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# Feeding and the Equilibrium Feeder Animal Price-Weight Schedule

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days on feed, energy use, feed ban, growth hormones, maintenance requirements, ration energy density, veal market

## **Disciplines**

Agricultural and Resource Economics | Agricultural Economics | Finance

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*JEL classification:* G1, Q1, Q4

# FEEDING AND THE EQUILIBRIUM FEEDER ANIMAL PRICE-WEIGHT SCHEDULE

## Introduction

Agricultural production is typically thought of as a biological process. In the case of crop agriculture, output and seed are living organisms. In the case of animal agriculture, parent stock, young stock, and feedstuffs are or recently were living organisms. Notwithstanding the rapid advances in biological sciences since 1950, our understanding of biological input-output relations outside the laboratory is not yet nearly as concrete as, say, our understanding of the physics of a diesel engine tractor in the field. Yet engines, plants, and animals are all ultimately governed by the same universal laws of energy. Problems with managing biological production relations have included the uniqueness of the organism due to sexual propagation, reliance on stochastic inputs such as rain and sunshine, and the absence of discerning laws regarding how energy converts in the organism.

Yet hard biological sciences are emerging, allowing growers, their advisors, and input suppliers to lean more heavily on some well-grounded, explicit production relations. In the case of genetics the hard science foundations are quite strong, while in other cases the foundations are less well understood. A notable instance of the latter that is of great importance to agriculture concerns the allometric scaling laws, that is, weight homogeneous scaling. The best known of these empirical regularities pertains to energy requirements for a body at rest, and so to the rate of biological growth on a given ration.

From an economics perspective, maintenance energy rates per unit of time are of interest because they may be viewed as a fixed but partially avoidable cost. The cost is fixed because, at least for the organism at hand, there is not much one can do about daily maintenance energy losses except discourage movement and provide shelter to avoid unnecessary heat loss, two major motives for animal confinement. The cost is avoidable to the extent that an accelerated growth regime, through feed management or other means, can bring the animal to maturity earlier. But rate of growth is not entirely a technical de-

cision because feedstuffs come in a variety of forms. Feedstuffs are bulky so that spatial price dispersion can be very high. The efficient husbander will look to trade off high growth rates, and so low lifetime expenditure on maintenance costs, against the reduced daily feed costs that lower density feeds can deliver.

Briefly, this article seeks to establish consequences for optimal feeding and live animal pricing of one widely affirmed allometric scaling law, Kleiber's law (1932), that when viewed in isolation can be seen as a scale economy. We use the animal scientist's accounting for energy uses in maintenance and growth to develop how Kleiber's law should affect feed ration decisions given the variety of ration compositions available to the grower. We also use the law to derive structure on the relationship between the mature animal price in equilibrium and the price of an immature feeder animal at any weight.

The paper should be of interest because active feeder markets, be they spot or contract, exist for hogs and cattle in most countries with a significant agricultural sector, while local specialty feeder markets exist for goats, horses, and other species. Apart from young stock, feed is generally the largest input in meat production. Feeder markets have undergone significant structural changes in the United States and elsewhere over the past fifty years. For example, steers are now being fed more intensively to be slaughtered younger and at a heavier weight (Elam and Preston 2004). In addition, many of the most significant innovations in animal agriculture, including confinement, genetic manipulation, hormone treatments, and nutrition innovations, have been used to improve feed conversion efficiency (fce). Among major policy issues concerning meat markets in recent years are bans on the use of some animal-derived materials in animal feed and the use of some growth promotants. To better understand evolving trends, new technologies, and the economic consequences of policy adjustments, it would be useful to have available a robust microeconomic model. This model should be detailed and explicit in representing the role of feed in meat production, and it should also incorporate equilibrium pricing. Existing models are very limited in these areas.

The most influential paper in the literature is by Jarvis (1974). He viewed cattle as assets with biological growth options such that slaughter occurs when growth potential has been exhausted. His model specified a single-harvest objective function, assuming also that daily feeding costs were weight and time invariant. Feeder animal prices were

imputed as Ricardian rent in perfect competition. The analysis found a withholding motive in response to higher retail level beef prices, suggesting a backward-bending supply curve. Slaughter weight was found as the weight that minimized the unit cost of meat, and it was suggested that the feeder animal price per pound should decline as weight increased to the slaughter weight.

Paarsch's (1985) set of models advocated, more realistically, that the grower be held to harvest sequentially (i.e., in rotation) according to Faustmann's paradigm. In addition, weight gain was allowed to depend on weight and the chosen feed regime was allowed to vary over time. Finally, the price of young stock was held to be exogenous to the meat price. This means that, in contrast with the Jarvis model, Paarsch's model can relate nothing about the feeder animal price-weight schedule (pws) in what typically are competitive feeder animal markets. He found that the rotation model reversed Jarvis's inference that there should be a delayed slaughter response to an increase in meat prices. Amer et al. (1994) have utilized Jarvis's economic insight, together with empirical equations from Fox, Sniffen, and O'Connor (1988), to ask when it is optimal to slaughter and how one should go about comparing performances across breeds with different growth characteristics. Much of the subsequent related work has focused on some peculiar dynamics of feeder animal markets, namely, inverted price responses and the role of dynamic biological restrictions in price cycle behavior (Rosen 1987; Rosen, Murphy, and Scheinkman 1994; Chavas 2000; Aadland 2004).

That literature has not addressed feeding decisions, the role of maintenance costs, or the structure of the feeder animal pws.<sup>1</sup> A second literature track has sought to identify the best feeding path in an optimal control framework. Chavas, Kliebenstein, and Crenshaw (1985) modeled swine feeding and marketing under rotational harvest. Talpaz et al. (1988, for broilers) and Cacho, Kinnucan, and Hatch (1991, for catfish) modeled other species without accounting for sequential harvest. These papers did account for maintenance costs and emphasized empirical simulations with appropriate available data rather than seeking to develop a conceptual model in the manner of Jarvis or Paarsch.<sup>2</sup> Later empirical simulation work, summarized in Hernández et al. 2003, has focused on fish production. The papers most relevant to the current research are by Arnason (1992) and by Heaps (1993), and they will be reviewed in due course. This optimal feeding lit-

erature has taken feeder animal prices as given without reference to equilibrium, and optimum feeding paths for realistic growth dynamics are difficult to interpret.

The intent here is to re-visit some theoretical issues raised by Jarvis and Paarsch, ones not addressed in any of the other theoretical or empirical literatures. By contrast with the Jarvis and Paarsch models, we model daily maintenance costs with specific dynamic technical production relations. We take Jarvis's Ricardian perspective on feeder animal prices but the Paarsch rotation assumption on the grower's objective function. We depart from both approaches in providing explicit specifications for animal growth and how growth depends on feed through the animal's life. We do so in part because findings in animal science allow for the technical structure we will impose, and in part because we seek a clearer understanding of the feeder animal pws and the determinants of optimal feed rations than has been provided in the literature to this point. Daily maintenance costs are crucial in this regard and have been ignored outside the empirical literature. We depart from the vast majority of the literature, Jarvis being the exception, by endogenizing prices paid for feeder animals. Rather than making it more difficult to interpret optimal feeding paths, our equilibrium pricing approach and growth specifications establish a simple and intuitive optimal feeding trajectory. This simplicity allows us to study such policy issues as regulations on hormone implants and the use of animal-derived materials in feedstuffs, and also to impute the equilibrium feeder animal pws. We also show that the insights we provide on the optimal feeding trajectory are quite general in that they do not depend upon the technical growth relations assumed in our main model.

