On relationships between the Lyapunov spectrum and the Morse spectrum

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On relationships between the Lyapunov spectrum and the Morse spectrum

by

Ruchira Majumdar

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Applied Mathematics
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We consider dynamical systems on manifolds and explore the relationship between the Lyapunov and the Morse spectra. The concept of Morse spectrum is based on a study of the exponential growth rates associated with the $e^{-T}$ chains in the chain recurrent components of the flow on the projective bundle. It is known that the Morse spectrum contains the Lyapunov spectrum and the Morse spectrum is a union of closed intervals whose boundary points are Lyapunov exponents. Here we present a treatment for the case of a flow on a two dimensional compact manifold under the assumption that the chain recurrent components coincide with the limit sets. For this case we investigate the equality of the two spectra and provide methods to calculate them in practice.

For planar flows there are only three possible types of nonwandering sets: fixed points, periodic orbits and cycles. We consider the case where these are isolated chain recurrent components. For these three types of sets we perform a case by case study by first linearizing the system over the solutions and then by computing the Lyapunov and the Morse spectra. The Lyapunov spectrum over a fixed point consists of the real parts of the eigenvalues of the Jacobian matrix of partial derivatives. For periodic orbit, they are the characteristic exponents. For a cycle we prove that the Lyapunov spectrum consists of the Lyapunov exponents for the fixed points in the cycle. Belgrade’s theorem gives a relationship between the chain recurrent components in the base and the chain recurrent components for the flow in the projective bundle. We use this theorem to find Morse spectrum. We prove that for an isolated fixed point, a connected set of fixed points and a periodic orbit the Lyapunov and the Morse spectra coincide.

ABSTRACT

We consider dynamical systems on manifolds and explore the relationship between the Lyapunov and the Morse spectra. The concept of Morse spectrum is based on a study of the exponential growth rates associated with the $e^{-T}$ chains in the chain recurrent components of the flow on the projective bundle. It is known that the Morse spectrum contains the Lyapunov spectrum and the Morse spectrum is a union of closed intervals whose boundary points are Lyapunov exponents. Here we present a treatment for the case of a flow on a two dimensional compact manifold under the assumption that the chain recurrent components coincide with the limit sets. For this case we investigate the equality of the two spectra and provide methods to calculate them in practice.

For planar flows there are only three possible types of nonwandering sets: fixed points, periodic orbits and cycles. We consider the case where these are isolated chain recurrent components. For these three types of sets we perform a case by case study by first linearizing the system over the solutions and then by computing the Lyapunov and the Morse spectra. The Lyapunov spectrum over a fixed point consists of the real parts of the eigenvalues of the Jacobian matrix of partial derivatives. For periodic orbit, they are the characteristic exponents. For a cycle we prove that the Lyapunov spectrum consists of the Lyapunov exponents for the fixed points in the cycle. Belgrade’s theorem gives a relationship between the chain recurrent components in the base and the chain recurrent components for the flow in the projective bundle. We use this theorem to find Morse spectrum. We prove that for an isolated fixed point, a connected set of fixed points and a periodic orbit the Lyapunov and the Morse spectra coincide.
cycle, when all the fixed points are hyperbolic, or when all of them have two distinct Lyapunov exponents, we get an attractor-repeller pair which gives two chain recurrent components. These correspond to two Morse intervals. For a fixed point in a cycle if both the Lyapunov exponents are the same then they can only take the value zero. When for at least one fixed point in a cycle both the Lyapunov exponents are zero we prove that the entire projective bundle is a single chain recurrent component. This gives a single Morse interval whose endpoints consist of the minimum and the maximum Lyapunov exponents. Hence we get intervals for Morse spectrum here whereas Lyapunov spectrum were just points. This result answers the fundamental question of whether or not the Lyapunov and Morse spectra coincide.
1 INTRODUCTION

In the book The Dynamics of Control [8] Colonius and Kliemann attempted a synthesis of concepts and ideas from dynamical systems and control or perturbation theory. They associated to a control or perturbation system a dynamical system over the space of control/perturbation functions. In this approach they used a variety of techniques from the theory of dynamical systems and ideas from control theory to analyze the associated dynamical systems. The key control theoretic notions were control sets, chain control sets, linearization and spectrum, while the basic concepts from the dynamical systems were topological and chain recurrence, flows on vector bundles and Lyapunov exponents.

Colonius and Kliemann starts with a dynamical system on a smooth manifold and then, given the range $U \subset \mathbb{R}^n$ of possible time-varying perturbations, they consider all possible perturbed systems from a given one. This gives a family of time-varying differential equations. To study the behavior of the solutions one approach is to analyze the possible limit-sets $\Omega$ and their stability behavior. To do this, one usually linearizes the system at $\Omega$ and obtains the stable and unstable subspaces of the linearization. These give an idea about the invariant manifolds of $\Omega$ locally. The central concepts of this analysis are the spectrum of the linearization, the induced decomposition of the tangent bundle and its projection onto the invariant manifolds ([8]pg 13). It is known for a long time that the Lyapunov spectrum [for detailed definition see Chapter 2] contains all the exponential growth rates of the perturbed system and hence the information about the (exponential) stability behavior of the solutions of the perturbed family of
Another approach to study the perturbed system is to consider it as one dynamical system. The properties of the flow obtained from this system can be used to get valuable information about the solutions of not only the perturbed system but also some properties related to the unperturbed system. In this context, again, Lyapunov exponents and stability are two important concepts people would like to understand. As they describe the exact exponential growth behavior of the solutions, Lyapunov exponents are a powerful tool for the analysis of asymptotic stability and stable subspaces and manifolds. Hence it would have been nice to actually be able to compute this spectrum of the perturbed system.

Not only the aforementioned problem, [8] describes several problems of control-perturbation and dynamical systems where the Lyapunov exponents appear with significant role but in each case either it is difficult to attack them directly or they are not sufficient to reach the desired information. To get rid of the difficulties that arise with the Lyapunov exponents and to study the systems more efficiently the authors of [8] developed a new spectrum called Morse spectrum [7]. This new spectrum is based on the Morse decompositions of a flow on the projective bundle. It consists of limits of (finite time) exponential growth rates of $(c - T)$ chains (for definitions of these see chapter 2).

1.1 Difficulties with the Lyapunov Spectrum

What is a spectrum? The spectrum of a constant matrix $A$ is the set of its eigenvalues. Its Lyapunov spectrum consists of the real parts of the eigenvalues. The Lyapunov spectrum of $A$ describes the stability behavior of the linear differential equation $\dot{x} = Ax$ in $\mathbb{R}^n$. For periodic matrix functions the Floquet spectrum is same as the Lyapunov spectrum. If $A : \mathbb{R} \rightarrow gl(d, \mathbb{R})$ is a general bounded, measurable function then the
exponential growth rates of the solutions of \( \dot{x} = A(t)x \) in \( \mathbb{R}^d \) is defined as follows:

Let \( \varphi(t, x) \) be a solution with \( \varphi(0, x) = x \in \mathbb{R}^d \). Define the Lyapunov exponent as

\[
\lambda(x) = \limsup_{t \to \infty} \frac{1}{t} \log |\varphi(t, x)|.
\]

Define the Lyapunov spectrum of the matrix function \( A \) as

\[
\Sigma_{Ly} = \{ \lambda_i, \lambda_i \text{ is a Lyapunov exponent of a solution } \varphi(\cdot, y) \text{ of } \dot{x} = A(t)x, \ i = 1, \ldots, d \}
\]

One problem with the Lyapunov exponents is that they do not, in general, have nice “regularity” properties with respect to system parameters.

Let \( A : \mathbb{R} \to gl(d, \mathbb{R}) \) be a bounded measurable matrix function and consider the spectra \( \Sigma_{Ly}(A) = \{ \lambda_1 \leq \ldots \leq \lambda_d \} \) and \( \Sigma_{Ly}(-A^T) = \{ \mu_1 \geq \ldots \geq \mu_d \} \). The matrix function \( A \) is called regular if \( \lambda_i + \mu_i = 0 \) for \( i = 1, \ldots, n \). Properties of regular matrices are mentioned in [15] section 64 and [25] section 79. Examples of regular matrices are constant and periodic matrices. Triangular matrices are regular iff \( \lim_\tau \frac{1}{t} \int_0^t a_i(\tau)d\tau \) exists \( \forall i = 1, \ldots, d \).

There exists nonregular matrices with arbitrary small entries. One example is

\[
A(t) = \epsilon \begin{pmatrix} \cos \log t & \sin \log t \\ \sin \log t & \cos \log t \end{pmatrix}
\]

One can obtain nonregular matrices as small perturbations of any constant matrix.

Consider the equation \( \dot{x} = X(x, t) \) in \( \mathbb{R}^d \) with fixed point at the origin. If the vector field \( X \) is smooth, this equation can be written as

\[
\dot{x} = A(t)x + f(x, t).
\]

If \( |f(x, t)| \leq c(x)^m \) for some \( m > 1, \ c(x) > 0 \), and if \( A \) is regular, then the second equation has a stable manifold of dimension \( \sum_{i=1}^t m_i \), where \( \lambda_i \) is the largest negative Lyapunov exponent of \( A \). If the largest exponent \( \lambda_k \) is negative, then the origin is asymptotically stable for the second equation, and hence for the first equation. In
general this result is not true without regularity as shown by an example in [8] pg 41 and also in [25].

Another problem the authors of [8] discuss is that of stabilization of bilinear systems. There they have shown that the Lyapunov spectrum of a system is not sufficient to determine the set of null controllable points. One needs to know the infimal exponent over each control set, more precisely, its closure. As the Lyapunov spectrum over different control sets may overlap, one has to develop a spectral characterization that separates the corresponding parts of the spectrum.

Lyapunov spectrum is very difficult to attack directly. Hence it is necessary to construct an 'outer' approximation or a 'rougher' spectral concept. One example is the dichotomy spectrum or dynamical spectrum introduced by Sacker and Sell, [31], [20], [32], [35], based on exponential dichotomy of subspaces. There is a topological concept introduced by Salamon and Zehnder [34] which the authors of [8] refer as topological spectrum. For smooth ergodic theory there is Oseledets spectrum [27].

In their study of persistence of attractors and spectra [8], pg 478, Colonius and Kliemann posed the following questions: (a) Which properties of a vector field $X_0$ extend to the time-varying perturbed vector fields (persistence)? (b) Which properties of the perturbed vector fields take on which form in limit $X_0$ (continuity)?

Study of these problems led to the Morse spectrum developed by Colonius and Kliemann. We quote from [8], pg 6:“The persistence and continuity properties of dynamical systems under time-varying perturbations lead to Morse sets and the Morse spectrum as “natural” objects for dynamical systems.”

### 1.2 Starting Point

In the theory developed by Colonius and Kliemann the connected components of the chain recurrent set of the projected flow over some connected chain recurrent set
in the base space are used to define a spectral concept via the growth rates of (finite
time) chains that lie in these components. These components correspond to a Morse
decomposition, hence they are Morse sets and this spectrum is named Morse spectrum.
It is shown in [7] that Morse spectrum consists of finitely many closed intervals, whose
boundary points are Lyapunov exponents. Also Morse spectrum for a linear flow \( \Psi \)
contains the entire Lyapunov spectrum of the flow \( \Psi \) for arbitrary initial values.

In their theory about persistence of attractors and spectra Colonius and Kliemann
introduced a real parameter \( \rho \geq 0 \) describing the size of control (perturbation) range. A
test for the appropriateness of their concepts is the study of the system behavior as \( \rho \to 0 \). They have shown that chain recurrent components of a vector field \( X_0 \) are persistent
and continuous under arbitrary time-varying perturbations in any finite-dimensional
neighborhood of \( X_0 \). The continuity result for the Morse spectrum over a Morse set of
the vector field \( X_0 \) is that the Morse spectrum is right continuous in \( \rho \). Also there are
persistence results for the Morse spectrum. In general the Lyapunov spectrum of a flow
does not depend continuously on parameters.

It is known that Morse spectrum contains the Lyapunov spectrum but it was not
known if the Lyapunov spectrum can be a proper subset of the Morse spectrum. More
precisely, it is an open problem to find out exactly under what conditions do the two
spectra agree. Then we can use the continuity and persistency properties of Morse
spectrum of a perturbed system to actually get the Lyapunov spectrum for the original
system and apply it for stability analysis.

There are computational results by Lars Grune, included as Appendix D of [8],
which cite about some computational approximation of the Lyapunov spectrum. Using
a refinement of Bowen's shadowing lemma it is shown in [7] that for vector fields with
hyperbolic projective flow Lyapunov and Morse spectra coincide. But so far, no general
theory is available.

In our investigation first we tried to find out whether Lyapunov spectrum can at
all be a proper subset of Morse spectrum. We considered a two dimensional compact manifold \( M \) and a chain recurrent component of a continuous time flow \( \Phi \) on \( M \). However, our proofs and results are done for systems in compact invariant sets in \( \mathbb{R}^2 \). We did a case by case study over three specific types of chain recurrent components, viz. fixed point, periodic orbit and cycle. For each of these cases we actually calculated both the Lyapunov and the Morse spectra. We found that although for fixed points and periodic orbits the two spectra are the same, in general they are not. Over a cycle, the Morse spectrum consists of intervals while the Lyapunov spectrum is just the boundary points.

Andronov's result indicates that in two dimensions the only possible nonwandering sets that can occur are the abovementioned three types. Beyond two dimensions we cannot do a case by case analysis like this. Besides fixed points and periodic orbits in three dimensions (and onward) we have attractors and chaotic sets and things get much more complicated. But even in two dimensions, we found it was not completely trivial to directly calculate the Morse spectrum. Besides using the Morse decomposition and chain recurrence at times we had to use intricate objects like attractors and their complementary repellers. Another approach might have been to use some version of Bowen's shadowing lemma. In this approach we could not find any convenient and practicable technique or theory in the available literature beyond what is mainly done by [9], [5], [10] and [11]; and hence we could not proceed any further than what is done in [8].

Though we could not find any generic criteria for equality of the two spectra we have shown that even in a simple situation in two-dimensions the two spectra can be different. Also we analyzed different techniques of finding the Morse spectrum.
1.3 Outline

In Chapter 2 we explain some background material, mostly from dynamical systems. Also here we mention some results and theorems which we are going to use extensively in our work. Chapter 3 deals with calculations of Lyapunov spectrum for the three different cases. For the case of fixed point and periodic orbit the results were known. For the sake of completeness we include the derivations. Chapter 4 devotes entirely on calculation of Morse spectrum. Here with the case of an isolated fixed point we analyze in detail a case for a connected set of fixed points. In each case we compare our results with the corresponding result for the Lyapunov spectrum found in Chapter 3.

