1991

Scaling behavior of deposition models with limited downward mobility

H. C. Kang
*Iowa State University*

James W. Evans
*Iowa State University, evans@ameslab.gov*

Follow this and additional works at: [http://lib.dr.iastate.edu/physastro_pubs](http://lib.dr.iastate.edu/physastro_pubs)

Part of the [Atomic, Molecular and Optical Physics Commons](http://lib.dr.iastate.edu/atomicmoloptphyscomm)

The complete bibliographic information for this item can be found at [http://lib.dr.iastate.edu/physastro_pubs/403](http://lib.dr.iastate.edu/physastro_pubs/403). For information on how to cite this item, please visit [http://lib.dr.iastate.edu/howtocite.html](http://lib.dr.iastate.edu/howtocite.html).

This Article is brought to you for free and open access by the Physics and Astronomy at Iowa State University Digital Repository. It has been accepted for inclusion in Physics and Astronomy Publications by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.
Scaling behavior of deposition models with limited downward mobility

Abstract
We have studied the interface-width scaling behavior of deposition models in which limited downward mobility is introduced. In all the models studied here, there is a maximum allowed slope for the interface at which the growth velocity is zero. The Kardar-Parisi-Zhang equation is resummed in order to show explicitly this slope constraint, but still incorporates parameters \( \lambda \) measuring the slope dependence of the growth velocity, \( \nu \) measuring the surface tension, and \( D \) measuring the noise amplitude. Increasing mobility naturally smooths the interface, and is associated with a decrease in the magnitude of \( \lambda_{\text{eff}} = \lambda - D^{1/2}/\nu^{3/2} \). A detailed study of a bridge-site deposition model, with one hop of probability \( p \), shows that \( \lambda \) increases with \( p \). From an independent assessment of noise-amplitude behavior, we can conclude that \( \nu \) must also increase with \( p \) to ensure the required \( \lambda_{\text{eff}} \) behavior. Direct determination of \( \nu \) via the Wolf-Tang procedure of imposing inhomogeneity on some length scale \( L \) supports this conclusion. (However \( \nu \) depends on the inhomogeneity strength, which should be chosen small, and on \( L \).) We comment on anticipated behavior of similar limited-mobility models in \( d \geq 2 \) dimensions, and compare with behavior of limited-mobility ballistic deposition and noise-reduced deposition models.

Disciplines
Atomic, Molecular and Optical Physics | Physics

Comments

This article is available at Iowa State University Digital Repository: http://lib.dr.iastate.edu/physastro_pubs/403
Scaling behavior of deposition models with limited downward mobility

H. C. Kang
Department of Chemistry and Ames Laboratory, Iowa State University, Ames, Iowa 50011

J. W. Evans
Department of Mathematics and Ames Laboratory, Iowa State University, Ames, Iowa 50011

(Received 28 February 1991; revised manuscript received 6 May 1991)

We have studied the interface-width scaling behavior of deposition models in which limited downward mobility is introduced. In all the models studied here, there is a maximum allowed slope for the interface at which the growth velocity is zero. The Kardar-Parisi-Zhang equation is resummed in order to show explicitly this slope constraint, but still incorporates parameters $\lambda$ measuring the slope dependence of the growth velocity, $\nu$ measuring the surface tension, and $D$ measuring the noise amplitude. Increasing mobility naturally smooths the interface, and is associated with a decrease in the magnitude of $\lambda_{\text{eff}}=\lambda D^{1/2}/\nu^{1/2}$. A detailed study of a bridge-site deposition model, with one hop of probability $p$, shows that $|\lambda|$ increases with $p$. From an independent assessment of noise-amplitude behavior, we can conclude that $\nu$ must also increase with $p$ to ensure the required $\lambda_{\text{eff}}$ behavior. Direct determination of $\nu$ via the Wolf-Tang procedure of imposing inhomogeneity on some length scale $L$ supports this conclusion. (However $\nu$ depends on the inhomogeneity strength, which should be chosen small, and on $L$.) We comment on anticipated behavior of similar limited-mobility models in $d \geq 2$ dimensions, and compare with behavior of limited-mobility ballistic deposition and noise-reduced deposition models.

I. INTRODUCTION

Far-from-equilibrium evolution of surface or interface profiles has been the subject of much recent study [1]. Often discrete (lattice) deposition models are invoked, which in the simplest case allow no restructuring or mobility of atoms between adsorption sites following deposition. Such models include irreversible random deposition at atom sites (for a simple-cubic geometry) [2], at bridge sites [for a one-dimensional substrate] [3], at fourfold-hollow (4hf) sites [for a fcc(100) substrate] [3]; and single-step models [4,5], restricted solid-on-solid (RSOS) models [6], and lattice models of ballistic deposition [1]. Not surprisingly these models produce "rough" interface profiles, as we quantify below.

Our interest here is in the smoothing effect that occurs if one modifies the above models to allow atoms to make a single hop laterally or downward to a lower adsorption site, with probability $p$, immediately following deposition. (Thus $p=0$ recovers the simple immobile models.) One might think of such motion as "transient mobility" associated with the inability of the deposited atom to instantaneously dissipate the kinetic energy gained upon formation of the atom-surface bond [7]. Alternatively, it might reflect the effects of the onset of thermal mobility in systems for which layer-by-layer growth is thermodynamically favored, and in which deposition is essentially immobile at low temperatures [7].

Monte Carlo simulation of these models provides direct information on interface evolution. However, to elucidate the observed behavior, it is invaluable to consider a "coarse-grained" continuum description of the process. In such a description, one postulates that the evolution of the interface height $h(r,t)$ at lateral position $r$ and time $t$ is described by a stochastic Kardar-Parisi-Zhang (KPZ) equation of the form [8,9]

$$\frac{\partial h}{\partial t}=\kappa+\lambda|\nabla h|^2/2+\nu\nabla^2 h+\cdots+\eta.$$  \hspace{2cm} (1)

Here we have absorbed the microscopic-model deposition rate into the time scale. Also $\lambda$, if nonzero, reflects a slope-dependence of the propagation velocity, $v$ represents a surface-tension induced by the deposition dynamics (rather than energetics), and implicit terms involve higher derivatives. The noise term $\eta$ satisfies $\langle \eta \rangle=0$ and $\langle \eta(r,t)\eta(r',t') \rangle=2\delta(r-r')\delta(t-t')$. Simple transformation of Eq. (1) shows that the behavior of the solution is determined by the single parameter $\lambda_{\text{eff}}=\lambda D^{1/2}/\nu^{1/2}$ [10].