The paper is laid out as follows. First is a discussion of the origin and grounding of the allometric scaling law of relevance to us, Kleiber's law (Smil 2000). This is followed by the baseline production model, and an analysis of model implications for pricing immature animals. Conditions such that incentives support the slaughter of young animals are also considered, as is closure of the model under long-run equilibrium. The roles of biological parameters in determining how the unit price of animals changes with weight are developed. After generalizing the model to apply for less structured assumptions on the growth technology, the paper concludes with a brief discussion.

## **Maintenance Costs**

In 1839, scientists F. Sarrus and J. Rameaux suggested a relationship between weight and energy expended by an organism at rest (also known as base metabolic rate, and hereafter labeled as  $r$ ).<sup>3</sup> Animals of all weights,  $w$ , are in approximate equilibrium with the world around them, and lose heat at the rate of body heat production. In addition, skin surface area grows in proportion to  $w^{2/3}$  and heat loss is in proportion to surface area. Therefore, they argued that  $r \propto w^{2/3}$  also. Max Kleiber (1932) sought to validate this by using observed metabolic rates for different species, but concluded that  $r \propto w^{3/4}$  was more consistent with the data. Later works by Brody (1945), by Hemmingsen (1960), by Bartels (1982), by Bennett and Harvey (1987), by Heusner (1991), and by numerous others have continued a vigorous debate on whether  $3/4$  or  $2/3$  or neither are the appropriate power relations.

Recent research by Dodds, Rothman and Weitz (2001) suggests that values close to  $3/4$  may be more appropriate for animals above 10 kg, for example, hogs, sheep, goats, and cattle, while  $2/3$  is more plausible for birds and smaller mammals. For the qualitative results in this paper, the exact number is not relevant except in that it should satisfy  $r \propto w^\alpha, \alpha \in (0,1)$ , so as to provide size economies with respect to maintenance costs.

Evidence on scaling is of interest to biologists because it must comply with Darwinian concepts of efficient energy use on the part of a species. Scaling relations should reveal much about the critical constraints on survival that species face, and on how efficiently a species converts energy. Among the first formal explanations proposed for Kleiber's law was due to McMahon (1973), who viewed it as a civil engineering problem.<sup>4</sup> More recent theories by West, Brown, and Enquist (1997, 1999) and Banavar, Maritan, and Rinaldo (1999) emphasize efficiency in the essential function of nutrient circulation. Whatever the reason,  $3/4$  power scaling has become embedded in practical science. Because many drugs are believed to clear the body according to a  $3/4$  scaling law (Mordenti 1986), anesthetics and other drug prescriptions are often scaled to body mass in this manner. In agriculture, and of direct relevance to this research,  $3/4$  scaling is assumed for maintenance energy and feed intake relations (National Research Council 2000).

## Production Model

An animal requires maintenance calories amounting to  $M$  per day. In order to fatten the animal, energy content of liveweight gained must amount to  $E$ . The animal's intake capacity is  $I$  units of feed per day. Both  $M$  and  $I$  depend upon the animal's liveweight, denoted as  $w$  and measured in pounds, in a manner that we will soon specify. For the moment, we consider only a small weight gain interval such that these attributes may be considered to be fixed. Feedlot incoming and outgoing weights are  $w_{in}$  and  $w_{ou}$ , respectively.

A choice available to the grower is the energy density of rations fed, and energy density  $b \in [0, \bar{b}]$  can be attained at cost per unit feed amounting to  $\theta c(b)$ , where  $c(b)$  is an increasing and convex function.<sup>5</sup> Here,  $\theta$  is a cost shift parameter that might change because of feed regulations (such as a ban on using animal grease as a feedstuff) or an increase in component prices due to increased demand for feed. Growers are homogeneous in the sense that all face cost  $\theta c(b)$ . Cost of feed per day is, therefore,  $I\theta c(b)$  for all growers. Let  $T$  be days on feed. Liveweight gain per day is  $(Ib - M)\lambda$ , where  $\lambda$  is the conversion of surplus energy  $Ib - M$  into pounds liveweight. Animals should be fed to capacity because conversion to liveweight is linear in calories surplus to maintenance. Therefore, it takes  $E / [(Ib - M)\lambda]$  days to grow an animal from  $w_{in}$  to  $w_{ou}$ , and days on feed must satisfy  $T = E / [(Ib - M)\lambda]$ .

Gross revenue across days on feed is  $\Delta R$ , and its composition will be explained shortly. Gross revenue per day is  $(\Delta R)(Ib - M)\lambda / E$ . Revenue per day net of feed costs is

$$V = \frac{(\Delta R)(Ib - M)\lambda}{E} - I\theta c(b). \quad (1)$$

Reservation utility will depend on the level of resources applied, which will depend in turn on the animal's intake through demands on labor, machinery, and buildings. We make the assumption that  $V = \kappa I$ , so that perfect competition ensures<sup>6</sup>

$$\kappa I = \frac{(\Delta R)(Ib - M)\lambda}{E} - I\theta c(b). \quad (2)$$

It remains to characterize  $\Delta R$ , the change in revenue. We will do so by making  $I$  and  $M$  functions of weight so that we need no longer confine attention to the small weight interval  $[w_{in}, w_{ou}]$ . To do this we need to specify growth dynamics. We fix the initial, that is, time  $t = 0$ , weight of the animal at  $w(0)$ . The animal's maturity weight is given at  $w_m$ , referred to as the maturity plateau and taken as given, but the maturity date depends on the animal's feeding regime. Weight at time  $t \geq 0$  is written as  $w(t)$ . According to Kleiber's law, the animal's instantaneous maintenance cost is  $M = [w(t)]^{0.75} \gamma_M$ ,  $\gamma_M > 0$ . We will assume that the animal has intake capacity  $I = [w(t)]^{0.75} \gamma_I$ ,  $\gamma_I > 0$ .<sup>7</sup> For future reference, we will describe the couple  $(\gamma_I, \gamma_M)$  as an animal's genetic profile. An animal on ration density  $b$  and consuming at capacity will consume at the rate  $[w(t)]^{0.75} \lambda_I b$  calories. This allows us to write (2) as

$$\Delta R = \frac{[\kappa + \theta c(b)] \gamma_I E}{(\gamma_I b - \gamma_M) \lambda}, \quad (2')$$

where  $[\kappa + \theta c(b)] \gamma_I / [(\gamma_I b - \gamma_M) \lambda]$  is cost per unit gain (and so is free of time units) and where  $E$  is the target gain in calories.

