Finite strain analysis in elastic theory

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FINITE STRAIN ANALYSIS IN ELASTIC THEORY

by

Donald Hill Rock

A Thesis Submitted to the Graduate Faculty
for the Degree of

DOCTOR OF PHILOSOPHY

Major Subject Applied Mathematics

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1939
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I. INTRODUCTION

A feature common to the mathematical formulation of a large class of physical problems is the linearity of the differential equations involved. This fact is of great importance to one attempting the solutions of these problems subject to various boundary or initial conditions, for linear problems can be handled either by rigorous mathematical processes or by well-known approximation methods. However this linearity is not usually an inherent physical characteristic but is an approximation imposed in the course of the mathematical derivation of the problem. An excellent example of this fact occurs in the elementary mathematical theory of elasticity. Here, as in many problems, a Taylor's expansion is used, and those terms involving squares and products of the space derivatives of the displacement vector are neglected. If the displacements are very small, it is readily seen that the linear terms compose the principal part of the expansion. This is a sufficiently good approximation of actual physical phenomena in many cases—a fact to which the large number of problems in elasticity handled satisfactorily by this elementary method testifies. Nevertheless, when the displacements are no longer small, there is a large variance between theoretical and experimental results.
If the terms composed of the squares and products of the space derivatives of the displacements are retained, a system of non-linear differential equations results. This non-linear problem presents great difficulty towards solution, but the demand for approximate solutions of this system is increasing. For example, the experimental evaluation of stresses in large and expensive machines is not economically feasible; thus the theoretical approach is highly desirable. In addition the problem of the aeronautical engineer to minimize the weight without subsequent weakening of structural members of aircraft demands an accurate and usable theory of the bending and stability of thin plates.

In Chapter II a sketch of the development of the general theory of large deformations (variously called large or finite strain theory) will be presented together with the development of the non-linear theory of thin plates. Chapter III will be concerned with particular analytical procedures to be employed in the investigation. Two problems will be treated in Chapter IV. The first will be an approximation of second order effects in the stability of a thin circular plate under edge thrust. The second problem concerns the development of the non-linear differential equations for the bending of a moderately thick circular plate under normal loading, and then the solution of these equations for the clamped plate under normal and uniform loading.
II. HISTORY OF THE NON-LINEAR THEORY OF ELASTICITY

The general non-linear theory of elasticity may be considered to have begun with an article by Kirchoff [1] in 1852 on the development of the equilibrium equations and surface traction conditions for an elastic body subject to large strains. Later, in 1872, Boussinesq [2] wrote a treatise on periodic waves in liquids to which he appended a note on the energy considerations of elastic media in which the strains are finite. At the end of the nineteenth century numerous papers were published whose subject matter covered different aspects of the non-linear theory. Various modifications of the form of the elastic potential were proposed, in general by the addition of third order terms. Explicit forms of the equilibrium equations and the strain displacement relations for particular coordinate systems and possible changes in the stress-strain relation (Hooke's law) were included in these papers. Attempts were made to develop the equations of the non-linear theory for any coordinate system whatsoever. Finally in 1906, Müller and Timpe [3] wrote a concise resume of the then current status of the general non-linear theory of elasticity. Shortly after the turn of the century Almansi [4] introduced the idea of employing the coordinates of an element after strain instead of the initial coordinates (classically, Eulerian
and Lagrangian coordinates, respectively).

The invention of the tensor calculus in recent years has provided a valuable tool for the systematic treatment of the theory of elasticity, for with its help it is possible to express the fundamental equations in a form which is independent of the choice of coordinate systems. In 1925 L. Brillouin [5] gave the tensor formulation of the elementary theory of elasticity. Later in the same year he treated part of the non-linear theory while investigating radiation pressure. More recently Murnaghan [6] in 1937, by employing tensors, has given a systematic derivation of the non-linear theory in which the Eulerian viewpoint was adopted. The fundamental stress-strain relations for an isotropic medium were derived in terms of an elastic energy density which was shown to be a function of the strain tensor through its invariants.

Now although the equations of the general theory have been completely developed, there remains the difficult task of applying them to concrete problems such as thin plates. As early as 1890 attempts were made to develop empirical formulas for thin plates undergoing large deflections. In 1905 A. Föppl [7] derived a pair of non-linear partial differential equations to treat the bending of thin plates of negligible rigidity. Later v.Karman [8] extended this system of equations to the case of non-negligible rigidity. In this same article he gave an outline of this theory and references to earlier investiga-
tions. The bending of circular plates having axial symmetry has been the subject of most of the investigations, for in this case the differential equations are ordinary. Timoshenko [9], Hencky [10], and Federhofer [11] published papers between 1910 and 1922 on various special cases. Nadai [12] in 1925 approximated the solutions for both the clamped and simply-supported circular plate. In 1928 Timoshenko [13] employed the minimization of the potential energy to obtain a solution, and later Vincent [14] used a method of successive approximations. In 1934 Way [15] derived an approximation by expanding the stresses and the slope function in terms of a dimensionless radial coordinate. The results of this latter investigation were in good agreement with those previously obtained by Nadai and Timoshenko.
III. ANALYTICAL PROCEDURES

A. An Approximation Method for Solving Non-linear Differential Equations

The method to be employed in the investigation in solving non-linear differential equations was developed for the perturbation theory of celestial mechanics [16]. It has been but rarely used in the treatment of technical problems [17]. The method in brief consists in the development of the desired solution in a power series of a suitably selected parameter.

Suppose there is given a non-linear differential equation of the second order,

\[(1) \quad y'' = \phi(x,y,y'),\]

where \( \phi \) is an integral and rational function of its argument. A solution of this equation is sought which is single-valued, continuous, and satisfying certain boundary conditions. It is assumed that the solution \( y \) can be represented in the form

\[(2) \quad y(x) = y_0(x) + y_1(x)\varepsilon + y_2(x)\varepsilon^2 + \cdots,\]

where the \( y_n(x) \) are unknown functions of \( x \) and \( \varepsilon \) is any parameter by which the solution is developable. The parameter need not appear in either the differential equation (1) or the boundary conditions. It may be a definite quantity which char-
acterizes the solution in the form (2), and it must be re-
stricted to a sufficiently small magnitude to insure the con-
vergence of the process.

