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Pricing mortgages with default and prepayment

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DEDICATION

I would like to dedicate this thesis to my father Tong Xueliang and to my mother Li Yanyan without whose support I would not have been able to complete this work. I would also like to thank my friends and family for their loving guidance during the writing of this work.

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ABSTRACT

This thesis develops a model to price the fixed-rate mortgage with default and prepayment as derivative assets, generally termed the option-pricing approach to mortgage valuation. The problem is considered in a stochastic environment when the house price follows a log-normal diffusion process. A highly accurate numerical scheme is presented to solve the partial differential equation of the value of the mortgage. In order to simplify the method, the interest rate is considered to be a constant within each month. Further, a discount factor is graded to make the model more suitable to current economic situation.

CHAPTER 1. INTRODUCTION

1.1 Types of Mortgage Loans

A mortgage (literally meaning a dead pledge) is a type of financial contract which is a kind of fixed-income product. It is a legal document by which a real estate asset is pledged as security for the repayment of a loan; the pledge is cancelled when the debt is paid in full. The mortgage industry has undergone a massive evolution since the great depression in the 1930s. Back then the type of mortgage loans given resembled balloon loans in which the principal was not amortized, or only partially amortized at the maturity date. Since the type of mortgage loan and the cash flow it carries with it has a significant effect on the overall performance of a mortgage pool and for that matter the securities embedded with them, we will look at some very common and widely traded mortgage instruments and how their characteristics affect their cash flows.

1.1.1 Fixed-Rate, Level-Payment, Fully Amortized Mortgage

The basic idea behind the design of the fixed-rate, level-payment, fully amortized mortgage is that the borrower pays interest and principal in equal installments over the term of the loan. Typically payments are done monthly and by the end of the loan term the mortgage is fully amortized. Each monthly payment for a level payment mortgage is due on the first day of each month and consists of interest of 1/12 of the fixed annual interest rate times the amount of outstanding mortgage balance at the beginning of the previous month and a payment of some fraction of the principal.

1.1.2 Adjustable-Rate Mortgages

An adjustable rate mortgage (ARM) is a loan in which the mortgage rate is retuned periodically in accordance with some appropriate chosen reference rate. This instrument was specifically developed to deal with mismatch between mortgage durations and other liabilities in a high interest rate environment. ARMs usually start with lower interest rates and are reset in accordance with some index rate, such as the U.S. Treasury securities. To encourage borrowers to accept ARM rather than fixed rate mortgages, originators generally offer an initial contract rate that is less than prevailing market mortgage rate. The cash flow for ARMs is more complicated than that of fixed-rate mortgages'.

1.1.3 Balloon Mortgages

In a balloon mortgage the borrower is given long-term financing by the lender but at specific future dates the mortgage rate is renegotiated. Many single-family balloon originated today carry fixed rate and a 30-year amortization schedule. They typically require a balloon payment of the principal outstanding on the loan at the end of 5 or more years. Balloon mortgages are attractive to borrowers because they offer mortgages rates that are significantly lower than generic 30-year mortgages. Nowadays many balloon mortgages contract are actually hybrids that contain provisions allowing the borrower to take out a new loan from the current lender to finance the balloon payment with minimum requalification requirements. As has being pointed out earlier, the growing complexity of lending and borrowing has led to the development of more complicated mortgage products to basically cater for specific individual needs and requirements. Most of these products are however prevalent in the secondary mortgage market.

1.2 Mortgage-Backed Securities

Mortgage-backed securities (MBS) are securities backed by a pool (collection) mortgage loans. In section one we looked at an overview of mortgage loans, which is the raw material for mortgage-backed securities. While any type of mortgage loans, residential or commercial, can be used as collateral for a mortgage-backed security, most are backed by residential mortgages.

Just as the value of any other type of security depends on the cash flow of the underlining asset, the value of mortgage-backed securities depends on the cash flow of the underlining mortgage loans. It suffice to say therefore that different types of mortgage loans comes with different cashflows and hence affect the value of the MBS differently.

1.3 Overview of the Model

The thesis only considers the value of fixed-rate mortgage. A mortgage can be treated as a derivative security. The value of the mortgage is determined by the underlying house price and the term structure of interest rates. For simplicity, the paper considers the interest rate to be fixed for each month. A mortgage is a prime example of a financial product that can be modeled and then valued using option-pricing theory.

The lender(usually a bank), who issues the contract, would like to know the value of the future cash flows that will be received as the result of the borrower making the scheduled monthly payments. However, the borrower may also terminate the contract prior to maturity by the action of prepayment or default. Prepayment means the borrower has the option to prepay the remainder of the outstanding balance owed if interest rates are financially favorable; this is an American call option which spans the whole mortgage. The borrower also has the option to default on the mortgage when a monthly payment falls due; this amounts to a series of linked monthly European options.

We maintain the traditional assumptions that capital markets are perfect and operate costless. Option pricing techniques lead to the value of the mortgage being described as the solution to a one dimensional partial differential equation (PDE) in backward time, whose terminal and boundary conditions embody the terms of the contract. Since default can rationally occur only at the time of required payment, both the amortization and default characteristics of a fixed-rate mortgage by treating it as compound option. When treated as a PDE, the problem is restarted at monthly intervals, with the previous month's valuation yielding the terminal conditions needed to value the subsequent month. However, for prepayment which can occur at any time and so must be treated as a free boundary condition, the device cannot be applied. So Newton Method is used to find the free boundary. In order to comply with the traditional numerical method, a substitution is applied to the PDE to convert it to a forward time.

As the mortgage is a contract between two parties, it is assumed that neither would enter into an agreement unless it was fair at the onset. This means that the value of the mortgage to the lender at origination (when the contract begins) must be equal to the amount lent to the borrower. So we have to further impose the condition that the initial value of the contract matches the value of loan to guarantee that the contract is in equilibrium at origination. Thus, the entire procedure just described must be repeated for different contract rates, until one is found that yields a normal rate of return on the loan. This contract rate would be the mortgage contract rate at origination. However, it may take a few days for the program to run in order to find a proper contract rate. So this paper will only provide one set of result of the value of the mortgage with determined (and reasonable) contract rate.

After presenting the model to value the fixed-rate mortgage, I move forward to add a discount factor to the model to make it more suitable to present economic situation.

CHAPTER 2. DESCRIPTION OF THE MODEL

2.1 Economic Environment

2.1.1 House Price

The house price H is assumed to follow the standard log-normal diffusion process:

$$dH = (\mu - \delta)Hdt + \sigma_H H dX_H$$

where

- μ is the instantaneous average rate of house-price appreciation,
- δ is the “dividend-type“ per unit service flow provided by the house,
- σ_H is the house price volatility,
- X_H is the standard Wiener process for the house price.

The house-price appreciation μ is analogous to the drift term for standard stock-price model. The service flow δ is analogous to a dividend on a stock as the borrower benefits from the underlying asset (the borrower is allowed to live in the real estate asset during the life of the mortgage contract). The borrower benefits from the asset, therefore the price must drop by this amount otherwise arbitrage would occur.

2.1.2 Term Structure

The term structure of interest rates is modeled using the single factor Cox-Ingersoll-Ross (CIR) mean-reverting square root process, where the spot rate r is driven by:

$$dr = \kappa(\theta - r)dt + \sigma_r \sqrt{r} dX_r$$

where

- κ is the speed of adjustment in the mean reverting process,
- θ is the long-term mean of the short-term interest rate r,
- σ_r is the interest-rate volatility,
- X_r is the standard Wiener process for the interest rate.

But for simplicity of the numerical scheme, the paper considers the interest rate to be fixed for each month.

2.1.3 Correlation

The stochastic elements of the house-price H process and the spot interest-rate r process which involve the standardized Wiener processes, X_H for house price and X_r for interest rate respectively, are correlated according to

$$dX_H dX_r = \rho dt,$$

where ρ is the instantaneous correlation coefficient between the two Wiener processes.