Efficiency requires that  $\Delta R$  be minimized over  $b \in [0, \bar{b}]$ . This is the critical assumption that differentiates our work from the optimal feeding literature reviewed earlier. If  $\Delta R$  is not minimized then an alternative energy regime will be more efficient, allowing the grower to pay more for a feeder animal in perfect competition and so drive  $\Delta R$  down.<sup>8</sup>

**PROPOSITION 1.** *Let an animal have Kleiber's law maintenance costs with three-fourths power intake and genetic profile  $(\gamma_I, \gamma_M)$ . Then the optimal choice of  $b$  satisfies*

$$b^* = \arg \min_{b \in [0, \bar{b}]} H(b; \gamma_I, \gamma_M, \theta), \quad H(b; \gamma_I, \gamma_M, \theta) = \frac{[\kappa + \theta c(b)] \gamma_I E}{(\gamma_I b - \gamma_M) \lambda}. \quad (3)$$

*It is weight invariant.*

To better understand the relationship in (3), write it as  $c(b)b^* - c(b^*)b \geq (b - b^*)\kappa / \theta + (c(b) - c(b^*))\gamma_M / \gamma_I \forall b \neq b^*$ . The left-hand side is positive if and only if  $c(b)/b \geq c(b^*)/b^*$ . If  $c(b)/b$  is decreasing in  $b$  then  $[c(b)/b - c(b^*)/b^*](b - b^*) < 0$ . Since  $[c(b)/b - c(b^*)/b^*](b - b^*) \geq \kappa(b - b^*)^2 / [\theta bb^*] + [c(b) - c(b^*)](b - b^*)\gamma_M / [\gamma_I bb^*]$ , it follows that for  $c(b)/b$  to decrease requires the corner solution  $b^* = \bar{b}$ . In that case, cost per unit of energy is decreasing while an increase in ration density also saves on both maintenance costs and the opportunity costs of committed resources. It is reasonable, however, to ignore this case because it is likely that  $c(\hat{b}) = 0$  for some  $\hat{b} > 0$ , where the ration is comprised of by-products (e.g., from bakeries) with insufficient nutrient density to clear maintenance costs. If  $c(b)$  is convex on  $(\hat{b}, \bar{b}]$ , then  $c(b)/b$  must be an increasing function. From this point on we will consider only interior solutions.

For interior solutions, implications of equation (3) include the following:<sup>9</sup>

PROPOSITION 2. *Under the growth technology assumptions in Proposition 1, the incentive to feed a high-energy ration increases in the value of (a)  $\gamma_M$ , (b)  $-\gamma_I$ , (c)  $-\rho = -\gamma_I / \gamma_M$ , (d)  $-\theta$ , and (e)  $\kappa$ .*

Part (a) demonstrates some consequences of higher daily maintenance costs. All else being equal, a higher value of  $\gamma_M$  will depress feeder animal prices and may induce early culling. We will return to this issue later. If the animal is not culled early, then a higher maintenance cost parameter will elicit a more intensive feeding regime in order to more efficiently gain beyond daily maintenance costs. A genetic innovation, of which the only effect is to push down the value of  $\gamma_M$  for a breed, should reduce the incentive to feed intensively. In part (b),  $\gamma_I$  can be considered to be determined by genetic endowments or through feed management practices that enhance palatability, for example, rolling grain and reducing dust (Ensminger 1987, chap. 30). Alternatively, non-use of growth-promoting implants (i.e., hormone implants but not ionophores) is held to decrease dry matter intake by about 6 percent (Fox et al. 1992).<sup>10</sup> An increase in  $\gamma_I$  reduces the need

to avoid daily maintenance costs through a costly high-energy regime. Part (c) is a complete characterization of the contents in parts (a) and (b). One may view  $\rho$  as an index of an animal's genetic potential for meat production.

Concerning part (d), the recent bans on ruminant-to-ruminant feed in the United States and elsewhere has raised the possibility that animal and poultry fats will be removed entirely from the animal and poultry feed markets. This will increase the cost of a high-energy diet. By contrast, an increase in opportunity costs of resources (e.g., high  $\kappa$  due to high labor costs, environmental regulations, or animal welfare laws) will strengthen the incentive to increase the marginal productivity of those resources by feeding to increase throughput.

The reader may find the effects in Proposition 2 to be quite intuitive. As we will show, intuition is more likely to fail regarding consequences for days on feed. Given weight homogeneity (degree 0.75, in our case) regarding growth, in order to establish the roles of  $\gamma_I$  (through, e.g., hormones, feed preparation methods, or genetics) and  $\gamma_M$  (through, e.g., confinement or genetics) for days on feed, we need only understand how  $\gamma_I b^* - \gamma_M$  changes.

*PROPOSITION 3. Make the growth technology assumptions in Proposition 1. Then there exist increasing, convex ration cost functions such that optimal days on feed either increase or decrease with an increase in either (a)  $\gamma_I$ , or (b)  $\gamma_M$ . But (c) optimal days on feed always increase with an increase in  $\theta$ .*

Intuition for part (a) is that the direct effect of an increase in  $\gamma_I$  on  $\gamma_I b^* - \gamma_M$  need not be so strong as to dominate the negative effect on the level of optimal rations. Intuition for part (b) is that a decrease in  $\gamma_M$  relaxes the incentive to feed intensively to clear daily maintenance costs and, again, the indirect effect can dominate the direct effect. An interesting feature of part (a) is that  $(db^* / d\gamma_I)|_{\gamma_M=0} = 0$  and so days on feed always increase with  $\gamma_I$  whenever  $\gamma_M = 0$ . The existence of daily maintenance costs has a qualitative effect on how optimal rations respond to intake innovations. In each of parts

(a) and (b), we believe that the direct effect will likely dominate. We lay out the other possibility in order to better illustrate how technologies and regulations can affect age at slaughter. Part (c) follows almost directly from Proposition 2 as there is not any direct effect on optimal days on feed. Part (c) is of interest because the ratio of corn to live cattle price in the United States has been, over 2000-2004, approximately half its level in the late 1940s. While official data have not been kept, it is widely believed that cattle slaughter ages have declined since World War II (Schroeder, Mintert, and Brester 1995; Elam and Preston 2004).

### Feeder Price-Weight Schedule

Having considered ration decisions, we turn to the feeder pws. Notice in (2') that homogeneity allowed us to write revenue growth independently of animal weight. Propositions 1 and 2 do not involve weight, so the feeder animal's ration density is weight separated. Thus, we can take  $b$  as given and time invariant when solving for physical growth and the feeder pws.

Our model set-up identifies the flow of calories converted to liveweight as  $\lambda\gamma_I b^* [w(t)]^{0.75} - \lambda\gamma_M [w(t)]^{0.75}$ . Therefore, the rate of change of weight is governed by

$$\frac{1}{[w(t)]^{0.75}} \frac{dw(t)}{dt} = \lambda\gamma_I b^* - \lambda\gamma_M. \quad (4)$$

Observe that (4) implies

$$\frac{d^2 w(t)}{dt^2} = \frac{3\lambda^2 (\gamma_I b^* - \lambda\gamma_M)^2}{4} [w(t)]^{0.5} > 0; \quad \frac{d^2 \text{Ln}[w(t)]}{dt^2} = -\frac{\lambda^2 (\gamma_I b^* - \lambda\gamma_M)^2}{4[w(t)]^{0.5}} < 0; \quad (5)$$

so that weight (with ration path endogenized) is convex in time before the maturity plateau but the log of weight is concave in time before that plateau. Convexity  $d^2 w(t) / dt^2 > 0$  is in contrast with the concavity assumptions made in Jarvis and in Paarsch but is consistent with the beef production literature (Ensminger, 1987, p. 838; Goodwin 1977, p. 158; Neumann 1977, p. 400; Owen 1991, p. 39).