By introducing the series (2) into the given differential
equation and considering the identity in $\varepsilon$ which results, as
many equations are obtained as there are terms in the expansion
(2). If these equations are linear and, outside of possibly
the initial equation, non-homogeneous, their integration may
be effected by elementary processes. In characteristic value
problems the characteristic value is given by the linear terms,
the solution itself remaining indeterminate up to an arbitrary
constant. By retaining the higher terms this indeterminacy is
removed and the complete solution is obtained for all values
of the parameter.

This method may be extended to systems of differential
equations. For a system of two equations it is convenient to
develop one dependent variable in even powers of the parameter
and the other in odd powers. The method then resembles that
of successive approximations in the theory of integral equa-
tions.
B. Garabedian's Parameter Method for the Treatment of Elastic Problems

In 1922 C. Garabedian [18] made use of a parameter expansion of the elastic displacements in order to obtain solutions of various plate problems. The method really originated with Birkhoff [19], but whereas he approached the problem from energy considerations, Garabedian made use of the equilibrium equations and the surface traction conditions. Only axially symmetric plates were handled, for as has been remarked earlier, in this case the differential equations are ordinary and not partial.

An expansion of the radial displacement $u$ and the axial displacement $w$ in terms of a "natural" parameter is assumed. The equilibrium equations and the surface traction conditions are then to be considered as identities in the parameter. In addition it is necessary to consider a similar expansion for the loading. The result of this procedure is a sequence of equilibrium equations and surface traction conditions. The equilibrium equations can then be integrated with respect to the axial coordinate and subjected to the proper surface traction conditions. Finally one arrives at a series of differential equations for the deflection $w$ ordered in the "natural" parameter. The solution of these equations is then sought for specific boundary conditions.
IV. INVESTIGATION

A. Stability of the Clamped Circular Plate

The von Kármán equations for the deflections of a thin plate when the deflections are of the same order of magnitude as the thickness of the plate are obtained by adding the effects of the stretching stresses in the middle place to the contribution of the bending stresses. If the mean stresses over the thickness of the plate

$$\overline{\sigma_x} = \frac{1}{h} \int_{-h/2}^{h/2} \sigma_x \, dz, \text{ etc.}$$

are expressed by means of an Airy function

$$\overline{\sigma_x} = \frac{\partial^2 F}{\partial y^2}, \quad \overline{\tau_{xy}} = -\frac{\partial^2 F}{\partial x \partial y}, \quad \overline{\sigma_y} = \frac{\partial^2 F}{\partial x^2},$$

then the von Kármán equations are

$$\Delta^4 F = -E \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right]$$

(1)

$$Na^4 w = p + h \left[ \overline{\sigma_x} \frac{\partial^2 w}{\partial x^2} + 2 \overline{\tau_{xy}} \frac{\partial^2 w}{\partial x \partial y} + \overline{\sigma_y} \frac{\partial^2 w}{\partial y^2} \right]$$

where

$$\Delta = \text{Laplacian operator}, \quad \Delta^4 = \Delta^2 \Delta^2$$

$N$ = plate rigidity = $Eh^3/12(1 - \nu^2)$

$E$ = Young's modulus

$\nu$ = Poisson's ratio

$h$ = plate thickness

$p$ = normal load on the plate

$w$ = vertical deflection.
In the case of a circular plate with axial symmetry where
\[ \overline{\sigma}_r = \frac{1}{r} \frac{\partial F}{\partial r}, \quad \overline{\sigma}_0 = \frac{3}{r^2} \frac{\partial^2 F}{\partial r^2}, \quad p = 0, \]
equations (1) take the form
\[ \Delta^4 F = -E \frac{1}{r} \frac{\partial w}{\partial r} \frac{\partial^2 w}{\partial r^2} \]
(2)
\[ Na^4 w = h \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial F}{\partial r} \frac{\partial w}{\partial r} \right]. \]
These equations constitute a characteristic value problem for the edge-loading \( P = -\overline{\sigma}_r \bigg|_{r=a} \) where \( a \) is the radius of the plate.

The above equations can be simplified by taking
\[ \frac{dw}{dr} = \sqrt{2N/hE} r q, \quad \frac{dp}{dr} = r \overline{\sigma}_r = -(rN/h)p, \]
where \( p \) and \( q \) are new functions. If the shear and the radial stress are required to be regular at the origin, equations (2) become
\[ \frac{d^2 p}{dr^2} + \frac{3}{r} \frac{dp}{dr} = q^2, \quad \frac{d^2 q}{dr^2} + \frac{3}{r} \frac{dq}{dr} + pq = 0. \]

Since \( dw/dr \) and the radial stress are even functions, then \( q \) is an odd function and \( p \) is an even function. Therefore the expansions of these in powers of a parameter \( \varepsilon \) are:
\[ p = p_0 + p_2 \varepsilon^2 + \cdots \]
(5)
\[ q = q_1 \varepsilon + q_3 \varepsilon^3 + \cdots. \]
When these two series are inserted in the differential equations (4), two sequences of equations result which are ordered in the parameter. These are given in the table below.

| $\varepsilon^0$ | $p''_0 + \frac{3}{r} p'_0 = 0$ | $\varepsilon^1$ | $q''_1 + \frac{3}{r} q'_1 + p_0 q_1 = 0$ |
| $\varepsilon^2$ | $p''_2 + \frac{3}{r} p'_2 = q_1$ | $\varepsilon^3$ | $q''_3 + \frac{3}{r} q'_3 + p_0 q_3 = -p_0 q_1$ |
| $\varepsilon^4$ | $p''_4 + \frac{3}{r} p'_4 = 2q_1 q_3$ | | etc. |

The boundary conditions for the clamped plate are:

$$r q = 0 \text{ at } r = 0, \quad r = a,$$

$$p \text{ is regular at } r = 0.$$ (6)