2.2 Derivation of the PDE

This section demonstrates a derivation of the asset valuation PDE using standard no arbitrage arguments. The PDE for the valuation of any asset $F = F(H, r, t)$ whose value is a function only of house price H, interest rate r, and time t, can be found as follows. House price is described by the standard log-normal diffusion process and stochastic interest rate follows the single factor Cox-Ingersoll-Ross (CIR) mean-reverting square root process. Using Ito's lemma for the two stochastic variables (see Ito, 1951, for the details) on the function $F(H, r, t)$, it can be shown that,

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial H}dH + \frac{\partial F}{\partial r}dr + \frac{1}{2}(\sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + 2\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H \partial r} + \sigma_r^2 r \frac{\partial^2 F}{\partial r^2})dt.$$

Now we construct a portfolio Π consisting of long one asset $F_1(H,r,t)$ with maturity T_1, short Δ_2 of an asset $F_2(H,r,t)$ with maturity T_2, and short Δ_1 of the underlying asset H. Thus,

$$\Pi = F1 - \Delta_2 F2 - \Delta_1 H$$

$$d\Pi = dF1 - \Delta_2 dF2 - \Delta_1 dH,$$

where Δ_1 and Δ_2 are constants during time dt. The effect of the service flow δ is to cause the price of the underlying asset H to drop in value by δH over a time dt. Therefore, the portfolio must change by an amount $-\delta H \Delta_1 dt$ during this time. Thus, the correct change in the value of the portfolio over a time dt is

$$d\Pi = dF_1 - \Delta_2 dF_2 - \Delta_1(dH + \delta H dt).$$

We choose $\Delta_2 = \frac{\partial F_1 \setminus \partial r}{\partial F_2 \setminus \partial r}$, $\Delta_1 = \frac{\partial F_1}{\partial H} - \Delta_2 \frac{\partial F_2}{\partial H}$. So

$$\begin{aligned} d\Pi = & \frac{\partial F_1}{\partial t}dt + \frac{1}{2}(\sigma_H^2 H^2 \frac{\partial^2 F_1}{\partial H^2} + 2\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F_1}{\partial H \partial r} + \sigma_r^2 r \frac{\partial^2 F_1}{\partial r^2})dt - \delta H \frac{\partial F_1}{\partial H}dt \\ & - \frac{\partial F_1 \setminus \partial r}{\partial F_2 \setminus \partial r}(\frac{\partial F_2}{\partial t}dt + \frac{1}{2}(\sigma_H^2 H^2 \frac{\partial^2 F_2}{\partial H^2} + 2\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F_2}{\partial H \partial r} + \sigma_r^2 r \frac{\partial^2 F_2}{\partial r^2})dt - \delta H \frac{\partial F_2}{\partial H}dt) \\ & = r(F_1 - \frac{\partial F_1 \setminus \partial r}{\partial F_2 \setminus \partial r}F_2 - \frac{\partial F_1}{\partial H}H + \frac{\partial F_1 \setminus \partial r}{\partial F_2 \setminus \partial r}\frac{\partial F_2}{\partial H}H)dt. \end{aligned}$$

No arbitrage arguments implies that the return on the portfolio to be $r\Pi dt$,since the growth of the portfolio in a time step dt is equal to the risk-free growth rate of the portfolio, as the portfolio is now completely deterministic(volatility term is zero). Dividing by dt and separating the F_1 and F_2 terms leads to,

$$\begin{aligned} & \frac{1}{\partial F_1 \setminus \partial r}(\frac{\partial F_1}{\partial t} + \frac{1}{2}\delta_H^2 H^2 \frac{\partial^2 F_1}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F_1}{\partial H \partial r}\frac{1}{2}\sigma_r^2 r \frac{\partial^2 F_1}{\partial r^2} + (r-\delta)H\frac{\partial F_1}{\partial H} - rF_1) \\ & = \frac{1}{\partial F_2 \setminus \partial r}(\frac{\partial F_2}{\partial t} + \frac{1}{2}\delta_H^2 H^2 \frac{\partial^2 F_2}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F_2}{\partial H \partial r}\frac{1}{2}\sigma_r^2 r \frac{\partial^2 F_2}{\partial r^2} + (r-\delta)H\frac{\partial F_2}{\partial H} - rF_2) \end{aligned}$$

Although this is one equation in two unknowns, the left-hand side is a function of T_1 but not of T_2 and the right-hand side is a function of T_2 but not of T_1. The only way for this to be possible is for both sides to be independent of maturity date. Thus, removing the subscript from F,

$$\frac{1}{\partial F \setminus \partial r}(\frac{\partial F}{\partial t} + \frac{1}{2}\delta_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H \partial r} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 F}{\partial r^2} + (r-\delta)H\frac{\partial F}{\partial H} - rF) = a(H,r,t),$$

is obtained for some function $a(H, r, t)$. It is convenient to write $a(H, r, t) = -\kappa(\theta - r)$ (this is a standard procedure in the literature, see Kau et al. 2002, 2003), which leads to the asset valuation PDE for $F(H, r, t)$,

$$\frac{1}{2}\delta_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H \partial r} + \frac{1}{2}\sigma_r^2 r\frac{\partial^2 F}{\partial r^2} + \kappa(\theta - r)\frac{\partial F}{\partial r} + (r - \delta)H\frac{\partial F}{\partial H} + \frac{\partial F}{\partial t} - rF = 0.$$

This PDE will be solved using a backward valuation procedure.It is necessary to start the process from the known information at maturity, referring to the known cashflows at the final moment of the contract. In the next section, we will explore the specific PDEs and boundary conditions of each component of the contract.

2.3 Restrictions of The PDE Derived from the Contract

For this section, a detailed explanation of the model is provided.

2.3.1 Formulae

To formalize the model, we introduce the following notation.

- L: Original loan amount
- c: Fixed yearly contract rate
- num: Term of the loan in months
- i: Payment date in months, $1 \leq i \leq num + 1$
- $\tau(i)$: Calendar time of the i_{th} month, i.e. $\tau(i) = i/12$
- MP: Monthly mortgage payment
- $PB(i)$: Unpaid principal after the i_{th} payment date
- $TD(\tau)$: Unpaid principal plus accrued interest for $\tau(i) < \tau \leq \tau(i + 1)$
- $A(r, \tau, i)$: Value at time τ of the promised mortgage payments from i to num

- $D(i,H,\tau,r)$: Value at time τ of the default option, when the next mortgage payment is due at time $\tau(i)$

- $C(i,H,\tau,r)$: Value at time τ of the prepayment option, when the next mortgage payment is due at time $\tau(i)$

- $V(i,H,\tau,r)$: Value at time τ of the contract, when the next mortgage payment is due at time $\tau(i)$

2.3.1.1 Value of Monthly Payments

To define the value of each monthly payment MP it is necessary to recognize that future value of the outstanding debt in the terminal period of the contract must be equal to the future value of all the payments, when this value is also referred to the terminal moment of the contract. Consequently,

$$L(1+\frac{c}{12})^{num} = MP(1+\frac{c}{12})^{num}[\frac{1-(1+\frac{c}{12})^{-num}}{\frac{c}{12}}],$$

which upon slightly simplifying yields,

$$L(1+\frac{c}{12})^{num} = MP[\frac{(1+\frac{c}{12})^{n}-1}{\frac{c}{12}}].$$

So

$$MP = \frac{L(1+\frac{c}{12})^{num}(\frac{c}{12})}{(1+\frac{c}{12}-1)},$$

gives the formula for the value of the monthly payments, where L is the amount initially loan to the borrower, c is the fixed yearly contract rate, and num is the life of the mortgage in months.

2.3.1.2 Value of the Principal Balance

Immediately after the i_{th} monthly payment has been made, the unpaid principle $PB(i)$ the borrower still has to repay can be expressed in the following way

$$PB(i) = \{L - MP[\frac{1-(1+\frac{c}{12})^{-i}}{\frac{c}{12}}]\}(1+\frac{c}{12})^{i}.$$

Substitute MP into previous expression yields and then simplifying gives

$$PB(i) = \frac{L[(1+\frac{c}{12})^{num} - (1+\frac{c}{12})^{i}]}{(1+\frac{c}{12})^{num} - 1},$$

which is the formula for the value of the outstanding balance $PB(i)$ after the i_{th} monthly payment has been made.