Since  $b^*$  is weight-independent, we may readily integrate (4) from initial weight  $w(0)$  to obtain the weight path as the formula

$$w(t) = \left( [w(0)]^{0.25} + 0.25(\gamma_I b^* - \gamma_M) \lambda t \right)^4. \quad (6)$$

With maturity weight  $w_m$ , the maturity date is given as  $T_m$ ,

$$T_m = \frac{4 \left( w_m^{0.25} - [w(0)]^{0.25} \right)}{(\gamma_I b^* - \gamma_M) \lambda}. \quad (7)$$

In general, the time on feed to weight  $w$  will be  $t_w = 4(w^{0.25} - [w(0)]^{0.25}) / [(\gamma_I b^* - \gamma_M) \lambda]$ .

For  $P(w)$  as the feeder animal pws when weight is  $w$ , equation (2'), upon taking to the infinitesimal limit as  $dR(w) / dw = P(w) + w dP(w) / dw = d(\text{Cost}) / dw$ , allows us to write

$$\int_{s=w}^{s=w_m} \frac{dR(s)}{ds} ds = P_m w_m - P(w)w = \int_{s=w}^{s=w_m} H(b^*; \gamma_I, \gamma_M, \theta) ds = H(b^*; \gamma_I, \gamma_M, \theta)(w_m - w). \quad (8)$$

Thus, growth rate homogeneity, as given by (4), ensures that feeder animal value  $P(w)w$  is linear in weight with value  $P(w)w = [P_m - H(\cdot)]w_m + H(\cdot)w$ . It is not linear in time. The unit price of feeder animals is

$$P(w) = H(b^*; \gamma_I, \gamma_M, \theta) + \frac{[P_m - H(b^*; \gamma_I, \gamma_M, \theta)]w_m}{w}. \quad (9)$$

Differentiation of (9) supports

$$\frac{dP(w)}{dw} = \frac{[H(b^*; \gamma_I, \gamma_M, \theta) - P_m]w_m}{w^2}, \quad \frac{d^2P(w)}{dw^2} = 2 \frac{[P_m - H(b^*; \gamma_I, \gamma_M, \theta)]w_m}{w^3}. \quad (10)$$

If  $P_m > H(b^*; \gamma_I, \gamma_M, \theta)$ , as we will demonstrate should be the case, then the first derivative is negative while the second derivative is positive.

PROPOSITION 4. *Make the growth technology assumptions in Proposition 1. If  $P_m > H(b^*; \gamma_I, \gamma_M, \theta)$ , then the feeder animal pws is decreasing and convex.*

That feeder animal prices are decreasing in weight is consistent with the idea that young animals have growth potential (Jarvis 1974) to be expended as the animal grows. Convexity only asserts that the growth potential component to value is most rapidly expended at lower weights. Since (10) shows that relative curvature is  $-[d^2P(w)/dw^2]/[dP(w)/dw] = 2/w$ , it is decreasing in weight and the proportional rate of decline in price with weight is larger at lower weights.

The pws should also be affected by the growing environment. In particular, we have the following.

PROPOSITION 5. *Make the growth technology assumptions in Proposition 1. Then  $\forall w \in [w(0), w_m]$ , (a)  $dP(w)/dP_m = w_m/w \geq 1$  and  $d^2P(w)/dP_m dw \leq 0$ ; (b)  $dP(w)/d\gamma_M \leq 0$  and  $d^2P(w)/d\gamma_M dw \geq 0$ ; (c)  $dP(w)/d\gamma_I \geq 0$  and  $d^2P(w)/d\gamma_I dw \leq 0$ ; (d)  $dP(w)/d\theta \leq 0$  and  $d^2P(w)/d\theta dw \geq 0$ .*

Part (a) may be viewed as a sensitivity result, where the Ricardian rent due the owner of young stock as a result of high meat prices becomes less significant on a per pound basis at larger weights. In contrast with part (a), parts (b) through (d) require an understanding of ration consequences. Parts (a) through (d) do, though, follow a similar theme. The effects of a parameter innovation that tends to push the feeder price up are spread along the schedule so that the price dampening value becomes less severe at higher weights.

## Veal Market

Meat from young animals is typically more tender and palatable, so the principal motive for growing an animal is to realize growth potential when feed costs are low. The intent of this section is to better understand growth potential when the option exists to slaughter early. For the sake of exposition, we assume that the maturity price is the unique live price for slaughtered animals, that is, there is not a premium for young meat.<sup>11</sup>

From (9) we have that  $P(w) \geq P_m$  if and only if

$$P_m \geq H(b^*; \gamma_I, \gamma_M, \theta). \quad (11)$$

In this case, a veal market will be absent for animals of type  $(\gamma_I, \gamma_M)$  because it will always be more profitable to feed or sell to a feedlot than to exercise the slaughter option before maturity. Notice that the condition is more likely to hold when  $P_m / [\kappa + \theta c(b^*)]$  is relatively high, that is, when the meat-corn price ratio is high. The condition is also more likely to hold when  $\gamma_M$  is relatively low, and when  $\gamma_I$  is relatively high. Thus, animals with low growth parameters, perhaps surplus calves from dairy herds or chicks from laying hen flocks, are more likely to be slaughtered early. From (11) and observation of  $H(b^*; \gamma_I, \gamma_M, \theta)$  in (3), we have the following.

*PROPOSITION 6. Make the growth technology assumptions in Proposition 1. If a genetic profile  $(\gamma_I, \gamma_M) = (\bar{\gamma}_I, \bar{\gamma}_M)$  is such that the grower does not slaughter the  $w = w(0)$  animal, then the grower does not slaughter the  $w = w(0)$  animal with genetic profile  $(v_1 \bar{\gamma}_I, v_2 \bar{\gamma}_M)$ ,  $v_1 \geq v_2 > 0$ . If a genetic profile  $(\gamma_I, \gamma_M) = (\bar{\gamma}_I, \bar{\gamma}_M)$  is such that the  $w = w(0)$  animal is slaughtered, then the  $w = w(0)$  animal with genetic profile  $(v_1 \bar{\gamma}_I, v_2 \bar{\gamma}_M)$ ,  $0 < v_1 \leq v_2$ , is slaughtered.*

Proposition 6 may be viewed as a scaling result. The decision to slaughter at the outset depends only on  $\rho = \gamma_I / \gamma_M$ . For the growth technology in question, there exists a  $P_m$ -dependent  $\rho$  value, call it  $\hat{\rho}(P_m)$ , such that one should keep to maturity if and only if  $\rho \geq \hat{\rho}(P_m)$ .

### **Closing the Model: The Long-Run**

To this point we have derived feeder animal prices from the mature animal price. However, feeder animals are a renewable resource and there will be a long-run supply response to a change in mature animal prices. In addition to the features considered in our model thus far,

exogenous determinants of the feeder animal supply function and of the demand for mature animals matter. This section will close the model by accounting for young stock supply and retail demand functions so that a long-run equilibrium is established.