1.4 Future Aspects

The problem can be extended to control/perturbation systems. The open problem of finding the exact conditions under which the Lyapunov and the Morse spectra are the same of course still remains.
2 PRELIMINARIES

In this chapter we briefly describe our set up, collecting some basic notions and facts on linear flows on vector bundles. This set up closely coincides with the one in [7]. Some definitions are from [29, 12, 37, 38].

Definition 2.1. A continuous time flow on a metric space \((S, d)\) is given by a continuous map \(\Phi : \mathbb{R} \times S \to S\) with \(\Phi(0, p) = p, \ \Phi(t, \Phi(s, p)) = \Phi(t + s, p) \ \forall t, s \in \mathbb{R}, p \in S\).

Other notations that we might use are \(\Phi_t(p)\) or \(\Phi(t)p\) or simply \(\Phi_t\). For \(p \in S\), the function \(\Phi(\cdot, p) : \mathbb{R} \to S\) defines a trajectory or orbit of the flow through \(p\) in \(S\).

Definition 2.2. A point \(y\) is an \(\omega\)-limit point of \(x\) for \(\Phi_t\) provided there exists a sequence of \(t_k\) going to infinity such that \(\lim_{k \to \infty} d(\Phi_{t_k}(x), y) = 0\). The set of all \(\omega\)-limit points of \(x\) for \(\Phi_t\) is denoted by \(\omega(x)\) and is called the \(\omega\)-limit set.

The \(\alpha\)-limit point of \(x\) is defined the same way but with \(t_k\) going to negative infinity. The set of all such points is denoted by \(\alpha(x)\) or \(\omega^*(x)\) and is called the \(\alpha\)-limit set.

Definition 2.3. For \(A \subseteq S\), the \(\omega\)–limit set of \(A\) is given by \(\omega(A) = \{q \in S; \text{ there are } p_k \in A, t_k \to \infty \text{ with } \Phi(t_k, p_k) \to q \text{ as } k \to \infty\}\) and similarly for the \(\alpha\)–limit set \(\omega^*(A)\) via \(t_k \to -\infty\).

Definition 2.4. The limit set for a flow \(\Phi_t\) on \(S\) is \(L(\Phi_t) = \overline{\bigcup_{x \in S} \omega(x) \cup \alpha(x)}\).

Limit sets represent asymptotic behavior of certain classes of solutions.

Definition 2.5. A point \(p\) is called a fixed point for the flow \(\Phi_t\) if \(\Phi_t(p) = p\) for all \(t\). Sometimes such a point is also called an equilibrium or singular point. If the flow
is obtained as a solution of a differential equation $\dot{x} = f(x)$, then a fixed point is a point for which $f(p) = 0$. A point $p$ is called a periodic point provided there is $T > 0$ such that $\Phi_T(p) = p$ and $\Phi_t(p) \neq p$ for $0 < t < T$. The time $T > 0$ which satisfies the above conditions is called the period of the orbit. The orbit of such a point $p$ is called a periodic orbit. They are also called closed orbits as the set of points on the orbit is a closed curve.

**Definition 2.6.** An invariant set $V$ for a flow $\Phi$ is a set such that $\Phi_t(x) \in V$ for $x \in V$ for all $t \in \mathbb{R}$.

The stable and unstable manifolds of a fixed point or periodic orbit provide examples of invariant sets.

**Definition 2.7.** A compact set $V$ is called isolated invariant, if it is invariant, and there exist a neighborhood $N$ of $V$ such that $\Phi(x, t) \subset N$ implies $x \in V$ for all $t \in \mathbb{R}$.

Another important set to the study of a long-term behavior is the nonwandering set. Fixed points and periodic orbits represent stationary or repeatable behavior. A generalization of these sets is the nonwandering set.

**Definition 2.8.** For a flow $\Phi_t$ on $S$, a point $p$ is called nonwandering provided for every neighborhood $U$ of $p$ and $T > 0$ there is a time $t > T$ such that $\Phi_t(U) \cap U \neq \emptyset$. Thus there is a point $q$ in $U$ with $\Phi_t(q) \in U$. The set of all nonwandering points for $\Phi_t$ is called the nonwandering set and is denoted by $\Omega(\Phi_t)$.

The global behavior of flows on compact metric spaces can be described via Morse decompositions, which are special collections of compact invariant subsets.

**Definition 2.9.** A Morse decomposition of a flow on a compact metric space is a finite collection $\{M_i, i = 1, \ldots, n\}$ of nonvoid, pairwise disjoint, and isolated compact invariant sets such that: (i) For all $x \in S$ one has $\omega(x), \omega^*(x) \subset \bigcup_{i=1}^n M_i$. (ii) Suppose
there are $\mathcal{M}_{j_0}, \mathcal{M}_{j_1}, \ldots, \mathcal{M}_{j_l}$ and $x_1, \ldots, x_l \in S \setminus \bigcup_{i=1}^n \mathcal{M}_i$ with $\omega(x_i) \subset \mathcal{M}_{j_{i-1}}$ and $\omega(x_i) \subset \mathcal{M}_{j_i}$ for $i = 1, \ldots, l$; then $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_l}$. The elements of a Morse decomposition are called Morse sets.

Morse decompositions can be constructed from attractors and their complementary repellers. We define them next.

**Definition 2.10.** For a flow on a compact metric space $S$ a compact invariant set $A$ is an **attractor** if it admits a neighborhood $N$ such that $\omega(N) = A$.

A repeller is a compact invariant set $R$ that has a neighborhood $N$ with $\omega(N) = R$.

A neighborhood $N$ as in the above definition is called an attractor neighborhood. The empty set is considered as an attractor. A repeller is an attractor for the time reversed flow. Also, if $A$ is an attractor in $S$ and $Y \subset S$ is a compact invariant set, then $A \cap Y$ is an attractor for the flow restricted to $Y$.

Next we cite the Lemma B.2.11 from [8] pg 543 which we are going to use in our proof.

**Lemma 2.1.** For an attractor $A$, the set $A^* = \{x \in S, \omega(x) \cap A = \emptyset\}$ is a repeller, called the complementary repeller. Then $(A, A^*)$ is called an attractor-repeller pair.

$A$ and $A^*$ are disjoint. There is always a trivial attractor-repeller pair $A = X, A^* = \emptyset$. In the time reversed system the complementary repeller of $A^*$ is $A$. The following is the Lemma B.2.12 from [8] pg 543.

**Lemma 2.2.** If $(A, A^*)$ is an attractor-repeller pair and $x \notin A \cup A^*$, then $\omega(x) \subset A^*$ and $\omega(x) \subset A$.

Trajectories starting in a neighborhood of an attractor leave the neighborhood in backwards time.

The next Theorem characterizes Morse decompositions via attractor-repeller sequences. For details see [33], [34] or [8] pg 544 Theorem B.2.15.
Theorem 2.1. For a flow on a compact metric space $S$ a finite collection of subsets $\{M_1, \ldots, M_n\}$ defines a Morse decomposition if and only if there is a strictly increasing sequence of attractors

$$\phi = A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = X,$$

such that

$$M_{n-i} = A_{i+1} \cap A_i^* \text{ for } 0 \leq i \leq n - 1.$$

We will now introduce the concept of chain recurrence. The idea of the nonwandering set incorporates a weak concept of recurrence. There is a still weaker idea of recurrence, called chain recurrence, which is useful in discussion of structural stability and is one of the basic ideas for this paper.

Definition 2.11. For $x, y \in S$ an $\epsilon - T$ chain $\zeta$ from $x$ to $y$ of $\Phi$ is given by $n \in \mathbb{N}$, $T_0, \ldots, T_{n-1} \geq T$ and $p_0 = x, p_1, \ldots, p_n = y \in S$, such that $d(\Phi(T_i, p_i), p_{i+1}) < \epsilon$ for $i = 0, \ldots, n - 1$.

An $\epsilon - T$ chain is called periodic if $p_n = p_0$.

Definition 2.12. A subset $Y \subset S$ is chain transitive if for all $x, y \in Y$ and all $\epsilon, T > 0$ there exists an $\epsilon - T$ chain from $x$ to $y$.

Definition 2.13. The chain recurrent set $R(\Phi)$ is defined as $R(\Phi) = \{p \in S; \text{ for all } \epsilon, T > 0 \text{ there is a periodic } \epsilon - T \text{ chain with } p = p_0 \}$

The chain recurrent set is closed and invariant.

The next Theorem gives a relationship between the chain recurrent set and attractors. This is from [8] pg 550 [Theorem B.2.26].

Theorem 2.2. The chain recurrent set $R$ satisfies

$$R = \bigcap \{A \cup A^*, \text{ } A \text{ is an attractor} \}.$$
In particular, there exists a finest Morse decomposition \( \{ \mathcal{M}_1, \ldots, \mathcal{M}_n \} \) if and only if the chain recurrent set \( R \) has only finitely many connected components. In this case, the Morse sets coincide with the chain recurrent components of \( R \) and the flow restricted to every Morse set is chain transitive and chain recurrent.

We use the standard definition of (real) vector bundles \( \pi : E \to S \) as given, e.g. in [21], chapter I or [34], appendix.

**Definition 2.14.** Real vector bundles are denoted by \( \pi : E \to S \) with \( S \) as the base space and \( \pi \), a continuous surjection such that the fibers \( E_p = \pi^{-1}(p) \), \( p \in S \), are \( d \)-dimensional real vector spaces and \( E \) is locally isomorphic to \( S \times \mathbb{R}^d \).

We assume that the base space \( S \) is a compact connected metric space. We fix a (Riemannian) metric on \( E \) and on any fiber we denote the norm by \( |\cdot| \).

**Definition 2.15.** The zero section \( Z \) in \( E \) is a continuous map \( Z : S \to E \) given by \( Z(p) = 0 \in E_p \).

Next we introduce the projective bundle.

**Definition 2.16.** The projective bundle \( \mathbb{P}E \) is given by \( \mathbb{P}E = (E\setminus Z)/\sim \) where \( \sim \) is the equivalence relation defined by \( e \sim e' \) iff \( \pi(e) = \pi(e') \) and \( \exists \alpha \in \mathbb{R}\setminus\{0\} \) such that \( e = \alpha e' \).

**Definition 2.17.** The canonical projection map will be denoted by \( \pi : E\setminus Z \to \mathbb{P}E \).

For \( A \subseteq E \) we write \( \mathbb{P}A = \{ \mathbb{P}e; e \in A\setminus Z \} \). There exists a unique projection \( \pi : \mathbb{P}E \to S \) such that the following diagram commutes

\[
\begin{array}{ccc}
E\setminus Z & \xrightarrow{\pi} & \mathbb{P}E \\
\downarrow & & \downarrow \mathbb{P}\pi \\
S & & \\
\end{array}
\]

\( \mathbb{P}E \) is a compact metric space iff \( S \) is compact.
Definition 2.18. A linear flow $\Psi$ on a vector bundle $\pi : E \to S$ is a flow on $E$ preserving fibers such that

$$\Psi(t, e_1 + e_2) = \Psi(t, e_1) + \Psi(t, e_2), \quad t \in \mathbb{R}, e_1, e_2 \in E_p$$

and

$$\Psi(t, \alpha e) = \alpha \Psi(t, e), \quad t \in \mathbb{R}, \alpha \in \mathbb{R}, e \in E_p.$$ 

$\Psi$ induces a flow $\pi \Psi$ on the base space $S$ and a flow $\pi \Psi$ on $\pi E$.

One of the main themes of this paper is Lyapunov spectrum. We define that next.

Definition 2.19. For points $e \in E \setminus Z$ the Lyapunov Exponent or exponential growth rate of the corresponding trajectory is given by

$$\lambda(e) = \limsup_{t \to \infty} \frac{1}{t} \log |\Psi(t, e)|$$

and the Lyapunov spectrum $\Sigma_{Ly}$ of the linear flow $\Psi$ is the set of all Lyapunov Exponents

$$\Sigma_{Ly} = \{ \lambda(e); e \in E \setminus Z \}$$

Next we introduce the Morse spectrum of linear flows on vector bundles. It is based on Morse decompositions of the flow on the projective bundle, and consists of limits of (finite time) exponential growth rates of $e - T$ chains. The Morse spectrum is the union of (not necessarily disjoint) intervals and contains the Lyapunov spectrum of the flow.

Before defining the Morse spectrum formally we state some results on the form of the chain recurrent components of the projective flow $\pi \Psi$ and their relationships to the chain recurrent components of the induced flow on the base space $S$. The following is Theorem 5.2.6 in [8] pg 153.

Theorem 2.3 (Selgrade/Kl & Col). Let $\Psi$ be a linear flow on a vector bundle $\pi : E \to S$ with projected flow $\pi \Psi$ on $\pi E$ : $\pi E \to S$. Let $M \subset S$ be a chain recurrent component of the induced flow on $S$. 

(1) The chain recurrent set of $\text{IP}\Psi|((\text{IP})^{-1}M$ has finitely many components (Morse sets) $\mathcal{M}_1, \ldots, \mathcal{M}_l$ with $1 \leq l = l(M) \leq d := \dim E_p, p \in S$.

(2) Every $\mathcal{M}_i$ defines a (continuous, constant dimensional) subbundle of $\pi^{-1}M$ via $\mathcal{V}_i := \{e \in \pi^{-1}M; e \notin Z \Rightarrow \text{IP}e \in \mathcal{M}_i\}$ and the following decomposition into a Whitney sum holds

$$\pi^{-1}M = \mathcal{V}_1 \bigoplus \mathcal{V}_2 \bigoplus \ldots \bigoplus \mathcal{V}_l.$$

(3) Conversely, every chain recurrent component $\mathcal{M}$ of $\text{IP}\Psi$ is of the form described in (2), in particular $\text{IP}(\mathcal{M})$ is a chain recurrent component in $S$.

(4) The chain recurrent sets $R(\text{IP}\Psi)$ and $R(\text{IP})$ satisfy $R(\text{IP}\Psi) = \text{IP}(\pi^{-1}(R(\Psi)))$.

We will now define exponential growth rates in finite time.

**Definition 2.20.** For any $e \in E \setminus Z$ and any time $t > 0$ we define the finite time exponential growth rate by

$$\lambda^t(e) := \frac{1}{t} \log \frac{|\Psi(t, e)|}{|e|}.$$

The Morse spectrum of a linear flow $\Psi$ will be defined as the set of limits of (finite time) exponential growth rates of $e - T$ chains in the chain recurrent components of $\text{IP}\Psi$.

Let $\Psi : \mathbb{R} \times E \rightarrow E$ be a linear flow on a vector bundle $\pi : E \rightarrow S$. For $e, T > 0$ an $e - T$ chain $\zeta$ of $\text{IP}\Psi$ is given by $n \in \mathbb{N}, T_0, \ldots, T_{n-1} \geq T, \text{IP}e_0, \ldots, \text{IP}e_{n-1} \in IP$ with $d(\text{IP}\Psi(T_i, \text{IP}e_i), \text{IP}e_{i+1}) < \epsilon$ for $i = 0, \ldots, n - 1$. [Here $d(\cdot, \cdot)$ is the induced metric on $\text{IP}E$].