2.3.1.3 Value of the Unpaid Principal Plus Accrued Interest

The unpaid principal plus accrued interest $TD(\tau)$ for $\tau(i) < \tau \leq \tau(i+1)$ is

$$TD(\tau) = PB(i)[1 + c(t - \tau(i))].$$

2.3.2 PDE and Terminal Conditions for Value of the Promised Mortgage Payments

Denote $A(r, \tau, i)$ to be the value of the promised mortgage payments from payment date i to num. Since A is unrelated to H, the PDE for A is

$$\frac{1}{2} r \sigma_r^2 \frac{\partial^2 A}{\partial^2 r} + \kappa(\theta - r)\frac{\partial A}{\partial r} + \frac{\partial A}{\partial \tau} - rA = 0.$$

Since we look at r to be constant $r(i)$ at i_{th} month, the PDE then becomes a ODE

$$\frac{\partial A}{\partial \tau} - r(i)A = 0.$$

According to the contract, at maturity i.e. when $\tau = \tau(num)$, the borrower only need to pay the last monthly payment, so

$$A(r_{num}, \tau(num), num) = MP.$$

Given next are the value of all the mortgage components for the other payment dates, namely at the end of month $1, 2, ..., num - 2, num - 1$. The value of the remaining future payments promised to the lender at these times is,

$$A(r(i), \tau(i), i) = A(r_{i+1}, \tau(i), i+1) + MP,$$

where $1 \leq i \leq num - 1$. With the above ODE and boundary conditions we will actually solve for A analytically in next chapter.

2.3.3 PDE and Terminal Conditions for Value of the Default Option

The value of at time τ of the default option when the next mortgage payment is due at $\tau(i)$ is denoted as $D(r, H, \tau, i)$. It satisfies the PDE we derived in section 2.2. So

$$\frac{1}{2}\delta_H^2 H^2 \frac{\partial^2 D}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 D}{\partial H \partial r} + \frac{1}{2}\sigma_r^2 r\frac{\partial^2 D}{\partial r^2} + \kappa(\theta - r)\frac{\partial D}{\partial r} + (r-\delta)H\frac{\partial D}{\partial H} + \frac{\partial F}{\partial t} - rD = 0.$$

Since again we look at r as constant for each month, above PDE becomes

$$\frac{1}{2}\delta_H^2 H^2 \frac{\partial^2 D}{\partial H^2} + (r-\delta)H\frac{\partial D}{\partial H} + \frac{\partial F}{\partial t} - rD = 0.$$

At maturity,

$$D(\tau(num), num) = \max\{0, MP - H\}.$$

This is because default would be of value in the extreme case where the value of the house H had fallen to less than the final monthly payment MP.

For earlier payment dates,

$$D(H, \tau(i), i) = \begin{cases} D(H, \tau(i), i+1) & \text{if no default,} \\ A(\tau(i), i) - H & \text{if default.} \end{cases}$$

Note that the value of default in the present, $D(\tau(i), i)$, equals the value of future default $D(\tau(i), i+1)$, when no default currently occurs, while $D(\tau(i), i)$ equals $A - H$ when default does occur. Nonetheless, we cannot simply compare these values to determine whether default does or does not occur, since we must also consider the value of future prepayment lost upon default.

2.3.4 PDE and Terminal Conditions for Value of the Prepayment Option

Denote $C(r, H, \tau, i)$ to be the value at time τ of the prepayment option when the next mortgage payment is due at $\tau(i)$. It satisfies the PDE we derived in section 2.2. So

$$\frac{1}{2}\delta_H^2 H^2 \frac{\partial^2 C}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 C}{\partial H \partial r} + \frac{1}{2}\sigma_r^2 r\frac{\partial^2 C}{\partial r^2} + \kappa(\theta - r)\frac{\partial C}{\partial r} + (r-\delta)H\frac{\partial C}{\partial H} + \frac{\partial F}{\partial t} - rC = 0.$$

Since again we look at r as constant for each month, above PDE becomes

$$\frac{1}{2}\delta_H^2 H^2 \frac{\partial^2 C}{\partial H^2} + (r-\delta)H\frac{\partial C}{\partial H} + \frac{\partial F}{\partial t} - rC = 0.$$

At maturity,

$$C(\tau(num), num) = 0.$$

This is because the borrower holds the house and has an obligation to make the final payment MP, but also has the options to default or prepay. Of course, prepayment could not be of any value at the maturity of the loan. For earlier payment dates,

$$C(H, \tau(i), i) = \begin{cases} C(H, \tau(i), i+1) & \text{if no default} \\ 0 & \text{if default} \end{cases}$$

2.3.5 PDE and Terminal Conditions for Value of the Mortgage

The value at time τ of the contract $V(r, H, \tau, i)$ when the next payment is due at time $\tau(i)$ is

$$V(r, H, \tau, i) = A(r, \tau, i) - D(r, H, \tau, i) - C(r, H, \tau, i).$$

The borrower may either pay the required monthly amount MP or default. The value of the mortgage to the lender immediately before the payment at maturity is the minimum of MP and the house value, i.e.

$$V(\tau(num), num) = \min\{MP, H\}.$$

For earlier payment dates,

$$V(\tau(i), i) = \min\{V(\tau(i+1)) + MP, H\}.$$

2.3.6 Boundary Conditions

Here the boundary conditions when $H = 0$ and $H \to \infty$ is discussed. If the house price becomes zero, the borrower will default and the mortgage is now worth the same as the house, and so

$$V(r, 0, \tau) = 0.$$

Prepayment at this point is worthless, thus

$$C(r, 0, \tau) = 0.$$

The option to default is now equal to the value of the remaining payments. Since $D = A - C - V$, then

$$D(r, 0, \tau) = A(r, \tau).$$

As $H \to \infty$ the value of the default option tends to zero. Therefore,

$$\lim_{H \to \infty} D(r, H, \tau) = 0.$$

The value of the prepayment option is constant as H tends to infinity, implying

$$\lim_{H \to \infty} \frac{\partial C}{\partial H}(r, H, \tau) = 0.$$

Thus

$$\lim_{H \to \infty} V(r, H, \tau) = A(r, \tau) - \lim_{H \to \infty} C(r, H, \tau).$$

2.3.7 Free Boundary Condition

As noted earlier, the option to prepay is American in type, in the sense that prepayment could occur at any time during the lifetime of the contract. This produces a free boundary which must be applied in the appropriate position. On one side of the free boundary it is financially optimal for the borrower to prepay and on the other it is not. The prepayment boundary condition is obtained by observing that at each moment in time the value of the mortgage to the lender can be no greater than the value of the total debt TD,

$$V(r, H, \tau) \leq TD(\tau),$$

otherwise the lender would choose to prepay the mortgage. This occurs when the value of the mortgage to the lender is equal to the total debt required to be paid by the borrower if the mortgage was chosen to be prepaid at that time,

$$V(r, H, \tau) \leq TD(\tau).$$

Clearly, it is important to position the free boundary accurately. In the next chapter the problem is treated using the Newton method.