Set the inverse supply function for weight  $w(0)$  animals as  $P[w(0)] = P_{ys}(q_{ys})$ , where  $q_{ys}$  is the quantity of young stock supplied. Were  $P[w(0)] \neq P_{ys}(q_{ys})$ , then there would be long-run disequilibrium in the feeder animal market and breeding herd adjustments would occur. Let  $P_m = P_m(q_{ms}; \omega)$  be the live price of matured stock at mature stock quantity flow  $q_{ms}$  and conditional on exogenous parameter  $\omega$ . Here,  $dP_m(\cdot)/d\omega > 0$ , that is,  $\omega$  represents an exogenous increase in demand for meat. Then (8) provides

$$\left( P_m(q_{ms}; \omega) - H(b^*; \gamma_I, \gamma_M, \theta) \right) w_m = \left( P_{ys}(q_{ys}) - H(b^*; \gamma_I, \gamma_M, \theta) \right) w(0). \quad (12)$$

One way to view (12) is to consider when young stock supply is perfectly elastic at  $P_{ys}(q_{ys})w(0) = F > 0$ . Then the price of live, mature animals will be independent of demand at  $P_m(q_{ms}; \omega) = (F/w_m) + H(b^*; \gamma_I, \gamma_M, \theta)(w_m - w(0))/w_m$ , the sum of a fixed cost for young stock and cumulated feeding costs. In general, for a perfectly elastic supply of young stock, the general equilibrium pws is

$$P(w) = \frac{F}{w} + H(b^*; \gamma_I, \gamma_M, \theta) \left( \frac{w - w(0)}{w} \right), \quad (13)$$

so that the fixed young stock cost is apportioned over weight. Note that if

$P_{ys}(q_{ys}) \equiv [F/w(0)] < H(b^*; \gamma_I, \gamma_M, \theta)$  then equilibrium price is monotone increasing in weight. But this is not possible in any event because the young stock would be slaughtered immediately. Using (10) and (13), we have that  $dP(w)/dw =$

$[H(b^*; \gamma_I, \gamma_M, \theta) - P_{ys}(q_{ys})]w(0)/w^2$ . Since  $H(b^*; \gamma_I, \gamma_M, \theta)$  is constant, the derivative must be negative in general equilibrium where the genetic profile  $(\gamma_I, \gamma_M)$  is not slaughtered. Convexity of the pws is assured also in this case.

The quantity flow of mature stock onto the retail market will depend upon the product of two factors. These are the quantity flow of young stock,  $q_{ys}$ , onto the feeder market and the inverse of time to maturity as given in (7), that is,

$$q_{ms} = \frac{(\gamma_I b^* - \gamma_M) \lambda q_{ys}}{4(w_m^{0.25} - [w(0)]^{0.25})}. \quad (14)$$

While Proposition 2 identifies no direct role for  $P_m$ , that is,  $\omega$ , in determining  $b^*$  there will be an indirect role in general equilibrium because ration costs will change in response to altered demand in feed markets. General equilibrium responses to a change in the value of  $\omega$  then involve two types of adjustments. Most directly there are adjustments in  $q_{ys}$  and  $q_{ms}$  subject to (14) and a fixed  $b^*$ . Indirectly, as recognized by Yver (1971), for a significant meat producing species there will be feed market adjustments due to a change in demand where we might view  $\theta = \Theta(\omega)$ ,  $d\Theta(\omega)/d\omega \geq 0$ . The demand-induced change in cost of rations will alter equilibrium feed decisions and so will shock (12) on both the right-hand side and the left-hand side, through (14). From Proposition 3 we already know that optimal time to maturity increases with an increase in  $\theta$ , so flow balance equation (14) suggests non-trivial system responses in the long run.

### Generalized Feeding Path

Suppose instead that  $I(w) = \gamma_I w^\sigma$ ,  $\sigma \in (0, 1]$ . Then work provided in Appendix B shows that the ration optimization problem resolves to

$$b^*(w) = \arg \min_{b \in [0, \bar{b}]} H[b; w, \gamma_I, \gamma_M, \theta]; \quad H[b; w, \gamma_I, \gamma_M, \theta] = \frac{[\kappa + \theta c(b)] \gamma_I E}{(\gamma_I b - \gamma_M w^{0.75-\sigma}) \lambda}; \quad (15)$$

under Ricardian rent extraction by young stock producers. From optimality condition

$$(\gamma_I b - \gamma_M w^{0.75-\sigma}) \theta c_b(b) = [\kappa + \theta c(b)] \gamma_I, \text{ it is clear that } db^* / dw =$$

$(0.75 - \sigma) w^{-0.25-\sigma} c_b(b^*) / [(\rho b^* - w^{0.75-\sigma}) c_{bb}(b^*)] = 0.75 - \sigma$ . That is, the optimal plane of nutrition increases (decreases) with animal weight (and time) if  $0.75 > (<) \sigma$ . For

$0.75 > \sigma$ , the intuition is that intake contracts relative to maintenance costs as weight increases. Ration density should increase at higher weights in order to accelerate the animal through the later stages of feeding. Anecdotal evidence suggests to the author that some beef growers at any rate tend to increase feed density close to slaughter, but this will depend on the animal type, season of slaughter whenever animal prices are seasonal, and prevailing feed market prices.

In addition,  $db^* / dw$  will tend to be small when the relative curvature of the cost function,  $c_{bb}(b^*) / c_b(b^*)$ , is large. We also have fce (growth/day over intake/day) as

$$\frac{1}{I(w)} \frac{dw(t)}{dt} = \lambda b^*[w(t)] - \lambda \rho^{-1}[w(t)]^{0.75-\sigma}. \quad (16)$$

If  $c_{bb}(b^*) / c_b(b^*)$  is sufficiently large that  $b^*(w)$  is quite weight-insensitive and if  $0.75 > \sigma$ , then the right-hand side of (16) will decline with weight so that fce declines with weight. There is fairly firm evidence that fce declines with animal weight, even at weights well below the mature weight (National Research Council 2000, p. 24).<sup>12</sup>

The feeding path described in (15) is simpler and differs from those derived by Paarsch (1985), by Arnason (1992), and by Heaps (1993), among others using optimal control methods because they do not view the price of feeder stock as a derivative. Feeding then is not just about cost minimization but also about throughput in order to capture economic profits on feeding. When positive economic profits exist per lot then there will be an incentive to accelerate feeding and incur a higher cost of gain in order to re-stock early. We argue that, while this will be true when positive economic rents exist, economic rents will be dissipated through competition so that the problem reduces to one of cost-minimized feeding.