Of primary interest here is the asymptotic behavior of the interface width or roughness

$$w=\langle (h-\langle h \rangle)^2 \rangle^{1/2} \sim t^\beta$$ or $\langle h \rangle$ as $t \to \infty$, \hspace{2cm} (2)

for one- or two-dimensional substrates of infinite extent (i.e., for $d=1+1$ or $d=2+1$ dimensional versions of the above models). To this end, we shall exploit the following result for Eq. (1). If $\lambda=\nu=0$, then trivially $\beta=\frac{1}{2}$ for all $d \geq 2$. If $\lambda=0$ but $\nu \neq 0$, Eq. (1) reduces to the linear Edwards-Wilkinson (EW) equation [11] from which it readily follows that $\beta=\frac{1}{4}$ when $d=1+1$, and $\beta=0$ ($w \sim \ln t$) for $d=2+1$. In contrast, if both $\lambda \neq 0$ and $\nu \neq 0$, then $\beta=\frac{1}{3}$ when $d=1+1$, and $\beta=\frac{1}{2}$ when $d=2+1$. At least in the latter case, one should question whether associated microscopic behavior is completely described by (1) since the microscopic models sometimes exhibit a transition to a smooth $\beta=0$ phase for
sufficiently small $|\lambda_{eff}|$, whereas Eq. (1) apparently does not [10]. We now return to the consideration of smoothing effects of transient mobility in deposition models.

Consider first trivial irreversible deposition at atom sites. This leads to the growth of independent columns with a Poisson distribution of heights. Here the surface is very rough with $\beta=\frac{1}{2}$ (corresponding to the case $\lambda=\nu=0$ above). Family [12] noted that introducing a single hop (with $p=1$) to this model produces dramatically smoother EW growth. The resultant coupling between columns implies $\nu\neq0$, but still $\lambda=0$, since the mean growth velocity is slope independent.

Next consider conventional ballistic deposition on a lattice with $p=0$ [1]. Here one finds that the compactness or density of the growing film decreases with increasing $Vh$. Thus $\lambda>0$, since the product of compactness and growth velocity is constant in this model. Consequently, the $p=0$ model exhibits KPZ behavior. Pelligrini and Juillien [13] have considered in detail the smoothing effect of allowing a downward hop with probability $0<p<1$, anticipating that $\lambda=\lambda(p)\geq0$ should decrease with increasing $p$. They found that the model exhibits a phase transition to smooth growth above $p_c=0.63$ (0.74) in $d=3$ (4). However, in agreement with known KPZ equation behavior, there is no phase transition in $d=2$ except for $p=1$, where $\lambda=0$. From a practical perspective, even for $d=2$, increasing $p$ results in a decrease in the effective $\beta$. Of course, at sufficiently long times, there must be a crossover to the higher KPZ value of $\beta=\frac{1}{2}$. This behavior for $d=2$ is displayed in more detail in Appendix A. We emphasize that the smoothing effects seen in this model naturally result from the decrease in $\lambda$ with increasing $p$.

In this contribution, we focus on the smoothing effects of hopping in models involving random deposition at bridge and 4fh sites, and in the RSOS models. We note that the single-step model [4,5] in $d=2$ ($d=3$) is equivalent to random deposition at bridge (4fh) sites with $p=0$ [14]. All of these models are distinguished by the fact that $\langle Vh \rangle$ has a maximum value (for each lateral direction). For a surface with the maximum slope, growth is not possible. From this observation, one can correctly anticipate that $\lambda<0$ for all these models. Thus $\beta$ is given by the larger KPZ value. In the following sections, we analyze in detail the effect of introducing one lateral or downward hop with probability $p>0$. One observes the anticipated smoothing effect, i.e., a decrease in the effective $\beta$ with $p$. However, we also find that $|\lambda|$ increases with $p$, in contrast to the ballistic deposition case. This behavior is elucidated in detail.

In Sec. II, we first discuss a resummed form of the KPZ equation appropriate to models with a slope constraint. Our emphasis is on the extraction of the parameters $\lambda$, $\nu$, and $D$ for various microscopic models. In Sec. III we provide a detailed analysis of the bridge-site model with one hop, and comment on the similar behavior of the $d=2$ RSOS model in Sec. IV. For comparison, we analyze a noise-reduced version of the bridge-site model with $p=0$ in Sec. V. Finally we discuss the behavior of corresponding $d=3$ models in Sec. VI, and summarize our findings in Sec. VII.

II. RESUMMED KPZ EQUATION FOR MODELS WITH A SLOPE CONSTRAINT

Since growth is completely eliminated for interfaces with the maximum slope in each lateral direction, one can resum the KPZ equation into the form

$$\frac{\partial h}{\partial t} = f(\nabla h)(1 + v\nabla^2 h / \kappa + \lambda_2 |\nabla h|^2 / 2\kappa + \cdots + \eta'),$$

(3)

where $f$ is even in $\nabla h$ and vanishes at the maximum slope for each lateral direction. If $f = \kappa + \lambda_1 |\nabla h|^2 / 2$, for small $|\nabla h|$, then Eq. (3) recovers Eq. (1) to the lowest order with the identification $\lambda = \lambda_1 + \lambda_2$. We recall that asymptotic scaling is determined only by the lowest-order terms made explicit in Eq. (1). Equation (3) correctly incorporates the property that the noise is also quenched as $\nabla h$ approaches its maximum value [15]. Our primary interest here is in the extraction of parameters $\lambda$, $\nu$, and $D$ from data from various specific lattice models.

The procedure for determination of $\lambda$ is relatively straightforward and has been implemented previously [16]. In the lattice model, we start with a perfect vicinal surface with nonzero constant slope $\nabla h_0$. Then $\langle \nabla h \rangle = \nabla h_0$ for all later times, and one can readily determine the asymptotic mean growth velocity in the form

$$\left\langle \frac{\partial h}{\partial t} \right\rangle = F(\langle \nabla h \rangle) = -\kappa' + \lambda' \langle \nabla h \rangle^2 / 2.$$

(4)

If ensemble fluctuations are small, as is the case for large coarse-graining lengths [15,17], then $\langle f(\nabla h) \rangle = f(\langle \nabla h \rangle)$, etc., so one can identify $\kappa = \kappa'$ and $\lambda = \lambda'$. See Ref. [18] for a slightly different approach.