2.3.8 Equilibrium Condition

As mentioned previously, the mortgage contract would not be agreed originally by the two counter parties unless it was fair. This means that at origination the contract must be in financial equilibrium, which is the case if the value of the mortgage to the bank is equal to the amount lent to the borrower. A generalized equilibrium condition for a generic mortgage loan is as follows

$$V(\tau = 0, c) = (1 - fee)L.$$

The bank's position in the contracts is $V = A - D - C$, i.e. the scheduled payments minus the sum of the value of the borrower's options to terminate the mortgage (D is the value of the default option and C is the value of the prepayment option). The borrower's position is the amount lent by the bank, which will be some percentage of the initial house value, minus an arrangement fee (for a UK contract) or the points (for a US loan) charged as a percentage of the L which is the loan amount. The equilibrium constraint is to avoid contractual arbitrage. The parameter c is used to balance the equilibrium condition.
The free parameter c can be found using an iterative process following Newton's method. Let $f(c)$ be a function of c only and it is given

$$f(c) = V(\tau = 0, c) - (1 - fee)L,$$

which must be zero to satisfy the equilibrium condition. An initial estimate for the value of c is made, let this estimate to be c_0. Then the values of the mortgage components involved in the equilibrium condition are calculated with the initial estimate c_0 used as the value of the free parameter. Next, a tolerance to which the absolute value of $f(c)$ must be less than is specified; once $f(c)$ is less than this tolerance the iterative process is terminated. An estimate is required for the initial increment change in c_0; call this increment Δ_0. The next potential equilibrium setting free parameter c is given by $c_1 = c_0 + \Delta_0$. Given this information it is then possible to calculate $f(c_1)$ and check if its absolute value is less than tolerance. If the absolute value of $f(c_1)$ and any further $f(c_i)$ is greater than the tolerance the new increment for the change in c is calculated as follows,

$$\Delta_{i+1} = -\frac{\Delta_i f(c_i)}{f(c_i) - f(c_{i-1})}.$$

CHAPTER 3. NUMERICAL METHOD

In this chapter, the PDE is first changed from backward to forward. The initial conditions are also changed to forward time. Then the Crank-Nicolson method is applied to solve the PDE under certain initial and boundary conditions numerically. Since the prepayment option is subjected to a free boundary, we apply Newton method to deal with the problem. At last, the algorithm and pseudo code is provided.

3.1 Forward PDE and Corresponding Change in the Model

3.1.1 Change in the Notations

In order to simplify the numerical schemes used to approximate the solution to the PDE, we perform the following change of notations:

$$t = \tau(num) - \tau.$$

Also, at this moment, the interest rate r is set to be fixed for each payment month. Because of these change, we have to modify the other notations in the model also:

- r_k: The interest rate at the $k_t h$ month, $1 \leq k \leq num + 1$.
- T_k: Calendar time of the $k_t h$ payment date from the matueity, i.e. $T_k = (k-1)/12, 1 \leq k \leq num$. $T_1 = 0$ is the maturity time of the loan.
- $PB(k)$: Unpaid principal after the $k_t h$ payment date.
- $TD(t)$: Unpaid principal plus accrued interest for $T_k \leq t < T_{k+1}$.
- $A(t, k)$: Value at time t of the promised mortgage payments from k to $num + 1$.

- $D(H, t, k)$: Value at time t of the default option, when the next mortgage payment is due at time T_k.
- $C(H, t, k)$: Value at time t of the prepayment option, when the next mortgage payment is due at time T_k.
- $V(H, t, k)$: Value at time t of the contract, when the next mortgage payment is due at time T_k.

The total debt, i.e. the unpaid principal plus accrued interest for $T_k \leq t < T_{k+1}$ becomes:

$$TD(t) = [1 + c(T_{k+1} - t)]PB(num - k).$$

3.1.2 Changes in the PDE and Terminal Conditions for Value of the Promised Mortgage Payments

After the change of variable, the valuation PDE for $A(t, k)$, i.e. the value at time t of the promised mortgage payments from k to $num + 1$, becomes:

$$\frac{\partial A}{\partial t} + r_k A = 0.$$

At maturity when $t = T_1 = 0$,

$$A(T_1, 1) = MP.$$

The terminal conditions at earlier payment dates when $t = T_{k+1}$, $1 \leq k \leq num$ for A now becomes:

$$A(T_{k+1}, k + 1) = A(r_k, T_{k+1}, k) + MP.$$

3.1.3 Changes in the PDE and Terminal Conditions for Value of the Default Option

The valuation PDE for the value of the default option $D(H, t, k)$ becomes

$$\frac{1}{2}\delta_H^2 H^2 \frac{\partial^2 D}{\partial H^2} + (r_k - \delta)H\frac{\partial D}{\partial H} - \frac{\partial F}{\partial t} - r_k D = 0.$$

At maturity, when $t = T_1 = 0$,

$$D(H, T_1, 1) = \max\{0, MP - H\}.$$

For earlier payment dates when $t = T_{k+1}, 1 \leq k \leq num$,

$$D(H, T_{k+1}, k+1) = \begin{cases} D(H, T_{k+1}, k) & \text{if no default,} \\ A(T_{k+1}, k+1) - H & \text{if default.} \end{cases}$$

3.1.4 Changes in the PDE and Terminal Conditions for Value of the Prepayment Option

The valuation PDE for the value of the prepayment option $C(H, t, k)$ at time t when the next mortgage payment is due at time T_k now becomes

$$\frac{1}{2}\delta_H^2 H^2 \frac{\partial^2 C}{\partial H^2} + (r_k - \delta) H \frac{\partial C}{\partial H} - \frac{\partial F}{\partial t} - r_k C = 0.$$

At maturity, when $t = T_1 = 0$,

$$C(H, T_1, 1) = 0.$$

For earlier payment dates,

$$C(H, T_{k+1}, k+1) = \begin{cases} C(H, T_{k+1}, k) & \text{if no default,} \\ 0 & \text{if default.} \end{cases}$$

3.1.5 Changes in the PDE and Terminal Conditions for Value of the Mortgage

The value at time t of the contract $V(H, t, k)$ when the next payment is due at time T_k is

$$V(H, t, k) = A(t, k) - D(H, t, k) - C(H, t, k).$$

At maturity,

$$V(H, T_1, 1) = \min\{MP, H\}.$$

At earlier payment dates,

$$V(H, T_{k+1}, k+1) = \min\{V(H, T_{k+1}, k) + MP, H\}.$$

3.2 Finite Difference Method to Solve the PDEs

In this section, a specific numerical scheme is given to solve the PDEs. I will first present the storage and

3.2.1 Storage of the Result and Discretion of the Intervals

When the interest rate r is fixed to be r_k in the r_{th} month, the domain of D, C and V is

$$[H_{min}, H_{max}] \times [T_1, T_{num+1}].$$

Here $H_{min} = 0$, $H_1 = 0$. H direction is divided into I intervals in the way that for $i = 1, ..., I+1$

$$\delta H = \frac{H_{max} - H_{min}}{I},$$

$$H_i = H_{min} + (i-1)\delta H$$

For the t direction, as mentioned before, the payment date T_k for $1 \leq k \leq num + 1$ is

$$T_k = \frac{k-1}{12}.$$

T_{num+1} denotes the origination of the loan. Each interval $[T_k, T_{k+1}]$ is discretized into N parts. Denote that

$$\delta t = \frac{T_{k+1} - T_k}{N},$$

$$t_n^k = T_k + (n-1)\delta t,$$

where $1 \leq n \leq N$. So in this way,

$$T_{k+1} = t_1^{k+1}.$$

when $T_k \leq t < T_{k+1}$, the value of the promised mortgage payments A is stored in a two dimensional matrix $A(n,k)$. In fact, since A is subject to a ODE, so it can be solve analytically. The values of the default option D, prepayment option C and the mortgage values V is each stored in a three dimensional matric $F(k,i,n)$, where $F = C, D$ or V. Again, $1 \leq i \leq I$ is used to express different values of H. $1 \leq k \leq num$ is used to denote the value in the k_{th} month, i.e. $T_k \leq t \leq T_{k+1}$. $1 \leq n \leq N$ is to express time t_n^k in the k_{th} month.