The insights in (3) and (15) actually allow a feeding path analysis for much less structured growth technology specifications. Suppose that there is a vector  $\vec{b}$  of feeding choices, energy density, protein density, and so on. Conditional on weight, the time rate of change in costs is written in dot form as  $\dot{C}(\vec{b}; w)$ . Here,  $\vec{b} \in B$  where  $B \subset \overline{\mathbb{R}}_+^N$ , that is, is non-negative, with  $B$  closed and bounded so that any maximum sought for a continuous function does exist. The growth equation is written in dot form as  $\dot{w}(\vec{b}; w)$ , and

feeder value as a function of weight is  $R(w)$  so that revenue growth with time is  $[dR(w)/dw]\dot{w}(\vec{b}; w)$  and profit per unit of time is  $[dR(w)/dw]\dot{w}(\vec{b}; w) - \dot{C}(\vec{b}; w)$ . If competition bids prices to the point of zero economic profits, then  $dR(w)/dw = \dot{C}(\vec{b}; w)/\dot{w}(\vec{b}; w)$  and the successful bidders will solve

$$\vec{b}^*(w) = \arg \min_{\vec{b} \in B} \text{Ln} [H(\vec{b}; w)], \quad H(\vec{b}; w) = \frac{\dot{C}(\vec{b}; w)}{\dot{w}(\vec{b}; w)}; \quad (17)$$

where the strictly increasing log transformation does not change the values of maximizing arguments.

It is reasonable to hold that the components of  $\vec{b}$  are technical complements because energy, proteins, and other nutrients are required in approximate proportion for muscle formation. As far as the unit cost of liveweight gain goes, this may be represented by the assumption that  $d^2H(\vec{b}; w)/db_i db_j \leq 0 \forall b_i, b_j \in \vec{b}$ . From a standard result on submodular cost functions (Topkis 1995), if  $d^2\text{Ln}[H(\vec{b}; w)]/db_i dw \leq 0 \forall b_i \in \vec{b}$  then  $\vec{b}^*(w)$  is component-wise increasing in weight while if  $d^2H(\vec{b}; w)/db_i dw \geq 0 \forall b_i \in \vec{b}$  then  $\vec{b}^*(w)$  is component-wise decreasing in weight. Now if  $\dot{C}(\vec{b}; w) = f(\vec{b})g(w)$  and  $\dot{w}(\vec{b}; w) = h(\vec{b})k(w)$  then  $\text{Ln}[H(\vec{b}; w)] = \text{Ln}[f(\vec{b})/h(\vec{b})] + \text{Ln}[g(w)/k(w)]$  with null cross-derivatives so that this form of separability ensures invariance of rations composition to weight in the optimal feeding schedule. Invariance failed in (15) because separation of energy density from weight was not possible. It was possible to establish monotonicity on  $b^*(w)$  because  $d^2\text{Ln}[H(b; w)]/db dw$  has a uniform sign, that of  $\sigma - 0.75$ .

### **Generalized Price-Weight Schedule**

To demonstrate that the insights arrived at are not driven by the specifics of the growth technology assumptions, let us solve the pws under some intake function  $I(w) = \gamma_l l(w)$ . From the analog to (15) we have<sup>13</sup>

$$\int_{s=w}^{s=w_m} \frac{dR(s)}{ds} ds = P_m w_m - P(w)w = \int_{s=w}^{s=w_m} H[b^*(s); s, \gamma_I, \gamma_M, \theta] ds. \quad (18)$$

Using derived feeder animal pricing it follows that  $P(w) =$

$(P_m w_m - \int_{s=w}^{s=w_m} H[b^*(s); s, \gamma_I, \gamma_M, \theta] ds) / w$  and, in general equilibrium,  $P(w) =$

$[P_{ys}(q_{ys})w(0) + \int_{s=w(0)}^{s=w} H[b^*(s); s, \gamma_I, \gamma_M, \theta] ds] / w$  so that

$$\frac{dP(w)}{dw} = \frac{\left( H[b^*(w); w, \gamma_I, \gamma_M, \theta] - \frac{\int_{s=w(0)}^{s=w} H[b^*(s); s, \gamma_I, \gamma_M, \theta] ds}{w} \right) - \frac{P_{ys}(q_{ys})w(0)}{w}}{w}. \quad (19)$$

To simplify, if  $w(0) = 0$  then a sufficient condition for the sign to be negative is that average feed costs per unit weight gain to weight  $w$ , or  $w^{-1} \int_{s=w(0)}^{s=w} H[b^*(s); s, \gamma_I, \gamma_M, \theta] ds$ , be

decreasing. A sufficient, but not necessary, condition for this to occur is that

$H[b^*(s); s, \gamma_I, \gamma_M, \theta]$  be decreasing. Conditions are readily identified such that the price derivative is positive, at least locally. For  $w(0) = 0$ , it suffices that

$w^{-1} \int_{s=w(0)}^{s=w} H[b^*(s); s, \gamma_I, \gamma_M, \theta] ds$  be increasing locally.

Convexity of the pws requires that

$$\frac{d^2 P(w)}{dw^2} = \frac{1}{w} \frac{dH[b^*(w); w, \gamma_I, \gamma_M, \theta]}{dw} - \frac{2}{w} \frac{dP(w)}{dw} \quad (20)$$

so that if  $dP(w)/dw \leq 0$  and  $dH[b^*(w); w, \gamma_I, \gamma_M, \theta]/dw \geq 0$  then the pws is convex. But, since we cannot preclude both  $dP(w)/dw > 0$  and  $dH[b^*(w); w, \gamma_I, \gamma_M, \theta]/dw < 0$  over a weight interval, we cannot rule out local concavity in the pws. The shape of the pws, while likely decreasing and convex as in Proposition 4, could be otherwise under growth technology assumptions that we are not in a position to rule out based on our review of the animal science literature.

## **Discussion**

The intent of this article has been to gain a better understanding of the animal feeding operation, with emphasis on rations maintenance costs and equilibrium pricing for feeder animals. Our model is inevitably a simplification. Among arguably significant omissions is the assumption that the interest rate is zero. There will of course be a downward shift in feeder animal prices when the opportunity cost of capital is included, and this shift will be more marked at earlier weights. A positive rate will also cause minor distortions in feeding incentives because early feeding becomes comparatively more costly.

A similar minor concern is that feed costs are held to depend only on the ration density. In the case of grain feed, this is not true because there must be inter-year price appreciation in order to motivate storage. For forage, ration costs vary seasonally given preservation costs and the high levels of spoilage over time. When ration costs are seasonal, then ration levels and feeder animal prices will be seasonal. Accommodating feed seasonality in the model would be challenging because of storage issues and because processing constraints in the face of supply seasonality will also affect feeder animal prices.

We believe that the insights our model provides will be robust to these and other modeling concerns, particularly when markets tend to be stable and non-seasonal in nature. An issue that might be more challenging for the data analyst concerns policy distortions in feeder animal markets. It has been policy across European Union countries for a long period prior to decoupling reform in the early years of the twenty-first century to make annual per head payments on feeder cattle over some of the animal's life. This presents a problem for testing the hypotheses provided in Propositions 4 through 6. But it also presents an opportunity to test for rationality regarding price formation consistent with the Ricardian rent assumption.