**Definition 2.21.** Define the exponential growth rate of $\zeta$ by

$$\lambda(\zeta) = \left( \sum_{i=0}^{n-1} T_i \right)^{-1} \sum_{i=0}^{n-1} (\log |\Psi(T_i, e_i)| - \log |e_i|)$$

with $e_i \in \text{IP}^{-1}(\text{IP}e_i)$

**Definition 2.22 (Morse spectrum).** Let $\Psi$ be a linear flow on a vector bundle. Define the Morse spectrum of $\Psi$ on a chain recurrent component $\mathcal{M}$ of the projected flow
\( \mathcal{IP}_\Psi \) as \( \Sigma_{M_0}(\mathcal{M}, \Psi) = \{ \lambda \in \mathbb{R} : \text{there are } \epsilon^k \to 0, T^k \to \infty \text{ and } (\epsilon^k, T^k)\text{-chains } \zeta^k \text{ in } \mathcal{M} \text{ with } \lambda(\zeta^k) \to \lambda \text{ as } k \to \infty \} \)

The Morse spectrum of \( \Psi \) (on \( \pi : E \to S \)) is

\[ \Sigma_{M_0}(\Psi) = \bigcup_{\mathcal{M}} \{ \Sigma_{M_0}(\mathcal{M}, \Psi) ; \mathcal{M} \text{ is a chain recurrent component of } \mathcal{IP}_\Psi \} \]

For a compact invariant set \( T \subset \mathcal{IP}_\Psi \) we define the Morse spectrum \( \Sigma_{M_0}(T, \Psi) \) of \( \Psi \) on \( T \) as the union of the Morse spectra of the chain recurrent components of \( \mathcal{IP}_\Psi | T \) defined as above.

The next Theorem is a very important result obtained by Colonius and Kliemann about the relationship between the two spectra. In our work, we will apply this theorem frequently.

**Theorem 2.4 (Kli & Col).** ([7], pg 4373) Let \( \Psi \) be a linear flow on a vector bundle \( \pi : E \to S \) and let \( \mathcal{M} \) be a chain recurrent component of the projected flow \( \mathcal{IP}_\Psi \). Then \( \Sigma_{M_0}(\Psi) \) consists of closed intervals of chain exponents and \( \Sigma_{\nu}(\Psi) \subset \Sigma_{M_0}(\Psi) = \bigcup_{\mathcal{M}} [\kappa^*(\mathcal{M}), \kappa(\mathcal{M})] \) whose boundary points are Lyapunov exponents.

For Linear flows on vector bundles a uniform exponential spectrum is defined in [13]. For a compact invariant set for the projected flow this spectrum is obtained by taking all accumulation points for the time tending to infinity of the union over the finite time exponential growth rates for all initial values in this set.

**Definition 2.23.** Let \( \mathcal{IP}_K \subset \mathcal{IP}_\Psi \) be a compact invariant set for \( \mathcal{IP}_\Psi \). We define the uniform exponential spectrum over \( \mathcal{IP}_K \) by

\[ \Sigma_{UE}(\mathcal{IP}_K) := \left\{ \mu \in \mathbb{R} \mid \text{there exist } t_k \to \infty \text{ and points } \mathcal{IP}_k \in \mathcal{IP}_K \text{ such that } \lim_{k \to \infty} \lambda^k(e_k) = \mu \right\} \]

Uniform exponential spectrum can be interpreted as a set valued extension of the Lyapunov exponent. The next two theorems are from [13].
Theorem 2.5. Let $\mathcal{F}K \subseteq \mathcal{I}PE$ be a connected compact invariant set for the flow $\mathcal{I}P\Psi$. Then there exists values $\gamma^*, \gamma \in \mathbb{R}$ such that

$$\Sigma_{UE}(\mathcal{F}K) = [\gamma^*, \gamma]$$

Furthermore there exists points $\mathcal{I}Pe^*, \mathcal{I}Pe \in \mathcal{F} K$ such that $\lambda^t(\mathcal{I}Pe^*) \leq \gamma^*$, $\lambda^t(\mathcal{I}Pe) \geq \gamma$ for all $t > 0$ and $\lim_{t \to \infty} \lambda^t(\mathcal{I}Pe^*) = \gamma^*$, $\lim_{t \to \infty} \lambda^t(\mathcal{I}Pe) = \gamma$.

The relationship of this spectrum with the Morse spectrum is given in the following theorem.

Theorem 2.6. Let $\mathcal{F}K \subseteq \mathcal{I}PE$ be a compact invariant set for the projected flow $\mathcal{I}P\Psi$ such that $\mathcal{I}P\Psi|_{\mathcal{F}K}$ is chain transitive. Then

$$\Sigma_{MO}(\mathcal{F}K) = \Sigma_{UE}(\mathcal{F}K)$$

With these background ideas we proceed with our work.
3 LYAPUNOV EXPONENTS

In this chapter we introduce our set up for the study of spectra in the nonlinear systems. For our purpose we will linearize the system and will consider the linear flow. Linear flows are a generalization of time-invariant linear differential equations. They are given by linear differential equations where the coefficients are determined by some dynamical system on a nonlinear base space. Section 3.1 describes the set up and assumptions. Section 3.2 introduces the notion of Lyapunov spectrum which is one of the main ideas for our work. We also describe some general theory and known results on the Lyapunov exponents. With these ideas we proceed to the case by case study. Section 3.3, Section 3.4 and Section 3.5 describe Lyapunov spectrum for a fixed point, a periodic orbit and a cycle respectively.

3.1 Set Up and Assumptions

We consider a two dimensional compact manifold $M$ and a chain recurrent component of a continuous time flow on $M$. Denote the tangent bundle of $M$ by $TM$. Thus $TM = \{(p, v) \in M \times \mathbb{R}^3 | v \in T_p M\}$ where $T_p M$ is the tangent space to $M$ at $p$. So the tangent bundle is the set of all possible tangent vectors to $M$, and $TM$ itself has the structure of a four-dimensional manifold.

Similarly, we denote the tangent bundle of $TM$ by $TTM$. Consider a dynamical system on $M$ given by

$$\dot{x}(t) = f(x(t)) \quad (3.1)$$
where \( f : M \to TM \) is a vector field. Here and onwards we assume that \( f \) is \( C^\infty \).

Denote by \( \Phi(t, x) \) the solutions of (3.1) with initial condition \( \Phi(0, x) = x \in M \), i.e., \( \Phi : \mathbb{R} \times M \to M \) is the flow corresponding to (3.1). Linearization along the trajectories gives a system on the tangent bundle \( TM \) described by

\[
\frac{d}{dt} Tx(t) = Tx(t) = Tf(Tx(t))
\]

where \( Tx = (x, v) \) with \( x \in M, v \in T_xM \); and, for a vector field \( f \) on \( M \) its linearization is denoted by \( Tf = (f, Df) \), i.e. \( Tf : TM \to TTM \)

or, \((x, v) \xrightarrow{Tf} ((x, v), p) \) with \( p \in T_{(x, v)}TM \).

Let us discuss what this means in local coordinates. Each point \( x \) in \( M \) possesses a neighborhood \( \mathcal{W} \) in \( M \) which is diffeomorphic to an open set \( \mathcal{U} \) of \( \mathbb{R}^2 \). A diffeomorphism \( \phi : \mathcal{W} \to \mathcal{U} \) is a coordinate system on \( \mathcal{W} \). When we write the map \( \phi \) in coordinates \( \phi = (x_1, x_2) \) the two smooth functions \( x_1, x_2 \) on \( \mathcal{W} \) are coordinate functions or local coordinates on \( \mathcal{W} \). \( (\mathcal{W}, \phi) \) is a chart for \( M \). The inverse diffeomorphism \( \phi^{-1} : \mathcal{U} \to \mathcal{W} \) is a parametrization of the neighborhood \( \mathcal{W} \).

Let \( f : M \to TM \) be locally given by \( \tilde{f} : \mathcal{U} \to \mathcal{U} \times \mathbb{R}^2 \) i.e. \( x \to (x, v) \) \( v \in T_xM \) where \( (\mathcal{U}, \phi) \) forms a chart for \( M \). We can express \( \tilde{f}(x) = \sum_{i=1}^2 \alpha_i(x) \frac{\partial}{\partial x_i} \), where \( \alpha_i \)'s are real (coefficient) functions, \( i = 1, 2 \).[[4] pg 110]. Denote the Jacobians of the coefficient functions by

\[
D_x\tilde{f}(x) = \left( \frac{\partial \alpha_k(x)}{\partial x_l} \right)_{i,k}
\]

Locally, \( Tf \) is described by \( T\tilde{f} : \mathcal{U} \times \mathbb{R}^2 \to (\mathcal{U} \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \mathbb{R}^2) \)

i.e.\( (x, v) \xrightarrow{Tf} ((x, v), \tilde{f}(x), (D_x\tilde{f}(x)v)) \)

or, \( T\tilde{f}(x, v) = ((x, v), \sum_{i=1}^2 \alpha_i(x) \frac{\partial}{\partial x_i}, \sum_{i=1}^2 (D_x\tilde{f}(x)v) \frac{\partial}{\partial y_i}) \) and the system is described by a pair of coupled differential equations given locally by

\[
\dot{x} = \tilde{f}(x(t)) \quad \text{in } TM
\]
\[
\dot{v} = D_x\tilde{f}(x(t))v(t) \quad \text{in } TTM
\]

(3.3)
$D_xf(x(t))$ is a linear map for each $x$. Thus, relative to a local coordinate system it would have a matrix representation. We denote that matrix by $A(x)$.

Let $T\Phi$ be the flow associated with the linearized system (3.2), $T\Phi: \mathbb{R} \times TM \rightarrow TM$ is locally given by $(\Phi, D\Phi)$ where $D\Phi$ (sometimes also written as $D\Phi_1$) corresponds to the solution of the second component of the linearization ($D\Phi$ is not a flow by itself).

Note that $T\Phi$ is a special case of the linear flow we denoted in Chapter 2 by $\Psi(2.18)$. From now on, for all the necessary descriptions we would use $T\Phi$ instead of $\Psi$.

As for planar flows, all the possible nonwandering sets fall into three classes, viz. (1) fixed points, (2) periodic orbits and (3) the unions of fixed points and trajectories connecting them, viz. cycles. This result is due to Andronov et al. [12] pg. 45.

We mentioned earlier in Chapter 2 that the limit set is a subset of the nonwandering set and the nonwandering set is a subset of the chain recurrent set. [12] pg 236 and [36] pg 180 mention examples where the nonwandering set is actually not equal to the chain recurrent set. Also in [12] pg 62 it is mentioned that if the phase space is planar then there is no limit set possible other than fixed points, closed orbits and cycles. For our purpose we will consider only cases where the chain recurrent components coincide with the limit sets, i.e., the only possible chain recurrent components in our case would be fixed points, periodic orbits and cycles. Also, our proofs and results are done for systems in compact invariant sets in $\mathbb{R}^2$.

### 3.2 Some Facts on the Lyapunov Spectrum

In this section we introduce the concept of Lyapunov spectrum. We also introduce the Lyapunov transformation and justify that we can use this transformation for our computations. We begin with the definition of Lyapunov spectrum.
Definition 3.1. We define the Lyapunov exponent for $x \in M$ in the direction $v \in T_x M$ as

$$\lambda(x, v) = \limsup_{t \to \infty} \frac{1}{t} \log |T\Phi(t, (x, v))|$$

$$= \limsup_{t \to \infty} \frac{1}{t} \log |D\Phi_t v|$$

and the Lyapunov spectrum for (3.1)

$$\Sigma_{Ly} = \{\lambda(x, v), (x, v) \in TM\}$$

Definition 3.2. If $\mathcal{M}$ be a chain recurrent component for the flow $\Phi$ on $M$, then the Lyapunov spectrum over $\mathcal{M}$ is defined as

$$\Sigma_{Ly}(\mathcal{M}) = \bigcup_{x_0 \in \mathcal{M}} \lambda(x_0, v) = \bigcup_{x_0 \in \mathcal{M}} \{\limsup_{t \to \infty} \frac{1}{t} \log |D\Phi_t v|, v \in T_{x_0} M\}$$

We denote the induced flow on the projective bundle $\mathbb{P}M$ by $\mathbb{P}\Phi$.

In Chapter 2 we defined the Morse spectrum of $T\Phi$ on a chain recurrent component $\mathcal{M}$ of the projected flow $\mathbb{P}\Phi$ (2.22). Also in Theorem (2.4) we found that the Lyapunov spectrum is actually contained in the Morse spectrum and the latter consists of closed intervals of chain exponents.

Selgrade's Theorem (2.3) gives us a correspondence between the chain recurrent sets for the flow in the base space and for the flow in the projective space. Thus for the chain recurrent component of the projected flow $\mathbb{P}\Phi$ we can actually think of the chain recurrent components of the flow $\Phi$ in the base space $M$ only. Thus, to calculate Morse spectrum it is enough to consider the chain recurrent components for the flow $\Phi$ in the base space $M$. We will calculate the Lyapunov spectrum over the same chain recurrent components in the base space $M$.

As mentioned in Section 3.1, for our case, the only possible chain recurrent components would be the fixed points, periodic orbits and cycles. We would linearize along each of these to calculate the Lyapunov and the Morse spectra explicitly.
Before we talk about the explicit calculations let us discuss the Lyapunov transformations. We will see that these transformations leave the Lyapunov spectrum of a system unchanged. This will enable us to transform our systems to a simpler form and perform the calculations on the transformed system. We mostly follow [[15], pg 311 and 303].

**Definition 3.3.** Let $S(t)$ be a bounded, invertible matrix with differentiable elements. Assume that matrices $S(t)^{-1}$ and $\dot{S}(t)$ are also bounded. Then the transformation $y = S(t)x$ is called a Lyapunov transformation.

The statements of the next two lemmas are from [15] pg 311.

**Lemma 3.1.** A Lyapunov transformation transforms a linear differential equation with bounded coefficients into a linear differential equation with bounded coefficients.

**Proof:** Let $\dot{x} = A(t)x$ be a differential equation with bounded coefficients. Consider a Lyapunov transformation $x = S(t)y$ with a bounded inverse $S^{-1}(t)$ and bounded $\dot{S}(t)$. Differentiating the equation of substitution, we obtain

$$\dot{x} = \dot{S}(t)y + S(t)\dot{y}$$

Substituting into the original equation we get

$$\dot{S}(t)y + S(t)\dot{y} = A(t)S(t)y$$

or,

$$\dot{y} = S(t)^{-1}(A(t)S(t) - \dot{S}(t))y$$

$$= (S(t)^{-1}A(t)S(t) - S(t)^{-1}\dot{S}(t))y$$

$$= D(t)y$$

where $D(t)$ is a matrix of bounded coefficients. Hence the proof. $\square$
Lemma 3.2. A Lyapunov Transformation does not change Lyapunov exponent.