 Determination of $\nu$ is more complicated. Wolf and Tang [18] developed a technique based on the idea of retarding growth at designated sites in $d=2$ (or lines of sites in $d=3$), $x = x_i$, say, where $x$ is some lateral coordinate. For an infinite system, they are naturally chosen to be periodically separated so $x_{i+1} - x_i = L$, say. The shape of the resulting periodic mean profile $\langle h \rangle(x)$ provides information on $\nu$ [see Fig. 1(a)]. Alternatively, one can retard growth at a single designated site $x_1$, say, on a finite lattice of size $L$ with periodic boundary conditions [Fig. 1(b)]. In both cases, one is imposing an inhomogeneity on a length scale $L$. Here we implement retardation by re-

![FIG. 1. Schematic of the mean interface profile for (a) an infinite system with deposition periodically retarded at $x = x_i$, and (b) a finite system with deposition retarded at $x_1$. The slope is zero at peak positions indicated by $x^*$.](image)
moving, with probability \( g \), any particle that ends up at a designated site (either indirectly after hopping, or by direct adsorption if failing to hop). For an alternative prescription see Ref. [19]. Thus \( g = 0 \) corresponds to unperturbed growth, and for \( g = 1 \) growth is pinned at all \( x = x_i \).

This has the effect of adding a term \( Q(g) \sum \delta (x - x_i) \) inside the parentheses in the resummed KPZ equation (3). Clearly \( Q(0) = 0 \) and \( Q(g) \) decreases monotonically with \( g \) to \( Q(1) = -2v|\nabla h_{\text{max}}|/\kappa \), where \( |\nabla h_{\text{max}}| \) denotes the maximum slope orthogonal to the lines of designated sites (see Appendix B). We note here that this choice of \( Q(1) \) guarantees that the profile described by the resummed KPZ equation evolves into the appropriate pinned V-shaped profile of maximum slope when \( g = 1 \). One other important observation based solely on our simulation data is that \( \langle h \rangle(x) \) appears to have a simple slope discontinuity [of \( \kappa Q(g)/v \)] at each \( x_i \) (see Fig. 1). This implies that \( v \neq 0 \), so the \( \nabla^2 h \) term is present, but that higher-order terms like \( \nabla^4 h, \nabla^6 h \ldots \) are absent or insignificant in Eq. (3).

The procedure for determining \( v \), and therefore its \( p \) dependence, from simulations is as follows. One first determines the difference \( \Delta \equiv 0 \) in the mean growth velocity between \( g = 0 \) (no retardation) and some other \( g < 1 \) (see below). From Eq. (3), this difference equals \( v(\partial^2 h / \partial x^2)(x^*) \), where \( x^* \) is some peak in the profile where \( \langle \partial h / \partial x \rangle = 0 \) (see Fig. 1). This result relies on the absence of \( \nabla^4 h, \nabla^6 h \ldots \) terms in Eq. (3), and again assumes small fluctuations in derivatives of \( h \) about their mean values. Thus \( v \) is determined by

\[
v = \frac{\Delta}{\langle \partial^2 h / \partial x^2 \rangle (x^*)} > 0.
\]  

Reference [18] provides a derivation of an analogous formula using (1). Our derivation has used the full equation (3), since strictly one can only neglect higher-order terms in a conventional asymptotic analysis.

A previously recognized feature of this procedure is that the resulting values for \( v \) depend on the length scale \( L \) of the imposed inhomogeneity [18]. This dependence on \( L \) occurs for fixed “coarse-grained” length and time scales (i.e., for \( \delta x = 1 \) corresponding to a fixed number of lattice spacings). It is distinct from the scaling of \( v \) with length and time scales renormalized using appropriate scaling exponents [8], or renormalized simply by a common multiplicative factor. (In both cases, \( \lambda \) is invariant.)

We shall describe below how estimates of \( v \) are “more corrupted” for smaller \( L \) if one uses an infinite system with periodically retarded growth sites, compared to a finite system with a single retarded site (and the same \( L \)). We shall also show that \( v \) depends on the strength of the imposed inhomogeneity \( g \), which should be chosen small.

Finally we emphasize that our primary goal is to estimate comparative values of \( \lambda_{\text{eff}} \) for various \( p \). Estimates for \( \lambda \) are straightforward, and for \( v \) are more problematic. Noise-amplitude behavior is determined from the steady-state value of the interface width in finite systems, as will be described in detail in the following sections for \( (d = 1 + 1) \)-dimensional models.

**III. THE \( d = 1 + 1 \) BRIDGE-SITE MODEL WITH ONE HOP**

In the immobile bridge-site model growth occurs by random deposition at bridge sites, i.e., only particles impinging upon bridge sites are adsorbed [3]. Smoothing effects can be included in the bridge-site model by allowing the particles to hop with probability \( p \) after impinging upon a bridge site [14]. We implement this as follows (see Fig. 2). If a hop is not attempted, the particle adsorbs where it lands. If a hop is attempted, the direction of the hop is decided as follows. If the interface slopes upward in both directions away from the bridge site, the attempted hop is not successful and the particle adsorbs where it lands. If the interface slopes downward or is flat in one or both directions, then such a direction is chosen at random and the particle migrates until it arrives at the nearest bridge site, where it is then adsorbed. As a result of this mobility it can be seen that the growth rate is higher at the bottoms of “valleys” than at the tops of “hills”. The interface is, thus, expected to be smoother when the probability for hopping \( p \) is higher. We examine this model in the language of the KPZ equation in order to elucidate the effects of hopping upon the parameters \( \lambda, \nu, \) and \( D \).

The dependence of the interface width upon the average interface height is plotted in Fig. 3. The value of \( p \) for each curve is indicated in the legend. It is found that as the value of \( p \) increases from zero the effective value of the exponent \( \beta \) changes from \( \frac{1}{2} \) toward \( \frac{1}{3} \). This indicates that as \( p \) increases, the parameter \( \lambda_{\text{eff}} = \lambda D^{1/2} / \nu^{3/2} \) decreases. However, it is not immediately clear how each of the factors in \( \lambda_{\text{eff}} \) changes with \( p \). From the \( d = 1 + 1 \) results of renormalization-group analysis of the KPZ equation (8), we expect a crossover to \( \beta = \frac{2}{3} \) for all values of \( p \) if the interface is allowed to grow for sufficiently long times.