It may be noted that the differential operator on the $p$'s may be written

$$\frac{1}{r} \frac{d}{dr} \left[ r^2 \frac{d}{dr} \right].$$

Proceeding to the solution of the first equation in Table I, it is easily seen that the only solution satisfying the requirement of regularity at the origin is

$$p_0 = a^2 = \text{a constant}. \quad (7)$$

This constant is undetermined; it is the characteristic value to be determined by the succeeding equation. The equation in
$q_1$ now reads:

\begin{equation}
 q'' + \frac{3}{r} q' + \alpha^2 q_1 = 0.
\end{equation}

The solution of this which is regular at the origin is

\begin{equation}
 q_1 = c \frac{J_1(\alpha r)}{r}
\end{equation}

where $J_1$ is the Bessel function of the first kind and the first order. The boundary condition at the periphery requires that

\begin{equation}
 J_1(\alpha a) = 0.
\end{equation}

This is true only if the $\alpha$'s take discrete values, that is, the $\alpha$'s must be zeros of the function. These characteristic values are:

\begin{equation}
 \lambda_0 = 0, \lambda_1 = 3.8317, \lambda_2 = 7.0156, \text{ etc.}, (\lambda = \alpha a).
\end{equation}

Therefore $p_0$ may take any of the permissible values

\begin{equation}
 p_0 = \frac{\lambda_n^2}{a^2}
\end{equation}

The lowest characteristic value corresponds to the plane position of the plate; the next one gives the least value of the edge loading for which buckling of the plate occurs. The higher values give the higher modes of buckling. The first three forms of the deflection are shown in Figure 1.

Fig. 1
The numerical value of the first buckling load is

\[ - \pi \sigma_r \bigg|_{r=a} = n \lambda_1^2/a^2 = 14,682 \text{ Eh}^3/12(1 - \nu^2). \]

Hereafter \( a \) shall refer to \( \lambda_1/a \). In other words the investigation will be concerned with the first mode of buckling.

Thus far the form of the deflected surface is determined only to within an arbitrary constant. This will be selected so that

\[ q_1(0) = a^2, \]

whence

\[ q_1(r) = 2a \frac{J_1(ar)}{r}. \]

The equation in \( \varepsilon^2 \) is now

\[ p'' + \frac{3}{r} p' = 4a^2 \frac{J_1^2(ar)}{r^2}, \]

and the solution of this which is regular at the origin is

\[ p_2 = \beta^2 - 2a^2 \left[ J_0^2(ar) + J_1^2(ar) - \frac{1}{ar} J_0(ar) J_1(ar) \right] \]

where \( \beta^2 \) is a constant still to be determined.

The next equation in the sequence is

\[ q'' + \frac{3}{r} q' + a^2 q = - p_2 q_1. \]

Since the homogeneous equation has a solution different from zero (see equations (9) and (9)), the necessary condition for the existence of a solution of (17) is \([20]:\)
(18) \[ \int_0^a p_2 q_1 r^3 dr = 0. \]

If this condition is satisfied, there exists a single infinity of solutions:

(19) \[ q_2 = k q_1 + q_s, \]

where \( q_1 \) is the solution of the homogeneous equation (8), \( k \) is an arbitrary constant, and \( q_s \) is a particular integral of (17).

Using the numerical values given in Table II, the value of \( \beta^2 \) is determined from equation (18) to be

(20) \[ \beta^2 = 0.6173 \alpha^2. \]

Now since \( p \sim -\sigma_r \), the edge-loading may be defined, apart from a proportionality factor, by

(21) \[ P_n = P_n \bigg|_{r=a} \quad \text{and} \quad P = P_o + P_2 + P_4 + \cdots. \]

Using the value of \( \beta^2 \) in (20) the value of \( P_2 \) is found to be

(22) \[ P_2 = 0.2929 \alpha^2; \]

and since \( P_o = \alpha^2 \) then

(23) \[ (P_o + P_2)/P_o = 1.2929. \]

As indicated in equation (19) it is necessary to determine a particular integral of equation (17). By the method
of variation of parameters it can easily be shown that this particular integral can be written in the form

\begin{equation}
Q = - \frac{J_{1}}{r} \int \frac{N_{1}Ds}{D} dr + \frac{N_{1}}{r} \int \frac{J_{1}Ds}{D} dr,
\end{equation}

where

\[ D = N_{1} \frac{d}{dr} \left( \frac{J_{1}}{r} \right) - J_{1} \frac{d}{dr} \left( \frac{N_{1}}{r} \right), \]

the integrals are indefinite, and \( N_{1} \) is the Bessel function of the second kind and first order. The argument of these functions is \( \alpha r \) in every case. By means of the relation between the Bessel functions of the first and second kinds, namely [21]

\begin{equation}
N_{0}(x)J_{1}(x) - N_{1}(x)J_{0}(x) = \frac{2}{\pi x},
\end{equation}

the particular integral may be written

\begin{equation}
Q = (\pi/2r) \left[ J_{1} \int r^{2}N_{1}Ds dr - N_{1} \int r^{2}J_{1}Ds dr \right].
\end{equation}

Inserting the values of \( q_{1} \) and \( p_{2} \) from equations (14) and (16) this becomes

\begin{equation}
Q = \left( \frac{\pi \alpha}{r} \right) \left[ J_{1} \int rN_{1}J_{1} dr - N_{1} \int rJ_{1} dr \right] + \left( \frac{2\pi}{r} \right) \left[ J_{1} \int rN_{1}J_{1}(J_{0}^{2} + J_{1}^{2} - \frac{1}{dr}J_{0}J_{1}) dr \right]
\end{equation}

\[ -N_{1} \int rJ_{1}(J_{0}^{2} + J_{1}^{2} - \frac{1}{dr}J_{0}J_{1}) dr ].

From Table II part of this can be integrated and the remainder simplified to give
Despite many different attempts the term in brackets could not be integrated. Whether or not it is possible to obtain a closed expression for this term remains an open question. It may be that these integrals can be handled only by series. In any case the succeeding steps in the approximation can be outlined.