Proof Consider the differential equation

\[ \dot{x} = A(t)x \]  

(3.5)

Set \( x(t) = S(t)y(t) \), a Lyapunov transformation. Then as above we get

\[ \dot{y} = D(t)y \]  

(3.6)

where \( D(t) = S(t)^{-1}A(t)S(t) - S(t)^{-1}\dot{S}(t) \).

If \( \Theta(t) \) is a fundamental matrix for (3.5) it will satisfy \( \dot{\Theta}(t) = A(t)\Theta(t) \).

Let the fundamental matrix of (3.6) be \( \hat{\Theta}(t) \). Then

\[ \Theta(t) = S(t)\hat{\Theta}(t) \]

and

\[ \hat{\Theta}(t) = S(t)^{-1}\Theta(t) \]

As \( S(t) \) is bounded, let the bound be \( B \), i.e. \( |S(t)| \leq |B| \).

Hence

\[ |\Theta(t)v_0| = |S(t)\hat{\Theta}(t)v_0| \leq |S(t)||\hat{\Theta}(t)v_0| \leq |B||\hat{\Theta}(t)v_0| \]

\[ \Rightarrow \frac{1}{t} \log |\Theta(t)v_0| \leq \frac{1}{t} \log |B| + \frac{1}{t} \log |\hat{\Theta}(t)v_0| \]

Thus

\[ \lambda(v_0) = \limsup_{t \to \infty} \frac{1}{t} \log |\Theta(t)v_0| \]

\[ \leq \limsup_{t \to \infty} \frac{\log |B|}{t} + \limsup_{t \to \infty} \frac{1}{t} \log |\hat{\Theta}(t,v_0)| \]

\[ = \limsup_{t \to \infty} \frac{\log |\hat{\Theta}(t)v_0|}{t} \]  

(3.7)

Similarly, as \( S(t)^{-1} \) is bounded, using \( \hat{\Theta}(t) = S(t)^{-1}\Theta(t) \),

\[ \limsup_{t \to \infty} \frac{1}{t} \log |\hat{\Theta}(t)v_0| \leq \limsup_{t \to \infty} \frac{\log |\Theta(t)v_0|}{t} \]  

(3.8)
Combining the inequalities (3.7) and (3.8) we can conclude that
\[
\limsup_{t \to \infty} \frac{1}{t} \log |\Theta(t)u_0| = \limsup_{t \to \infty} \frac{1}{t} \log |\hat{\Theta}(t)|u_0
\]
i.e. Lyapunov transformation does not change the Lyapunov exponent.

The detailed calculations for Lyapunov spectrum over the three types of chain recurrent components follow next.

3.3 Case 1: Lyapunov Spectrum for a Fixed Point

Lyapunov spectrum of a fixed point was a well known result. For the sake of completeness we include the derivation.

Let \( p \) be an isolated fixed point of (3.1) which by itself is a chain recurrent component. The system is given by \( \dot{x}(t) = f(x(t)) \). Linearization gives

\[
\begin{align*}
\dot{x}(t) &= f(x(t)) \\
\dot{v}(t) &= D_x f(x(t))v(t)
\end{align*}
\]

(3.9)

At the fixed point \( p, f(p) = 0 \), i.e.

\[ D_x f(p) v = Av, \]

(3.10)

where \( A \) is a matrix with constant coefficients.

In the Lemma 3.2 above it was shown that a Lyapunov transformation does not change the Lyapunov exponents. Thus, if we choose \( S \) to be the nonsingular matrix which reduces \( A \) to its Jordan canonical form \( B \) then the transformed equation (3.6) will look like

\[ \dot{y} = S^{-1} A S y = By \]

(3.11)

and the systems with linear component (3.9) or (3.11) will have the same Lyapunov exponents. This means, for the purpose of calculation of Lyapunov Exponents, we can actually think of the matrix \( A \) in its Jordan canonical form.
Possible Jordan forms for $A$ are
\[
\begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}, \quad \begin{bmatrix}
\lambda_1 & 1 \\
0 & \lambda_1
\end{bmatrix}
\]
where $\lambda_1, \lambda_2$ denote the eigenvalues of $A$. Then the solution looks like
\[
\begin{bmatrix}
e^{\lambda_1 t} & 0 \\
0 & e^{\lambda_2 t}
\end{bmatrix} v_0, \text{ or } \begin{bmatrix}
e^{\lambda_1 t} & 1 & t \\
0 & 1
\end{bmatrix} v_0
\]
respectively.

If
\[
v_0 = \begin{bmatrix}
v_{01} \\
v_{02}
\end{bmatrix}
\]
the solution is
\[
\begin{bmatrix}
e^{\lambda_1 t}v_{01} \\
\lambda_2 e^{\lambda_2 t}v_{02}
\end{bmatrix}, \text{ or } \begin{bmatrix}
e^{\lambda_1 t}v_{01} + t v_{02} \\
v_{02}
\end{bmatrix}
\]
respectively.

For the first case [using supnorm](for $i = 1$ or 2)
\[
\frac{1}{t} \log |e^{\lambda_i t}v_{0i}| = \frac{1}{t} \log |e^{\lambda_i t}| + \frac{1}{t} \log |v_{0i}|
\]
\[
= \frac{1}{t} \log e^{\Re(\lambda_i)t} + \frac{1}{t} \log |v_{0i}|
\]
\[
= \Re(\lambda_i) + \frac{1}{t} \log |v_{0i}|
\]
\[
\Rightarrow \limsup_{t \to \infty} \frac{1}{t} \log |e^{\lambda_i t}v_{0i}| = \Re(\lambda_i)
\]

For the second case, first note that the Lyapunov exponent of a power function is zero, since
\[
\limsup_{t \to \infty} \frac{1}{t} \log t^m = \lim_{t \to \infty} m \frac{\log t}{t} = 0
\]

Thus if $p_m(t)$ be a polynomial in $t$ with constant coefficients of degree $m$, then
\[
\limsup_{t \to \infty} \frac{1}{t} \log p_m(t) = 0. \text{ Similarly } \liminf_{t \to \infty} \frac{1}{t} \log p_m(t) = 0.
\]

What we get here are $p_0(t) = v_{02}$ and $p_1(t) = v_{01} + t v_{02}$. Thus, for $i = 0$ or 1,
\[
\limsup_{t \to \infty} \frac{1}{t} \log |e^{\lambda_i t}(p_i(t))| \leq \limsup_{t \to \infty} \frac{1}{t} \log |e^{\lambda_i t}| + 0 = \Re(\lambda_i)
For the other direction of the inequality
\[
\frac{1}{t} \log |e^{\lambda t}| = \frac{1}{t} \log |e^{\lambda t}p_i(t)| - \frac{1}{t} \log |p_i(t)| \quad (i = 0 \text{ or } 1)
\]
Hence, for \( i = 0 \text{ or } 1 \),
\[
\limsup_{t \to \infty} \frac{1}{t} \log |e^{\lambda t}| \leq \limsup_{t \to \infty} \frac{1}{t} \log |e^{\lambda t}p_i(t)| - \liminf_{t \to \infty} \frac{1}{t} \log |p_i(t)| = \limsup_{t \to \infty} \frac{1}{t} \log |e^{\lambda t}p_i(t)|
\]
Thus \( \lambda(v_0) = Re(\lambda_1) \).

i.e. the Lyapunov spectrum \( \Sigma_{Ly}(p) \) consists of the real parts of the eigenvalues of \( A \). \qed

3.4 Case 2: Lyapunov Spectrum for a Periodic Orbit

It was a known result that Lyapunov spectrum for a periodic orbit is same as its Floquet spectrum [e.g. [15] pg 311]. For the sake of completeness we cite a derivation here. We also discuss some stability results for periodic systems.

Let \( p \) be a nonconstant, \( T \)-periodic solution of (3.1). The corresponding linearized system is
\[
\dot{z}(t) = f(z(t))
\]
\[
\dot{v} = D_z f(p(t))v
\]
(3.12)
As a periodic orbit is compact it has a finite number of charts. Corresponding to each of these charts \( D_z f(p(t)) \) has a local matrix representation \( A(t) \). Overlapping charts satisfy a certain compatibility condition so that it is possible to express a matrix \( A(t) \) in terms of the other matrices in other local coordinate systems. [For details vide [37] pg 35]. \( A(t) \) is a matrix of period \( T \).

The following result is based on a discussion in [26] pg 116.

**Lemma 3.3.** There exists a Lyapunov transformation which reduces the linear component of (3.12) to \( \dot{z} = Rz \), an equation with constant coefficients.
Proof: Consider the linear component of (3.12)

\[ \dot{v} = A(t)v \quad (3.13) \]

If \( \Theta(t) \) is a fundamental matrix for (3.13), then so is \( \Theta(t + T), \ \forall t \in \mathbb{R} \). Moreover, by Floquet's theorem, corresponding to every \( \Theta \), there exists a nonsingular matrix \( P \) which is also periodic with period \( T \) and a constant matrix \( R \), such that

\[ \Theta(t) = P(t)e^{tR} \quad (3.14) \]

[We refer to [26] pg 113, & [6] pg 79]. In particular, \( P(t) \) is given by \( \Theta(t)e^{-tR} \). Being periodic and nonsingular, both \( P(t) \) and \( P(t)^{-1} \) are bounded. Also

\[ \dot{P}(t) = \dot{\Theta}(t)e^{-tR} - \Theta(t)e^{-tR}R \]
\[ = A(t)\Theta(t)e^{-tR} - P(t)R \]
\[ = A(t)P(t) - P(t)R \quad (3.15) \]

Thus, \( |\dot{P}(t)| \leq |A(t)P(t)| + |P(t)R| \). As both \( A(t) \) and \( P(t) \) are bounded and \( R \) is a constant matrix \( \dot{P}(t) \) is bounded.

Put \( v = P(t)z \) which is a Lyapunov transformation.

Then \( z = P^{-1}(t)v \)

\[ \dot{z} = (\dot{P}^{-1})v + P^{-1}\dot{v} \]
\[ = -(P^{-1})\dot{P}P^{-1}Pz + P^{-1}APz \]
\[ = -P^{-1}\dot{P}z + P^{-1}APz \]
\[ = (P^{-1})(AP - \dot{P})z \]
\[ = (P^{-1})PRz \text{ from (3.15)} \]
\[ = Rz \quad (3.16) \]
It is known [e.g. in [26] pg 114, 115] that although $\Theta$ does not determine $R$ uniquely, the set of all fundamental matrices of (3.13) and hence $A$ determines uniquely all quantities associated with $R$ which are invariant under a similarity transformation. In particular the set of all fundamental matrices of $A$ determine a unique set of eigenvalues $\mu_1, \mu_2$ of the matrix $e^{TR}$, which are the multipliers associated with $A$ or the Floquet multipliers. None of these vanish as $\mu_1\mu_2 = det(e^{TR}) \neq 0$. If the eigenvalues of $R$ be $\rho_1, \rho_2$, then $\mu_i = e^{T\rho_i}$. Thus, even though the $\rho_i$ are not uniquely determined, their real parts are. The eigenvalues of $R$ are called the characteristic exponents.

**Theorem 3.1.** Let $p$ be a nonconstant, $T$-periodic solution of (3.1), if the linearization about $p$ gives the linear component of the linearized system as $\dot{v} = A(t)v$ as explained above, then the Lyapunov exponents of (3.1) are given by the real parts of the characteristic exponents, i.e., the real parts of the eigenvalues of $R$, where $R$ is given as above in (3.14).

**Proof:** In Lemma 3.2 it is shown that a Lyapunov transformation does not change the Lyapunov exponents. Hence (3.13) and (3.16) will have the same Lyapunov exponents. As proved above in Case 1, the real parts of the eigenvalues of $R$, i.e. the characteristic exponents will be the Lyapunov exponents in this case. On a compact manifold, the Lyapunov spectrum does not depend on the charts we choose. [For reference see [37] pg 49, Proposition 1.3.4]. Thus, the real parts of the eigenvalues of an $R$, from any chart, will give the Lyapunov spectrum for the entire periodic orbit. □

**Theorem 3.2.** One of the eigenvalues of $R$ must be zero.

**Proof:** Let $p(t)$ be a periodic orbit for $\dot{x} = f(x(t))$.

$$\Rightarrow \dot{p}(t) = f(p(t))$$

$$\Rightarrow \ddot{p}(t) = Df(p(t))\dot{p}(t) = A(t)\dot{p}(t)$$
\[ \dot{v} = A(t)v \]

(i.e. \( \dot{p}(t) \) is a solution of the linearized component)

Let \( \Theta(t) \) be a fundamental solution of (3.17).

Hence \( p(t) = \Theta(t)C \).

Now \( \dot{p}(0) = f(p(0)) = f(0) \), say, \( \Rightarrow p(0) = \Theta(0)C = f(0) \Rightarrow C = f(0) \)

Thus \( \dot{p}(t) = \Theta(t)f(0) \)

\( \Rightarrow \dot{p}(T) = \Theta(T)f(0) = f(0) \)

Therefore \( \Theta(T) \) has an eigenvalue 1 with eigenvector \( f(0) \).

Also \( \Theta(t) = P(t)e^{tR} \)

\[ \Theta(T) = P(T)e^{RT}. \] [Now, \( P(T) = P(0) = \Phi(0) = I \)]

\[ = e^{RT} \]

Therefore \( R \) has one eigenvalue 0.

This means that one Lyapunov exponent for a periodic orbit is zero.

Before discussing the other Lyapunov exponent let us explain some of the terminology which we shall require.[Ref:[29], pg 106 and [26]]

**Definition 3.4.** The orbit of a point \( p \) is (Lyapunov) stable for a flow \( \psi_t \) provided that given any \( \epsilon > 0 \) there is a \( \delta(\epsilon) > 0 \) such that if \( d(x,p) < \delta(\epsilon) \), then \( d(\psi_t(x),\psi_t(p)) < \epsilon \) for all \( t \geq 0 \). If \( p \) is a fixed point, then the condition can be written as \( d(\psi_t(x), p) < \epsilon \).

**Definition 3.5.** The orbit of \( p \) is called asymptotically stable provided it is (Lyapunov) stable and there is a \( \delta_1 > 0 \) such that if \( d(x,p) < \delta_1 \) then \( d(\psi_t(x),\psi_t(p)) \) goes to zero as \( t \) goes to infinity. If \( p \) is a fixed point then it is asymptotically stable provided it is (Lyapunov) stable and \( \delta_1 > 0 \) such that if \( d(x,p) < \delta_1 \) then \( \omega(x) = \{p\} \).