By performing the simulations on a sloped surface, it is easy to obtain the dependence of the mean growth velocity upon the average slope \( \langle \nabla h \rangle \). Since particles impinge only upon bridge sites, the asymptotic mean growth velocity \( \langle \partial h / \partial t \rangle_{\infty} \) is simply the bridge-site density. We have defined the length scales so that the velocity of a microscopically flat surface is unity and the maximum slope is unity. In Fig. 4, we plot the mean growth velocity \( \langle \partial h / \partial t \rangle_{\infty} \) as a function of the squared average interface slope \( \langle \nabla h \rangle^2 \) for \( p \) equal to \( 0, \frac{1}{2}, \) and \( 1 \). It should be noted that for the case in which \( p \) is equal to zero, one has the exact behavior \( \langle \partial h / \partial t \rangle_{\infty} = (1 - \langle \nabla h \rangle^2)^2 / 2 \) [15,20]. Our simulation data agree well with this result. For each

![FIG. 2. Schematic of the bridge-site model with one hop.](image-url)
value of the slope \(\langle \nabla h \rangle\), the mean growth velocity increases with increasing \(p\), since this corresponds to smoother interfaces and, consequently, higher bridge-site densities. However, at the maximum allowed slope the mean growth velocity must be zero regardless of the value of \(p\). Assuming that the mean growth velocity is a monotonic function of \(\langle \nabla h \rangle^2\), we expect these plots to have progressively more negative slopes at \(\langle \nabla h \rangle = 0\) for progressively larger values of \(p\), i.e., \(\lambda\) should become increasingly negative with increasing \(p\). This behavior is confirmed in Fig. 4, from which we find that \(-\lambda = 0.5, 0.79,\) and 1.10 for \(p = 0, \frac{1}{2},\) and 1, respectively. Thus, unlike the ballistic deposition model, changes in \(\lambda\) do not facilitate the decrease of \(|\lambda_{\text{eff}}|\) as \(p\) increases.

The behavior of the noise amplitude \(D\) can be elucidated by exploiting the observation that the steady-state value of \(w^2\), for a finite system of \(L\) sites, is proportional to \(LD/\nu\) in \(d = 1+1\) dimensions [21]. Thus steady-state behavior is independent of \(\lambda\). Simulations produce steady-state values for \(w^2/L\) in the ratio of 1:0.39:0.20 for \(p\) values 0:1:1. Thus using our above results for \(\lambda\), one finds values for \(-\nu\lambda_{\text{eff}} = -\lambda(D/\nu)^{1/2}\) in the ratio of 1:0.99:0.98 for \(p\) values 0:1:1. Consequently, \(\nu\) must increase with \(p\) to achieve the desired decrease in \(\lambda_{\text{eff}}\).

As noted in Sec. II, the surface tension \(\nu\) can be obtained from simulations of a system in which the deposition rate is not homogeneous. This is done as follows. We use lattices of \(L\) sites with \(L\) ranging from 10 to 50. Periodic boundary conditions are imposed. The deposition rate for the one designated site on the lattice is retarded relative to the deposition rate for the other sites by removing, with a probability \(g\), each particle that ends up there. We used \(g = 0.125, 0.5,\) and 0.875. By measuring the curvature of the interface at \(x = x^*\) and the average growth velocity, we can determine the surface tension \(\nu\) using Eq. (5). The results of these simulations are summarized in Tables I–III. Note that these \(\nu\) estimates, together with the above data for \(w^2/L \ll D/\nu\), show that \(D\) does not increase with \(p\).

When \(g\) is equal to unity, there is complete retardation of the growth rate at the designated site. Since there is a maximum slope in the systems that are studied here, it is clear that zero growth at any one site pins the entire interface and causes growth to cease completely when the interface reaches a V shape. Thus for the case where \(g = 1\), the curvature is infinite at \(x^*\) and the surface tension is zero. We have shown that \(\nu\) is a function of \(g\) in Fig. 5. For any value of \(g\), the surface tension increases

**TABLE I.** Results of simulations with periodically retarded deposition rate \((g = 0.125)\) for the bridge-site model with one hop. Uncertainties in \(\Delta\) and \(\nu\) estimates are large here. The average interface height is approximately 2500 monolayers in all cases.

<table>
<thead>
<tr>
<th>(L)</th>
<th>(-\left(\frac{\partial^2 h}{\partial x^2}\right)(x^*))</th>
<th>(-\Delta)</th>
<th>(\nu = \Delta/\left(\frac{\partial^2 h}{\partial x^2}\right)(x^*))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p = 0, \lambda = -0.50)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.0058</td>
<td>0.005</td>
<td>0.9</td>
</tr>
<tr>
<td>20</td>
<td>0.0018</td>
<td>0.003</td>
<td>1.6</td>
</tr>
<tr>
<td>25</td>
<td>0.0015</td>
<td>0.001</td>
<td>0.7</td>
</tr>
<tr>
<td>40</td>
<td>0.00068</td>
<td>0.001</td>
<td>1.5</td>
</tr>
<tr>
<td>50</td>
<td>0.00064</td>
<td>0.001</td>
<td>1.6</td>
</tr>
<tr>
<td>(p = \frac{1}{2}, \lambda = -0.79)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.0027</td>
<td>0.020</td>
<td>7</td>
</tr>
<tr>
<td>20</td>
<td>0.0017</td>
<td>0.012</td>
<td>12</td>
</tr>
<tr>
<td>25</td>
<td>0.00083</td>
<td>0.006</td>
<td>7</td>
</tr>
<tr>
<td>40</td>
<td>0.00041</td>
<td>0.004</td>
<td>10</td>
</tr>
<tr>
<td>50</td>
<td>0.00039</td>
<td>0.004</td>
<td>10</td>
</tr>
<tr>
<td>(p = 1, \lambda = -1.1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.0016</td>
<td>0.028</td>
<td>18</td>
</tr>
<tr>
<td>20</td>
<td>0.00057</td>
<td>0.010</td>
<td>18</td>
</tr>
<tr>
<td>25</td>
<td>0.00057</td>
<td>0.009</td>
<td>16</td>
</tr>
<tr>
<td>40</td>
<td>0.00029</td>
<td>0.004</td>
<td>14</td>
</tr>
<tr>
<td>50</td>
<td>0.00025</td>
<td>0.003</td>
<td>12</td>
</tr>
</tbody>
</table>
TABLE II. Results of simulations with periodically retarded deposition rate \( g = 0.5 \) for the bridge-site model with one hop. The average interface height is approximately 2500 monolayers in all cases.