Thus far the form of \( q_3 \) is determined only to within a multiple of \( q_1 \). The constant \( k \) in equation (19) will be selected so as to give the best possible solution of the problem. Now \( q \) is the solution of a certain minimum problem. If \( q = q_1 \varepsilon + (kq_1 + q) \varepsilon^3 \), this integral expression may be minimized with respect to \( k \). The resulting condition for a minimum is a function of \( \varepsilon \); this may be developed in powers of \( \varepsilon \) and the lowest term taken [22].

With \( k \) fixed, \( p_4 \) and \( q_5 \) are to be determined. The edge-loading \( P_4 \) is fixed by the condition that \( q_5 \) have a solution, and \( q_5 \) like \( q_3 \) is determined only to within a multiple of \( q_1 \). Again this multiple factor can be selected in the same way as that for \( k \).

The convergence of the process of approximation may be established by recourse to the theory of non-linear integral equations [23].
Table II

\[ \int_0^a r J_1^2 dr = \frac{r^2}{2} \left[ J_0^2 + J_1^2 - \frac{2}{ar} J_o J_1 \right]_0^a = \frac{a^2}{2} J_o(aa) \]

\[ \int r N_1 J_1 dr = \frac{r^2}{2} \left[ N_o J_o + N_1 J_1 - \frac{1}{ar} (N_o J_1 + N_1 J_o) \right] \]

\[ \int J_o J_1^3 dr = \frac{ar^2}{4} (J_o^2 + J_1^2) (J_o^2 + J_1^2 - \frac{2}{ar} J_o J_1) \]

\[ \int N_2 J_o J_1^2 dr = \frac{ar^2}{4} \left[ (N_o J_o + N_1 J_1) (J_1^2 - \frac{2}{ar} J_o J_1) + N_o J_o (J_o^2 + J_1^2) \right] \]

\[ + \frac{1}{2\pi a} J_o^2 - \frac{r J_o J_1}{2\pi} \]

\[ \int r J_o J_1^2 dr = \frac{r^2}{6} (J_o^2 + J_1^2)^2 - \frac{r J_o^3 J_1}{3a} + \frac{1}{3} \int r J_1^4 dr \]

\[ \int r N_1 J_o J_1^2 dr = \frac{r^2}{6} (J_o^2 + J_1^2) (N_o J_o + N_1 J_1) - \frac{r J_o^3 N_1}{3a} + \frac{1}{3} \int r N_1 J_1^3 dr \]

\[ \int r^2 J_1 J_1^2 dr = \frac{r^2}{2a} J_1^2 (J_o^2 + J_1^2) \]

\[ - \frac{r^2}{4a} J_1^4 + \frac{1}{2a} \int r J_1^4 dr \]

\[ \int r J_1^3 dr = \frac{r^2}{4a} J_1^4 + \frac{1}{2a} \int r J_1^4 dr \]

\[ \int r J_1^4 dr = \frac{r^2}{2} (J_o^2 + J_1^2) + \int r J_1^4 dr \]

\[ \int_0^a r J_1^4 dr = 0.0204 a^2 \quad \text{(by numerical integration)} \]

\[ J_o(aa) = -0.40276 \]
B. The Bending of Moderately Thick Plates

The first part of this section will be concerned with the derivation of the differential equation of bending of the moderately thick circular plate when the strains are large. Later this differential equation will be solved by approximation methods for the case of clamped edges. It will be assumed throughout that the plate, which is shown in Figure 2, is axially symmetric.

![Fig. 2](image)

If the unit strains are

\[ e_r = \frac{3u}{3r} + \frac{1}{2} w', \quad e_\theta = \frac{u}{r}, \quad e_z = \frac{\partial w}{\partial z}, \quad \gamma_{rz} = \frac{1}{2} \left[ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right], \]

then the equilibrium equations as given by Love [24] must be modified. These will be:

\[ (\lambda + 2\mu)u'' + (\lambda + \mu)\frac{\partial w'}{\partial z} + \mu \frac{\partial^2 u}{\partial z^2} + \frac{\lambda + 2\mu}{2} (w')^2 + \frac{\mu}{r} w'' = 0, \]

\[ (\lambda + 2\mu)\frac{\partial^2 w}{\partial z^2} + (\lambda + \mu)\frac{\partial w'}{\partial z} + \mu \frac{\partial w''}{\partial z} + \frac{\lambda}{2} \frac{\partial}{\partial z} (w')^2 = 0, \]

where \( \lambda \) and \( \mu \) are Lame's elastic constants, \( u \) and \( w \) are the radial and axial displacements, and primes denote differentiation with respect to \( r \). In addition the star means:

\[ f^* = \frac{1}{r} \frac{\partial}{\partial r} (rf') = f' + \frac{f}{r}, \quad \text{and} \quad f''^* = \Delta^2 f. \]
Now let $z$ be replaced by $\zeta$, $t$ being a parameter, and also let the displacements and the applied surface tractions be expanded in power series in the parameter:

$$u(r, \zeta) = u_0(r, \zeta) + u_1(r, \zeta)t + u_2(r, \zeta)t^2 + \cdots,$$
$$w(r, \zeta) = w_0(r, \zeta) + w_1(r, \zeta)t + w_2(r, \zeta)t^2 + \cdots,$$
$$S|_{-h} = S^0 + S^1t + S^2t^2 + \cdots,$$
$$S|_{+h} = S^0 + S^1t + S^2t^2 + \cdots.$$

If these expansions together with the change of variable are used, the equilibrium equations and the surface traction conditions on the plate are:

$$(E_r) \quad (\lambda + 2\mu)u'' + \frac{\lambda + \mu}{t} \frac{\partial w'}{\partial \zeta} + \frac{\mu}{t^2} \frac{\partial^2 u}{\partial \zeta^2} + \frac{\lambda + 2\mu}{r} (w'^2) + \frac{\mu}{r} w'^2 = 0,$$

$$(E_z) \quad \frac{\lambda + 2\mu}{t^2} \frac{\partial^2 w}{\partial \zeta^2} + \frac{\lambda + \mu}{t} \frac{\partial u'}{\partial \zeta} + \frac{\mu}{t^2} w'' + \frac{\lambda}{2t} \frac{\partial}{\partial \zeta} (w'^2) = 0,$$

$$(R) \quad \tau_{rz} = \frac{1}{t} \frac{3u'}{\partial \zeta} + w' \bigg|_{-h} = 0,$$

$$(Z) \quad \sigma_z = \lambda u'' + \frac{\lambda + 2\mu}{t} \frac{\partial w'}{\partial \zeta} + \frac{\lambda}{2} \frac{w'^2}{t} \bigg|_{-h} = S^0_n.$$

The superscript on the applied traction in $(Z)$ denotes the applied traction on the top face of the plate, the subscript that on the bottom face.