**Definition 3.6.** The orbit of a point \( p \) is exponentially stable for a flow \( \psi_t \) if there exists an \( \alpha > 0 \), and for any \( \beta > 0 \), there exists a \( k(\beta) > 0 \) such that if \( |\xi| = d(x,p) < \beta \), then \( d(\psi_t(x),\psi_t(p)) < k(\beta)|\xi|e^{-\alpha t} \) for all \( t \geq 0 \).
**Definition 3.7.** The orbit of a point is unstable if it is not (Lyapunov) stable.

Recall that if \( \rho_i \) were eigenvalues of \( R \) we called \( \mu_i = e^{Tp_i} \) the Floquet multipliers, \( i = 1 \) to \( n \). As one eigenvalue of \( R \) is \( 0 \) one Floquet multiplier, say \( \mu_n \) is \( 1 \). The rest \( n - 1 \) Floquet multipliers are called the characteristic multipliers of the periodic orbit.

We include the following theorem from [29] pg 168.

**Theorem 3.3.** (a) Let \( \gamma \) be a periodic orbit for which all the \( (n - 1) \) characteristic multipliers \( \mu_j \) satisfy \( |\mu_j| < 1 \). Then \( \gamma \) is asymptotically stable.

(b) Let \( \gamma \) be a periodic orbit for which there is at least one characteristic multiplier \( \mu_k \) with \( |\mu_k| > 1 \). Then \( \gamma \) is not Lyapunov stable.

**Theorem 3.4.** For the case of a periodic orbit in two dimensions, the second Lyapunov exponent, if not zero, may be negative or positive depending on whether the periodic solution is asymptotically stable or unstable.

**Proof:** Suppose the periodic solution is asymptotically stable. If the second (nonzero) Lyapunov exponent is not negative, then it must be positive. This means that there is a characteristic multiplier \( \mu > 1 \). [\( \mu \) must be real as one Lyapunov exponent is zero]. Then by Theorem 3.3 part(b) the periodic orbit is not Lyapunov stable and hence it cannot be asymptotically stable, which is a contradiction. Hence for asymptotic stability the nonzero Lyapunov exponent must be negative.

If the periodic solution is unstable then the nonzero Lyapunov exponent cannot be negative, because, if it is so, then by part(a) of Theorem 3.3, the periodic orbit must be asymptotically stable, which is a contradiction. Thus in this case the second Lyapunov exponent must be positive.

For the other direction, suppose the second Lyapunov exponent is positive. We will like to show that the periodic orbit is unstable. If not, then the periodic orbit is (Lyapunov) stable. But this contradicts Theorem 3.3 part(b). If the second Lyapunov
exponent is negative, then Theorem 3.3 part(a) implies that the periodic orbit will be asymptotically stable.

The second Lyapunov exponent can be zero too.

**Theorem 3.5.** If a periodic orbit is Lyapunov stable but not asymptotically stable then the second Lyapunov exponent is zero.

**Proof:** If the second eigenvalue is not zero it must be either positive or negative.

If it is negative then by Theorem 3.3 the periodic orbit must be asymptotically stable which we assumed to be not true.

If it is positive then by the same theorem part(b) the periodic orbit is not stable, which is not the case here.

Thus the second Lyapunov exponent must be zero.

**Example 3.1.** Let a system be given by

\[
\begin{align*}
\dot{x} &= -y \\
\dot{y} &= x
\end{align*}
\]

The polar form for this system is

\[
\begin{align*}
\dot{r} &= 0 \\
\dot{\theta} &= 1
\end{align*}
\]

This is a stable but not asymptotically stable system. \( r = 1 \) is a periodic orbit. Linearization along this periodic orbit gives the linearized component

\[
\begin{align*}
\dot{v}_1 &= -v_2 \\
\dot{v}_2 &= v_1
\end{align*}
\]

A fundamental matrix for this is

\[
\Omega(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} = P(t)e^{2\pi R}
\]
where

\[
e^{2\pi R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

This shows that both eigenvalues of \( R \) are 0.

We say that there is an exponential growth in a neighborhood of a periodic orbit \( \gamma(t) \) if for all \( \xi_0 \) in that neighborhood the solution \( \psi_t(\xi_0) \) of \( \dot{x} = f(x) \) satisfies \( d(\psi_t(\xi_0), \gamma(t)) \leq Ce^{\alpha t} \) for some constants \( \alpha \) and \( C \) and \( t \geq 0 \). We say there is a polynomial growth if \( d(\psi_t(\xi_0), \gamma(t)) \leq Ct^n \).

**Theorem 3.6.** Let a periodic orbit be asymptotically stable. If there is no exponential growth in a neighborhood of the periodic orbit (there might be polynomial growth e.g.) then the second Lyapunov Exponent is zero.

**Proof:** If possible, let the second Lyapunov exponent be nonzero. As we assume the orbit to be asymptotically stable, this nonzero Lyapunov exponent must be negative (by Theorem 3.3 (b)). According to our notations this means the matrix \( R \) has eigenvalues 0 and \( \rho \) with \( Re(\rho) < 0 \). Let \( \rho < \alpha / T < 0 \). Then, by Theorem 11.1, [17] pg.254, \( \exists \) a \( \delta > 0 \) and a constant \( L > 0 \) with the property that for each \( \xi_0 \) on the open set disc \( d(\xi_0, C) < \delta \), where \( C : \xi = \gamma(t), 0 \leq t \leq T \), there is an asymptotic phase \( t_0 \) such that the solution \( \psi_t(\xi_0) \) of 3.1 satisfies \( d(\psi_{t+t_0}(\xi_0), \gamma(t)) \leq Le^{\alpha t/T} \) for \( t \geq 0 \).

But this shows that there is an exponential growth in the neighborhood of the periodic orbit which is a contradiction. Thus the second Lyapunov exponent must be zero too.

Example 3.2. Consider

\[
\begin{align*}
\dot{r} &= r(1 - r^2)^3 \\
\dot{\theta} &= 1
\end{align*}
\]
Periodic orbit is $r = 1$ which is asymptotically stable. The system in Cartesian coordinates is

$$
\dot{x} = x(1 - x^2 - y^2)^3 - y
$$

$$
\dot{y} = y(1 - x^2 - y^2)^3 + x
$$

Linearization over the periodic orbit gives the linearized component

$$
\dot{\mathbf{v}}_1 = -v_2
$$

$$
\dot{\mathbf{v}}_2 = v_1
$$

A fundamental matrix of this is given by

$$
\Omega(t) = \begin{bmatrix}
\cos t & -\sin t \\
\sin t & \cos t
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = P(t)e^{2\pi R}
$$

Thus both Lyapunov exponents are zero. □

The Jordan curve theorem states [ref [16] pg 366]

**Theorem 3.7.** A closed curve in a plane which does not intersect itself separates the plane into two connected components, one bounded, which is called the interior of the curve, and the other unbounded, which is called the exterior of the curve.

Thus a periodic orbit in a plane has essentially two sides, inside and outside. A periodic orbit $p(t)$ is attracting if there is a neighborhood of the orbit such that if $\xi$ belongs to that neighborhood then $d(\psi_t(\xi), p(t)) \to 0$ as $t \to \infty$. The periodic orbit is repelling if it is attracting in the time reversed sense, i.e. if there is a neighborhood such that if $\xi$ belongs to that neighborhood then $d(\psi_t(\xi), p(t)) \to 0$ as $t \to -\infty$. The periodic orbit is attracting (repelling) from one side if there is a $\delta > 0$ such that if $d(\xi, p(t)) < \delta$ where $\xi$ lies only on one side of $p(t)$, i.e. either in the interior or in the exterior, then $d(\psi_t(\xi), p(t)) \to 0$ as $t \to \infty$ (as $t \to -\infty$).
Theorem 3.8. In two dimensions, if a periodic orbit is attracting from one side and repelling from the other side then both the Lyapunov exponents are zero.

[This is a semistable case as defined in [39] pg 6, Def 1.4]

Proof: Nonzero Lyapunov exponent means in a neighborhood of the periodic orbit there is either an exponential growth or an exponential decay depending on whether the Lyapunov exponent is positive or negative respectively. So either the periodic orbit should be repelling or it should be attracting (from both sides). It cannot attract from one side and repel from the other, as is the situation here. Thus the second Lyapunov Exponent must be zero.

Example 3.3. Let

\[
\dot{r} = r(r^2 - 1)^2 \\
\dot{\theta} = 1.
\]

Here \( r = 1 \) is a semistable periodic orbit. Linearization about the periodic orbit gives the linearized component

\[
\dot{v}_1 = -v_2 \\
\dot{v}_2 = v_1
\]

A fundamental matrix for this is

\[
\Omega(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = P(t)e^{2\pi R}
\]

Thus both Lyapunov exponents are zero.

3.5 Case 3: Lyapunov Spectrum for a Cycle

In this Section we calculate the Lyapunov spectrum for a cycle. Let \( x_1, x_2 \) be two fixed points and \( \gamma \) a trajectory. We say the trajectory \( \gamma \) is connecting \( x_1, x_2 \) if for
all \( y \in \gamma \), \( \omega^*(y) = x_1 \) and \( \omega(y) = x_2 \). A cycle is a finite set of fixed points and trajectories connecting them. Let \( x_1, x_2, \ldots, x_n \) be the fixed points and \( \gamma_1, \gamma_2, \ldots, \gamma_n \) be the connecting trajectories. We call \( \{x_1, \gamma_1, x_2, \gamma_2, \ldots, x_n, \gamma_n\} \) a cycle if for \( y_1 \in \gamma_1, y_2 \in \gamma_2, \ldots, y_n \in \gamma_n \) we have \( \omega^*(y_1) = x_1, \omega(y_1) = x_2, \omega^*(y_2) = x_2, \omega(y_2) = x_3, \ldots, \omega^*(y_n) = x_n, \omega(y_n) = x_1 \).

For simplicity, we consider only two isolated fixed points \( x_1, x_2 \) and trajectories \( \gamma_1 \) and \( \gamma_2 \) connecting them. The results we get here can be extended for larger cycles. [See Figure 3.1].

Let the system

\[
\dot{x}(t) = f(x(t))
\]  (3.18)

have a cycle given by \( \Gamma\{x_1, \gamma_1, x_2, \gamma_2\} \) [see Figure 3.1]. If \( \Phi_t \) is a solution for 3.18 starting at \( y \) on the trajectory joining \( x_1 \) and \( x_2 \) then

\[
\lim_{t \to -\infty} \Phi(t, y) = x_1
\]  (3.19)

\[
\lim_{t \to \infty} \Phi(t, y) = x_2
\]  (3.20)
This is true for any $y$ on the connecting trajectory.

Let $\Phi(t, x_0)$ be a solution to the nonlinear system in the base space $M$. Linearization about $\Phi$ gives

\[
\begin{aligned}
\dot{x} &= f(x), \\
\dot{v} &= D_x f(\Phi(t))v
\end{aligned}
\]

$D_x f(\Phi(t))$ will have local matrix representations as explained before in section 3.4.

Let $T\Phi(t)$ be the flow associated with the linearized system above. For any solution $\Phi(t, x_0)$ let

\[
\lambda(x_0, v_0) = \limsup_{t \to \infty} \frac{1}{t} \log |T\Phi(t, (x_0, v_0))|
\]

**Definition 3.8.** Define the Lyapunov spectrum for the cycle as $\Sigma_{Ly}(cycle) = \{\lambda(x_0, v_0), \text{ s.t. } \Phi(t_0, x_0) \text{ is a solution in the cycle, } v_0 \in T_{x_0} M\}$.

First we develop a few ideas following [25] pg 348.

Consider a linear system in $\mathbb{R}^n$,

\[
\dot{y} = B(t)y
\]

where $B(t)$ is a time varying square matrix of dimension $n$.

Along with it consider another system

\[
\dot{y} = (B(t) + \xi(t))y
\]

The coefficient matrices $B(t)$ and $\xi(t) (\equiv \xi_{s\ell}(t)) \text{ s, } j = 1, \ldots, n$ are assumed to be bounded and continuous for $t \geq 0$.

Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the Lyapunov exponents of the system (3.21) and $\lambda'_1 \geq \lambda'_2 \geq \ldots \geq \lambda'_n$ be the Lyapunov exponents for the system (3.22).

**Definition 3.9.** The Lyapunov exponents $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the system (3.21) are called stable if for any $\epsilon > 0$ we can find a positive number $\eta(\epsilon)$ such that the Lyapunov exponents $\lambda'_i$ of the system (3.22) satisfy the inequalities

\[
|\lambda'_i - \lambda_i| < \epsilon \quad i = 1, 2, \ldots, n
\]
for any selection of functions $x_{s,j}$ which, for $t \geq 0$, satisfy

$$|x_{s,j}(t)| \leq \eta \quad (s,j = 1,2,\ldots,n) \quad (3.24)$$

If the Lyapunov exponents of the system (3.21) are stable the inequalities (3.23) will remain in force when the inequalities (3.24) are satisfied not necessarily for all $t \geq 0$, but only for $t \geq T$, where $T$ is any large number. This follows from the fact that the Lyapunov exponents of a linear system defined by the behavior of its solutions as $t \to \infty$ depend only upon the form of the coefficient matrix for $t \geq T$. For the next theorem can compare Theorem 4 in [25] pg 356.

**Theorem 3.9.** If the Lyapunov exponents of the system (3.21) are stable and if

$$\lim_{t \to \infty} x_{s,j}(t) = 0 \quad (s,j = 1,2,\ldots,n) \quad (3.25)$$

then the Lyapunov exponents of the system (3.22) coincide with the Lyapunov exponents of the system (3.21).

**Proof:** If (3.25) is satisfied, then for any $\eta > 0$ we can find a sufficiently large $T$ such that $\forall t \geq T$

$$|x_{s,j}(t)| \leq \eta.$$

Also, as the Lyapunov exponents of (3.21) are stable, for any $\epsilon > 0$ we can find an $\eta(\epsilon)$ such that if $|x_{s,j}(t)| \leq \eta$ for $t \geq T$ then $|\lambda_i' - \lambda_i| < \epsilon$.

As $\epsilon$ was an arbitrary positive number we must have $\lambda_i' = \lambda_i$. \hfill \Box

Also we refer to Theorem 3 in [25] pg.356, which in our context reads as:

**Theorem 3.10.** The Lyapunov exponents of a system of linear equations with constant coefficients are always stable.

With the above ideas we proceed with the computation of the Lyapunov spectrum for a cycle.
Theorem 3.11. The Lyapunov spectrum for a cycle consists of the Lyapunov exponents for the fixed points in the cycle.

Proof: Let \( \Phi(t, x_0) \) be a solution of the system \( \dot{x} = f(x) \) which lies on the cycle, i.e. \( \Phi(t, x_0) \) can be either one of the fixed points or can lie on the trajectory joining them.

