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Haifeng Liu
California Independent System Operator Corporation

Leigh Tesfatsion
Iowa State University, tesfatsi@iastate.edu

Ali A. Chowdhury
California Independent System Operator Corporation

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Haifeng Liu
California Independent System Operator Corporation, 151 Blue Ravine Road, Folsom, CA 95630, USA; email: hliu@caiso.com

Leigh Tesfatsion
Department of Economics, Iowa State University, Ames, IA 50011-1070, USA; email: tesfatsi@iastate.edu

Ali A. Chowdhury
California Independent System Operator Corporation, 151 Blue Ravine Road, Folsom, CA 95630, USA; email: achowdhury@caiso.com

Although locational marginal pricing (LMP) plays an important role in many restructured wholesale power markets, the detailed derivation of LMP as it is actually used in industrial practice is not readily available. This lack of transparency greatly hinders the efforts of researchers to evaluate the performance of these markets. In this paper, different alternating current and direct current optimal power flow models are presented to help us understand the derivation of LMP. As a byproduct of this analysis, we are able to provide a rigorous explanation of the basic LMP and LMP-decomposition formulas (neglecting real power losses) that are presented without derivation in the business practice manuals of the US Midwest Independent System Operator.

1 INTRODUCTION

In an April 2003 white paper the US Federal Energy Regulatory Commission proposed a market design for common adoption by US wholesale power markets. Core features of this proposed market design include central oversight by an independent market operator; a two-settlement system consisting of a day-ahead market supported by a parallel real-time market to ensure continual balancing of supply and demand for power; and management of grid congestion by means of locational marginal pricing (LMP), ie, the pricing of power by the location and timing of its injection into, or withdrawal from, the transmission grid.

Versions of the Federal Energy Regulatory Commission's market design have been implemented (or are scheduled for implementation) in US energy regions in the Midwest (MISO), New England (ISO-NE), New York (NYISO), the mid-Atlantic states (PJM), California (CAISO), the Southwest (SPP) and Texas (ERCOT). Nevertheless, strong criticism of the design persists (Joskow (2006)). Part of this criticism stems from the concerns of non-adopters about the suitability of the design for their regions due to distinct local conditions (eg, hydroelectric power in the Northwest). Even in regions adopting the design, however, criticisms continue to be raised about market performance.

One key problem underlying these latter criticisms is a lack of full transparency regarding market operations under the Federal Energy Regulatory Commission's design. Due in great part to the complexity of the market design in its various actual implementations, the business practices manuals and other public documents released by market operators are daunting to read and difficult to comprehend. Moreover, in many energy regions (eg, MISO), data is only posted in partial and masked form with a significant time delay (Dunn (2007)). The result is that many participants are wary regarding the efficiency, reliability and fairness of market protocols (eg, pricing and settlement practices). Moreover, university researchers are hindered from subjecting the Federal Energy Regulatory Commission's design to systematic testing in an open and impartial manner.

One key area in which lack of transparency prevents objective assessment is determination of LMPs. For example, although MISO's *Business Practices Manual 002: Energy Markets* (MISO (2008a)) presents functional representations for LMPs as well as an LMP decomposition for settlement purposes, derivations of these formulas are not provided. In particular, it is unclear whether the LMPs are derived from solutions to an alternating current (AC) optimal power flow (OPF) problem or from some form of direct current (DC) OPF approximation. Without knowing the exact form of the optimization problem from which the LMPs are derived, it is difficult to evaluate the extent to which pricing in accordance with these LMPs ensures efficient and reliable market operations.

This paper focuses careful attention on the derivation of LMPs for the operation of wholesale power markets. Section 2 presents a "full-structured" AC OPF model for LMP calculation. The LMPs are derived from the full-structured AC OPF model based on the definition of an LMP and the envelope theorem. Section 3 first derives a "full-structured" DC OPF model from the full-structured AC OPF model, together with corresponding LMPs. A "reduced-form" DC OPF model is then derived from the full-structured DC OPF model, and it is shown that the LMPs derived from the reduced-form DC OPF model are the same as those derived from the full-structured DC OPF model. As a byproduct of this analysis, we are able to provide a rigorous explanation of the basic LMP and LMP-decomposition formulas (neglecting real power losses) presented without derivation in MISO's *Business Practices Manual 002: Energy Markets*. Section 4 concludes.

2 LMP CALCULATION UNDER AC OPF

The concept of an LMP (also called a spot price or a nodal price) was first developed by Schweppe *et al* (1998). LMPs can be derived using either an AC OPF model or a DC OPF model (Momoh *et al* (1999)).

The AC OPF model is more accurate than the DC OPF model, but it is prone to divergence. Also, the AC OPF model can be up to 60 times slower than the DC OPF model (Overbye *et al* (2004)). The DC OPF model (or the linearized AC OPF model) has been widely used for LMP calculation for power market operation (Ott (2003); Litvinov *et al* (2004)). Several commercial software tools for power market simulation, such as Ventyx Promod IV®, ABB GridView™, Energy Exemplar PLEXOS® and PowerWorld, use the DC OPF model for power system planning and LMP forecasting (Clayton and Mukerji (1996); Yang *et al* (2003); Li (2007)).

There are two forms of DC OPF models: "full structured" (Sun and Tesfatsion (2007a,c)) and "reduced form" (Ilic *et al* (1998); Shahidehpour *et al* (2002); Ott (2003); Litvinov *et al* (2004); Li (2007); Li and Bo (2007)). The full-structured DC OPF model has a real power balance equation for each bus. This is equivalent to imposing a real power balance equation for all but a "reference" bus, together with a "system" real power balance equation consisting of the sum of the real power balance conditions across all buses. The reduced-form DC OPF model solves out for voltage angles using the real power balance equations at all but the reference bus, leaving the system real power balance equation.

In this paper, real power load and reactive power load are assumed to be fixed and a particular period of time is taken for the OPF formulations, eg, an hour. Given a power system with N buses, $G_{ij} + jB_{ij}$ is the ijth element of the bus admittance matrix, Y, of the power system. (See Appendix A for the details of the bus admittance matrix.) Let the bus voltage in polar form at bus i be denoted as follows:

$$\dot{V}_i = V_i \angle \theta_i \tag{1}$$

where V_i denotes the voltage magnitude and θ_i denotes the voltage angle.

The N buses are renumbered as follows for convenience.

- Non-reference buses are numbered from 1 to $N - 1$.
- The reference bus is numbered as bus N. Only the differences of voltage angles are meaningful in power flow calculation. Therefore, following standard practice, the voltage angle of the reference bus is set to 0.

2.1 Power balance constraint

The power flow equations (equality constraints) in the AC OPF problem formulation are as follows:

$$f_{pk}(x) + [\xi_k + D_k] - \sum_{i \in I_k} p_i = 0 \quad \text{for } k = 1, \dots, N \tag{2}$$

$$f_{qk}(x) + Q_{\text{load}\,k} - \sum_{i \in I_k} q_i = 0 \quad \text{for } k = 1, \dots, N \tag{3}$$

Here, we have the following:

- $x = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_{N-1} & V_1 & V_2 & \cdots & V_N \end{bmatrix}^{\mathrm{T}}$ is a vector of voltage angles and magnitudes.
- $f_{pk}(x)$ is the real power flowing out of bus k:

$$f_{pk}(x) = \sum_{i=1}^{N} V_k V_i [G_{ki} \cos(\theta_k - \theta_i) + B_{ki} \sin(\theta_k - \theta_i)] \tag{4}$$

- $f_{qk}(x)$ is the reactive power flowing out of bus k:

$$f_{qk}(x) = \sum_{i=1}^{N} V_k V_i [G_{ki} \sin(\theta_k - \theta_i) - B_{ki} \cos(\theta_k - \theta_i)] \tag{5}$$

- I_k is the set of generators connected to bus k.
- p_i is the real power output of generator i.
- D_k is the given real power load at bus k.
- $Q_{\mathrm{load}\,k}$ is the given reactive power load at bus k.
- q_i is the reactive power output of generator i.
- ξ_k is an auxiliary parameter associated with bus k that is set to zero. Changes in ξ_k will later be used to parameterize the real load increase at bus k in order to derive the real power LMP at bus k.