\[
L \quad -\left[\frac{\partial^2 h}{\partial x^2}\right](x^*) \quad -\Delta \quad v = \Delta \left[\frac{\partial^2 h}{\partial x^2}\right](x^*)
\]

\[(p = 0, \lambda = -0.50)\]

<table>
<thead>
<tr>
<th></th>
<th>(-\left<a href="x%5E*">\frac{\partial^2 h}{\partial x^2}\right</a>)</th>
<th>-(\Delta)</th>
<th>(v = \Delta \left<a href="x%5E*">\frac{\partial^2 h}{\partial x^2}\right</a>)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.090</td>
<td>0.051</td>
<td>0.6</td>
</tr>
<tr>
<td>20</td>
<td>0.047</td>
<td>0.037</td>
<td>0.8</td>
</tr>
<tr>
<td>25</td>
<td>0.036</td>
<td>0.035</td>
<td>1.0</td>
</tr>
<tr>
<td>40</td>
<td>0.028</td>
<td>0.031</td>
<td>1.1</td>
</tr>
<tr>
<td>50</td>
<td>0.022</td>
<td>0.029</td>
<td>1.3</td>
</tr>
</tbody>
</table>

\[(p = \frac{1}{2}, \lambda = -0.79)\]

<table>
<thead>
<tr>
<th></th>
<th>(-\left<a href="x%5E*">\frac{\partial^2 h}{\partial x^2}\right</a>)</th>
<th>-(\Delta)</th>
<th>(v = \Delta \left<a href="x%5E*">\frac{\partial^2 h}{\partial x^2}\right</a>)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.020</td>
<td>0.053</td>
<td>2.7</td>
</tr>
<tr>
<td>20</td>
<td>0.013</td>
<td>0.034</td>
<td>2.6</td>
</tr>
<tr>
<td>25</td>
<td>0.011</td>
<td>0.026</td>
<td>2.4</td>
</tr>
<tr>
<td>40</td>
<td>0.0071</td>
<td>0.028</td>
<td>3.9</td>
</tr>
<tr>
<td>50</td>
<td>0.0066</td>
<td>0.021</td>
<td>3.2</td>
</tr>
</tbody>
</table>

\[(p = 1, \lambda = -1.1)\]

<table>
<thead>
<tr>
<th></th>
<th>(-\left<a href="x%5E*">\frac{\partial^2 h}{\partial x^2}\right</a>)</th>
<th>-(\Delta)</th>
<th>(v = \Delta \left<a href="x%5E*">\frac{\partial^2 h}{\partial x^2}\right</a>)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.0105</td>
<td>0.058</td>
<td>5.5</td>
</tr>
<tr>
<td>20</td>
<td>0.0047</td>
<td>0.027</td>
<td>5.7</td>
</tr>
<tr>
<td>25</td>
<td>0.0041</td>
<td>0.022</td>
<td>5.4</td>
</tr>
<tr>
<td>40</td>
<td>0.0026</td>
<td>0.014</td>
<td>5.4</td>
</tr>
<tr>
<td>50</td>
<td>0.0025</td>
<td>0.013</td>
<td>5.2</td>
</tr>
</tbody>
</table>

with the hop probability \( p \). This result shows that the decrease in \( \lambda_{\text{eff}} \) as \( p \) increases is driven by an increase in the surface tension \( v \). From Fig. 3, it can be seen that for \( p = \frac{1}{2} \) the effective value of the exponent \( \beta \) is already very close to \( \frac{1}{2} \). It might thus be expected that \( \lambda_{\text{eff}} \) for both \( p = \frac{1}{2} \) and 1 are much smaller than for \( p = 0 \). As we have seen earlier the product \( v\lambda_{\text{eff}} \) is, to a good approximation, independent of \( p \). Thus we expect that \( v \) for both \( p = \frac{1}{2} \) and 1 to be much larger than for \( p = 0 \). From the results presented in Tables I–III and Fig. 5, we note that this trend is clear for \( g = 0.125 \) but not for \( g = 0.5 \) or 0.875. This suggests that the correct values of \( v \) to be used for calculating \( \lambda_{\text{eff}} \) should correspond to the \( g \rightarrow 0 \) zero retardation limit. However, as a practical matter, as \( g \) decreases, it becomes increasingly difficult to accurately estimate the vanishing quantities whose ratio determines \( v \).

It is thus useful to discuss directly anticipated behavior for small \( g \). One expects that the magnitude of the curvature at \( x^* \) should decrease with \( p \) for any value of \( g \), as our data confirm. Thus if we can argue that \( \Delta \) must increase in magnitude with \( p \), for small \( g \), we can conclude that the same is true for \( v \). This, in fact, seems quite reasonable. The propensity for removal of particles at the designated site should be enhanced by the introduction of mobility, since more particles should hop into, rather than out of, any site that is a local minimum. The increase in the \( g = 0 \) unperturbed growth velocity with \( p \) could also enhance \( |\Delta| \). For larger \( g \), the trend in \( |\Delta| \) with \( p \) should reverse, as is clear for \( g = 1 \). (If one uses the alternate retardation procedure [19] for small \( g \), then \( |\Delta| \) does not increase, at least to the same extent, with \( p \). However, the decrease in the curvature is correspondingly faster.)

Two other significant issues should be considered when calculating \( v \). Firstly, there is a finite-size correction to the growth velocity, \( v = \left.< \frac{\partial h}{\partial t} >_m \right> [15] \), and secondly there is a distinction between ensembles with a fixed number of adsorption attempts and ensembles with a fixed number of adsorbed atoms. In performing these simulations we used relatively small lattices (\( L = 10 \) to 50). For these lattice sizes the finite-size correction to the growth velocity is significant, particularly since we need the difference in the growth velocities \( \Delta \) between the retarded and the unretarded cases. Thus, when using Eq. (5), \( \Delta \) should be taken to be \( v - v_{0,L} \), where \( v_{0,L} \) is the
growth velocity without retardation for a lattice of size $L$. This $v_{0,L}$ is determined exactly by $[1 + (2L - 1)^{-1}] / 2$, when $p = 0$ \[15\], and estimated by simulations otherwise.