Linearization along the solution \( \Phi(t, x_0) \) gives

\[
\begin{align*}
\dot{x} &= f(x), & (3.26) \\
\dot{v} &= D_x f(\Phi(t, x_0)) v. & (3.27)
\end{align*}
\]

If \( \Phi(t, x_0) = x_1 \) or \( x_2 \), then

\[
\dot{v} = D_x f(x_i)v, \quad i = 1, 2.
\]

Choosing charts at \( x_1 \) and \( x_2 \) and picking matrix representations, \( D_x f(x_i)v \) are two matrices. This was our first case for fixed points for which the Lyapunov exponents are the real parts of the eigenvalues of the matrix \( D_x f(x_i) \). So, Lyapunov exponents of the fixed points are contained in the Lyapunov spectrum for the cycle.

If \( \Phi(t, x_0) \) lies on \( \gamma_i \), then the second component of the linearization becomes

\[
\dot{v} = D_x f(\Phi(t, x_0)) v
\]

Pick a chart at \( x_2 \). Then we can choose a matrix representation \( A(t) \) for \( D_x f(\Phi(t, x_0)) \) locally around \( x_2 \). \( A(t) \) is a time-varying matrix.

As \( t \to \infty \), \( \Phi(t, x_0) \to x_2 \), and as \( f \) was assumed to be \( C^\infty \),

\[
A(t) \rightarrow_{t \to \infty} A = D_x f(x_2)
\]

, a constant matrix.

Consider

\[
\dot{v} = A(t)v \quad (3.28)
\]
to be the perturbed system as in (3.22) above. Take

\[ \dot{v} = Av \]  

(3.29)

to be the unperturbed system as in (3.21) above.

Since \( A(t) \to A \) as \( t \to \infty \) we can write \( A(t) = A + \xi(t) \) with \( \xi(t) \to 0 \) as \( t \to \infty \), or \( \xi_{st}(t) \to 0 \) as \( t \to \infty \) \( s, j = 1, 2 \).

System (3.29) is a system with constant coefficients, thus Lyapunov exponents of (3.29) are stable. Also \( \xi_{st}(t) \to 0 \) as \( t \to \infty \) \( s, j = 1, 2 \). Thus by Theorem (3.10) the Lyapunov exponents of (3.28) will coincide with the Lyapunov exponents of (3.29), which are the real parts of the eigenvalues of the matrix \( A = Dxf(x_2) \).

The above conclusion is true independent of the initial point. This is true because no matter where \( x_0 \) is, if \( x_0 \in \gamma_1 \) then \( \omega(x_0) = x_2 \) and \( \omega^*(x_0) = x_1 \); i.e. for all points on \( \gamma_1, x_2 \) is the \( \omega \)-limit point. So, in the discussion above, for all \( x_0 \in \gamma_1, \)

\[ A(t) \to t \to \infty A = Dxf(x_2) \]

Hence, no matter where the the trajectory \( \Phi_t \) starts on \( \gamma_1 \) the Lyapunov spectrum for the cycle will be just the Lyapunov exponents for the fixed point \( x_2 \). The same arguments would work for any trajectory on \( \gamma_2 \) where we obtain the Lyapunov exponents for the fixed point \( x_1 \).

We can extend the above arguments for a finite number (more than two) of fixed points in a similar fashion and conclude that the Lyapunov spectrum for the cycle will be the Lyapunov exponents for the fixed points in the cycle, i.e. the real parts of the eigenvalues of \( Dxf(x_i) \).
4 MORSE SPECTRUM

In this chapter we calculate the Morse spectrum for a continuous time flow in a compact invariant set \( M \) in \( \mathbb{R}^2 \). As explained at the beginning of Chapter 3 we will consider only cases where the chain recurrent components coincide with the limit sets, i.e., the only possible chain recurrent components will be an isolated fixed point, a connected set of fixed points, periodic orbits and cycles. We begin with some overview about Morse spectrum and flow on a projective bundle. Here we mention some important theorems we are going to use. Section 4.2, Section 4.3, Section 4.4 and Section 4.5 describe derivation of Morse spectrum for an isolated fixed point, for a connected set of fixed points, a periodic orbit and for a cycle respectively. As we will find, though the Morse the Lyapunov spectra agree for the first three cases they are not the same for the last one.

4.1 Overview

Recall from Definition 2.22 that the Morse spectrum is defined as the set of limit points of (finite time) exponential growth rates of \((\epsilon - T)\) chains in the chain recurrent components of \( \mathbb{P} \). More precisely [Ref [S] Pg 159], let \( \Phi : \mathbb{R} \times M \to M \) be a flow on \( M \). Let \( T \Phi : \mathbb{R} \times TM \to TM \) be the corresponding linear flow on the vector bundle \( \pi : TM \to M \). For \( \epsilon, T > 0 \) an \((\epsilon - T)\) chain \( \zeta \) of \( \mathbb{P} \) is given by \( n \in \mathbb{N}, T_0, \ldots, T_{n-1} \geq T, p_0, \ldots, p_n \in \mathbb{P}M \) (the projective bundle) with \( d(\mathbb{P}(T_i, p_i), p_{i+1}) < \epsilon \) for \( i = 0, \ldots, n - 1 \).
Define the exponential growth rate of $\zeta$ by

$$
\lambda(\zeta) = \left( \sum_{i=0}^{n-1} T_i \right)^{-1} \sum_{i=0}^{n-1} (\log |T\Phi(T_i, e_i)| - \log |e_i|)
$$

with $e_i \in \mathbb{P}^{-1} p_i$.

Let $\mathcal{L} \subset \mathbb{P}M$ be a compact invariant set for the induced flow $T\Phi$ on $\mathbb{P}M$ and assume that $\mathbb{P}\Phi|\mathcal{L}$ is chain transitive. Then the Morse spectrum over $\mathcal{L}$ is

$$
\Sigma_{M_0}(\mathcal{L}, T\Phi) = \{ \lambda \in \mathbb{R}, \text{there are } \epsilon^k \to 0, \ T^k \to \infty \}
$$

and $(\epsilon^k, T^k)$ - chains $\zeta^k$ with $\lambda(\zeta^k) \to \lambda$ as $k \to \infty$.

For a compact invariant set $L \subset M$, the Morse spectrum over $L$ is defined as

$$
\Sigma_{M_0}(L, T\Phi) = \bigcup \Sigma_{M_0}(\mathcal{M}),
$$

where the union is taken over all chain recurrent components $\mathcal{M}$ of $\mathbb{P}\Phi|\mathbb{P}^{-1} L$.

Finally, the Morse spectrum of the flow $T\Phi$ is defined as

$$
\Sigma_{M_0}(T\Phi) = \Sigma_{M_0}(\mathbb{P}M, T\Phi) = \Sigma_{M_0}(M, T\Phi).
$$

To calculate the Morse spectrum we need the chain recurrent components of the induced flow in the projective bundle. As explained in [8] pg 142, in order to motivate the construction and the associated decomposition into invariant subbundles, we consider the special case of constant coefficients. Consider a time-invariant linear differential equation in $\mathbb{R}^2$ of the form

$$
\dot{v}(t) = Av(t), \ v(0) = v_0 \neq 0
$$

with $A \in \mathbb{R}^{2 \times 2}$. We can project the linear equation down to the unit sphere $S^1$ by defining $s(t) = \frac{v(t)}{|v(t)|}$. Applying chain rule we get

$$
\dot{s}(t) = [A - s(t)^T As(t)].I]s
$$
Hence the projected trajectories are the trajectories of a nonlinear differential equation defined on the unit sphere leaving the unit sphere invariant. The original equation induces a differential equation in projective space $\mathbb{P}^d$ obtained by identifying opposite points on the sphere.

Consider a real eigenvalue $\lambda$ of the matrix $A$ and a corresponding eigenvector $v$, i.e. $Av = \lambda v$.

$$\left[ A - s^T A s. I \right] s = \left[ A - \frac{v^T v}{|v|^2} I \right] \frac{v}{|v|} = \left[ A - \lambda I \right] \frac{v}{|v|} = 0$$

Thus projection of $v$ onto the unit sphere is an equilibrium point of the differential equation (4.2). Conversely, every equilibrium point on the sphere corresponds to an eigenvector for a real eigenvalue $\lambda$.

For general equations of the form $\dot{v} = Av$, the sums of (generalized) eigenspaces corresponding to eigenvalues with equal real part is described topologically in Proposition 5.1.2, pg 142, which is as follows.

**Theorem 4.1.** Let $W$ be the sum of all real (generalized) eigenspaces of $A$ corresponding to all eigenvalues $\lambda_i$ with equal real part. Then the projection $\mathbb{P}W$ of $W$ onto the projective space is a chain recurrent component of the induced differential equation and every such component is of this form.

**Proof:** For proof see [8], Pg 142. □

We also mention Lemma 5.3.1 in [8] pg 159, which is as follows:

**Lemma 4.1.** For a linear flow $T\Phi$ the following assertions hold:

(i) The chain recurrent set $\mathcal{R}(\Phi) \subset M$ of the base flow $\Phi$ satisfies

$$\Sigma_{M_0}(T\Phi) = \Sigma_{M_0}(\pi^{-1}(\mathcal{R}(\Phi)), T\Phi)$$

(ii) It holds that $\Sigma_{M_0}(T\Phi) = \bigcup_{M_0} \Sigma_{M_0}(\pi^{-1}M, T\Phi)$, where the union is taken over all chain recurrent components $M$ of $\Phi$. 
(iii) If $\{M_1, M_2, \ldots, M_n\}$ is a Morse decomposition of the base flow $\Phi$, then $\Sigma_{M_0}(T\Phi) = \bigcup \Sigma_{M_0}(\pi^{-1}M_j, T\Phi)$. In particular, if $\Phi$ has a finest Morse decomposition $\{M_1, M_2, \ldots, M_n\}$, then $\Sigma_{M_0}(T\Phi) = \bigcup_{j=1}^n \bigcup_{i=1}^{l(M_j)} \Sigma_{M_0}(M_{ji}, T\Phi)$ where $M_{ji}, i = 1, \ldots, l(M_j)$, denotes the chain recurrent components of $\Phi((\pi)^{-1}M_j)$.

It is proved in [8] pg.160 that it is actually sufficient to consider periodic chains in the definition of the Morse spectrum, i.e., the exponents of chains from a point to itself.

Also we will use Theorem 2.4 extensively in the following.

4.2 Case 1: Morse Spectrum for an Isolated Fixed Point

In this section we consider the case of an isolated fixed point, i.e. for a fixed point that is in itself a chain recurrent component.

For the procedure we will actually mention two approaches and show that the first one breaks down at a point and we cannot use it. Nonetheless, we mention the procedure and explain why it does not work.

Let $x^*$ be an isolated fixed point for the system $\dot{x} = f(x)$ in $M$. As before, we assume $f$ to be $C^\infty$. Linearization of the system about the fixed point gives

$$\begin{cases} \dot{x} = f(x) & \text{in } M \\ \dot{v} = D_x f(x^*) v = A(x^*) v = Av & \text{in } T_{x^*}M. \end{cases}$$

(4.3)

To calculate the Morse spectrum one possible approach might have been to consider the matrix $D_x f(x^*) = A(x^*)$ in its Jordan canonical form. If we can show that the Morse spectrum does not change under linear coordinate change then we can choose $A$ in its Jordan canonical form which might help us to calculate the Morse exponents directly (as then we could work with just four types of specific $2 \times 2$ matrices). There is a theorem in [7] stating that if there is a "cohomology" (explained later) between two linear flows then the two flows will have the same Morse spectrum. In the next few pages we try to
check if a linear change of coordinates is a cohomology. First we discuss what is meant by a cohomology.

The following definition is from [8] pg.168.

**Definition 4.1.** A cohomology $F$ between a linear flow $\Phi_t$ on $\pi_1 : E_1 \to S$ and a linear flow $\Psi_t$ on $\pi_2 : E_2 \to S$ is a fiber preserving homeomorphism $F : E_1 \to E_2$ such that the induced maps $F_p$ on the fibers are linear, and the diagram

$$
\begin{array}{ccc}
E_1 & \xrightarrow{F} & E_2 \\
\Phi_t \downarrow & & \Psi_t \downarrow \\
E_1 & \xrightarrow{F_p} & E_2 \\
\end{array}
$$

commutes.

A cohomology $F$ between $\Phi_t$ and $\Psi_t$ induces a homeomorphism $IPF$ on the associated projective bundle such that

$$
\begin{array}{ccc}
IP E_1 & \xrightarrow{IPF} & IP E_2 \\
I\Phi_t \downarrow & & I\Psi_t \downarrow \\
IP E_1 & \xrightarrow{IPF} & IP E_2 \\
\end{array}
$$

commutes for all $t \in \mathbb{R}$.

We cite next a proposition from [7].

**Proposition 4.1.** Let $F$ be a cohomology between the linear flows $\Phi$ on $\pi_1 : E_1 \to S$ and $\Psi$ on $\pi_2 : E_2 \to S$. Then

(i) $IPF(\mathcal{R}(I\Phi)) = \mathcal{R}(I\Psi)$

(ii) $\Sigma_{Mo}(\Phi) = \Sigma_{Mo}(\Psi)$

(iii) $\Sigma_{Ly}(\Phi) = \Sigma_{Ly}(\Psi)$

□
Thus if we can show that a linear change of coordinates is a cohomology $F : TM \to TM$ then we can apply the above Proposition (4.1) and work with the Jordan canonical forms of $A$ instead of working with the actual matrix $A$ to calculate the Morse spectrum.

We try to define a cohomology $F$ on the tangent bundle which on the base component is identity (to ensure that the base point remains unchanged under this map) and on the linear component it is chosen in such a way so that it transforms the matrix $A$ to its Jordan canonical form. $F : TM \to TM$ preserves fibers. If $\pi F$ is the induced map on the base then define $\pi F : M \to M$ as $\pi F(p) = p$. This leaves the base point unchanged under $F$, only change would be in the linear component.