3.2.2 Discretization of the PDE and Boundary Conditions

To solve for A, D, C and V, we look at the interval $T_k \leq t < T_{k+1}$. First of all, the equation for A is

$$\frac{dA}{dt} = -rA.$$

So $A = Ce^{-rt}$. On the interval $[T_k, T_{k+1})$, the initial condition at T_k is determined by prepayment and default decision based on its value at t_N^{k-1}, i.e. $A(N, k-1)$. So

$$A(1,k) = Ce^{r_k T_k}.$$

Thus

$$C = A(1,k)e^{r_k T_k}.$$

So we have

$$A(n,k) = A(1,k)e^{-r_k(t_n^k - T_k)}.$$

The other components of the mortgage are subjected to a PDE. So a numerical method is to be applied to approximate the solutions. In the paper, we choose the Crank-Nicolson method. In the interval of $[T_k, T_{k+1})$, the discretization of the PDE is

$$\frac{F_i^{n+1}}{\delta t} = \frac{1}{2}H_i^2\sigma_H^2[\frac{F_{i+1}^{n+1} - 2F_i^{n+1} + F_{i-1}^{n+1}}{2(\delta H)^2} + \frac{F_{i+1}^{n} - 2F_i^{n} + F_{i-1}^{n+1}}{2(\delta H)^2}]$$
$$+ (r-\delta)H_i[\frac{F_{i+1}^{n+1} - F_{i-1}^{n+1}}{4\delta H} + \frac{F_{i+1}^{n} - F_{i-1}^{n}}{4\delta H}] - r\frac{F_i^{n+1} + F_i^n}{2},$$

where $2 \leq i \leq I$, $1 \leq n \leq N-1$, $F_i^n = F(r_k, H_i, t_n^k)$. For the simplicity of the code, we denote

$$a_i = \frac{\delta t H_i}{4\delta H}[\frac{H_i\sigma_H^2}{\delta H} - (r-\delta)]$$
$$b_i = \frac{\delta t}{2}[\frac{H_i^2\sigma_H^2}{\delta H^2} + r]$$
$$c_i = \frac{\delta t H_i}{4\delta H}[\frac{H_i\sigma_H^2}{\delta H} + (r-\delta)],$$

where $2 \leq i \leq I$. So the Crank-Nicolson schemes becomes:

$$a_iF_{i-1}^{n+1} + (1+b_i)F_i^{n+1} - c_iF_{i+1}^{n+1} = a_iF_{i-1}^{n} + (1-b_i)F_i^{n} + c_iF_{i+1}^{n},$$

where $2 \le i \le I$, $1 \le n \le N-1$. Since the values of C, D are known when $H = 0$ and $H = H_{max}$, i.e. F_1^n and F_{I+1}^n are known, the first equation in the system becomes:

$$(1+b_2)F_2^{n+1} - c_2 F_3^{n+1} = a_2 F_1^n + (1-b_2)F_2^n + c_2 F_3^n + a_2 F_1^{n+1}.$$

The last one becomes

$$-a_I F_{I-1}^{n+1} + (1+b_I)F_I^{n+1} = a_I F_{I-1}^n + (1-b_I)F_I^n + c_I F_{I+1}^n + c_I F_{I+1}^{n+1}.$$

By solving the system of equations, we can get the value of $F(k, i, n+1)$ from previous level $F(k, i, n)$, where $1 \le n \le N$. However, we still need the value of $F(k, i, 1)$ for $1 \le i \le I+1$ as an initial condition. In fact, such value is determined by the terminal conditions for C and D at payment dates.

3.3 Terminal Conditions at Payment Dates

At each payment dates T_k we have to check whether the borrower will default or not. Based on his choice, we can then determine the initial value of F at T_k.

In fact, when at T_1, we need to check the relation between H_i and MP, $1 \le i \le I+1$. When $H_i < MP$, which means a extreme situation when the house price drops below the monthly payment before the last payment, a rational borrower will choose to default. So the value of C, D and V are choose to be value correspond to the default situation at maturity when $t = T_1$. On the other hand, when $H_i > MP$, a rational borrow would choose to pay the last monthly payment. So at this time the value of C, D and V are set to be the value correspond to the no default condition at maturity.

At previous payment dates T_k when $2 \le k \le num$, the default condition is different. We have to compare the value of $V(k-1, i, N) + MP$ with H_i for $1 \le i \le I+1$. When $H_i < V(k-1, i, N) + MP$, a rational borrower will choose to default. So the value of C, D and V are choose to be value correspond to the default situation at maturity when $t = T_k$, i.e. t_1^k. On the other hand, when $H_i > V(k-1, i, N) + MP$, a rational borrow would choose to pay the last monthly payment. So at this time the value of C, D and V are set to be the value correspond to the no default condition at $t = T_k = t_1^k$.

3.4 Newton Method to Find the Free Boundary

As mentioned in chapter 1, it is very important to notice that the prepayment option is similar the a series of American option which can be exercised any time before maturity. So the prepayment option is subjected to the free boundary:

$$V(k, i, n) \leq TD(t_n^k),$$

where $1 \leq k \leq num$, $1 \leq i \leq I + 1$, and $1 \leq n \leq N$. This means that at each time level t_n^k, When $V(k, i, n) < TD(t_n^k)$, a rational borrower will choose not to prepay the loan. So the values of C, D and V are subjected to the PDE and boundary conditions we listed in previous sections. However, when $V(k, i, n) > TD(t_n^k)$, a rational borrow would choose to prepay the loan. So at this time the value of the mortgage $V = C = TD(t_n^k)$ and $D = 0$.

How to find the free boundary is one of the most difficult problem in this model. For previous literature people tend to use linear complimentary method the find the free boundary. A main contribution of this paper is to apply Newton Method to tackle the free boundary problem.

In fact, the general idea of Newton Method is at the time level t_n^k, we first give a initial guess of i_0. Set $F(i) = V(k, i, n) - TD(t_n^k)$. n, k are fixed. For $1 \leq i \leq i_0$, we use the PDEs to solve for $C(k, i, n), D(k, i, n)$ and $V(k, i, n)$ from the previous level. For the $i_0 < i \leq I + 1$, we set $V(k, i, n) = TD(t_n^k)$. Next we find another i_1 by the iteration equation

$$H_{i_1} = H_{i_0} - \frac{F(i_0)}{dF_{i_0}}.$$

Then set $i_0 = i_1$. Keep doing this procedure until we find the i_0 such that $F(i_0) = 0$. For the last step, For $1 \leq i \leq i_0$, we use the PDEs to solve for $C(k, i, n), D(k, i, n)$ and $V(k, i, n)$ from the previous level. For the $i_0 < i \leq I+1$, we set $V(k,i,n) = TD(t_n^k)$.

3.5 Algorithm

For this section, the algorithm used to solve the model is presented. Four programs are needed to solve the model. The first program is called *mainmbs*, which is used to calculate V at each time level for a given contract rate c. The algorithm for *mainmbs* is:

1. Fix a initial contract rate c_1, and discretize the domain as described before.

2. Input $A(n,k)$ by the solution given previously and $TD(n,k)$, $PB(k)$ by definition for $1 \leq N$ and $1 \leq k \leq num$.

3. Based on default condition, give values to C and D before the last payment, i.e. $C(1,i,1)$ and $D(1,i,1)$ for $1 \leq i \leq I+1$.

4. Use the result of (3) and Newton Method to determine the value of C, D and V of the next level, i.e. $C(1,i,2)$, $D(1,i,2)$ and $V(1,i,2)$.

5. Repeat step (4) $N-1$ times to solve all $C(1,i,n)$, $D(1,i,n)$ and $V(1,i,n)$ for $2 \leq n \leq N$ and $1 \leq i \leq I+1$.

6. Use the result of step (5), i.e. $V(1,i,N)$ to determine the value of $C(2,i,1), D(2,i,1)$ according to both default and prepayment condition.

7. Repeat step $(4) - (6)$, find the values of $V(k, i, n)$ for all $1 \leq k \leq num$, $1 \leq i \leq I + 1$ and $1 \leq n \leq N$.

8. Use iteration defined by Newton Method to find the next c, i.e. c_2. Iteration stops until find the proper c such that $V(1, H_0, num + 1) = (1 - fee)L$, which is the equilibrium condition at origination.

A detailed algorithm for step (4) is also provided:

1. Give an initial guess H_{i0} for point that is on the free boundary at the next level.

2. Based on the initial guess, find the value of V at the next level.

3. Use Newton Method to find H_{i1} and repeat 2. Iteration stops when the H is found to satisfy $V(H) = TD$.

CHAPTER 4. RESULTS

In this chapter, the result provided by the program is presented in forms of graphs. To save time, the maturity is set to be one year, i.e $num = 12$. As mentioned before $c = 5.5\%$ is also fixed. Each month is divided into 20 levels. The result turns out that the prepayment option is zero at every point. The reason for this is probably that the length of maturity num we choose is too small. In such a short time the house price may not be able to be large enough for the borrower to sell the house and prepay the loan.

4.1 Graph of V with Respect to H for Fixed time

In the section, the time is fixed and the relation between the mortgage value V and the house price H is graphed.