2.2 Network constraints

In general, the network constraints for an AC OPF problem formulation include:

- branch (transmission line and transformer) power flow limits, and
- voltage magnitude and angle limits.

The complex power flowing from bus i to bus j on the branch ij is:

$$\begin{aligned}
\tilde{S}_{ij} &= P_{ij} + jQ_{ij} = \dot{V}_i \overset{*}{I}_{ij} = \dot{V}_i \left[\frac{\dot{V}_i - \dot{V}_j}{r_{ij} + jx_{ij}} \right]^* \\
&= \dot{V}_i \frac{\overset{*}{V}_i - \overset{*}{V}_j}{r_{ij} - jx_{ij}} = \frac{[V_i^2 - \dot{V}_i \overset{*}{V}_j][r_{ij} + jx_{ij}]}{r_{ij}^2 + x_{ij}^2} \\
&= \frac{[V_i^2 - V_i V_j \cos\theta_{ij} - jV_i V_j \sin\theta_{ij}][r_{ij} + jx_{ij}]}{r_{ij}^2 + x_{ij}^2}
\end{aligned} \tag{6}$$

where I_{ij} is the current flowing from bus i to bus j, $\theta_{ij} = \theta_i - \theta_j$, and r_{ij} and x_{ij} are the resistance and reactance of branch ij, respectively. Therefore, the real power flowing from bus i to bus j is:

$$P_{ij}(x) = \frac{[V_i^2 - V_i V_j \cos\theta_{ij}]r_{ij} + [V_i V_j \sin\theta_{ij}]x_{ij}}{r_{ij}^2 + x_{ij}^2} \tag{7}$$

The reactive power flowing from bus i to bus j is:

$$Q_{ij}(x) = \frac{[V_i^2 - V_i V_j \cos\theta_{ij}]x_{ij} - [V_i V_j \sin\theta_{ij}]r_{ij}}{r_{ij}^2 + x_{ij}^2} \tag{8}$$

The magnitude of the complex power flowing from bus i to bus j is:

$$S_{ij}(x) = |\tilde{S}_{ij}(x)| = \sqrt{P_{ij}^2(x) + Q_{ij}^2(x)} \tag{9}$$

The power system operating constraints include the following.

Branch power flow constraints:

$$0 \leqslant S_{ij}(x) \leqslant S_{ij}^{\max} \quad \text{for each branch } ij \tag{10}$$

Bus voltage magnitude constraints:

$$V_k^{\min} \leqslant V_k \leqslant V_k^{\max} \quad \text{for } k = 1, 2, \dots, N \tag{11}$$

To simplify the illustration, a general form of constraints is used to represent the above specific inequality constraints (10) and (11), as follows:

$$g_m^{\min} \leqslant g_m(x) \leqslant g_m^{\max} \quad \text{for } m = 1, \dots, M \tag{12}$$

2.3 Generator output limits

Generator real power output limits for the submitted generator supply offers are assumed to take the following form:

$$p_i^{\min} \leqslant p_i \leqslant p_i^{\max} \quad \forall i \in I \tag{13}$$

Similarly, generator reactive power output limits are assumed to take the following form:

$$q_i^{\min} \leqslant q_i \leqslant q_i^{\max} \quad \forall i \in I \tag{14}$$

2.4 The objective function of the market operator

According to MISO's business practices manuals and tariff (MISO (2005, 2008b)), the supply (resource) offer curve of each generator in each hour h must be either a step function or a piecewise linear curve consisting of up to 10 price–quantity blocks, where the price associated with each quantity increment (MW) gives the minimum price (US dollars per megawatt hour) the generator is willing to accept for this quantity increment. The blocks must be monotonically increasing in price and they must cover the full real power operating range of the generator.

Let $C_i(p_i)$ denote the integral of generator i's supply offer from $p_i^{\min}$ to p_i. For simplicity of illustration, $C_i(p_i)$ will hereafter be assumed to be strictly convex and non-decreasing over a specified interval.

In this study the independent system operator (ISO) is assumed to solve a centralized optimization problem in each hour h to determine real power commitments and LMPs for hour h conditional on the submitted generator supply offers and given loads (fixed demands) for hour h; price-sensitive demand bids are not considered. As will be more carefully explained below, this constrained optimization problem is assumed to involve the minimization of total reported generator operational costs defined as follows:

$$\sum_{i \in I} C_i(p_i) \tag{15}$$

where $C_i(p_i)$ is generator i's reported total cost of supplying real power p_i in hour h and I is the set of generators. Since for each generator supply offer the unit of the incremental energy cost is US dollars per megawatt hour and the unit of the operating level is MW, the unit of the objective function (15) is US dollars per hour.

2.5 The AC OPF problem

The overall optimization problem is as follows:

$$\min_{p_i, q_i, x} \sum_{i \in I} C_i(p_i) \tag{16}$$

such that:

- Real power balance constraints for buses $k = 1, \ldots, N$:

$$f_{pk}(x) + [\xi_k + D_k] - \sum_{i \in I_k} p_i = 0 \tag{17}$$

- Reactive power balance constraints for buses $k = 1, \ldots, N$:

$$f_{qk}(x) + Q_{\text{load}\,k} - \sum_{i \in I_k} q_i = 0 \tag{18}$$

- Power system operating constraints for $m = 1, \ldots, M$:

$$g_m^{\min} \leqslant g_m(x) \leqslant g_m^{\max} \tag{19}$$

- Generator real power output constraints for generators $i \in I$:

$$p_i^{\min} \leqslant p_i \leqslant p_i^{\max} \tag{20}$$

- Generator reactive power output constraints for generators $i \in I$:

$$q_i^{\min} \leqslant q_i \leqslant q_i^{\max} \tag{21}$$

The endogenous variables are p_i, q_i and x. The exogenous variables are ξ_k, D_k and $Q_{\text{load}\,k}$. The above optimization problem is also called the *AC OPF problem*.