Given a vector field $f : M \to TM$ let $(p, v) \in TM$. Let $(\mathcal{W}, \phi)$ be a chart on $M$ and $\mathcal{U} = \phi(\mathcal{W}) \in \mathbb{R}^d$. Then $\phi^{-1} : \mathcal{U} \to \mathcal{W}$ is a parametrization of the neighborhood $\mathcal{W}$ of $p$. [For reference see [14] pgs 8-11].

\[
p \in M \xrightarrow{f} TM \ni (p, f(p)) \\
\phi \downarrow \quad \downarrow (\phi, d\phi) \\
\mathcal{U} \xrightarrow{f} \mathcal{U} \times \mathbb{R}^d
\]

\[
(p, v) \in TM \xrightarrow{Tf} TT M \ni ((p, v), f(p), Df(v)) \\
(\phi, d\phi) \downarrow \quad \downarrow (\phi, d\phi, d(\phi, d\phi)) \\
\mathcal{U} \times \mathbb{R}^d \xrightarrow{Tf} (\mathcal{U} \times \mathbb{R}^d) \times (\mathbb{R}^d \times \mathbb{R}^d)
\]

\[
\hat{f}(\phi(p)) = (\phi(p), d\phi(f(p))) \\
\hat{T}f(\phi(p), d\phi(v)) = (\phi(p), d\phi(v), d\phi(f(p)), d(d\phi)Df(v))
\]

$d(d\phi)Df$ is a linear map for each $p$, which is the Jacobian of $f$. $d(d\phi)Df(v)$ is a point in $\mathbb{R}^d$, and $\phi$ defines a standard coordinate system. This coordinate system has a basis and $d(d\phi)Df$ will be a matrix $A(p)$ relative to this basis. In the schematic diagram below we indicate this choice of basis as $\delta(\phi)$ and $E^d_\phi$ is the space $\mathbb{R}^d$ under this choice.
of basis. Once we get the matrix $A(p)$ we can transform it to its real Jordan canonical form $B(p)$ via the nonsingular matrix of generalized eigenvectors $J(p)$. Essentially, we are making a change of coordinates both in the domain and range of $A(p)$. $E^d_{J(p)}$ is the space $J(p)E^d_{\phi}$. We keep in mind that $A(p)$, $B(p)$ and $J(p)$ depends on $\phi$.

\[
\begin{array}{ccc}
\mathbb{R}^d & \xrightarrow{d(\phi)D} & \mathbb{R}^d \\
\iota(\phi) \downarrow & & \iota(\phi) \downarrow \\
E^d_{\phi} & \xrightarrow{A(p)} & E^d_{\phi} \\
J(p) \downarrow & & J(p) \downarrow \\
E^d_{J(p)} & \xrightarrow{B(p)} & E^d_{J(p)}
\end{array}
\]

Now we can define $F$ (locally).

\[
\begin{array}{ccc}
TM & \xrightarrow{F} & TM \\
(\phi,d\phi) \downarrow & & \downarrow (\phi,d\phi) \\
\mathcal{U} \times \mathbb{R}^d & \xrightarrow{\tilde{F}} & \mathcal{U} \times \mathbb{R}^d
\end{array}
\]

We define $\tilde{F} : \mathcal{U} \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d$ as $\tilde{F}(\phi(p), d\phi(v)) = (\phi(p), J_{\phi}(\phi)d\phi(v))$.

First we show that this definition leaves $F$ independent of a specific parametrization. Consider another parametrization $(\mathcal{V}, \xi^{-1})$. To stress the specific parametrization dependence we write $J_{\phi}(p)$ and $J_{\xi}(p)$ instead of $J(p)$ in this proof for independence of specific parametrization. For the rest of the discussion we will simply write $J(p)$ keeping in mind that it actually depends on $\phi$. With respect to the second parametrization $F$ is defined locally through $\tilde{G} : \mathcal{V} \times \mathbb{R}^d \to \mathcal{V} \times \mathbb{R}^d$ given by $\tilde{G}(\xi(p), d\xi(v)) = (\xi(p), J_{\xi}(p)d\xi(v))$.

\[
\begin{array}{ccc}
\mathcal{V} \times \mathbb{R}^d & \xrightarrow{G} & \mathcal{V} \times \mathbb{R}^d \\
(\xi,d\xi) \uparrow & & \uparrow (\xi,d\xi) \\
TM & \xrightarrow{F} & TM \\
(\phi,d\phi) \downarrow & & \downarrow (\phi,d\phi) \\
\mathcal{U} \times \mathbb{R}^d & \xrightarrow{\tilde{F}} & \mathcal{U} \times \mathbb{R}^d
\end{array}
\]
For the first parametrization \((\mathcal{U}, \phi^{-1})\):

\[ F(p, v) = (p, d\phi^{-1} J\phi(p)d\phi(v)). \]

For the second parametrization \((\mathcal{V}, \xi^{-1})\):

\[ F(p, v) = (p, d\xi^{-1} J\xi(p)d\xi(v)). \]

Now,

\[ d\phi^{-1} J\phi(p)d\phi(v) = d\phi^{-1} d\phi d\xi J\xi(p) d\xi d\phi^{-1} d\phi(v) = d\xi^{-1} J\xi(p)d\xi(v) \]

Hence \( F \) is independent of a specific parametrization.

Now that \( F \) is well defined we proceed to prove that \( F \) is a cohomology.

First we show that the following diagram (notations are explained in the following) commutes.

\[
\begin{array}{ccc}
TM & \xrightarrow{F} & TM \\
\downarrow T\Phi_i & & \downarrow T(\Theta)\Phi_i \\
TM & \xrightarrow{F} & TM
\end{array}
\]

Let \( A(p) \) be the matrix representation of \( d(d\phi)Df \) given by a specific parametrization \((\mathcal{U}, \phi^{-1})\). Depending on this chart let \( F \) be locally represented by \( \tilde{F} \) defined as before.

We also showed that \( F \) is independent of any specific local representation. Also let us identify \( d\phi(v) \) and \( v \), i.e., what we denoted by \( d\phi(v) \) previously, we will simply denote that by \( v \). Thus we get a system given locally by

\[
\begin{align*}
\dot{z} &= f(x) \\
\dot{v} &= A(p)v
\end{align*}
\]

This system gives a linear flow \( T\Phi_i \) from \( TM \) to \( TM \) starting at \( v_0 \). Call \( B(p) \) the real Jordan form of \( A(p) \); i.e. \( \exists \) nonsingular \( J(p) \) such that \( J(p)A(p) = B(p)J(p) \). Writing \( y = J(p)v \) we get from (4.4),

\[
\dot{y} = J(p)A(p)J(p)^{-1}y = B(p)y
\]
This gives a second system, system II

\[
\begin{align*}
\dot{x} &= f(x) \\
\dot{y} &= B(p)y
\end{align*}
\] (4.5)

\(B(p)\) depends on the specific parametrization \((U, \phi^{-1})\) and \(y\) is in local coordinates. Let the linear flow associated with this system be \(T(B)\Phi_t\) which starts at \(y_0 = J(p)v_0\). So \(T(B)\Phi_t\) satisfies the (local) linear component of system II:

\[
T(B)\Phi_t = B(p)T(B)\Phi_t
\]

\[
T(B)\Phi_0 = J(p)v_0
\]

and \(T\Phi\) satisfies

\[
T\Phi_t = A(p)T\Phi_t
\]

or \(J(p)T\Phi_t = J(p)A(p)T\Phi_t\)

or \(J(p)T\Phi_t = B(p)(J(p)T\Phi_t)\)

Also, \(J(p)T\Phi_0 = J(p)v_0\). So, \(J(p)T\Phi_t\) is the trajectory of the flow corresponding to 4.5 with \(T\Phi_0 = v_0\), i.e. the trajectory of \(J(p)T\Phi_t\) starting at \(v_0\) is same as \(T(B)\Phi_t\) starting at \(J(p)v_0\).

Start with \((\phi(p), v_0)\).

\[
\tilde{F}(\phi(p), v_0) = (\phi(p), J(p)v_0).
\]

\[
T(B)\Phi_t(\phi(p), J(p)v_0) = (\phi(p), T(B)\Phi_t, J(p)v_0)
\]

\[
= (\phi(p), J(p)T\Phi_t v_0)
\]

by the above discussion.

\[
\tilde{F}^{-1}(\phi(p), J(p)T\Phi_t v_0) = (\phi(p), J(p)^{-1}J(p)T\Phi_t v_0) = (\phi(p), T\Phi_t v_0)
\] (4.6)

(4.7)
and the diagram commutes.

The next step is to show that the induced map on the fibers are linear. For the fiber $T_p M$ denote the induced map as $F_p$.

\[
\begin{array}{ccc}
T_p M & \xrightarrow{F_p} & T_p M \\
(\sigma, d\sigma) \downarrow & & \downarrow (\sigma, d\sigma) \\
U \times \mathbb{R}^d & \xrightarrow{F_p} & U \times \mathbb{R}^d
\end{array}
\]

\[
F_p(p, v_1 + v_2) = (p, d\phi^{-1} J(p) d\phi(v_1 + v_2))
\]

\[
= (p, d\phi^{-1} J(p) d\phi(v_1)) + (p, d\phi^{-1} J(p) d\phi(v_2))
\]

(as $T_p M$ has the structure of a vector space.)

\[
= F_p(p, v_1) + F_p(p, v_2).
\]

Thus $F$ is linear on the fibers.

Lastly, we have to show that $F$ is a homeomorphism.

1-1: Let $(p_1, v_1) \neq (p_2, v_2)$ in $T\phi^{-1}[U]$. Then $\phi(p_1) \neq \phi(p_2)$ as $\phi$ is 1-1 and $d\phi(v_1) \neq d\phi(v_2)$ as $d\phi$ is 1-1.

\[
\Rightarrow (\phi(p_1), J(p_1) d\phi(v_1)) \neq (\phi(p_2), J(p_2) d\phi(v_2)).
\]

[Although the second components might be the same, the first components are not, hence they are unequal.]

\[
\Rightarrow (p_1, d\phi^{-1} J(p_1) d\phi(v_1)) \neq (p_2, d\phi^{-1} J(p_2) d\phi(v_2))
\]

Also, if $(p, v_1) \neq (p, v_2)$, then

\[
(\phi(p), J(p) d\phi(v_1)) \neq (\phi(p), J(p) d\phi(v_2))
\]

[the second component is not the same.]

\[
\Rightarrow (p, d\phi^{-1} J(p) d\phi(v_1)) \neq (p, d\phi^{-1} J(p) d\phi(v_2)).
\]
Thus $F$ is 1-1.

**Onto:** Consider the point $(\phi(p), d\phi(v)) \in U \times \mathbb{R}^d$. As $J(p)$ is nonsingular $(\phi(p), d\phi(v))$ has a preimage with respect to $\tilde{F}$ in $U \times \mathbb{R}^d$, the preimage is $(\phi(p), J^{-1}(p)d\phi(v))$. Then the point $(p, d\phi^{-1}J^{-1}(p)d\phi(v))$ will be a preimage with respect to $F$ of the point $(p, v) \in TM$. Thus $F$ is onto.

Next we have to show that $F$ is continuous.

There is no problem in the first component as $\phi$ is a diffeomorphism and the identity map is continuous. But there is a problem with respect to the second component as Jordan form of a matrix need not be a continuous function of the entries of the matrix. An example to show this is in [19] pg 127. Consider

$$A_\epsilon = \begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix}, \quad \epsilon \neq 0$$

$$A_\epsilon = S_\epsilon J_\epsilon S_\epsilon^{-1}$$

with

$$S_\epsilon = \begin{bmatrix} 0 & \epsilon \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad J_\epsilon = \begin{bmatrix} 0 & 0 \\ 0 & \epsilon \end{bmatrix}.$$ 

If we let $\epsilon \to 0$, then

$$J_\epsilon \to \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which cannot be a Jordan form of the nonzero matrix

$$A_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$ 

This happens because though eigenvalues of a matrix depend continuously upon its entries, eigenvectors in general, do not [for reference see [22] Pg 100,111]. Thus the function $J(p)$ is not necessarily continuous, i.e. if $A(p_n) \to A(p)$ then $J(p_n)$ may not tend to $J(p)$ where $J(p)$ is the Jordan form corresponding to $A(p)$. Thus $F$ is not continuous, hence not a homeomorphism and this approach breaks down at this point.
As we couldn't prove that $F$ is a cohomology, we cannot actually work with the Jordan canonical forms of $A$. We approach the problem from a different direction, viz., we use Theorem 4.1 and find the chain recurrent components for the flow $\Phi$ and use Theorem 2.4 to find the Morse spectrum. First we state another theorem based on Theorem 5.4.12, [8] pg 176, which will be useful.

**Theorem 4.2.** Let $M$ be a chain recurrent component in $\mathbb{P}M$. Let $[\kappa^*(M), \kappa(M)]$ be the interval of Morse spectrum over $M$. Then there exist two points $P_v^*$ and $P_v$ in $M$ such that

\[
\kappa^*(M) = \lim_{t \to \infty} \frac{1}{t} \log |T\Phi(t, v^*)| \tag{4.8}
\]
\[
\kappa(M) = \lim_{t \to \infty} \frac{1}{t} \log |T\Phi(t, v)| \tag{4.9}
\]

**Proof:** Here $T\Phi$ is a linear flow on a vector bundle $\pi : TM \to M$ and as $M$ is a chain recurrent component in $\mathbb{P}M$ it is a compact $\mathbb{P}\Phi$-invariant set. Then by Theorem 5.4.12 in [8], pg 176, there exists ergodic $\mathbb{P}\Phi$-invariant probability measures $\mu^*(M)$ and $\mu(M)$ with support in $M$ such that $\kappa^*(M) = \lim_{t \to \pm \infty} \frac{1}{t} \log |T\Phi(t, v)|$ for $\mu^*(M)$-almost all $P_v \in \mathbb{P}M$ and $\kappa(M) = \lim_{t \to \pm \infty} \frac{1}{t} \log |T\Phi(t, v)|$ for $\mu(M)$-almost all $P_v \in \mathbb{P}M$.

As the support of a measure is contained in a closed set there exists at least one point $P_v^*$ in $M$ for which (4.8) is true and at least one point $P_v$ in $M$ for which (4.9) is true. \hfill \Box

Let $\mathcal{M}$ be a compact chain recurrent component in $M$ and $\mathcal{M}$ a chain recurrent component in $\mathbb{P}M$ with projection $\mathbb{P}_\pi(M) = \mathcal{M}$. The theorem above tells us that the boundaries of the Morse interval over $M$, $\kappa^*(M)$ and $\kappa(M)$, are contained in $\sum_{L_\pi}(\mathcal{M})$.

**Theorem 4.3.** If a fixed point is an isolated chain recurrent component then its Morse spectrum coincides with the Lyapunov spectrum.

**Proof:** Let $x^*$ be an isolated fixed point for the system

\[
\dot{x} = f(x) \tag{4.10}
\]
in M. Linearization about the fixed point gives

\[
\begin{align*}
\dot{x} &= f(x) \quad \text{in } M \\
\dot{v} &= Df(x^*)v = Av \quad \text{in } T_xM.
\end{align*}
\]  

(4.11)

where $A$ is a constant matrix.

**Case 1:** Let $A$ have two distinct eigenvalues $\lambda_1$ and $\lambda_2$. For each $\lambda_i$ ($i = 1, 2$) we would have an eigenspace $W_i$ of $A$. Then by Theorem (4.1) the projection $\mathbb{P}W_i$ of $W_i$ onto the projective space is a chain recurrent component of the induced differential equation. So there are two chain recurrent components for the flow in the projective space. As the Morse spectrum over each chain recurrent component for the flow in the projective space is an interval with boundary points being Lyapunov exponents (by Theorem 2.4 and Theorem 4.2), the only possibility for the Morse spectrum in this case is $\{\lambda_1\}$ and $\{\lambda_2\}$.