During simulation runs, snapshots of the interface can be taken either after a fixed number of adsorption attempts or after a fixed number of adsorbed particles. For systems of finite size these yield different ensembles and, in particular, different values for the growth velocity. This is clear if we consider a lattice of size $L = 2$ (with periodic boundary conditions). Starting from a flat surface, if snapshots are taken at integral (half integral) monolayer coverages, the surface obtained would always contain two (one) bridge sites. Thus the value of $v$ obtained would be either 1 or $\frac{1}{2}$. For an ensemble built from snapshots taken after a large fixed number of adsorption attempts, the value of $v$ obtained is $\frac{1}{2}$, in agreement with the analytic result [15]. In the former case, simple averaging of the values of $\frac{1}{2}$ and 1 for $v$ does not correctly account for the lifetime for each interface configuration. The effect of this discrepancy decreases with lattice size, but for the sizes we used it leads to significant errors in the growth velocity difference $\Delta$.

As noted in Sec. II, the simulations determining $v$ might be performed differently. For instance, instead of retarding growth at one site on a lattice of $L = 50$ sites, one could retard growth at every $50$th site on an (essentially) infinite lattice. This, however, yields different results. For the simulations on the large lattice, the heights of any two consecutive sites at which the adsorption rate is retarded can differ. The $L = 50$ lattice simulation, on the other hand, corresponds roughly to the case where the heights at these sites are forced to be equal. The fluctuation in height at the retarded sites for the case of the large lattice simulations causes the position of the peaks in the interface to fluctuate over a larger range of $x$. The peak position is more "smeared out." For this system, the curvature of the interface decreases as we move away from the peak [18]. Hence, average profiles from simulations on the large lattice will result in a "corrupted" smaller curvature at the peak ($x = x^*$) of the interface. From such simulations with $g = 0.5$ in which the rate is retarded at every $50$th site, we find that $-\langle \partial^2 h / \partial x^2 \rangle(x^*) = 0.0021, 0.0015, 0.0001, -\Delta = 0.036, 0.021, 0.011$, and $v = 17, 14, 11$, for $p = 0, \frac{1}{2}, 1$, respectively. Clearly, the resulting trend in $\lambda_{\text{eff}}$ is opposite that required.

IV. THE $d = 1 + 1$ RESTRICTED SOLID-ON-SOLID MODEL WITH ONE HOP

In the restricted solid-on-solid model growth occurs by deposition on top of particles that have two nearest neighbors [6]. We can also allow the deposited particle to hop with probability $p$ to another RSOS adsorption site analogously to the bridge-site model with one hop (see Fig. 6). In Fig. 7, we plot the interface width as a function of the average interface height for a range of values of $p$. As expected, the exponent $\beta$ extracted from these plots decreases from $\frac{1}{4}$ towards $\frac{1}{2}$ as $p$ increases. In Fig. 8, we plot the mean growth velocity, normalized to the mean density of RSOS adsorption sites, as a function of the average squared interface slope. The dependence of $\langle \partial h / \partial t \rangle$ on $\langle \nabla h \rangle^2$ is not linear for $p = 0$, unlike the bridge-site model, and the value of approximately 0.42 for $\langle \nabla h \rangle = 0$ is apparently not a simple fraction (contrasting the observation of Ref. [22]). The value of $\lambda$ again becomes more negative with increasing $p$ for the same reason as in the bridge-site model. Although no detailed analysis of this model was performed, we expect that it should behave similarly to the bridge-site model, i.e., an increase of $v$ should drive the decrease of $\lambda_{\text{eff}}$ (and thus $\beta$) with $p$.

V. THE $d = 1 + 1$ NOISE-REDUCED BRIDGE-SITE MODEL

In Sec. III we have shown that allowing the impinging particle to hop after deposition in the bridge-site model smooths the film and reduces $\lambda_{\text{eff}}$ not by reducing $|\lambda|$ or the noise magnitude $D$ in the KPZ equation, but rather by increasing $v$. Hence it is reasonable to investigate a situation where what is regarded as the noise in the microscopic model can be controlled directly. In order to accomplish this, we first return to the bridge-site model, but instead of allowing the particles to hop, we use a conventional noise-reduction algorithm to produce a smoothing effect [6,23]. This is done as follows. Deposition
FIG. 8. Dependence of the interface growth velocity upon the squared average interface slope for the RSOS model with one hop.

occurs randomly at bridge sites. However, a bridge site is allow to adsorb a particle only after \( M - 1 \) particles have previously impinged upon it. It this criterion is not met, the impinging particle is removed. We performed simulations in which the values of \( M \) varied from 1 to 51. The basic bridge-site model is recovered when \( M \) is equal to unity.

In Fig. 9, we plot the dependence of the interface width \( w \) upon the average interface height. The values of the noise-reduction parameter \( M \) are indicated in the plot. A smoothing effect is clearly observed as \( M \) is increased. The effective value of the exponent \( \beta \) decreases from \( \frac{1}{3} \) as \( M \) increases. It is clear from the plot for \( M = 51 \) that two crossover regimes are possible: up to a coverage of approximately 40 layers, the growth exhibits layer-by-layer character and the slope of the plot is almost zero; between coverages of 50 and 1000 layers the effective value of the exponent \( \beta \) is approximately \( \frac{1}{3} \); presumably, at an even higher coverage \( \beta = \frac{1}{3} \) can be observed. For smaller values of \( M \), the layer-by-layer character gets weaker and the crossover between layer-by-layer growth and the smoother \( \beta = \frac{1}{3} \) behavior is not as clearly seen.

We also performed simulations for sloped interfaces in order to extract the dependence of the mean growth velocity upon the average interface slope. The results are plotted in Fig. 10. We have again defined the length scales such that the velocity of a microscopically flat surface is equal to unity, and the maximum slope is unity. From the plots in Fig. 10, we obtain the values of \( -\lambda = 0.5, 0.60, 0.76, \) and 0.86 for \( M = 1, 3, 11, \) and 51, respectively. Thus we find that the magnitude of \( \lambda \) increases with \( M \) as the interface becomes smoother (analogous to Secs. III and IV).