First, before making the last payment, that is to say, when atT_1, the graph of V with respect to H is:

From the graph, V first linear increases with respect to H and then become flat. This is because at T_1, the only option for the borrower is either to default or not. When the house price is below certain value, the borrower will choose to default so the value of the mortgage is the same as the house price. This explains the linear part of the graph. However when the house price exceeds the default point, the value of the mortgage is equal to the monthly payment, i.e. MP in our notation. This explains the flat part of the graph.

The next graph is when we look at the value of the mortgage when $k = 8$ and $n = 15$. This means we look at the 8_{rd} level of the 5_{th} month before maturity. For each house value $H(i)$,

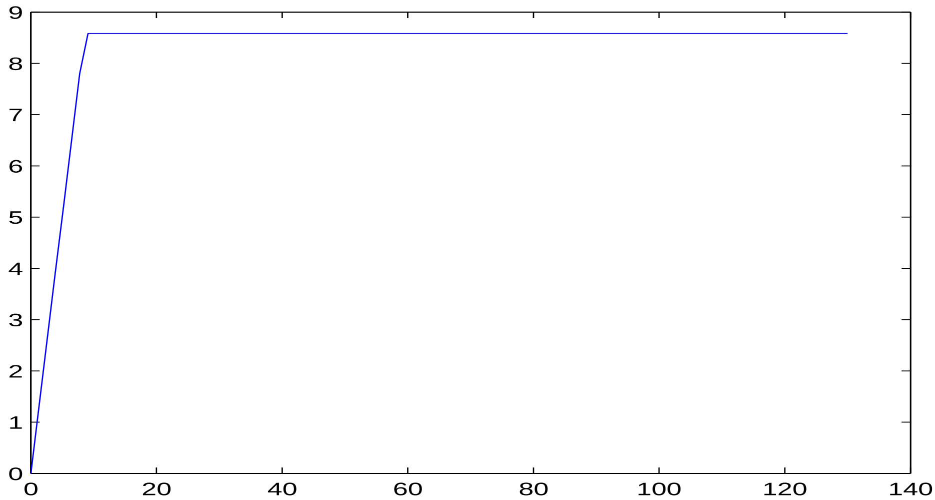

Figure 4.1 at T_1 H and V

$1 \leq i \leq 100$, we have the following graph:

This graph contains two linear parts at the left side and a flat part. This is because of the default and prepayment point we find.

4.2 Graph of V with Respect to t for Fixed House Price

In the section, the house price is fixed and the relation between the mortgage value V and time in a month is graphed.

The first graph is when we look at the value of the mortgage when $i = 50$ and $k = 8$. This means we look at the 8_{th} month before maturity and the house price is fixed to be $H(50)$. For each house value $H(i)$, $1 \leq i \leq 100$, we have the following graph:

4.3 Graph of the V with Respect to Both t and H in a Certain Month

In the section, a surface of V with respect to both t and H is graphed for each month. For instance, the surface at the 9th month is:

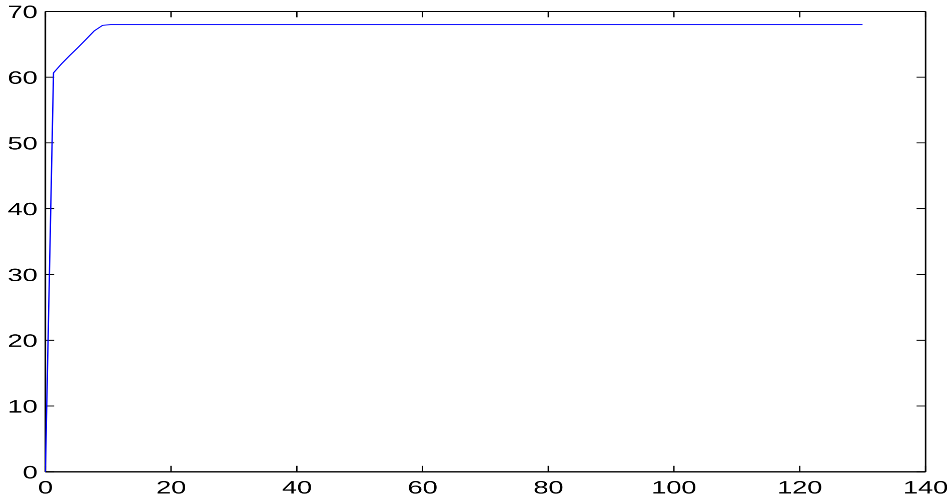

Figure 4.2 at t(15,8) H and V

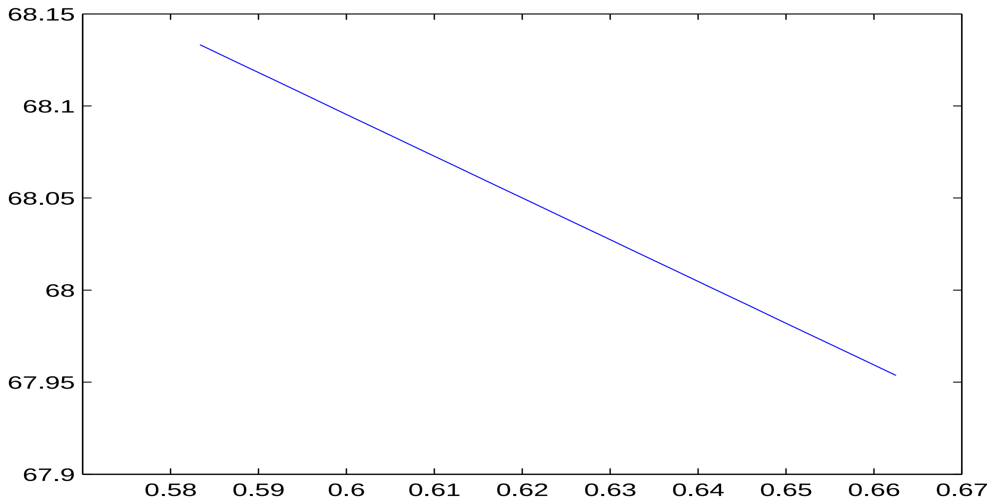

Figure 4.3 at H(50) k(8) t and V

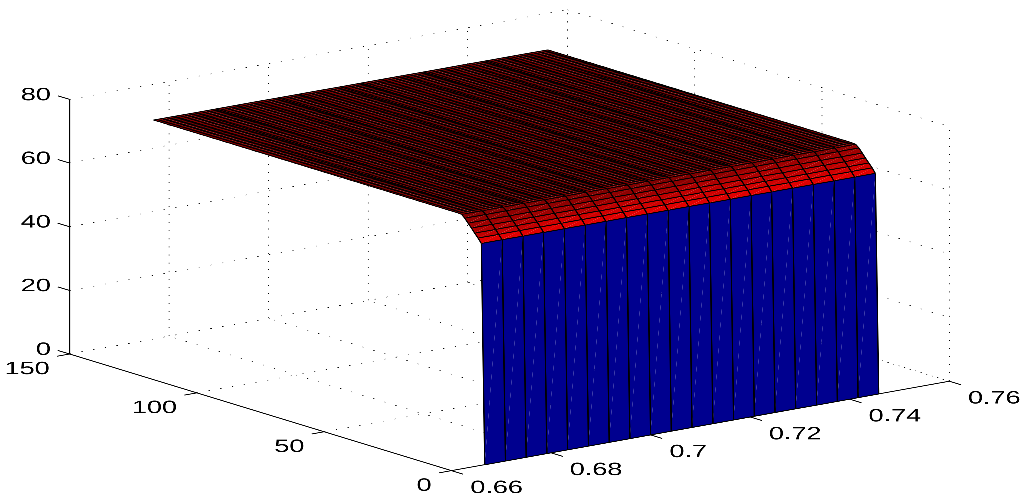

Figure 4.4 at k(9) t,H and V

CHAPTER 5. ADDING A DISCOUNT FACTOR TO THE MODEL

To incorporate directly the current economic conditions, the previous model is modified by "discounting" the value of the mortgage to the lender:

$$V(H, r, \tau; i) = \Big[A(r, \tau; i) - D(H, r, \tau; i) - C(H, r, \tau; i)\Big] \cdot \lambda(H, r, \tau; i),$$

where

$$0 \leq \lambda(H, r, \tau; i) \leq 1,$$

and $\tau(i)$ denotes the $i_t h$ payment date in years, $1 \leq i \leq N_m$.
λ is decomposed into three components:

$$\lambda = \lambda_{cr} \cdot \lambda_{liq} \cdot \lambda_{fin},$$

where , λ_{fin} is tied to the conditions of the economy, λ_{liq} is tied to the liquidity of the mortgage market market, and λ_{cr} is tied the quality of the mortgage. These discount factors are defined on the interval $(\tau(i-1), \tau(i)]$, $1 \leq i \leq N_m$. The above three parts are set in the following way.