2.6 LMP calculation based on AC OPF model

The Lagrangian function for the AC OPF problem is as follows:

$$
\begin{aligned}
\ell = {} & \sum_{i\in I} C_i(p_i) && \text{total cost} \\
& - \sum_{k=1}^{N} \pi_k \Big[-[\xi_k + D_k] - f_{pk}(x) + \sum_{i\in I_k} p_i \Big] \\
& && \text{active power balance constraint} \\
& - \sum_{k=1}^{N} \lambda_k \Big[-f_{qk}(x) - Q_{\text{load}\,k} + \sum_{i\in I_k} q_i \Big] \\
& && \text{reactive power balance constraint} \\
& - \sum_{m=1}^{M} \hat{\mu}_m [g_m^{\max} - g_m(x)] && \text{power system operating constraint upper limit} \\
& - \sum_{m=1}^{M} \check{\mu}_m [g_m(x) - g_m^{\min}] && \text{power system operating constraint lower limit} \\
& - \sum_{i\in I} \hat{\tau}_i [p_i^{\max} - p_i] && \text{generator real power output upper limit} \\
& - \sum_{i\in I} \check{\tau}_i [p_i - p_i^{\min}] && \text{generator real power output lower limit} \\
& - \sum_{i\in I} \hat{\omega}_i [q_i^{\max} - q_i] && \text{generator reactive power output upper limit} \\
& - \sum_{i\in I} \check{\omega}_i [q_i - q_i^{\min}] && \text{generator reactive power output lower limit}
\end{aligned}
\tag{22}
$$

DEFINITION 2.1 (LMP) The locational marginal price of electricity at a location (bus) is defined as the least cost to service the next increment of demand at that location consistent with all power system operating constraints (MISO (2005); CAISO (2006)).

Assume that the above AC OPF problem has an optimal solution and that the minimized objective function J^* (exogenous variables) is a differentiable function of ξ_k for each $k = 1, \ldots, N$. Using the envelope theorem (Varian (1992)), the LMP at each bus k can then be calculated as follows:

$$
\text{LMP}_k = \frac{\partial J^*}{\partial \xi_k} = \left. \frac{\partial \ell}{\partial \xi_k} \right|_{\chi^*} = \pi_k \quad \text{for } k = 1, 2, \ldots, N \tag{23}
$$

Here, we have the following:

- J^* is the minimized value of the total cost objective function (15), also referred to as the indirect objective function or optimal value function.

- χ^* is the solution vector consisting of the optimal values for the decision variables.

It follows from (23) that the real power LMP at each bus k is simply the Lagrange multiplier associated with the real power balance constraint for that bus.

3 LMP CALCULATION AND DECOMPOSITION UNDER DC OPF

3.1 DC OPF approximation in full-structured form

The AC OPF model involves real and reactive power flow balance constraints and power system operating constraints, which constitute a set of non-linear algebraic equations. It can be time consuming to solve AC OPF problems for large power systems, and convergence difficulties can be serious. The DC OPF model has been proposed to approximate the AC OPF model for the purpose of calculating real power LMPs (Overbye *et al* (2004)).

In the DC OPF formulation, the reactive power flow equation (3) is ignored. The real power flow equation (2) is approximated by the DC power flow equations under the following assumptions (Wood and Wollenberg (1996); Kirschen and Strbac (2004); Overbye *et al* (2004); Sun and Tesfatsion (2007b)).

a) The resistance of each branch r_{km} is negligible compared with the branch reactance x_{km} and can therefore be set to zero.

b) The bus voltage magnitude is equal to one per unit.[1]

c) The voltage angle difference $\theta_k - \theta_m$ across any branch is very small, so that $\cos(\theta_k - \theta_m) \approx 1$ and $\sin(\theta_k - \theta_m) \approx \theta_k - \theta_m$.

Purchala *et al* (2005) show that the resulting DC OPF model is acceptable in real power flow analysis if the branch power flow is not very high, the voltage profile is sufficiently flat and the r_{km}/x_{km} ratio is less than 0.25. The DC OPF model itself does

[1] In power system calculations, quantities such as voltage, current, power and impedance are usually expressed in per unit (pu) form, ie, as a percentage of a specified base value. The pu quantity is calculated as the actual quantity divided by the base value of the quantity, where the actual quantity is the value of the quantity in the actual units. The base value has the same units as the actual quantity. Thus pu quantity is dimensionless. Specifying two independent base quantities determines the remaining base quantities. The two independent quantities are usually taken to be base voltage and base apparent power. Manufacturers usually specify the impedances of machines and transformers in pu terms. The advantages of the pu system include: a) simplification of the transformer equivalent circuit; b) allowance of rapid checking of pu impedance data for gross errors; and c) reduction of the chances of numerical instability. For a detailed and careful discussion of base value determinations and pu calculations, see Chapter 5 of Bergen and Vittal (2000).

not include the effect of the real power loss on the LMP due to assumption a). Li and Bo (2007) propose an iterative approach to account for the real power loss in the DC OPF-based LMP calculation. In the present study, however, real power loss is neglected, in conformity with standard DC OPF treatments.

From (2) and (4) we have:

$$\sum_{m=1}^{N} V_k V_m [G_{km} \cos(\theta_k - \theta_m) + B_{km} \sin(\theta_k - \theta_m)] + D_k - \sum_{i \in I_k} p_i = 0 \quad \text{for } k = 1, \ldots, N \quad (24)$$

where, as explained in Appendix A, G_{km} and B_{km} are elements of the bus admittance matrix.

Given assumption a), it follows that:

$$G_{km} = -\frac{r_{km}}{r_{km}^2 + x_{km}^2} = 0 \quad \text{for } k \neq m$$

$$G_{kk} = \frac{r_{k0}}{r_{k0}^2 + x_{k0}^2} + \sum_{m=1, m \neq N}^{N} \frac{r_{km}}{r_{km}^2 + x_{km}^2} = 0$$

$$B_{km} = \frac{x_{km}}{r_{km}^2 + x_{km}^2} = \frac{1}{x_{km}} \quad \text{for } k \neq m$$

$$B_{kk} = \sum_{m=1}^{N} \frac{-x_{km}}{r_{km}^2 + x_{km}^2}$$

Given assumption b), it follows that $V_k = V_m = 1$. Given assumption c), it follows that $\sin(\theta_k - \theta_m) \approx \theta_k - \theta_m$. Therefore, (24) reduces to:

$$\sum_{m=1, m \neq k}^{N} \left[\frac{1}{x_{km}} (\theta_k - \theta_m) \right] + D_k - \sum_{i \in I_k} p_i = 0 \quad \text{for } k = 1, \ldots, N \quad (25)$$

Equation (25) can be re-expressed as:

$$\sum_{m=1, m \neq k}^{N} \left[\frac{1}{x_{km}} (\theta_k - \theta_m) \right] = P_k - D_k \quad \text{for } k = 1, \ldots, N \quad (26)$$

Therefore, the net injection $P_k - D_k$ of real power flowing out of any bus k can be approximated as a linear function of the voltage angles.

From (7), the real power flowing from bus k to bus m is as follows:

$$P_{km}(x) = \frac{[V_k^2 - V_k V_m \cos \theta_{km}] r_{km} + [V_k V_m \sin \theta_{km}] x_{km}}{r_{km}^2 + x_{km}^2} \quad (27)$$

Based on the assumptions a), b) and c):

$$P_{km}(x) = \frac{\theta_k - \theta_m}{x_{km}} \quad (28)$$

Therefore, this branch real power flow can be approximated as a linear function of the voltage angle difference between bus k and bus m.

From (8), the reactive power flowing from bus k to bus m is as follows:

$$Q_{km}(x) = \frac{[V_k^2 - V_k V_m \cos\theta_{km}]x_{km} - [V_k V_m \sin\theta_{km}]r_{km}}{r_{km}^2 + x_{km}^2} \tag{29}$$

Based on the assumptions a), b) and c):

$$Q_{km}(x) = 0 \tag{30}$$

From (9), the magnitude of the complex power flow $S_{km}(x)$ is:

$$S_{km}(x) = \sqrt{P_{km}^2(x) + Q_{km}^2(x)} = \sqrt{P_{km}^2(x)} \tag{31}$$

Therefore, the branch power flow constraint becomes:

$$F_{km}^{\min} \leqslant P_{km}(x) \leqslant F_{km}^{\max} \tag{32}$$

There are no voltage magnitude constraints because all voltage magnitudes are assumed to be 1.0 pu.

