\alpha})$$
$$F_{t1}^{*} = t - \ddot{\alpha}(t-1) - \tilde{\psi}_1 - \cdots - \tilde{\psi}_{p-1}$$
$$F_{t1}^{*} = \ddot{y}_{t-1}$$

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where

$$\ddot{y}_t = Y_t - \ddot{\mu} - \ddot{\beta} t.$$

Also let $e^{\ast}$ be defined by

$$
\begin{aligned}
(e_1^{\ast}, \ldots, e_p^{\ast}) &= (\ddot{y}_1, \ldots, \ddot{y}_p) & \ddot{\alpha} < 1 \\
&= (\ddot{y}_1, \Delta\ddot{y}_1, \ldots, \Delta\ddot{y}_{p-1}) & \ddot{\alpha} = 1
\end{aligned}
$$

and

$$e_t = e_t^{\ast} = \ddot{y}_t - \ddot{\alpha}\ddot{y}_{t-1} - \tilde{\psi}_1\Delta\ddot{y}_{t-1} - \cdots - \tilde{\psi}_{p-1}\Delta\ddot{y}_{t-p+1}, \quad t = p+1, \ldots, T.$$

Let V be the $p \times p$ covariance matrix of the first p observations of the estimated stationary process when $\ddot{\alpha} < 1$. Let V_1 be the $(p-1) \times (p-1)$ covariance matrix of the first $p-1$ observations of the estimated stationary process for ΔY_t when $\ddot{\alpha} = 1$. Let

$$V_2 = Block\ diag[1, V_1].$$

Let $\mathbf{F}_{(1)}^{\ast}$ be the first $p \times 3$ block of $\mathbf{F}^{\ast}$ and let

$$(F_{t1}, F_{t2}, F_{t3}) = (F_{t1}^{\ast}, F_{t2}^{\ast}, F_{t3}^{\ast})$$

for $t = p+1, \ldots, T.$

Define

$$
\begin{aligned}
F_{(1)} &= V^{-1/2}F_{(1)}^{\ast} & \ddot{\alpha} < 1 \\
F_{(1)} &= V_2^{-1/2}F_{(1)}^{\ast} & \ddot{\alpha} = 1
\end{aligned}
$$

Then the Gauss-Newton regression equations are defined by

$$e_t = F_{t1}\Delta\ddot{\mu} + F_{t2}\Delta\ddot{\beta} + F_{t3}\Delta\ddot{\alpha} + \nu_t \tag{A.1}$$

where

$$
\begin{aligned}
(e_1, \ldots, e_p)' &= V^{-1/2}(e_1^{\ast}, \ldots, e_p^{\ast})' & \ddot{\alpha} < 1 \\
(e_1, \ldots, e_p)' &= V_2^{-1/2}(e_1^{\ast}, \ldots, e_p^{\ast})' & \ddot{\alpha} = 1
\end{aligned}
$$

The standard error for the estimated generalized least squares estimator of β is the usual standard error of $\Delta\tilde{\beta}$ from the ordinary least squares fit of (A.1).

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Table 1: T×Mean Squared Errors multiplied by T of Feasible Generalized Least Squares Estimators of the Trend Parameter $\beta(=0)$ in the Linear Trend with AR(1) Error Model (T = 100)

	Estimator			t-Statistic		
α	$\hat{\beta}_{OLS}$	$\hat{\beta}_{WS}$	$\tilde{\beta}$	$t_{.975}(\hat{\beta}_{OLS})$	$t_{.975}(\hat{\beta}_{WS})$	$t_{.975}(\tilde{\beta})$
1.00	1.06572	1.05015	1.02044	8.76	8.36	5.15
0.98	0.47596	0.47807	0.45478	6.00	5.81	3.70
0.97	0.34807	0.33199	0.32509	5.27	5.11	3.28
0.95	0.19492	0.19685	0.20350	4.47	4.31	2.93
0.90	0.07662	0.07580	0.08608	3.43	3.21	2.40
0.80	0.02438	0.02387	0.02472	2.59	2.67	2.23
0.70	0.01156	0.01157	0.01169	2.36	2.38	2.10
0.60	0.00686	0.00688	0.00692	2.30	2.26	2.08
0.40	0.00327	0.00328	0.00328	2.13	2.17	2.05
0.00	0.00120	0.00120	0.00120	2.00	2.02	1.99

Table 2: The 97.5 Percentiles of the Distribution of Studentized Statistics for $\hat{\beta}$ as a Function of α (T $= 100$)

α	Statistic			
	$t_{.975}(\hat{\beta}_{OLS})$	$t_{.975}(\hat{\beta}_{WS})$	$t_{.975}(\tilde{\beta})$	$t_{.975}(\tilde{\beta}_{GN})$
1.00	8.76	8.36	5.15	2.35
0.98	6.00	5.81	3.70	1.81
0.97	5.27	5.11	3.28	1.76
0.95	4.47	4.31	2.93	1.86
0.90	3.43	3.21	2.40	1.95
0.80	2.59	2.67	2.23	2.11
0.70	2.36	2.38	2.10	2.08
0.60	2.30	2.26	2.08	2.06
0.40	2.13	2.17	2.05	2.05
0.00	2.00	2.02	1.99	1.98

Table 3: Properties of Trend Estimator for AR(2) model (T = 100)

(α, ψ_1)	$\dfrac{MSE(\hat{\beta}_{WS})}{MSE(\hat{\beta})}$	$t_{.975}(\hat{\beta}_{WS})$	$t_{.975}(\hat{\beta}_{GN})$
(1.00, 0.80)	1.094	10.53	3.19
(0.98, 0.80)	0.965	3.05	1.69
(0.95, 0.80)	0.986	2.48	1.97
(0.90, 0.80)	1.008	2.26	2.00
(0.80, 0.80)	1.002	2.15	2.00
(1.00, 0.50)	1.040	9.06	2.80
(0.98, 0.50)	1.058	3.67	1.50
(0.95, 0.50)	0.978	2.95	1.67
(0.90, 0.50)	1.013	2.51	1.91
(0.80, 0.50)	1.016	2.27	2.02
(1.00, 0.00)	1.040	8.95	2.79
(0.98, 0.00)	1.076	4.75	1.57
(0.95, 0.00)	1.060	3.41	1.47
(0.90, 0.00)	1.016	2.90	1.74
(0.80, 0.00)	1.022	2.53	1.91
(1.00,-0.50)	1.029	8.80	2.71
(0.98,-0.50)	1.073	5.38	1.87
(0.95,-0.50)	1.112	3.80	1.55
(0.90,-0.50)	1.081	3.18	1.65
(0.80,-0.50)	1.023	2.79	1.90
(1.00,-0.80)	1.042	8.62	2.84
(0.98,-0.80)	1.071	5.70	2.00
(0.95,-0.80)	1.113	4.07	1.55
(0.90,-0.80)	1.099	3.25	1.56
(0.80,-0.80)	1.024	2.79	1.82