We elucidate the behavior of the noise amplitude \( D \) using the strategy of Sec. III. For finite systems of \( L \) sites, the steady-state value of \( w^2/L \) is in the ratio of 1:0.72:0.40:0.39, for \( M \) values 1:3:11:51. Thus using our above results for \( \lambda \), one finds values for \( -\nu \lambda \text{eff} = -\lambda(D/v)^{1/2} \) in the ratio of 1:1.0:0.96:1.1 for \( M \) values 1:3:11:51. Consequently \( \nu \) must increase with \( M \) to achieve the desired decrease in \( \lambda \text{eff} \).

Simulations on a lattice of size \( L = 50 \), with deposition retarded at one site, were also performed to determine the values of \( \nu \). The value of \( g \) was 0.5. The mean growth velocity for these interfaces and the value of \( \langle \partial^2 h/\partial x^2 \rangle \langle x \rangle \) are easily obtained from the simulation results. Then from Eq. (5), we determine the value of \( \nu \) for each \( M \). These are shown in Table IV. From these \( \nu \) estimates and the above data for \( w^2/L \propto D/v \), one can readily check that \( D \) does not decrease with \( M \). Finally we note that simulations on a large lattice with growth retarded every 50th site yielded corrupted curvature estimates of about half those in Table IV (cf. Sec. III).

A study of the noise-reduced RSOS model has been performed with similar results; i.e., a smaller effective exponent \( \beta \) was found for larger values of the noise-reduction parameter \( M \) [6]. It was suggested that an increase in the surface tension could be responsible for the smaller effective exponent [6]. Our results for the bridge-site model support this proposition.

FIG. 9. Dependence of the interface width upon the interface height for the noise-reduced bridge-site model. The value of the noise-reduction parameter \( M \) is shown for each curve, as is the exponent \( \beta \) obtained from a best fit of the data for interface heights between 100 and 1000.

FIG. 10. Dependence of the mean growth velocity upon the squared average interface slope for the noise-reduced bridge-site model.
TABLE IV. Results of simulations with periodically retarded deposition rate ($g = 0.5$) for the noise-reduced bridge-site model. The average interface height is approximately 10,000 monolayers and $L = 50$ in all cases.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$-\left(\frac{\partial^2 h}{\partial x^2}\right)(x^*)$</th>
<th>$-\Delta$</th>
<th>$\nu = -\Delta \left(\frac{\partial^2 h}{\partial x^2}\right)(x^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.022</td>
<td>0.029</td>
<td>1.3</td>
</tr>
<tr>
<td>3</td>
<td>0.0071</td>
<td>0.089</td>
<td>13</td>
</tr>
<tr>
<td>11</td>
<td>0.0064</td>
<td>0.173</td>
<td>28</td>
</tr>
<tr>
<td>51</td>
<td>0.0061</td>
<td>0.249</td>
<td>41</td>
</tr>
</tbody>
</table>

VI. THREE-DIMENSIONAL DEPOSITION MODELS

In this section we present the results for random deposition at fourfold-hollow sites, starting from a fcc(100) substrate. This model is the three-dimensional analog of the bridge-site model. In this case, only particles impinging upon fourfold-hollow sites are adsorbed [3]. We expect that the mean growth velocity decreases with $|V h|$, as is the case in the bridge-site model. Hence $\lambda$ is negative in this model, and we expect the exponent $\beta$ to be given by the KPZ value of $\frac{1}{2}$ [14]. This behavior has been confirmed in a previous simulation study of the equivalent ($d = 2 + 1$)-dimensional single-step model [4].

It has been seen in Sec. III that a smoothing effect can be obtained in the bridge-site model by allowing the adsorbing particle to undergo a hop, with probability $p$, laterally or downward. An analogous three-dimensional model also shows this smoothing effect [14]. Simulation results for such a model, with $p = 0$ (no hop), $p = \frac{1}{2}$, and $p = 1$, are plotted in Fig. 11. For nonzero values of $p$ the curves indicate that the interface width $w$ increases with a much smaller effective exponent than $\beta = \frac{1}{2}$. In the inset of Fig. 11, we plot $w^2$ as a function of $\log_{10}(h)$ for $p = \frac{1}{2}$ and 1. For $d = 2 + 1$ we have argued that the growth behavior observed is not the true asymptotic behavior due to slow crossover effects. In $d = 2 + 1$, however, it may be the case that the logarithmic growth of the interface width $w$ with $\langle h \rangle$ is the asymptotic behavior for $p$, above some critical value $p_c$. For the bridge-site model with one hop, we have argued that an increase in $\nu$ with $p$, rather than a reduction in $|\lambda|$ or $D$, is responsible for the smoother interface and lower $|\lambda_{eff}|$ and effective $\beta$.

We have not performed simulations to determine the behavior of $\lambda$, $D$, and $\nu$ for this ($d = 2 + 1$)-dimensional model. However it is clear that $\lambda$ will become more negative with increasing $p$, for the same reason as in the analogous ($d = 1 + 1$)-dimensional models. Assuming that $D$ does not decrease significantly with $p$, as in $d = 1 + 1$, we conclude that an increase in $\nu$ is responsible for the smoothing effect in three dimensions.

The behavior of the three-dimensional RSOS model with one hop with probability $p$ is also expected to be similar. For $p = 0$ (no hop), clearly again $\lambda$ is negative, and Kim and Kosterlitz [6] have shown that the KPZ value of $\beta = \frac{1}{2}$ is obtained. Presumably the effective $\beta$ will decrease with $p$ driven by an increase in the corresponding $\nu$.

VII. DISCUSSION AND CONCLUSIONS

We have analyzed the smoothing effect of introducing limited downward mobility into various deposition mod-

![Fig. 11](image-url)
els with a slope constraint in \( d = 1 + 1 \) and \( d = 2 + 1 \) dimensions. The effective exponent \( \beta \) characterizing the scaling of the interface width with height naturally decreases with increasing hop probability. This is associated with a corresponding decrease in magnitude of the parameter \( \lambda_{\text{eff}} = \lambda D^{1/2} / \nu^{3/2} \). Here \( \lambda \) measures sensitivity of the growth velocity to interface slope, \( \nu \) measures surface tension, and \( D \) the noise amplitude.