5.1 Conditions of the Economy: $\lambda_{fin}(\tau)$

The Index of Leading Economic Indicators (LEI Index) is designed to predict the economy's direction. This index is a composite of a select group of economic statistics that are known to swing up or down well in advance of the rest of the economy. This indicator is released monthly by The Conference Board, a private business research group. The report is published three weeks after the end of the reporting month. Revisions to this index are usually minor, but more significant at other times.
The factor λ_{fin} by means of the LEI index is that On the interval $(\tau(i-1), \tau(i)]$, let Δ(%LEI)

denote the reported percent change of LEI index on the month prior to the payment date $\tau(i-1)$. Then define

$$\lambda_{fin}(\tau; i) = \begin{cases} 1 & \text{if } \Delta(\%\text{LEI}) \text{ is positive,} \\ 1 - \frac{\Delta(\%\text{LEI})}{100} & \text{if } \Delta(\%\text{LEI}) \text{ is negative and } |\Delta(\%\text{LEI})| < 1, \\ 0 & \text{if } \Delta(\%\text{LEI}) \text{ is negative and } |\Delta(\%\text{LEI})| \geq 1, \end{cases}$$

for $1 \leq i \leq num$.

5.2 Liquidity of the mortgage market market: $\lambda_{liq}(\tau)$

The Housing Market Index (HMI) assesses the current market for new single-family home sales along with builder expectations of future trends. The HMI is published monthly by the National Association of Home Builders and Wells Fargo. It is released in the same month it reports on. Revisions to this index tend to be minor. The HMI has a scale from 0 to 100, where 0 means that virtually everyone agreed conditions where poor, while 100 indicates that everyone believed the conditions were good. The index is adjusted for seasonal factors.
The factor λ_{liq} by means of the Housing Market Index: On the interval $(\tau(i-1), \tau(i)]$, let (HMI) denote the index reported on the month prior to the payment date $\tau(i-1)$. Then define

$$\lambda_{liq}(\tau; i) = \frac{(\text{HMI})}{100},$$

for $1 \leq i \leq num$.

5.3 Quality of the mortgage: λ_{cr}

The discount factor λ_{cr} is chosen to be a constant for the entire life of the loan. The probability of default is the likelihood that the loan will not be repaid and will fall into default. It is calculated for each borrower a group of borrowers with similar attributes. The credit score of the borrower is taken into account when calculating this probability. The simplest approach, taken by many banks, is to use external ratings agencies. For the credit score, mortgage lenders usually use the FICO score, which is usually intended to show the likelihood that a borrower will default on a loan. The FICO score was developed by th Fair Isaac Corporation. It is today's most commonly used scoring system. FICO scores range from 300-850 (higher FICO scores are better). Lenders buy FICO scores from three national credit reporting agencies (also called credit bureaus): Equifax, Experian and TransUnion.

At origination ($\tau = 0$) define $\lambda_{cr}(0)$ to be equal to the probability of default of the borrower as evidenced by his/her credit score. Then, on the interval $(\tau(i-1), \tau(i)]$, define

$$\lambda_{cr}(\tau; i) = \lambda_{cr}(0),$$

for $1 \leq i \leq num$.

APPENDIX
CODE

The code corresponding to the algorithm in chapter 3 is presented as follows:

1. No arbitrage equilibrium condition

$$c(1) = 0.05; c(2) = 0.1;$$

$$toll = 1e-3; err = toll + 1; nmax = 50; i = 2;$$

$$while(i < nmax \&\& err > toll)$$

$$[f(i-1), V] = mainmbs(c(i-1));$$

$$[f(i), V] = mainmbs(c(i));$$

$$df(i) = (f(i) - f(i-1)) \setminus (c(i) - c(i-1));$$

$$c(i+1) = c(i) - f(i) \setminus df(i);$$

$$err = abs(c(i+1) - c(i));$$

$$i = i + 1;$$

$$end$$

$$[f, V] = mainmbs(c(i));$$

$$savevaluesofV.txtV - asci;$$

2. Main program

$function[f, V] = mainmbs(c)$

%%%%%Inputs:

$I = 500;$

$N = 30;$

$Hmax = 1000;$

$num = 300;$

$deltaH = Hmax \setminus I;$

$deltat = 1 \setminus (12 * N);$

$L = 100;$

$fee = 0.015;$

$initial = 11;$ %%index of H(0)

%%%%%%Payment dates:

$T = zeros(num + 1, 1);$

$T(1) = 0;$

$fork = 2 : num + 1$

$T(k) = T(1) + (k - 1) \setminus 12;$

end

%%%%%%%%%Time discretization:

$t = zeros(N, num);$

$fork = 1 : num$

$forn = 1 : N$

$t(n, k) = (n - 1) * deltat + T(k);$

end

end

%%%%%%Monthly constant interest rates:

$r = zeros(num, 1);$

$for k = 1 : num$

$r(k) = 0.002 + k \setminus 1000;$

end

%%%%%%%%House price discretization:

$H = zeros(I, 1);$

$H(1) = 0;$

$for i = 2 : I + 1$

$H(i) = (i - 1) * deltaH;$

end

%%%%%%%%Monthly payments:

$MP = (L * (1 + c/12)^n * (c \setminus 12)) \setminus ((1 + c \setminus 12)^n - 1);$

%%%%%%%Principal balance:

$PB = zeros(num, 1);$

$for k = 1 : num$

$PB(k) = L * ((1 + c \setminus 12)^n - (1 + c \setminus 12)^k) \setminus (1 + c \setminus 12)^n - 1;$

end

%%%%%%%%Promised payments A

$A = zeros(N, num);$

%Initial condition given for T1 = 0 is different, so for the first month:

$A(1, 1) = MP;$

$A(2 : N, 1) = A(1, 1) * exp(-r(1) * (t(2 : N, 1) - T(1)));$

%Initial conditions given on payment-dates, so for subsequent months:

$for k = 2 : num$

$A(1, k) = A(N, k - 1) + MP;$

$A(2 : N, k) = A(1, k) * exp(-r(k) * (t(2 : N, k) - T(k)));$

end

%%%%%%Total debt TD:

$TD = zeros(N, num);$

$fork = 1 : num$

$TD(:, k) = (1 + c * (T(k + 1) - t(1 : N, k))) * PB(num - k + 1);$

end

% find the $H(i)$ that is smaller than and closest to $d(1) - - - dd(1)$

%% $dd(i)$ is index

$d(1) = MP;$

$i = 1;$

$whileH(i) \leq d(1)$

$dd(1) = i;$

$i = i + 1;$

end

%%%%give initial value based on default condition

$C = zeros(num, I + 1, N);$

$D = zeros(num, I + 1, N);$

$V = zeros(num, I + 1, N);$

$fori = 1 : dd(1)$

$C(1, i, 1) = 0;$

$D(1, i, 1) = MP - H(i);$

$V(1, i, 1) = H(i);$

end

%%%%%%not default

$fori = dd(1) + 1 : I$

$C(1, i, 1) = 0;$

$D(1, i, 1) = MP - H(i);$

$V(1, i, 1) = H(i);$

end

%check prepayment while get the next level

$p = zeros(N, num);$

$fork = 1 : num$

$forn = 2 : N$

$S[p(n, k), C(k, 1 : I + 1, n), D(k, 1 : I + 1, n)]$

$= newtonmb(C(k, 1 : I + 1, n - 1), D(k, 1 : I + 1, n - 1), TD(n - 1, k),$

$A(n, k), dd(k), H, deltaH, I, deltat, r(k));$

$V(k, 1 : I + 1, n) = A(n, k) - C(k, 1 : I + 1, n) - D(k, 1 : I + 1, n);$

end

% find the $H(i)$ that is smaller than and closest to $V(k, t(N, k)) + MP - - - dd(k + 1)$