The sequence of equations obtained by considering $(E_r)$, $(E_z)$, $(R)$ and $(Z)$ as identities in $t$ are given in Table III below.
Table III

\[
\begin{align*}
\frac{\partial^2 u_0}{\partial z^2} & = 0 \\
\mu \frac{\partial^2 u_1}{\partial z^2} + (\lambda + \mu) \frac{\partial w_1}{\partial z} & = 0 \\
\mu \frac{\partial^2 u_2}{\partial z^2} + (\lambda + \mu) \frac{\partial w_2}{\partial z} + (\lambda + 2\mu)(u_0^* + \frac{1}{2} w_0^{'^2}) + \mu w_0^{^2} & = 0 \\
\mu \frac{\partial^2 u_3}{\partial z^2} + (\lambda + \mu) \frac{\partial w_3}{\partial z} + (\lambda + 2\mu)(u_1^* + w_1^{'^2}) + 2w_1^{'^2} & = 0 \\
\mu \frac{\partial^2 u_4}{\partial z^2} + (\lambda + \mu) \frac{\partial w_4}{\partial z} + (\lambda + 2\mu)(u_2^* + w_2^{'^2} + \frac{1}{2} w_1^{'^2}) + \mu (w_1^{^2} + 2w_2^{^2}) & = 0 \\
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 w_0}{\partial z^2} & = 0 \\
(\lambda + \mu) \frac{\partial^2 w_1}{\partial z^2} + (\lambda + \mu) \frac{\partial u_0}{\partial z} + \lambda w_1^{'^2} & = 0 \\
(\lambda + \mu) \frac{\partial^2 w_2}{\partial z^2} + (\lambda + \mu) \frac{\partial u_1}{\partial z} + \mu w_2^{'^2} + \lambda (w_0^{'^2} + \frac{1}{2} w_1^{'^2}) & = 0 \\
(\lambda + \mu) \frac{\partial^2 w_3}{\partial z^2} + (\lambda + \mu) \frac{\partial u_2}{\partial z} + \mu w_3^{'^2} + \lambda (w_0^{'^2} + w_1^{'^2} + w_2^{'^2}) & = 0 \\
(\lambda + \mu) \frac{\partial^2 w_4}{\partial z^2} + (\lambda + \mu) \frac{\partial u_3}{\partial z} + \mu w_4^{'^2} + \lambda (w_0^{'^2} + w_1^{'^2} + w_2^{'^2}) & = 0 \\
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u_0}{\partial z} \bigg|_{\Phi} & = 0 & \frac{\partial w_0}{\partial z} \bigg|_{\Phi} & = 0 \\
\frac{\partial u_1}{\partial z} + w_1^{'^2} \bigg|_{\Phi} & = 0 & (\lambda + \mu) \frac{\partial w_1}{\partial z} + \lambda u_0^* + \frac{\lambda}{2} w_0^{^2} \bigg|_{\Phi} & = S_0 \\
\frac{\partial u_2}{\partial z} + w_2^{'^2} \bigg|_{\Phi} & = 0 & (\lambda + \mu) \frac{\partial w_2}{\partial z} + \lambda u_1^* + \lambda w_1^{'^2} \bigg|_{\Phi} & = S_1 \\
\frac{\partial u_3}{\partial z} + w_3^{'^2} \bigg|_{\Phi} & = 0 & (\lambda + \mu) \frac{\partial w_3}{\partial z} + \lambda u_2^* + \lambda (w_1^{'^2} + 2w_2^{^2}) \bigg|_{\Phi} & = S_2 \\
\frac{\partial u_4}{\partial z} + w_4^{'^2} \bigg|_{\Phi} & = 0 & (\lambda + \mu) \frac{\partial w_4}{\partial z} + \lambda u_3^* + \lambda (w_0^{'^2} + w_1^{'^2} + w_2^{'^2}) \bigg|_{\Phi} & = S_3 \\
\frac{\partial u_5}{\partial z} + w_5^{'^2} \bigg|_{\Phi} & = 0 & (\lambda + \mu) \frac{\partial w_5}{\partial z} + \lambda u_4^* + \lambda (w_0^{'^2} + w_1^{'^2} + w_2^{'^2}) \bigg|_{\Phi} & = S_3 \\
\end{align*}
\]
The method to be followed in the derivation of the differential equation of bending consists of the integration of the equilibrium equations in $u_n$ and $w_n$ from $(E_r)$ and $(E_z)$ and then the application of the surface traction conditions to the results. At the outset it should be noted that the integration process yields the general forms:

\[
\begin{align*}
    u_n &= u_{n0} + u_{n1}\zeta + \cdots + u_{nn}\zeta^n, \\
    w_n &= w_{n0} + w_{n1}\zeta + \cdots + w_{nn}\zeta^n,
\end{align*}
\]

where the double subscripts indicate functions of $r$ alone. Unless otherwise noted the single subscript will denote functions of $r$ and $\zeta$.

The first set of equations in $(E_r)$ and $(E_z)$ gives

\[
\begin{align*}
    u_0 &= u_{00} + u_{01}\zeta, \\
    w_0 &= w_{00} + w_{01}\zeta,
\end{align*}
\]  

(1)

and the application of the surface traction conditions shows that $u_{01}$ and $w_{01}$ must vanish. Therefore $u_0$ and $w_0$ are functions of $r$ only, and to simplify the notation these will be written with the single subscript.