## Endnotes

1. An empirical literature, reviewed in Marsh 2001, has estimated determinants of feeder prices.
2. These models, together with Amer et al. 1994 and much of the large production science literature on empirical animal growth models, are similar to ours in two features of relevance to our work. Both animal maintenance costs and animal dry matter intake are held to increase in proportion to  $(weight)^\alpha$  where  $\alpha$  is some number close to 0.75.
3. To be clear, many slightly different technical versions of  $r$  exist, but here it is intended to represent energy expenditure on vital biological functions other than feeding and motion. For the mature animal,  $r$  will not include an allocation for non-forced growth. For young animals, it may do so. If it does, then  $r$  should decline with age.
4. Animals must be engineered to withstand limb fracture under stress, and the rest follows from manipulating known mechanical laws.
5. We use energy as a representative growth-limiting requirement in order to economize on model notation. Inclusion of other constraints, protein balance for example, would not change the main messages our paper provides.
6. We have ignored interest rates. Like Jarvis (1974, p. 492), we are of the opinion that interest rates are of minor importance in determining feeder animal prices. See Table 1 in Marsh (2001) for empirical support on this.
7. See page 88 in National Research Council 2000.
8. An optimal control proof of weight invariance is provided in Appendix B. The setup there is more general than the one presently under consideration, so the reader might want to defer inspection. The proof clarifies how  $\min_{b \in [0, \bar{b}]} \Delta R$  simplifies the optimal feeding path problem.
9. Proofs of Propositions 2, 3, and 5 are provided in Appendix A.
10. This 6 percent figure has been adopted by the National Research Council Subcommittee on Beef Cattle Nutrition.

11. An age discount could be readily introduced. The analysis would become messy without providing additional insights.
12. As previously mentioned, the standard assumption is that  $\sigma = 0.75$ , but the data to support this are less firm than the data to support the 0.75 exponent on maintenance costs.
13. Divide through by  $l(w)$  rather than by  $w^\sigma$  to obtain  $H[b; w, \gamma_I, \gamma_M, \theta] = [\kappa + \theta c(b)] \gamma_I E \lambda^{-1} / [\gamma_I b - \gamma_M w^{0.75} / l(w)]$ .

## Appendix A

### Proofs of Propositions

*Proof of Proposition 2.* Write the first-order condition arising from (3) as

$$(\gamma_I b^* - \gamma_M) \theta c_b(b^*) - [\kappa + \theta c(b^*)] \gamma_I = 0. \quad (\text{A1})$$

Second-order conditions will be satisfied locally, and that is all that is necessary, if the cost function is strictly convex in  $b$ . Concerning part (d), differentiate (A1) and then use (A1) to obtain  $db^*/d\theta = -\gamma_I \kappa / [(\gamma_I b^* - \gamma_M) \theta^2 c_{bb}(b^*)] < 0$ . Part (e) follows from  $db^*/d\kappa = \gamma_I / [(\gamma_I b^* - \gamma_M) \theta c_{bb}(b^*)] > 0$ . For parts (a), and (b), observe that  $db^*/d\gamma_M = c_b(b^*) / [(\gamma_I b^* - \gamma_M) c_{bb}(b^*)] > 0$ , and  $db^*/d\gamma_I = -\gamma_M c_b(b^*) / [(\gamma_I b^* - \gamma_M) \gamma_I c_{bb}(b^*)] < 0$ . For part (c), write (A1) as  $(\rho b^* - 1) \theta c_b(b^*) = [\kappa + \theta c(b^*)] \rho$  and differentiate:  $db^*/d\rho = -c_b(b^*) / [(\rho b^* - 1) \rho c_{bb}(b^*)] < 0$ . ■

*Proof of Proposition 3.* In each case we will provide a proof by construction.

Part (a): From the first-order condition (A1) and proof of Proposition 2, write

$$\frac{d(\gamma_I b^* - \gamma_M)}{d\gamma_I} \stackrel{\text{sign}}{=} (b^*)^2 c_{bb}(b^*) \rho - b^* c_{bb}(b^*) - c_b(b^*). \quad (\text{A2})$$

This expression will certainly be positive if  $(b^*)^2 c_{bb}(b^*) \rho - b^* c_{bb}(b^*) - (1 + \varepsilon) c_b(b^*) = 0$ ,  $\varepsilon > 0$ . We will construct an increasing and convex cost function that satisfies this relation for all values of  $b$ , and so for the optimal value. Set  $y = c_b(b)$  and  $x = b$  so that the condition reduces to

$$\frac{1}{y} \frac{dy}{dx} = \frac{1+\varepsilon}{\rho x^2 - x} = \frac{(1+\varepsilon)\rho}{\rho x - 1} - \frac{(1+\varepsilon)}{x}, \quad \varepsilon > 0. \quad (\text{A3})$$

Integrate and substitute to obtain

$$c_b(b) = A_0 \left( \frac{\rho b - 1}{b} \right)^{1+\varepsilon}. \quad (\text{A4})$$

Because  $\rho b > 1$  is required for growth, (A4) is positive if  $A_0 > 0$ , while (A4) is increasing in  $b$  if both  $A_0 > 0$  and  $\varepsilon > -1$ . We have already imposed the last condition through  $\varepsilon > 0$ .

In order to establish a case in which  $(b^*)^2 c_{bb}(b^*)\rho - b^* c_{bb}(b^*) - c_b(b^*) < 0$ , we need only identify an acceptable cost function such that  $(b^*)^2 c_{bb}(b^*)\rho - c_b(b^*) = 0$ . Re-write this as the condition  $\rho x^2 (dy/dx) - y = 0$ , or  $(1/y)(dy/dx) = 1/(\rho x^2)$  with solution

$$c_b(b) = A_0 e^{-1/[\rho b]}. \quad (\text{A5})$$

Integrate over  $b \in [\xi, \bar{b}]$ ,  $\xi > 0$  to obtain  $c(b) = A_1 + A_0 \int_{\xi}^b e^{-1/[\rho s]} ds$  on  $b \in [\xi, \bar{b}]$ ,  $\xi > 0$ .

Assume  $A_1 > 0$ ,  $A_0 > 0$ , and then splice any acceptable cost function with requisite smoothness on over  $b \in [0, \xi)$  to obtain a function that is increasing and convex while satisfying  $b^2 c_{bb}(b)\rho - b c_{bb}(b) - c_b(b) < 0$  on  $b \in [\xi, \bar{b}]$ ,  $\xi > 0$ .

Part (b): We have

$$\frac{d(\gamma_I b^* - \gamma_M)}{d\gamma_M} \stackrel{\text{sign}}{=} \rho c_b(b^*) - \rho b^* c_{bb}(b^*) + c_{bb}(b^*). \quad (\text{A6})$$

The sign is positive if  $c_{bb}(b)/c_b(b) = (\rho - \varepsilon)/(\rho b - 1)$  and  $\rho > \varepsilon > 0$ , i.e., if  $c_b(b) = A_0(\rho b - 1)^{(\rho - \varepsilon)/\rho}$ . For  $\rho > \varepsilon > 0$ , this cost function is positive and increasing on any energy density level that clears maintenance requirements. The sign in (A6) is negative if  $\rho c_b(b) - \rho b c_{bb}(b) + c_{bb}(b) = -\rho c_b(b)$ , so that the differential equation to be solved is  $(1/y)(dy/dx) = 2\rho/(\rho x - 1)$ . The equation solves as

$$c_b(b) = A_0(\rho b - 1)^2, \quad (A7)$$

a positive and increasing function whenever energy density is sufficient to grow the animal.