For a power system consisting of N buses, the DC power flow equation for each bus k is shown in (26). The corresponding matrix form for the full system of equations is as follows:

$$\boldsymbol{P} - \boldsymbol{D} = \boldsymbol{B}\theta \tag{33}$$

Here, we have the following:

- $\boldsymbol{P} = \begin{bmatrix} P_1 & P_2 & \cdots & P_N \end{bmatrix}^{\mathrm{T}}$ is the $N \times 1$ vector of nodal real power generation for buses $1, \ldots, N$.
- $\boldsymbol{D} = \begin{bmatrix} D_1 & D_2 & \cdots & D_N \end{bmatrix}^{\mathrm{T}}$ is the $N \times 1$ vector of nodal real power load for buses $1, \ldots, N$.
- $\boldsymbol{B}$ is an $N \times N$ matrix (independent of voltage angles) that is determined by the characteristics of the transmission network as follows: $B_{kk} = \sum_m 1/x_{km}$ for each diagonal element kk, and $B_{km} = -1/x_{km}$ for each off-diagonal element km.
- $\theta = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_N \end{bmatrix}^{\mathrm{T}}$ is the $N \times 1$ vector of voltage angles for buses $1, \ldots, N$.

The system of equations (33) is called the *full-structured DC power flow model*.

The voltage angle at the reference bus N is usually normalized to zero since the real power balance constraints and the real power flow on any branch are only dependent on voltage angle differences, as seen from (26) and (28). We follow this convention here, therefore:

$$\theta_N = 0 \tag{34}$$

Given (34), the system of real power balance equations for buses $1, \ldots, N-1$ (33) can be expressed in reduced matrix form as follows:

$$\boldsymbol{P}' - \boldsymbol{D}' = \boldsymbol{B}'\theta' \tag{35}$$

Here, we have the following:

- $\boldsymbol{P}' = \begin{bmatrix} P_1 & P_2 & \cdots & P_{N-1} \end{bmatrix}^{\mathrm{T}}$ is the $(N-1) \times 1$ vector of real power generation for buses $1, \dots, N-1$.

- $\boldsymbol{D}' = \begin{bmatrix} D_1 & D_2 & \cdots & D_{N-1} \end{bmatrix}^{\mathrm{T}}$ is the $(N-1) \times 1$ vector of real power load for buses $1, \dots, N-1$.

- $\boldsymbol{B}'$ is the "B-prime" matrix of dimension $(N-1) \times (N-1)$, independent of voltage angles, that is determined by the characteristics of the transmission network. The $\boldsymbol{B}'$ matrix is derived from the $\boldsymbol{B}$ matrix by omitting the row and column corresponding to the reference bus.

- $\theta' = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_{N-1} \end{bmatrix}^{\mathrm{T}}$ is the $(N-1) \times 1$ vector of voltage angles for buses $1, \dots, N-1$.

For later reference, it follows from (B.16) in Appendix B that the real power balance equation at the reference bus N can be expressed as follows:

$$P_N - D_N = -\boldsymbol{e}^{\mathrm{T}}[\boldsymbol{P}' - \boldsymbol{D}'] \tag{36}$$

Here, $\boldsymbol{e}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$ is a $1 \times (N-1)$ row vector with each element equal to 1.

In the DC OPF model, the real power flow on any branch km is given in (28). Letting M denote the total number of distinct transmission network branches for the DC OPF model, it follows that the real power flow on all M branches can be written in a matrix form as follows:

$$\boldsymbol{F} = \boldsymbol{X}\theta \tag{37}$$

Here, we have the following:

- $\boldsymbol{F} = \begin{bmatrix} F_1(x) & F_2(x) & \cdots & F_M(x) \end{bmatrix}^{\mathrm{T}}$ is the $M \times 1$ vector of branch flows.

- $\boldsymbol{X} = \boldsymbol{H} \times \boldsymbol{A}$ is an $M \times N$ matrix, which is determined by the characteristics of the transmission network.

- $\boldsymbol{H}$ is an $M \times M$ matrix whose non-diagonal elements are all zero and whose kkth diagonal element is the negative of the susceptance of the kth branch.

- $\boldsymbol{A}$ is the $M \times N$ adjacency matrix. It is also called the node–arc incidence matrix, or the connection matrix. See Appendix C for the details of the development of the adjacency matrix $\boldsymbol{A}$.

Inverting (35) yields:

$$\theta' = [\boldsymbol{B}']^{-1}[\boldsymbol{P}' - \boldsymbol{D}'] \tag{38}$$

Substitution of (38) into (37) yields:

$$\begin{aligned} \boldsymbol{F} &= \boldsymbol{X}\theta \\ &= \boldsymbol{X}\begin{bmatrix} \theta' \\ \theta_n \end{bmatrix} \\ &= \boldsymbol{X}\begin{bmatrix} \boldsymbol{B}'^{-1}[\boldsymbol{P}' - \boldsymbol{D}'] \\ 0 \end{bmatrix} \\ &= \boldsymbol{X}\begin{bmatrix} \boldsymbol{B}'^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{P}' - \boldsymbol{D}' \\ P_N - D_N \end{bmatrix} \\ &= \boldsymbol{X}\begin{bmatrix} \boldsymbol{B}'^{-1} & 0 \\ 0 & 0 \end{bmatrix}[\boldsymbol{P} - \boldsymbol{D}] \end{aligned} \tag{39}$$

Let:

$$\boldsymbol{T} = \boldsymbol{X}\begin{bmatrix} \boldsymbol{B}'^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{40}$$

Here, we have the following:

- $\boldsymbol{T}$ is an $M \times N$ matrix.
- $T_{mN} = 0$ for $m = 1, \dots, M$.

Therefore, the branch power flows in terms of bus net real power injections can be expressed as:

$$\boldsymbol{F} = \boldsymbol{T}[\boldsymbol{P} - \boldsymbol{D}] \tag{41}$$

The system of equations (41) is called the *reduced-form DC power flow model* because it directly relates branch real power flows to bus net real power injections.

The real power flow on branch l in (41) is as follows:

$$F_l = \sum_{k=1}^{N} T_{lk}[P_k - D_k] = \sum_{k=1}^{N-1} T_{lk}[P_k - D_k] \quad \text{for } l = 1, \dots, M \tag{42}$$

Assume that P_k is increased to $P_k + \Delta P_k$ while $P_1, P_2, \dots, P_{k-1}, P_{k+1}, \dots, P_{N-1}$ and $D_1, D_2, \dots, D_N$ remain fixed. Then, according to (42), the increase in the real power flow on branch l, ΔF_l, is as follows:

$$\Delta F_l = T_{lk}\Delta P_k \tag{43}$$

By (36), note that the change in the real power injection at bus k is exactly compensated by an opposite change in the real power injection at the reference bus N, given by $P_N - \Delta P_k$. Therefore, T_{lk} in (43) is a *generation shift factor*.

More precisely, it is clear from (39) that the branch power flows are explicit functions of nodal net real power injections (generation less load) at the non-reference buses.