Detailed analysis of the effect of hopping on the bridge-site model reveals that enhanced smoothness is not driven by a decrease in \( |\lambda| \) or \( D \). Instead, it is associated with an increase in \( \nu \). This is true even in the corresponding noise-reduced model, where a significant increase in \( \nu \) with \( M \) enhances film smoothness. For the bridge-site model this increase in \( \nu \) with \( p \) is not difficult to understand. One should first recognize that \( \nu \) is appropriately calculated in the regime where the strength of the imposed inhomogeneity is small. Here we are able to argue that the magnitude of the velocity difference \( \Delta \) should increase with \( p \). With the larger \( |\Delta| \) and a smaller interface curvature at the peak (expected since larger \( p \) gives smoother films), the surface tension \( \nu \) must increase with \( p \).

We argue that analogous behavior should be seen as a result of introducing hopping or noise reduction to the fourfold-hollow site model in \( d = 2 + 1 \), and RSOS models for \( d \geq 2 \). Increasing the hop probability \( p \) or noise-reduction parameter \( M \) makes the film smoother locally. Thus the concentration of deposition sites, and \( \langle \nabla h \rangle \), increases for \( \langle \nabla h \rangle \rightarrow 0 \). However, \( \langle \nabla h / \delta t \rangle \) is always zero at the fixed maximum slope, so one anticipates that \( \nu \) will become more negative (cf. Figs. 4, 8, and 10). The above arguments for the increase of \( \nu \) with \( p \), at least when calculated in the limit of small imposed inhomogeneity strength, should also apply for these models.

In conclusion, we are able to correlate the smoothing effect on our deposition models of introduced limited downward mobility (or noise reduction) with the behavior of the KPZ equation parameters. Specifically smoothing is associated with an increase in \( \nu \) rather than a decrease in \( |\lambda| \). Our results contrast the behavior for the ballistic deposition model where introducing limited mobility enhances smoothness via decreasing \( \lambda \).

**ACKNOWLEDGMENTS**

The authors would like to acknowledge a useful discussion with Joachim Krug. This work was supported by National Science Foundation Grant No. CHE-9014214, and was performed at Ames Laboratory. Ames Laboratory is operated for the U.S. Department of Energy by Iowa State University, under Contract No. W-7405-Eng-87.

**APPENDIX A: THE \( d = 1 + 1 \) BALLISTIC DEPOSITION MODEL WITH ONE HOP**

Simulations were also performed for the previously studied \( d = 1 + 1 \) ballistic deposition model with both "sticky" and "sliding" particles, where the "sliding" particles are allowed to take one hop [13]. Our aim was to elucidate the behavior of parameter \( \lambda \) as a function of the probability \( p \) of depositing sliding particles. From simulations, we find that \( w \approx 0.87 \langle h \rangle^{0.33} \) when \( p = 0 \), and \( w \approx 0.79 \langle h \rangle^{0.26} \) when \( p = 1 \), for \( 10^2 \langle \dot{h} \rangle \leq 5 \times 10^3 \). Thus, as the fraction of sliding particles increases, the interface becomes smoother. The effective value of the exponent \( \beta \) decreases from \( 1 \) for \( p = 0 \) to approximately \( \frac{1}{4} \) for \( p = 1 \). In Fig. 12, we plot the mean growth velocity as a function of the squared average slope of the interface. For \( p = 0 \) and \( \frac{1}{4} \), it can be seen that the mean growth velocity increases as the interface slope increases. This is because the compactness of the film, to which the growth velocity is inversely proportional, decreases with the average slope. For \( p = 1 \), however, there are no holes in the growing film and the mean growth velocity is independent of the slope. From these results we can conclude that the growth exponent \( \beta = \frac{1}{4} \) for \( p \) not equal to unity, although there are probably crossover effects for \( p \) approaching unity. For \( p \) equal to unity, the growth exponent \( \beta = \frac{1}{4} \).

**APPENDIX B: RETARDED GROWTH**

Assume that no \( \nabla^{d+2m} h \) terms occur in Eq. (3). Then, ignoring fluctuations, the equation for the mean profile \( \langle h \rangle = H(x) \) in the case where deposition is retarded at \( x = x_i \) becomes

\[
\frac{\partial H}{\partial t} = f(H') \left[ 1 + vH'' / \kappa + d(H') + Q(g) \sum_i \delta(x - x_i) \right],
\]

where \( d(0) = 0 \) and \( d \) is an even function. We set \( H = 0 \) at \( t = 0 \). Clearly the \( \delta \) function generates a slope discontinuity at \( x = x_i \), and by symmetry \( H'_+ = H'_- \), where + (−) indicates evaluation slightly above (below) \( x_i \). Thus \( f(H') \) is still continuous at \( x_i \), and the above equation can be integrated to obtain

\[
\nu(H'_+ - H'_-) = 2vH'_+ = \kappa Q(g).
\]

The results described in Sec. II now follow immediately.


[9] The more general slope dependence $\mathbf{v} \cdot h^{T} - \lambda \mathbf{v} h$ can be reduced to $\lambda \mathbf{v}^{2} h^{T} h$ by transformation to an orthogonal coordinate system wherein the symmetric tensor $\lambda$ becomes diagonal, thereafter rescaling one coordinate.


[17] The height correlation scaling for a microscopic model with zero mean slope can be written as $\left\langle \left( [h_{m}\right. \left.-h_{n}]^{2} / (m-n)^{2} \right) \right\rangle \approx (m-n)^{2(\alpha-1)}$. Next identify $\mathbf{v}_{m-n} \cdot h = (h_{m}-h_{n}) / (m-n)$ as the discrete approximation to the continuum slope on a coarse-grained length scale $L \equiv m-n$. Then $\left\langle \mathbf{v}_{m} \cdot h \right\rangle \approx 0$ and $\left\langle \mathbf{v}_{m} \cdot h \right\rangle \approx \left\langle \mathbf{v}_{m} \cdot h \right\rangle^{2} + O(L^{2(\alpha-1)}) \rightarrow \left\langle \mathbf{v}_{m} \cdot h \right\rangle^{2}$, as $L \rightarrow \infty$ (since $\alpha < 1$). The exact calculations of Ref. [15], for nonzero slopes, agree with this result.


[19] Alternatively one could only remove particles with probability $g$ that impinge directly on the designated sites. This leads to smaller $|Q(g)|$ and it is no longer clear that $g = 1$ produces a pinned state if $p > 0$.