Part (c): We have  $d(\gamma_I b^* - \gamma_M) / d\theta = db^* / d\theta < 0$  from Proposition 2. ■

*Proof of Proposition 5.* Part (a): From (8),  $dP(w) / dP_m = w_m / w$  and

$d^2P(w) / dP_m dw = -w_m / w^2$ . Parts (b)-(d): From (8),

$$\begin{aligned} \frac{dP(w)}{d\gamma_M} &= \frac{(w - w_m)}{w} \frac{\partial H(b^*; \gamma_I, \gamma_M, \theta)}{\partial \gamma_M} \leq 0; & \frac{d^2P(w)}{d\gamma_M dw} &= \frac{w_m}{w^2} \frac{\partial H(b^*; \gamma_I, \gamma_M, \theta)}{\partial \gamma_M} \geq 0; \\ \frac{dP(w)}{d\gamma_I} &= \frac{(w - w_m)}{w} \frac{\partial H(b^*; \gamma_I, \gamma_M, \theta)}{\partial \gamma_I} \geq 0; & \frac{d^2P(w)}{d\gamma_I dw} &= \frac{w_m}{w^2} \frac{\partial H(b^*; \gamma_I, \gamma_M, \theta)}{\partial \gamma_I} \leq 0; \\ \frac{dP(w)}{d\theta} &= \frac{(w - w_m)}{w} \frac{\partial H(b^*; \gamma_I, \gamma_M, \theta)}{\partial \theta} \leq 0; & \frac{d^2P(w)}{d\theta dw} &= \frac{w_m}{w^2} \frac{\partial H(b^*; \gamma_I, \gamma_M, \theta)}{\partial \theta} \geq 0; \end{aligned} \quad (A8)$$

where the envelope theorem has been employed. ■

## Appendix B

### Ration Optimization Problem

Pose the problem as a control path optimization,

$$\begin{aligned} \text{Max}_{\{b(t), T\}} J[b(t), T] &= \text{Max}_{\{b(t), T\}} \frac{P_m w_m - P(w(0))w(0) - \int_{t=0}^T [\kappa + \theta c(b(t))] \gamma_I E[w(t)]^\sigma dt}{T}; \\ \text{subject to } \frac{dw(t)}{dt} &= \lambda \gamma_I b(t) [w(t)]^\sigma - \lambda \gamma_M [w(t)]^{0.75}; \\ b(t) \geq 0 \quad \forall t \in [0, t]; \quad w(t) \geq 0 \quad \forall t \in [0, t]; \quad T \geq 0; \quad w(T) &= w_m. \end{aligned} \quad (\text{B1})$$

In our case it is sufficient to develop the solution to the constrained optimization over  $b(t)$  with  $T$  fixed at the optimum (transversality conditions will establish optimum  $T$ ).

Fixing  $T$ , the Hamiltonian may be written as

$$\mathcal{H}(w(t), b(t), \phi(t)) \equiv -[\kappa + \theta c(b(t))] \gamma_I E[w(t)]^\sigma + \phi(t) \left( \lambda \gamma_I b(t) [w(t)]^\sigma - \lambda \gamma_M [w(t)]^{0.75} \right), \quad (\text{B2})$$

Necessary conditions for interior solutions involve

$$(\text{B3.1}) \quad \frac{dw(t)}{dt} = \lambda \gamma_I b(t) [w(t)]^\sigma - \lambda \gamma_M [w(t)]^{0.75},$$

$$(\text{B3.2}) \quad \phi(t) = \frac{\theta E}{\lambda} \frac{dc(b(t))}{db}, \quad (\text{B3})$$

$$(\text{B3.3}) \quad \frac{1}{[w(t)]^{\sigma-1}} \frac{d\phi(t)}{dt} = [\kappa + \theta c(b(t))] \gamma_I E \sigma - \phi(t) \lambda \left( \gamma_I b(t) \sigma - 0.75 \gamma_M [w(t)]^{0.75-\sigma} \right).$$

The system simplifies upon making the well-known observation that  $\phi(t)$  is the value of incremental stock (Léonard and Long 1992, p. 154; Conrad and Clark 1987, p. 36), where stock is in this case animal weight. In our case, it is  $\Delta R$  at the infinitesimal. But the cost of the increment is cost per incremental time over weight gain per incremental time, i.e.,

$$\phi(t) = \frac{[\kappa + \theta c(b^*(t))] \gamma_I [w(t)]^\sigma E}{(\lambda \gamma_I b^*(t) [w(t)]^\sigma - \lambda \gamma_M [w(t)]^{0.75-\sigma})} = \frac{[\kappa + \theta c(b^*(t))] \gamma_I [w(t)]^\sigma E}{(\lambda \gamma_I b^*(t) - \lambda \gamma_M [w(t)]^{0.75-\sigma})}. \quad (\text{B4})$$

Under Ricardian rent extraction, incremental value must equal incremental cost because otherwise there would be positive or negative economic rent over feeding operation, a contradiction. Therefore, (B3.2) resolves to

$$\frac{[\kappa + \theta c(b(t))] \gamma_I E}{(\gamma_I b(t) - \gamma_M [w(t)]^{0.75-\sigma}) \lambda} = \frac{\theta E}{\lambda} \frac{dc(b(t))}{db}, \quad (\text{B5})$$

i.e., the minimizer of  $H[b; w, \gamma_I, \gamma_M, \theta] = [\kappa + \theta c(b)] \gamma_I E / [(\gamma_I b - \gamma_M w^{0.75-\sigma}) \lambda]$ . In this light, (B3.3) becomes (qualitatively)

$$\frac{d\phi(t)}{dt} \stackrel{\text{sign}}{=} 0.75 - \sigma, \quad (\text{B6})$$

i.e., the unit cost of gain increases (decreases) with time, and so with an increase in weight, if  $0.75 > (<) \sigma$ . The case of Proposition 1 is when  $\sigma = 0.75$  so that the unit cost of gain is weight invariant. Another way of viewing (B4) under derived feeder animal pricing is that  $\int_{w(0)}^{w_m} \phi(w(t)) dw = P_m w_m - P(w(0))w(0)$  while

$$\begin{aligned} \int_{w(0)}^{w_m} \frac{[\kappa + \theta c(b^*(t))] \gamma_I [w(t)]^\sigma E}{(\lambda \gamma_I b^*(t) - \lambda \gamma_M [w(t)]^{0.75-\sigma})} dw &= \int_0^T \frac{[\kappa + \theta c(b^*(t))] \gamma_I [w(t)]^\sigma E}{(\lambda \gamma_I b^*(t) - \lambda \gamma_M [w(t)]^{0.75-\sigma})} \left[ \frac{dw}{dt} \right]^{-1} dt \\ &= \int_0^T [\kappa + \theta c(b^*(t))] \gamma_I [w(t)]^\sigma E dt, \end{aligned} \quad (\text{B7})$$

where  $dw/dt$  is the growth rate under optimal rations. But the right-most term in (B7) equals  $P_m w_m - P(w(0))w(0)$  under Ricardian rent extraction on the part of young stock suppliers.

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