The next set of equations yields on integration:

\[
\begin{align*}
    u_1 &= u_{10} + u_{11}\zeta, \\
    w_1 &= w_{10} + w_{11}\zeta,
\end{align*}
\]  

(2)

and the application of the next surface traction conditions
shows that

\[ u_{11} = -w_0', \quad w_{11} = -\frac{\lambda}{\lambda + 2\mu}(u_0^* + \frac{1}{2} w_0^{'2}). \]

Besides these relations it is found that the applied surface tractions must either be the same on both faces of the plate, \( S_0 = S_0 \), or both must be zero. Since only plates with loading on one face will be considered, then \( S_0 = 0 \).

The integration of the next equilibrium equations together with the next surface traction conditions results in the equations:

\[ u_2 = u_{30} + u_{31}x + u_{32}x^2, \quad w_2 = w_{30} + w_{31}x + w_{32}x^2, \]

\[ u_{21} = -w_{10}', \quad w_{21} = -\frac{\lambda}{\lambda + 2\mu}(u_{10}^* + w_0^{'w_{10}}'), \]

\[ u_{22} = -\frac{1}{2} w_{11}', \quad w_{22} = -\frac{\lambda}{2(\lambda + 2\mu)}(u_{11}^* + w_0^{'w_{11}}'), \]

with the additional conditions:

\[ S_{1}^1 = 0 \quad w_{11} = -\frac{\lambda}{4(\lambda + \mu)} \frac{w_0^{'2}}{r}. \]

This procedure then yields on the next set of equations the results:

\[ u_3 = u_{30} + u_{31}x + u_{32}x^2 + u_{33}x^3, \]

\[ w_3 = w_{30} + w_{31}x + w_{32}x^2 + w_{33}x^3, \]

with
\[ u_{31} = -w_{30} - h^2 \left[ \frac{4(\lambda + \mu)}{\lambda} w_{32} - \frac{w_{0} w_{11}}{r} \right] , \]

\[ u_{32} = -\frac{1}{2} w_{31} , \]

\[ u_{33} = \frac{3\lambda + 4\mu}{3\lambda} w_{32} - \frac{1}{3} \frac{w_{0} w_{11}}{r} , \]

\[ (7) \]

\[ w_{31} = -\frac{\lambda}{\lambda + 2\mu}(u_{30} + w_{0} w_{30} + \frac{1}{2} w_{10}) , \]

\[ w_{32} = -\frac{\lambda}{2(\lambda + 2\mu)}(u_{31} + w_{0} w_{31} + w_{10} w_{11}) , \]

\[ w_{33} = -\frac{\lambda}{3(\lambda + 2\mu)}(u_{32} + w_{0} w_{32} + \frac{1}{2} w_{11}) , \]

and the additional conditions:

\[ (8) \quad S_{2}^2 = 0, \quad w_{31}^i = \frac{\lambda}{2(\lambda + \mu)} \frac{w_{0} w_{10}}{r} . \]

Repetition of the above process gives \( u_{4} \) and \( w_{4} \).

Here it is found that \( S_{3} \) need not equal \( S_{3} \), and therefore the (7) equation yields significant results. The functions \( u_{4} \) and \( w_{4} \) are:

\[ (9) \]

\[ u_{4} = u_{40} + u_{41} \zeta + u_{42} \zeta^2 + u_{43} \zeta^3 + u_{44} \zeta^4 , \]

\[ w_{4} = w_{40} + w_{41} \zeta + w_{42} \zeta^2 + w_{43} \zeta^3 + w_{44} \zeta^4 , \]

with

\[ u_{41} = -w_{30} - h^2 \left[ \frac{4(\lambda + \mu)}{\lambda} w_{32} - \frac{w_{0} w_{11}}{r} \right] , \]

\[ u_{42} = \frac{3\lambda + 4\mu}{2\lambda} w_{31} - \frac{2w_{0} w_{30} + w_{10}}{2r} , \]

\[ u_{43} = \frac{3\lambda + 4\mu}{3\lambda} w_{32} - \frac{w_{0} w_{31} + w_{10} w_{11}}{3r} , \]
\[ u_{44} = \frac{3\lambda + 4\mu}{4\lambda} w_{33} - \frac{2w_{3}^2 + w_{11}^2}{12r} \]

\[ w_{41} = -\frac{\lambda}{\lambda + 2\mu} (u_{30}^* + w_{30}^* + w_{10}^* w_{30}) + \frac{\mu}{2(\lambda + 2\mu) S^3} \]

\[ w_{42} = -\frac{\lambda}{2(\lambda + 2\mu)} (u_{31}^* + w_{31}^* + w_{10}^* w_{31}) - \frac{\mu}{(\lambda + 2\mu)} (u_{31}^* + w_{30}^*) \]

\[ w_{43} = -\frac{\lambda}{3(\lambda + 2\mu)} (u_{32}^* + w_{32}^* + w_{10}^* w_{32} + w_{11}^* w_{31}) \]

\[ w_{44} = -\frac{\lambda}{4(\lambda + 2\mu)} (u_{33}^* + w_{33}^* + w_{11}^* w_{33}) - \frac{\mu}{4(\lambda + 2\mu)} (u_{33}^* + \frac{1}{3} w_{33}^*) \]

and the additional relations:

\[ w_{31} = \frac{\lambda}{4(\lambda + \mu)} \frac{2w_{3}^2 + w_{10}^2}{r} \]

\[ -h^2 \left[ w_{33} - \frac{2w_{3}^2 + w_{11}^2}{12(\lambda + 2\mu)} \right] \]

\[ (u_{31}^* + w_{30}^*) h + (u_{33}^* + \frac{1}{3} w_{33}^*) h^3 = \frac{1}{2\mu} S^3. \]

Since negative values of \( S \) correspond to pressure and \( P \) is the uniform normal loading on the plate, then \( S^3 \) must be replaced by \( -P \).