$i = 1;$

$while(i <= I + 1\&\&H(i) \leq V(k, i, N) + MP)$

$dd(k + 1) = i;$

$i = i + 1;$

end

%default $(1 - -- > H(dd(k + 1))$

$fori = 1 : dd(k + 1)$

$C(k + 1, i, 1) = 0;$

$D(k + 1, i, 1) = A(1, k + 1);$

$A(k + 1, i, 1) = A(N, k) + MP;$

$V(k + 1, i, 1) = H(i);$

end

%not default

$fori = dd(k + 1) + 1 : I$

$C(k + 1, i, 1) = C(k, i, N);$

$D(k + 1, i, 1) = A(1, k + 1);$

$A(k + 1, i, 1) = A(N, k) + MP;$

$V(k + 1, i, 1) = V(k, i, N) + MP;$

end

%check prepayment boundary at $t(1, k+1)$

$[p(1, k+1), C(k+1, 1:I+1, 1), D(k+1, 1:I+1, 1)]$

$= newtonmb(C(k, 1:I+1, N), D(k, 1:I+1, N), TD(N, k),$

$A(N, k), dd(k+1), H, deltaH, I, deltat, r(k));$

$V(k+1, 1:I+1, 1) = A(1, k+1) - C(k+1, 1:I+1, 1) - D(k+1, 1:I+1, 1);$

end

%%%Creates f:

$f = V(1, initial, num+1) - (1 - fee) * L;$

$functionF = MC(C, M, a, b, c, I);$

3. Newton Method

$function[p, Cnew, Dnew] = newtonmb(C, D, TD, A, d, H, deltaH, I, deltat, r)$

$p0 = d - 1;$

$nit = 0;$

$nmax = 100;$

%Parameters from the house price SDE from Sharp thesis:

$sigmaH = 0.05;$

$delta = 0.075;$

%Coefficients from the Crank Nicolson scheme:

$a = zeros(I, 1);$

$b = zeros(I, 1);$

$c = zeros(I, 1);$

$fori = 2 : I$

$a(i) = (H(i) * sigmaH^2 \setminus deltaH - r + delta) * deltat * H(i) \setminus (4 * deltaH);$

$b(i) = (H(i)^2 * sigmaH^2/(deltaH^2) + r) * deltat \setminus 2;$

$c(i) = (H(i) * sigmaH^2 \setminus deltaH + r - delta) * deltat * H(i) \setminus (4 * deltaH);$

end

%Lower, central, upper diagonals of the system of equations matrix:

$ldiag = -a(3 : I);$

$mdiag = 1 + b(2 : I);$

$udiag = -c(2 : I - 1);$

%Matrix for the system of equations:

$M = diag(ldiag, -1) + diag(mdiag) + diag(udiag, 1);$

$fori = 1 : I + 1$

$AA(i) = A;$

end

$Cnew = zeros(I + 1, 1);$

$Dnew = zeros(I + 1, 1);$

$Vnew = zeros(I + 1, 1);$

% initial guess, p0 is a index, calculate Vnew

$Cnew = MC(C, M, a, b, c, I);$

$Dnew = MD(D, M, a, b, c, I);$

$Vnew = AA - Cnew - Dnew;$

$fori = p0 + 1 : I + 1$

$Vnew(i) = TD;$

end

%%%define F and dF

$fori = 1 : I + 1$

$F(i) = Vnew(i) - TD;$

end

$fori = 1 : I$

$dF(i) = F(i + 1) - F(i);$

end

$dF(I + 1) = dF(I);$

$toll = 1e - 3;$

$err = toll + 1;$

$while(nit < nmax\&\&err > toll)$

$nit = nit + 1;$

$if(abs(dF(p0)) < 1e - 6)$

disp 'newton method failed';

$else$

$qnew = H(p0) - F(p0) \setminus dF(p0);$

$err = abs(qnew - H(p0));$

$i = 1;$

%from qnew find pnew(index)

$whileH(i) \leq qnew$

$pnew = i;$

$i = i + 1;$

end

$fori = 1 : pnew;$

$Cnew = MC(C, M, a, b, c, I);$

$Dnew = MD(D, M, a, b, c, I);$

$Vnew = A - Cnew - Dnew;$

end

$fori = pnew + 1 : I + 1$

$Vnew(i) = TD;$

end

%define F and dF

$fori = 1 : I + 1$

$F(i) = Vnew(i) - TD;$

end

$fori = 1 : I$

$dF(i) = F(i + 1) - F(i);$

end

$dF(I + 1) = dF(I);$

$p0 = pnew;$

end

end

$p = p0;$

4. The program to solve the PDE for C.

$functionF = MD(D, M, a, b, c, I)$

$N = 2;$

%Initialization of the solution matrix, columns represent time levels.

$F = sparse(I + 1, N);$

%Boundary Conditions for the first rows of F only:

$F(:, 1) = C;$

%Initialization of right hand side vector:

$G = sparse(I - 1, N - 1);$

$forn = 1 : N - 1$

%Right-hand side vector:

$G(1, n) = a(2) * F(1, n) + (1 - b(2)) * F(2, n) + c(2) * F(3, n) + a(2) * F(1, n + 1);$

%Different last entry, due to the Neumann type boundary condition

$G(I - 1, n) = a(I) * F(I - 1, n) + (1 - b(I) + c(I)) * F(I, n) + c(I) * F(I + 1, n);$

$fori = 2 : I - 2$

$G(i, n) = a(i + 1) * F(i, n) + (1 - b(i + 1)) * F(i + 1, n) + c(i + 1) * F(i + 2, n);$

end

%Solve the matrix system

$F(2 : I, n + 1) = M\ G(:, n);$

end

$F = F(:, 2)';$

5. The program to solve the PDE for D.

$functionF = MD(D, M, a, b, c, I)$

$N = 2;$

%Initialization of the solution matrix, columns represent time levels.

$F = sparse(I + 1, N);$

%Boundary Conditions provide the first and last rows of F:

$F(:, 1) = D;$

$G = sparse(I - 1, N - 1);$

$forn = 1 : N - 1$

%Right-hand side vector:

$G(1, n) = a(2) * F(1, n) + (1 - b(2)) * F(2, n) + c(2) * F(3, n) + a(2) * F(1, n + 1);$

$G(I - 1, n) = a(I) * F(I - 1, n) + (1 - b(I)) * F(I, n) + c(I) * F(I + 1, n) + c(I) * F(I + 1, n + 1);$

$fori = 2 : I - 2$

$G(i, n) = a(i + 1) * F(i, n) + (1 - b(i + 1)) * F(i + 1, n) + c(i + 1) * F(i + 2, n);$

end

%Solve the matrix system

$F(2 : I, n + 1) = M \setminus G(:, n);$

end

$F = F(:, 2)';$

BIBLIOGRAPHY

James Kau, Donald Keenan, Walter Muller(1995). The valuation at origination of fixed rate mortgages with default and prepayment. *Journal of Real Estate Finance and Economics, 11: 5-36*

Nicholas SSharp(2006). Advances in mortgage valuation: an option-theoretic approach. *A thesis submitted to the University of Manchester for the degree of Doctor of Philosophy.*

Bernard Baumohl(2008). The Secrets of Economic Indicators: hidden clues to future economic trends and investment opportunities.*New Jersey: Wharton School Publishing.*

Wikipedia(March 12th, 2009). Probability of default. *http://en.wikipedia.org/wiki/Probability_of_default.*

Fico(April 8th, 2009). Consumer federation of America and fair Isaac corporation "Your Credit Scores". *http://www.pueblo.gsa.gov/cic_text/money/creditscores/your.htm* .