It follows from (36) that the generation change at bus k will be compensated by the generation change at the reference bus N assuming the net real power injections at other buses remain constant. Thus, the lkth element T_{lk} in the matrix $\boldsymbol{T}$ in (41) is equal to the generation shift factor a_{lk} as defined on page 422 of Wood and Wollenberg (1996), which measures the change in megawatt power flow on branch l when a 1 MW change in generation occurs at bus k compensated by a withdrawal of 1 MW at the reference bus.

The full-structured DC OPF model is derived from the full-structured AC OPF model in Section 2 based on the three assumptions a), b) and c) in Section 3.1, as follows:

$$\min_{p_i,\theta_k} \sum_{i \in I} C_i(p_i) \tag{44}$$

such that:

- Real power balance constraint for each bus $k = 1, \dots, N$:

$$\sum_{i \in I_k} p_i - [\xi_k + D_k] = \sum_{m=1, m \neq k}^{N} \left[\frac{1}{x_{km}} (\theta_k - \theta_m) \right] \quad \text{for } k = 1, \dots, N \tag{45}$$

- Real power flow constraints for each distinct branch km:

$$\frac{1}{x_{km}} [\theta_k - \theta_m] \leqslant F_{km}^{\max} \tag{46}$$

$$\frac{1}{x_{km}} [\theta_k - \theta_m] \geqslant F_{km}^{\min} \tag{47}$$

- Real power generation constraints for each generator:

$$p_i^{\min} \leqslant p_i \leqslant p_i^{\max} \quad \forall i \in I \tag{48}$$

The endogenous variables are p_i and θ. The exogenous variables are D_k and ξ_k.

The optimal solution is determined for the particular parameter values $\xi_k = 0$ in (45). Changes in these parameter values are used below to generate LMP solution values using envelope theorem calculations.

The Lagrangian function for the optimization problem is:

$$\begin{aligned} \ell = \sum_{i \in I} C_i(p_i) - \sum_{k=1}^{N} \pi_k \Bigg[\sum_{i \in I_k} p_i - \sum_{m=1, m \neq k}^{N} \left[\frac{1}{x_{km}} (\theta_k - \theta_m) \right] - [\xi_k + D_k] \Bigg] \\ - \sum_{km} \hat{\mu}_{km} \left[F_{km}^{\max} - \frac{1}{x_{km}} [\theta_k - \theta_m] \right] \\ - \sum_{km} \check{\mu}_{km} \left[\frac{1}{x_{km}} [\theta_k - \theta_m] - F_{km}^{\min} \right] \\ - \sum_{i \in I} \hat{\tau}_i (p_i^{\max} - p_i) \\ - \sum_{i \in I} \check{\tau}_i (p_i - p_i^{\min}) \end{aligned} \tag{49}$$

Assume that the above DC OPF problem has an optimal solution and the optimized objective function J^* (exogenous variables) is a differentiable function of ξ_k for each $k = 1, \ldots, N$. Based on the envelope theorem and using the auxiliary parameter ξ_k, we can calculate the LMP at each bus k as follows:

$$\text{LMP}_k = \frac{\partial J^*}{\partial \xi_k} = \left.\frac{\partial \ell}{\partial \xi_k}\right|_{\chi^*} = \pi_k \quad \forall k \tag{50}$$

It follows from (50) that the LMP at each bus k is the Lagrange multiplier corresponding to the real power balance constraint at bus k, evaluated at the optimal solution.

As depicted in Figure 1, we use a three-bus system with two generators and one fixed load to illustrate LMP calculations based on the full-structured DC OPF model. For the purpose of illustration, assume that:

1) the reactance of each branch is equal to 1 pu;

2) the capacity of branch 2–1 is 50 MW;

3) there are no capacity limits on branches 2–3 and 3–1;

4) the demand at bus 1 is fixed at 90 MW;

5) the real power operating capacity limit for generator 2 and generator 3 is 100 MW;

6) the indicated marginal costs US\$5 per megawatt hour and US\$10 per megawatt hour for generator 2 and generator 3 are constant over their real power operating capacity ranges;

7) the time period assumed for the DC-OPF formulation is one hour; and

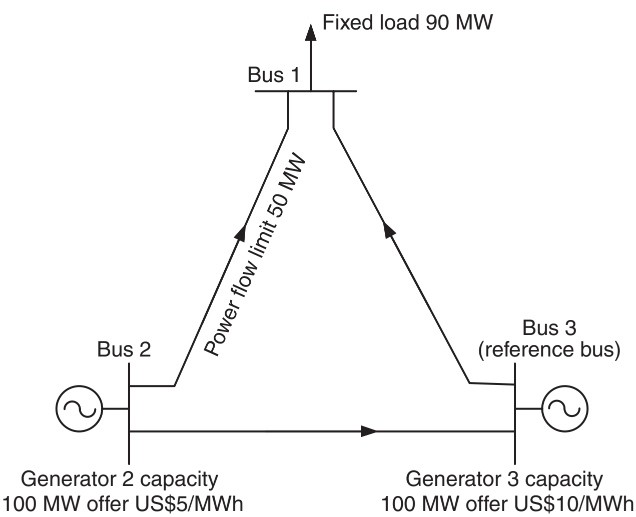
8) the objective of the market operator is the constrained minimization of the *total variable costs of operation* (US dollars per hour), ie, the summation of the variable costs of operation (marginal cost times real power generation) for generator 2 and generator 3.

In the following calculations, all power amounts (generator outputs, load demand and branch flows) and impedances are expressed in per unit form. The base power is chosen to be 100 MW. The objective function for the DC OPF problem is expressed in per unit terms as well as the constraints. The variable cost of each generator i is expressed as a function of per unit real power $P_{\text{G}i}$, ie, as $100 \times \text{MC}_i \times P_{\text{G}i}$, where MC_i denotes the marginal cost of generator i. Note that the per unit, adjusted total variable cost function is then still measured in US dollars per hour.

Given the above assumptions, the market operator's optimization problem is formulated as follows:

$$\min_{\theta_1,\theta_2,P_{\text{G}2},P_{\text{G}3}} 500 P_{\text{G}2} + 1{,}000 P_{\text{G}3} \tag{51}$$

FIGURE 1 A three-bus power system.



such that:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.9 \\ P_{G2} \\ P_{G3} \end{bmatrix} \tag{52}$$

$$\begin{bmatrix} F_{21}^{\min} \\ F_{31}^{\min} \\ F_{23}^{\min} \end{bmatrix} \leqslant \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ 0 \end{bmatrix} \leqslant \begin{bmatrix} F_{21}^{\max} \\ F_{31}^{\max} \\ F_{23}^{\max} \end{bmatrix} \tag{53}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \leqslant \begin{bmatrix} P_{G2} \\ P_{G3} \end{bmatrix} \leqslant \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{54}$$

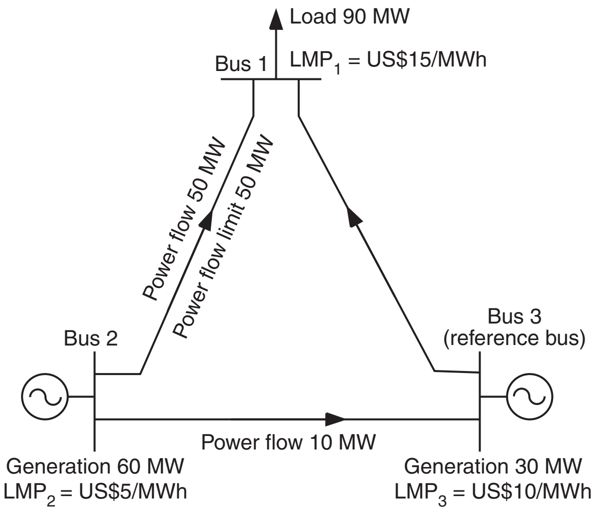
The solution to this optimization problem yields the following scheduled power commitments for generators 2 and 3 and LMP values for buses 1–3:

- $P_{G2} = 0.6$ pu $= 60$ MW, $P_{G3} = 0.3$ pu $= 30$ MW;
- $\text{LMP}_1 = \text{US\$15/MWh}$, $\text{LMP}_2 = \text{US\$5/MWh}$, $\text{LMP}_3 = \text{US\$10/MWh}$.