By reducing the order of the subscripts in (12) one obtains

\[ (w_{0}^* - w_{0} w_{11})^* = -\frac{\lambda + 2\mu}{2(\lambda + \mu)} \left[ \frac{w_{3}^2 w_{11}^2}{r} \right]^* = -\frac{5(\lambda + 2\mu)}{3\mu(\lambda + \mu) h^3} S^3 = \frac{P}{N}, \]

where \( N \) is the modulus of rigidity.
The elastic constants $\lambda$ and $\mu$ are related to $E$ and $\nu$ by:

$$\lambda = \frac{E \nu}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)}.$$

The transposition of terms and the changing of the elastic constants in equation (13) gives:

$$\text{(14)} \quad w''_o = \frac{P}{N} + \frac{1}{r} \frac{d}{dr} \left[ (1 - \nu)w'_o w_{11} + r(w'_o w_{11})' \right].$$

The other non-linear relationship is given by equation (5) which with the change of elastic constants becomes

$$\text{(15)} \quad w'_{11} = \nu \frac{w'_o}{2}.$$

If the non-linear term in the unit strain were omitted, then equations (14) and (15) would reduce to the elementary differential equation of bending:

$$N\Delta^4 w = P.$$

Comparison with v.Karman's non-linear differential equations:

$$\Delta^4 F = -E \frac{1}{r} \frac{dw}{dr} \frac{d^2 w}{dr^2},$$

$$N\Delta^4 w = P + h \frac{1}{r} \frac{d}{dr} \left[ \frac{1}{r} \frac{dF}{dr} \frac{dw}{dr} \right],$$

shows that equation (15) is exactly equivalent to the first of these. However the system (14) and (15) possesses an advantage over v.Karman's in that the former may be reduced to an equation in $w_o$ directly. Thus the system for moder-
ately thick plates can be put in the form

\[ \Delta w_0 = \frac{P}{N} + \frac{\nu}{2r} \frac{d}{dr} \left[ (1 - \nu) \frac{w_0^3}{r} + r \left( \frac{w_0^{13}}{r^2} \right) \right]. \]

The solution of equation (16) gives the deflection of the middle plane of the plate and therefore by equation (15) the value of \( w_{11} \). This latter leads to \( u_0 \), the stretching of the middle plane, by reason of equation (3). Higher approximations can be gotten by carrying the process further. For instance, the differential equation for \( w_{10} \) can be obtained by continuing the procedure to \( w_s \) and \( u_s \).

In order to solve (16) one must resort to approximation methods. Suppose that \( w_0 \) is replaced by \( r q \), where \( q \) is an odd function. If \( q \) is developed in powers of a parameter \( \varepsilon \),

\[ q = q_1 \varepsilon + q_3 \varepsilon^3 + \cdots, \]

and this expansion put in equation (16), a series of equations ordered in the parameter results. If (16) is integrated once and the constant of integration taken as zero in order to satisfy the condition that the shear must be finite at the origin, the sequence of equations is:

\[ \varepsilon^1 \left( r q_1 \right)' = \frac{Pr}{2N}, \]

\[ \varepsilon^3 \left( r q_3 \right)' = \frac{\nu(1 - \nu)}{2} r q_1^3 + \frac{\nu}{2} (r^2 q_1^3)', \]

\[ \varepsilon^5 \left( r q_5 \right)' = \frac{\nu(1 - \nu)}{2} 3r q_1^2 q_3 + \frac{3\nu}{2} (r^2 q_1 q_3)', \]

etc.
It is necessary to assume that \( P \) is of order \( \varepsilon^1 \) so that the first approximation shall give the solution of the elementary equation of bending. The differential operator on \( rq \) may be written

\[
(rq)'' = \frac{1}{r} \frac{d}{dr} \left[ r^3 \frac{dq}{dr} \right].
\]

The solution of the \( \varepsilon^1 \) equation in (18) is

\[
(19) \quad q_1 = \frac{Fr^2}{16N} - \frac{c_1}{2r^2} + c_2.
\]

Since the boundary conditions for the clamped plate are

\[
(20) \quad w'(0) = w'(a) = 0 \quad \text{and} \quad w(a) = 0,
\]

then the first approximation becomes

\[
(21) \quad q_1 = \frac{P}{16N}(r^2 - a^2).
\]

The second equation in (18) is now

\[
(22) \quad (r\dot{q}_3)'' = \frac{\nu}{2} \left[ \frac{P}{16N} \right]^3 \left[ (3-\nu)r(r^2 - a^2)^3 + 6r^3(r^2 - a^2)^3 \right],
\]

and the solution satisfying the boundary conditions is

\[
(23) \quad q_3 = \frac{\nu}{160 \left[ \frac{P}{16N} \right]^3} \left[ 9r^6 - 35a^2 r^6 + 50a^4 r^4 - 30a^6 r^2 + 6a^6 \\
- \nu(r^6 - 5a^2 r^6 + 10a^4 r^4 - 10a^6 r^2 + 4a^6) \right].
\]

Now if \( rq \) is replaced by \( w'_0 \), then using the boundary condition \( w(a) = 0 \) and letting \( \varepsilon \) take the value unity, the second approximation for \( w_0 \) is
\[(24) \quad w_0 = \frac{P}{64N}(r^2 - a^2)^2 + \frac{v}{160[16N]}\left[ \frac{9}{10}a^6 r^6 - \frac{25}{2}a^4 r^4 + 15a^2 r^2 \right.
- \left( \frac{1}{10}a^6 r^6 + \frac{2}{5}a^4 r^4 - \frac{5}{2}a^2 r^2 \right)
- \frac{43 - 77v}{120} a^6] .\]

The deflection at the center of the plate for \( v = 0.3 \) is

\[(25) \quad w_0(0) = \frac{Pa^4}{64N} - \frac{v(43 - 77v)}{300 a^2} \left[ \frac{Pa^4}{64N} \right]^3 .\]

The solution of the third equation in (18) gives rise to a polynomial of the fourteenth degree in \( r \). The general solution for \( w \) can be put in the form

\[(26) \quad w_0 = a f_4(r) + \beta f_{10}(r) + \gamma f_{16}(r) + \cdots ,\]

\( a, \beta, \gamma \) being coefficients dependent on the elastic constants and the loading, and \( f_n \) denoting a polynomial of degree \( n \). Successive approximations increase the degree of the polynomial by six. Vincent [14] used a method of successive approximations to solve v.Karman's equations for the simply supported thin plate. His results have the form

\[(27) \quad w = a'f_4(r) + \beta'f_{12}(r) + \gamma'f_{20}(r) + \cdots ,\]

each successive approximation increasing the degree of the polynomial by eight.