The power flow on branch 2–1 is 50 MW, which is at the capacity limit of the branch. The power flow on branch 2–3 is 10 MW and the power flow on branch 3–1 is 40 MW. Figure 2 depicts these results.

Recall that the LMP at a location (bus) of a transmission network is defined to be the minimal additional system cost required to supply an additional increment of electricity to this location. We now verify that the LMP solution values indicated in Figure 2 do indeed satisfy the definition of an LMP.

FIGURE 2 LMPs, generator scheduled power commitments and branch power flows.



Consider bus 2, which currently has zero load. Suppose an additional megawatt of load is now required at bus 2. It is clear that this additional load should be supplied by generator 2. This follows because the marginal cost of generator 2 is lower than the marginal cost of generator 3 and the current output (60 MW) of generator 2 is strictly lower than its operating capacity limit (100 MW). The transmission network has no impact on the LMP at bus 2 because the additional megawatt of power is produced and consumed locally. The LMP at bus 2 is therefore US$5 per megawatt hour, which is equal to the marginal cost of generator 2.

Determination of the LMP values at buses 3 and 1 is more complicated because network externalities are involved. Consider first the most efficient way to supply an additional megawatt of power at bus 3. This additional megawatt of power cannot be provided by generator 2, although it has the lowest marginal cost and is not at maximum operating capacity, because this would overload branch 2–1. The next-cheapest option is to increase the output of generator 3. Because generator 3 is located at bus 3, the additional megawatt of power will not flow through the transmission network. The LMP at bus 3 is therefore US$10 per megawatt hour, which is equal to the marginal cost of generator 3.

Consider instead the most efficient way to supply an additional megawatt of power at bus 1. It is not feasible to do this by increasing the output of generator 2 alone, or by increasing the output of generator 3 alone, because either option would overload branch 2–1. The only feasible option is to simultaneously increase the output of generator 3 and decrease the output of generator 2. The required changes in the outputs of generator 2 and generator 3 can be calculated by solving the following equations:

$$\Delta P_{G2} + \Delta P_{G3} = 1 \text{ MW} \tag{55}$$

$$\tfrac{2}{3}\Delta P_{G2} + \tfrac{1}{3}\Delta P_{G3} = 0 \text{ MW} \tag{56}$$

where (56) is Kirchhoff's circuit laws applied to the three-bus system at hand, for which the reactance on each branch is assumed to be equal. Solving these two equations, we get:

$$\Delta P_{G2} = -1 \text{ MW}$$

$$\Delta P_{G3} = 2 \text{ MW}$$

Supplying an additional megawatt of power at bus 1 at minimum cost therefore requires that we increase the output of generator 3 by 2 MW and reduce the output of generator 2 by 1 MW. The system cost of supplying this megawatt, and hence the LMP at bus 1, is thus given by:

$$\text{LMP}_1 = 2 \times \text{MC}_3 - 1 \times \text{MC}_2 = 2(\text{US\$10/MWh}) - 1(\text{US\$5/MWh}) = \text{US\$15/MWh}$$

In summary, we observe from this three-bus system illustration that:

- the MC of generator 2 determines the LMP of US\$5 per megawatt hour at bus 2;
- the MC of generator 3 determines the LMP of US\$10 per megawatt hour at bus 3;
- a combination of the marginal costs for generators 2 and 3 determines the LMP of US\$15 per megawatt hour at bus 1.

3.2 The DC OPF approximation in reduced form

The reduced-form DC OPF model can be derived directly from the full-structured DC OPF model in Section 3.1 by applying the following three steps.

1) Replace the real power balance equation at the reference bus N by the sum of the real power balance equations across all N buses. This is an equivalent formulation that will not change the optimal solution of the DC OPF problem. Since there is no real power loss in the DC power flow model, the sum of the net real power injections across all buses is equal to zero; see (B.16) in Appendix B. Therefore, the system real power balance constraint (in parameterized form) can be expressed as in (58) below.

2) Solve the voltage angles at the $N - 1$ non-reference buses as functions of the net real power injections at the $N - 1$ non-reference buses as shown in (38).

3) Replace the voltage angles in the branch flow constraints as functions of the net real power injections at the non-reference buses as shown in (39) and (42).

Since the above transformation is based on equivalency and only eliminates internal variables (ie, voltage angles at non-reference buses), the optimal solution and the corresponding Lagrange multipliers of the branch power flow constraints are the same for the two DC OPF models.

The resulting *reduced-form DC OPF model* is then as follows:

$$\min_{p_i} \sum_{i \in I} C_i(p_i) \tag{57}$$

such that:

- System real power balance constraint:

$$\sum_{k=1}^{N} (P_k - (D_k + \xi_k)) = 0, \quad \text{where } P_k = \sum_{i \in I_k} p_i \tag{58}$$

- Branch real power flow constraint for each branch l:

$$\sum_{k=1}^{N-1} T_{lk}[P_k - D_k - \xi_k] \leqslant F_l^{\max} \quad \text{for } l = 1, \ldots, M \tag{59}$$

$$F_l^{\min} \leqslant \sum_{k=1}^{N-1} T_{lk}[P_k - D_k - \xi_k] \quad \text{for } l = 1, \ldots, M \tag{60}$$

- Real power output constraint for each generator i:

$$p_i^{\min} \leqslant p_i \leqslant p_i^{\max} \quad \forall i \in I \tag{61}$$

The Lagrangian function for this optimization problem is:

$$\begin{aligned} \ell = {} & \sum_{i \in I} C_i(p_i) \\ & - \pi \sum_{k=1}^{N} [P_k - D_k - \xi_k] \\ & - \sum_{l=1}^{M} \hat{\mu}_l \left[F_l^{\max} - \sum_{k=1}^{N-1} T_{lk}[P_k - D_k - \xi_k] \right] \\ & - \sum_{l=1}^{M} \check{\mu}_l \left[\sum_{k=1}^{N-1} T_{lk}[P_k - D_k - \xi_k] - F_l^{\min} \right] \\ & - \sum_{i \in I} \hat{\tau}_i [p_i^{\max} - p_i] \\ & - \sum_{i \in I} \check{\tau}_i [p_i - p_i^{\min}] \end{aligned} \tag{62}$$

Assume that the reduced-form DC OPF problem has been solved. Based on the envelope theorem, using the auxiliary parameter ξ_k, we can calculate the LMPs for all buses as follows:

$$\begin{aligned}\text{LMP}_k &= \frac{\partial J^*}{\partial \xi_k} = \frac{\partial \ell}{\partial \xi_k}\bigg|_{\chi^*} = \pi - \sum_{l=1}^{M} \hat{\mu}_l T_{lk} + \sum_{l=1}^{M} \check{\mu}_l T_{lk} \\ &= \text{MEC}_N + \text{MCC}_k \quad \forall k \neq N \end{aligned} \tag{63}$$

$$\begin{aligned}\text{LMP}_k &= \frac{\partial J^*}{\partial \xi_k} = \frac{\partial \ell}{\partial \xi_k}\bigg|_{\chi^*} = \pi \\ &= \text{MEC}_N, \quad k = N \end{aligned} \tag{64}$$

Here, we have the following:

- $\text{MEC}_N = \pi$ is the LMP component representing the marginal cost of energy at the reference bus N.

- $\text{MCC}_k = -\sum_{l=1}^{M} \hat{\mu}_l T_{lk} + \sum_{l=1}^{M} \check{\mu}_l T_{lk}$ is the LMP component representing the marginal cost of congestion at bus k relative to the reference bus N.

The derived marginal cost of energy, MEC, in (63) and (64) is the same as that in (4-1) and (4-2) on page 35 of MISO's *Business Practices Manual 002: Energy Markets* (MISO (2008a)). Recall that T_{lk} is equal to the "generation shift factor" (GSF_{lk}), which measures the change in megawatt power flow on flowgate (branch) l when a 1 MW change in generation occurs at bus k compensated by a withdrawal of 1 MW at the reference bus. From (62), $\hat{\mu}_l - \check{\mu}_l$ is the "flowgate shadow price" (FSP_l) on flowgate l, which is equal to the reduction in minimized total variable cost that results from an increase of 1 MW in the capacity of the flowgate l. Therefore, the marginal congestion component MCC can be expressed as:

$$\text{MCC}_k = -\sum_{l=1}^{M} \text{GSF}_{lk} \times \text{FSP}_l \tag{65}$$

The derived marginal cost of congestion, MCC, in (65) is the same as that in (4-3) on page 36 of MISO's *Business Practices Manual 002: Energy Markets* (MISO (2008a)).

In the following example, we use the same three-bus system as in Section 3.1 to illustrate the calculation of LMP solution values based on the reduced-form DC OPF model. First, the optimization problem is formulated as follows:

$$\min_{P_{\text{G2}}, P_{\text{G3}}} \; 500 P_{\text{G2}} + 1{,}000 P_{\text{G3}} \tag{66}$$

such that:

$$P_{G2} + P_{G3} - 0.9 = 0 \tag{67}$$

$$\begin{bmatrix} F_{21}^{\min} \\ F_{31}^{\min} \\ F_{23}^{\min} \end{bmatrix} \leqslant \begin{bmatrix} -1/3 & 1/3 & 0 \\ -2/3 & -1/3 & 0 \\ 1/3 & 2/3 & 0 \end{bmatrix} \begin{bmatrix} -0.9 \\ P_{G2} \\ P_{G3} \end{bmatrix} \leqslant \begin{bmatrix} F_{21}^{\max} \\ F_{31}^{\max} \\ F_{23}^{\max} \end{bmatrix} \tag{68}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \leqslant \begin{bmatrix} P_{G2} \\ P_{G3} \end{bmatrix} \leqslant \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{69}$$

The optimal real power commitments for generators 2 and 3 are the same as those obtained for the full-structured DC OPF model:

- $P_{G2} = 0.6$ pu $= 60$ MW, $P_{G3} = 0.3$ pu $= 30$ MW.

The Lagrange multiplier corresponding to the system real power balance constraint, π, is US\$10 per megawatt hour and the Lagrange multiplier corresponding to the inequality constraint for branch 2–1, μ, is US\$15 per megawatt hour. The LMPs can then be calculated based on (63) and (64) as:

$$\text{LMP}_1 = \text{MEC}_3 + \text{MCC}_1 = \pi - \mu(T_{11}) = 10 - 15(-\tfrac{1}{3}) = \text{US\$15/MWh} \tag{70}$$

$$\text{LMP}_2 = \text{MEC}_3 + \text{MCC}_2 = \pi - \mu(T_{12}) = 10 - 15(\tfrac{1}{3}) = \text{US\$5/MWh} \tag{71}$$

$$\text{LMP}_3 = \text{MEC}_3 = \pi = \text{US\$10/MWh} \tag{72}$$

These LMP solution values are the same as those obtained using the full-structured DC OPF model. Moreover, the marginal cost of congestion at bus 1 relative to the reference bus 3, MCC_1, is US\$5 per megawatt hour, and the marginal cost of congestion at bus 2 relative to the reference bus 3, MCC_2, is −US\$5 per megawatt hour.

Consider instead the calculation of the shadow price of branch 2–1 directly from its definition. Recall that the shadow price of a branch is the reduction in minimized total variable cost that results from an increase of 1 MW in the capacity of the branch. For the example at hand, suppose the capacity of branch 2–1 is increased by 1 MW. The minimized total variable cost can then be reduced by simultaneously increasing the output of generator 2 and decreasing the output of generator 3, since the marginal cost of generator 2 is less than the marginal cost of generator 3. The required changes in the outputs of generator 2 and generator 3 can be calculated by solving the following equations:

$$\Delta P_{G2} + \Delta P_{G3} = 0 \text{ MW} \tag{73}$$

$$\tfrac{2}{3}\Delta P_{G2} + \tfrac{1}{3}\Delta P_{G3} = 1 \text{ MW} \tag{74}$$

Solving these equations, we get:

$$\Delta P_{G2} = 3 \text{ MW}$$
$$\Delta P_{G3} = -3 \text{ MW}$$

Therefore the shadow price of branch 2–1, μ, is:

$$\mu = 3(\text{US\$10/MWh} - \text{US\$5/MWh}) = \text{US\$15/MWh}$$

4 CONCLUSION

Locational marginal pricing plays an important role in many recently restructured wholesale power markets. Different AC and DC optimal power flow models are carefully presented and analyzed in this study to help understand the determination of LMPs. In particular, we show how to derive the full-structured DC OPF model from the full-structured AC OPF model, and the reduced-form DC OPF model from the full-structured DC OPF model. Simple full-structured and reduced-form DC OPF three-bus system examples are presented for which the LMP solutions are first derived using envelope theorem calculations and then derived by direct definitional reasoning. We also use these examples to illustrate that LMP solution values derived for the full-structured DC OPF model are the same as those derived for the reduced-form DC OPF model. As a byproduct of this analysis, we are able to provide a rigorous explanation of the basic LMP and LMP-decomposition formulas (neglecting real power losses) presented without derivation in MISO's *Business Practices Manual 002: Energy Markets*.

APPENDIX A: THE BUS ADMITTANCE MATRIX

Let bus k and bus m be connected by a branch km. The impedance of the branch is $r_{km} + jx_{km}$, where r_{km} is the resistance and x_{km} is the reactance. The admittance of the branch is $g_{km} + jb_{km}$, where g_{km} is the conductance and b_{km} is the susceptance. The bus admittance matrix $\boldsymbol{Y}$ can be constructed as follows (see page 295 of Bergen and Vittal (2000) for details).

a) The bus admittance matrix $\boldsymbol{Y}$ is symmetric.

b) $Y_{kk} = G_{kk} + jB_{kk}$ is the kth diagonal element of the admittance matrix $\boldsymbol{Y}$ and is equal to the sum of the admittances of all the branches connected to the kth bus.

c) $Y_{km} = G_{km} + jB_{km}$ is the kmth off-diagonal element of the admittance matrix $\boldsymbol{Y}$ and is equal to the negative of the admittance of all branches connecting bus k to bus m. If more than one such branch exists, the equivalent admittance of the branches is obtained before calculating this element in the bus admittance matrix.