Now the ratio of the center deflection to the plate
thickness for $h/a = 10^{-1}$ is from (25)

$$w_c(0)/h = 0.17 \frac{Pa^4}{Eh^4} - 10^{-6}\left[\frac{Pa^4}{Eh^4}\right]^3.$$  

Therefore $Pa^4/Eh^4$ has to be of the order $10^2$ before any substantial correction to the elementary solution occurs. On the other hand, when $Pa^4/Eh^4$ is approximately 1 there is an appreciable difference between the elementary solution and the solution of the v. Karman equations for thin plates.

From equations (3) and (5) the differential equation for $u_c$ is found to be

$$u_c^{xx} = -\frac{1}{2}(w_c^{12})^2 + \frac{\nu}{2} \frac{w_c^{12}}{r}.$$

As a first approximation the value of $w_c$ obtained from the elementary solution will be used. Then (29) becomes

$$u_c^{xx} = -\frac{1}{2}\left[\frac{P}{16N}\right]^2[(3-\nu)r(r^2-a^2)^2 + 4r^3(r^2-a^2)].$$

The general solution of this equation is

$$u_c = -\frac{1}{96}\left[\frac{P}{16N}\right]^2[(3-\nu)(r^6 - 4a^2r^4 + 6a^4r^2) + \frac{c_1}{r^2} + c_2 + 4r^6 - 3a^2r^4],$$

where $c_1$ and $c_2$ are constants of integration to be selected to make $u_c$ satisfy the boundary conditions. These are

$$u(0) = 0 = u(a).$$

It is readily seen that one must take $c_1 = 0$ because
u must be finite at the center of the plate, but \( c_2 \) can be determined so as to satisfy either one of the boundary conditions but not both. From the physical standpoint it is more reasonable to satisfy the condition at the center of the plate than that at the edge. Therefore \( c_2 \) must vanish, and \( u_o \) is then given by (31) with both constants zero. The result of this selection of the boundary conditions is equivalent to the assumption that rigid clamping of the edge of the plate against lateral displacements is impossible.
V. DISCUSSION

The study of the stability of the clamped thin plate, though not complete, offers an interesting result. Aside from a proportionality factor the function \( p \) represents the radial compressive membrane stress. From Figure 3 it may be observed that on the second approximation the compressive stress decreases at the center of the plate and increases at the edge. In fact, Friedrichs [22] has found in the case of the simply supported plate that eventually the stress at the center becomes tension while at the edge the stress becomes an increased compression.

The form of the deflection surface for the second approximation depends on the evaluation of

\[
\int r^n_1 j^p_1 dr \quad \text{and} \quad \int r j^p_1 dr,
\]

and as mentioned before these were not integrated. As a last resort one might use the series representations for the integrands and integrate these term by term. However, a
rough calculation shows that it would be necessary to take
about fifteen terms of the expansion for \( J_1^* \) before satisfac-
tory convergence could be obtained. Since the result of such
a procedure is to be used in evaluating the constant \( k \) and
later the functions \( p_4 \) and \( q_4 \), and higher terms, the use
of the series representations would entail a great deal of
laborious computation. Until this is done, however, very
little can be said concerning the convergence of the method of
approximation and the ultimate form of the deflection surface.

The second problem, the bending of moderately thick cir-
cular plates, presents some interesting conclusions. Signif-
icantly the same equation for the radial displacement is ob-
tained as in v.Karman's derivation for the thin plate. From
this it may be concluded that the horizontal static equilibrium
of a plate element is included in the three-dimensional ap-
proach to the problem of the moderately thick plate. In
v.Karman's equations the vertical resolution of the forces
includes the vertical components of the membrane stresses, and
it is this fact which explains to a great extent that thin
plates can bear much greater loads for a given deflection than
is predicted by the elementary theory. This consideration of
the vertical components of the membrane stresses is lacking in
the formulation for the moderately thick plate. It appears
as though the terms additional to the elementary solution for
the bending of the middle plane in the case of the moderately
thick plate were due entirely to the non-linear term in the strains, any contributions by the vertical shearing stresses being of higher order. An equation for the bending of the middle plane which included the effects of the non-linear strains and the vertical components of the membrane stresses can be gotten by superposing the result of v.Karman's equations on that for the moderately thick plate.
VI. SUMMARY

1. The stability of the clamped circular thin plate is investigated beginning with v. Karman's non-linear differential equations for large strains. This non-linear system is developed into two linear sequences of equations ordered in powers of a parameter. The first approximation is found to yield the elementary solution of the problem. The second approximation of the radial membrane stress is shown to decrease the compressive stress in the center of the plate and increase it at the edge. The method for removing the indeterminacy of the second approximation to the slope function is outlined but not carried out because the explicit form of the particular integral free of integrals was not obtained.

2. A parameter expansion, due to Garabedian, is employed to order the three-dimensional equilibrium equations and surface traction conditions for the moderately thick circular plate. From these equations the differential equations for the deflection and the stretching of the middle plane of the plate are derived for large strains. A method of successive approximations is then used to solve the non-linear differential equation for the case of the clamped plate with uniform loading. The correction to the elementary solution for the deflection is found to be significant only for extremely large
values of the loading. The solution for the stretching of the middle plane of the plate, using the elementary value of the deflection, is found to be possible only under the assumption that rigid clamping of the edge of the plate against lateral displacements is impossible.
VII. LITERATURE CITED


22. Friedrichs, K. Private communication. 1939.