APPENDIX B: REAL POWER LOSS CALCULATION

Let bus i and bus j be connected by a branch ij. The voltage at bus i is $\dot{V}_i = V_i \angle \theta_i = V_i \cos\theta_i + jV_i \sin\theta_i$, the voltage at bus j is $\dot{V}_j = V_j \angle \theta_j$, the impedance of the branch is $r_{ij} + jx_{ij}$, the admittance of the branch is $g_{ij} + jb_{ij} = (r_{ij} - jx_{ij})/(r_{ij}^2 + x_{ij}^2)$, the current on the branch is $\dot{I}_{ij}$, the complex power flowing out of bus i to bus j is $S_{ij} = P_{ij} + jQ_{ij}$, and the complex power flowing out of bus j to bus i is $S_{ji} = P_{ji} + jQ_{ji}$.

The real and reactive power loss along the transmission line ij is:[2]

$$\begin{aligned} P_{\text{loss}\,ij} + jQ_{\text{loss}\,ij} &= [\dot{V}_i - \dot{V}_j]\overset{*}{I}_{ij} = [\dot{I}_{ij}[r_{ij} + jx_{ij}]]\overset{*}{I}_{ij} \\ &= I_{ij}^2[r_{ij} + jx_{ij}] = I_{ij}^2 r_{ij} + jI_{ij}^2 x_{ij} \end{aligned} \tag{B.1}$$

where $\overset{*}{I}_{ij}$ is the complex conjugate of $\dot{I}_{ij}$. Therefore, the real power loss on the transmission line ij is:

$$P_{\text{loss}\,ij} = I_{ij}^2 r_{ij} \tag{B.2}$$

Since:

$$\dot{I}_{ij} = \frac{\dot{V}_i - \dot{V}_j}{r_{ij} + jx_{ij}} \tag{B.3}$$

The magnitude of the current $\dot{I}_{ij}$ is:

$$I_{ij} = |\dot{I}_{ij}| = \frac{\sqrt{[V_i \cos\theta_i - V_j \cos\theta_j]^2 + [V_i \sin\theta_i - V_j \sin\theta_j]^2}}{\sqrt{r_{ij}^2 + x_{ij}^2}} \tag{B.4}$$

Therefore:

$$I_{ij}^2 = \frac{V_i^2 + V_j^2 - 2V_i V_j \cos\theta_{ij}}{r_{ij}^2 + x_{ij}^2} \tag{B.5}$$

Therefore:

$$P_{\text{loss}\,ij}(x) = I_{ij}^2 r_{ij} = \frac{V_i^2 + V_j^2 - 2V_i V_j \cos\theta_{ij}}{r_{ij}^2 + x_{ij}^2} r_{ij} \tag{B.6}$$

where the elements of the state vector x are the $N-1$ voltage angles for the $N-1$ non-reference buses and the N voltage magnitudes for all N buses.

The complex power flowing from bus i to bus j is:

$$\begin{aligned} S_{ij} &= P_{ij} + jQ_{ij} = \dot{V}_i \overset{*}{I}_{ij} = \dot{V}_i \left[\frac{\dot{V}_i - \dot{V}_j}{r_{ij} + jx_{ij}}\right]^* \\ &= \dot{V}_i \frac{\overset{*}{V}_i - \overset{*}{V}_j}{r_{ij} - jx_{ij}} = \frac{[V_i^2 - \dot{V}_i \overset{*}{V}_j][r_{ij} + jx_{ij}]}{r_{ij}^2 + x_{ij}^2} \\ &= \frac{[V_i^2 - V_i V_j \cos\theta_{ij} - jV_i V_j \sin\theta_{ij}][r_{ij} + jx_{ij}]}{r_{ij}^2 + x_{ij}^2} \end{aligned} \tag{B.7}$$

where $\theta_{ij} = \theta_i - \theta_j$. From (B.7) we have the real power flowing from bus i to bus j:

$$P_{ij}(x) = \frac{[V_i^2 - V_i V_j \cos\theta_{ij}]r_{ij} + [V_i V_j \sin\theta_{ij}]x_{ij}}{r_{ij}^2 + x_{ij}^2} \tag{B.8}$$

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[2] See (2.18)–(2.20) in Section 2.2 of Bergen and Vittal (2000) for the basic principles of complex power.

From (B.7), the reactive power flowing from bus i to bus j is:

$$Q_{ij}(x) = \frac{[V_i^2 - V_i V_j \cos\theta_{ij}]x_{ij} - [V_i V_j \sin\theta_{ij}]r_{ij}}{r_{ij}^2 + x_{ij}^2} \tag{B.9}$$

The real power flowing out of bus j to bus i is:

$$P_{ji}(x) = \frac{[V_j^2 - V_i V_j \cos\theta_{ij}]r_{ij} + [V_i V_j \sin\theta_{ji}]x_{ij}}{r_{ij}^2 + x_{ij}^2} \tag{B.10}$$

Therefore:

$$P_{ij}(x) + P_{ji}(x) = \frac{[V_i^2 + V_j^2 - 2V_i V_j \cos\theta_{ij}]r_{ij}}{r_{ij}^2 + x_{ij}^2} \tag{B.11}$$

From (B.6) and (B.11), we have:

$$P_{\text{loss}\,ij}(x) = P_{ij}(x) + P_{ji}(x) \tag{B.12}$$

implying that the loss on each transmission line ij is a function of the state vector x.

The total system real power loss, which is the sum of the real power loss along each transmission line ij, is then given by:

$$P_{\text{loss\_sys}}(x) = \sum_{ij} P_{\text{loss}\,ij}(x) = \sum_{ij}[P_{ij}(x) + P_{ji}(x)] = \sum_{i=1}^{N} f_{pi}(x) \tag{B.13}$$

where N is the total number of buses and $f_{pi}(x)$ is the total real power flowing out of bus i. The latter expression denotes the sum of all the real power flowing out of the ith bus along the transmission lines connected to the ith bus, which can be represented as follows:

$$f_{pi}(x) = \sum_{k=1}^{n} V_i V_k (G_{ik}\cos(\theta_i - \theta_k) + B_{ik}\sin(\theta_i - \theta_k)) \tag{B.14}$$

where $G_{ik} + jB_{ik}$ is the ikth element of the bus admittance matrix $\boldsymbol{Y}$.

If branch resistance is neglected, ie, if we set $r_{ij} = 0$ for each transmission line ij, then from (B.2) we have:

$$P_{\text{loss}\,ij} = I_{ij}^2 r_{ij} = 0 \tag{B.15}$$

for each ij. From (B.13) we then have:

$$\sum_{i=1}^{N}[P_i - D_i] = \sum_{i=1}^{N} f_{pi}(x) = \sum_{ij} P_{\text{loss}\,ij}(x) = 0 \tag{B.16}$$

APPENDIX C: ADJACENCY MATRIX

The row-dimension of the adjacency matrix $\boldsymbol{A}$ is equal to M, the number of branches, and the column-dimension of $\boldsymbol{A}$ is equal to N, the number of buses. The kjth element of $\boldsymbol{A}$ is 1 if the kth branch begins at bus j, -1 if the kth branch terminates at bus j, and 0 otherwise. A branch k connecting a bus j to a bus i is said to "begin" at bus j if the power flowing across branch k is defined to be positive for a direction *from* bus j *to* bus i. Conversely, branch k is said to "terminate" at bus j if the power flowing across branch k is defined to be positive for a direction *to* bus j *from* bus i.

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