**Case 2:** Let $A$ have a single eigenvalue $\lambda_1$, but $A$ is diagonalizable, i.e., $A$ has two distinct eigendirections corresponding to $\lambda_1$. $W$ in Theorem (4.1) is the sum of all eigenspaces of $A$ corresponding to all eigenvalues with equal real part, $\lambda_1$. Then the projection $\mathbb{P}W$ of $W$ onto the projective space gives a single chain recurrent component for the projective flow. Thus there will be a single interval of Morse spectrum, which is $\{\lambda_1\}$ as $\lambda_1$ is the only Lyapounov exponent in this case.

**Case 3:** Let $A$ have a single eigenvalue $\lambda_1$ corresponding to which there is one eigenvector and one generalized eigenvector. By Theorem (4.1) there is a single chain recurrent component for the flow in the projective space. Thus the Morse spectrum is $\lambda_1$, same as the Lyapunov spectrum.

**Case 4:** Let $A$ have a pair of complex eigenvalues $\alpha + i\beta$ and $\alpha - i\beta$. The Lyapunov exponent is $\alpha$. $W$ is the sum of real eigenspaces of $A$ corresponding to the two eigenvalues (they have the same real part). Projection $\mathbb{P}W$ of $W$ is a single chain recurrent component for the flow in the projective space. Thus the Morse spectrum will be $\{\alpha\}$. 
4.3 Case 2: Morse Spectrum for a Connected Set of Fixed Points

The next case we consider is when there is a connected set of fixed points for the system (4.10).

Let $F$ be a connected set of fixed points. $F$, being a closed subset of compact $M$, is compact. For each point $x \in F$ we have a projective bundle $\{x\} \times \mathbb{P}(x)M$. Define the projective bundle $\mathbb{P}F$ as $\mathbb{P}F = \bigcup_{x \in F} \{x\} \times \mathbb{P}(x)M \subset \mathbb{P}M$. $\mathbb{P}F$ is connected and compact.

**Definition 4.2.** A linear system $\dot{x} = Ax$ is nonsimple if at least one of the eigenvalues of $A$ is zero. A fixed point of a nonlinear system is said to be nonsimple if the corresponding linearized system is nonsimple.

An example of such a system in two dimensions:[(2], pp.46, 82]

**Example 4.1.**

\[
\begin{align*}
\dot{x}_1 &= x_1 - x_2^2 \\
\dot{x}_2 &= x_2(x_1 - x_2^2)
\end{align*}
\]

*Fixed points lie on the parabola $x_2^2 = x_1$. A typical fixed point $(k^2, k)$, $k$ real, has linearization $\dot{v} = Av$ where*

\[
A = \begin{bmatrix} 1 & -2k \\ k & -2k^2 \end{bmatrix}
\]

*Clearly, $\det(A) = 0 \ \forall \ k$ and so the fixed points are all non-simple.*

We state the inverse function theorem here which we will use to prove the next theorem.
Theorem 4.4 (Inverse Function Theorem). [For reference see [18], pg 337] Let $W$ be an open set in a vector space $E$, and let $f : W \mapsto E$ be a $C^1$ map. Suppose $x_0 \in W$ is such that $D_x f(x_0)$ is an invertible linear operator on $E$. Then $x_0$ has an open neighborhood $V \subset W$ such that $f|V$ is a diffeomorphism onto an open set.

**Theorem 4.5.** Let $F$ be a connected set of fixed points of a system $\dot{x} = f(x)$. Linearization about a fixed point $x^* \in F$ gives the linearized system

$$
\dot{x} = f(x) \\
\dot{v} = D_x f(x^*)v = A_{x^*} v
$$

where $A_{x^*}$ is a constant matrix. Then one eigenvalue of $A_{x^*}$ will be zero.

**Proof:** Assume $A_{x^*}$ is invertible. Let $f : M \mapsto N$, $(U, \phi)$ a chart on $M$ and $(U^*, \psi)$ a chart on $N$. Let $x^* \in U$, an open set in $M$. Then $\phi(U)$ is an open set in $\mathbb{R}^2$ and locally $\tilde{f} : \phi(U) \mapsto \mathbb{R}^2$. We assume $\tilde{f}$ to be $C^\infty$.

Applying the Inverse Function Theorem 4.4 under our assumption that $A_{x^*}$ is invertible, $\phi(x^*)$ has an open neighborhood $V$ such that $\tilde{f}|V$ is a diffeomorphism onto an open set.

But in the neighborhood $\phi^{-1}(V)$ of $x^*$, there exists at least another fixed point $x_0$ of $f$. Hence,

$$
\tilde{f}(\phi(x^*)) = \psi \circ f \circ \phi^{-1}(\phi(x^*)) = \psi(0) = \psi \circ f \circ \phi^{-1}(\phi(x_0)) = \tilde{f}(\phi(x_0)).
$$

Thus $\tilde{f}|V$ cannot be $1-1$ unless $x^*$ is an isolated fixed point, which is not true in this case. This gives a contradiction and $x^*$ cannot be a simple fixed point, i.e. $\det D_x f(x^*)$ must be zero $\Rightarrow$ at least one Lyapunov exponent is always zero in this case. $\square$

For this case there are two possible situations:

1. For all $x^* \in F$, $A_{x^*}$ has two distinct eigenvalues $\lambda_1(x^*)$, $\lambda_2(x^*)$, one of which is zero.
2. For at least one $x^* \in F$, $A_{x^*}$ has both eigenvalues zero.

In the following subsections we analyze these two situations.
4.3.1 Morse Spectrum when $A_x$ has Two Distinct Eigenvalues

We consider the system $\dot{x} = f(x)$.

Linearization over a fixed point $x \in F$ gives

$$\dot{x} = f(x)$$
$$\dot{v} = A_x v$$

$f$ being $C^\infty$, $A_x$ is continuous in $x$, thus $\lambda(x)$ is continuous. As $\lambda_1(x) \neq \lambda_2(x)$ for each $x$, we have two distinct eigenvectors $e_1(x)$ and $e_2(x)$ which are continuous [[17] pp.69-70].

Consider $E_1 = \bigcup_{x \in F} \{ x \} \times \mathbb{P}e_1(x)$ and $E_2 = \bigcup_{x \in F} \{ x \} \times \mathbb{P}e_2(x)$. $E_1$ and $E_2$ are continuous in $\mathbb{P}F$. Over each $x \in F$ we have a projective bundle $\{ x \} \times \mathbb{P}\{ x \}$ and an induced flow in the projective space. According to Theorem 4.1 $\{ x \} \times \mathbb{P}e_1(x)$ and $\{ x \} \times \mathbb{P}e_2(x)$ gives the fixed points of this induced flow in the projective space.

**Proposition 4.2.** Let $x \in F$ and $A_x$ be the Jacobian matrix of partial derivatives for the linearized system about $x$. Let $A_x$ have two distinct eigenvalues $\lambda_1(x)$ and $\lambda_2(x)$ with $\lambda_1(x) > \lambda_2(x)$. Let $e_1(x)$ and $e_2(x)$ be the corresponding eigendirections respectively. Then $\{ x \} \times \mathbb{P}e_1(x)$ will be the attracting fixed point for the induced flow in the projective bundle and $\{ x \} \times \mathbb{P}e_2(x)$ will be the repelling one.

**Proof** First of all, let us transform $A_x$ to its diagonal form

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where $\lambda_1 \equiv \lambda_1(x)$ and $\lambda_2 \equiv \lambda_2(x)$;

i.e. $A_x = TDT^{-1}$ where $T$ is a constant matrix for fixed $x$. Consider the system $\dot{y} = Dy$ with $y = T^{-1}v$.

Project this equation down to the unit sphere $S^1$ by defining $s(t) = \frac{y(t)}{|y(t)|}$ where $|y(t)| = <y(t), y(t)>^{1/2}$ denotes the Euclidean norm of $y(t)$. 

Then the system in the projective space is given by

\[ \dot{s}(t) = (D - s^TDsI)s \]

or,

\[ \dot{s}_1 = (\lambda_1 - \lambda_2)s_1s_2^2 \]
\[ \dot{s}_2 = - (\lambda_1 - \lambda_2)s_1^2s_2 \]

where \( s_1^2 + s_2^2 = 1 \) and \( \lambda_1 > \lambda_2 \).

Fixed points for this flow are \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) corresponding to the eigenvectors of \( D \) from eigenvalues \( \lambda_1 \) and \( \lambda_2 \) respectively. In the following we will use Lyapunov's Stability Theorem ([23] pg 101) to analyze the nature of these fixed points.

Consider the fixed point \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

Let \( V(s) = (s_1 - 1)^2 + s_2^2 \).

Then \( V \) is a continuously differentiable function on a neighborhood \( \Delta \) of \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) such that

\[ V[(1,0)^T] = 0 \]

and \( V(s) > 0 \) in \( \Delta - \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \)

\[ \dot{V}(s) = 2(s_1 - 1)(\lambda_1 - \lambda_2)s_1s_2^2 - 2s_2(\lambda_1 - \lambda_2)s_1^2s_2 \]
\[ = -2(\lambda_1 - \lambda_2)s_1s_2^2 \leq 0 \text{ in } \Delta \]

and \( \dot{V}(s) < 0 \) in \( \Delta - \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \).
Then by Lyapunov's Stability Theorem ([23] pg 101) the fixed point \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) is attracting.

For the other fixed point \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) let \( V(s) = s_1^2 + (s_2 - 1)^2 \).

\( V \) is a continuously differentiable function on a neighborhood \( \Delta \) of \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) such that

\[
V[(0,1)^T] = 0 \quad \text{and} \quad V(s) > 0 \quad \text{in} \quad \Delta - \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.
\]

\[
\dot{V}(s) = 2s_1(\lambda_1 - \lambda_2)s_1s_2^2 - 2(s_2 - 1)(\lambda_1 - \lambda_2)s_1s_2
\]

\[
= 2(\lambda_1 - \lambda_2)s_1s_2^2 \geq 0
\]

and \( \dot{V}(s) > 0 \) in \( \Delta - \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \).

Then by Lyapunov's first instability theorem ([26] pp.213,214) the fixed point \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) is completely unstable or repelling.

Hence, the fixed point corresponding to \( \lambda_1 \) is an attracting one and the fixed point corresponding to \( \lambda_2 \) is a repelling one where \( \lambda_1 > \lambda_2 \).

The system, in the projective space for matrix \( A_x \), can be obtained by a linear coordinate change \( x = Ty \). We will get a rotated, squeezed or expanded system, but the fixed points would still be given by the projections of eigenfunctions corresponding to \( \lambda_1 \) and \( \lambda_2 \) [see Figure 4.1]. Also multiplication by a constant matrix would not change the attracting and repelling properties of those fixed points. This can be seen as follows:

Consider an autonomous system \( \dot{x} = g(x) \) with an attracting fixed point \( \bar{x} \), i.e. if \( X(t) \) be a solution of the system then \( \lim_{t \to \infty} X(t) = \bar{x} \).

Let \( y = Tx \) where \( T \) is a constant matrix.
Then \( y = T \hat{x} = T g(x) = T g(T^{-1}y) = T \circ g \circ T^{-1}(y) \).

If \( Y(t) \) be the solution of this system then \( T \circ g \circ T^{-1}(T \hat{x}) = 0 \) and \( \lim_{t \to \infty} Y(t) = \lim_{t \to \infty} T \Xi(t) = T(\lim_{t \to \infty} \Xi(t)) = T \hat{x} \).

Thus \( T \hat{x} \) remains an attracting fixed point.

For the case of a repelling fixed point replace \( t \to \infty \) by \( t \to -\infty \).

Hence the fixed point corresponding to the larger eigenvalue will be attracting and the other one repelling.

Thus over each \( x \in F \) we have a flow in the projective bundle \( \{x\} \times \mathbb{P}(x) \). If \( \lambda_1(x) \) and \( \lambda_2(x) \) be distinct eigenvalues of \( A_x \) with \( \lambda_1(x) > \lambda_2(x) \), and eigenvectors \( e_1(x) \) and \( e_2(x) \) respectively, then the fixed point \( \{x\} \times \mathbb{P}e_1(x) \) will be the attracting one and the fixed point \( \{x\} \times \mathbb{P}e_2(x) \) will be the repelling one for the flow in the projective space [see Figure 4.2]. We will show that for Case (1) the chain recurrent components are \( E_1 \) and \( E_2 \) and they are disjoint.

**Theorem 4.6.** Let \( F \) be a connected set of fixed points. For each \( x \in F \) let \( A_x \) be the Jacobian matrix of partial derivatives for the linearized component. Assume that \( \forall x, A_x \) has two distinct eigenvalues \( \lambda_1(x) \) and \( \lambda_2(x) \) with \( \lambda_1(x) > \lambda_2(x) \). Let \( e_1(x) \) and \( e_2(x) \) be the eigenvectors corresponding to \( \lambda_1(x) \) and \( \lambda_2(x) \) respectively. Consider \( E_1 = \)
$\bigcup_{x \in F}\{x\} \times \mathbb{P}e_1(x)$ and $E_2 = \bigcup_{x \in F}\{x\} \times \mathbb{P}e_2(x)$, $x \in F$. Then the chain recurrent components for the flow in the projective space are $E_1$ and $E_2$ and they are disjoint.

**Proof:** Let us first show that $E_1$ is a chain recurrent component. $F$ being compact (as a closed subset of a compact set) and connected, so is $E_1$. If $p$ and $q$ are any two points on $E_1$, given any $\epsilon$ and $T > 0$ we would like to construct an $\epsilon - T$ chain from $p$ to $q$.

Let $\gamma: [0,1] \mapsto E_1$ be a parametrization of a portion of $E_1$ with $\gamma(0) = p$, $\gamma(1) = q$. $\gamma$ is compact, hence every open cover of $\gamma$ has a finite subcover. Thus, if we construct an open cover of $\gamma$ with balls of radius $\epsilon$ there exists a finite number, $n$, of balls of radius $\epsilon$ covering $\gamma$. Let us denote them in order by $B_1 = B(p_1, \epsilon)$, $B_2 = B(p_2, \epsilon)$, ..., $B_n = B(p_n, \epsilon)$; $p = p_1, p_2, \ldots, p_n = q \in \gamma$ and $p_i < p_{i+1}$.

Choose $T_1 = T$, then $\mathbb{P}\Phi(T, p_1) = p_1$, $p_1$ being a fixed point [or, as the base point $x_p$ is a fixed point and $e_1(x_p)$ being eigendirection is invariant.]

The chain is formed like this:

For time $T_1 = T$, stay at $p_1 \in B_1$, then choose the next point in the chain from points contained in the intersection of $B_1$ and $B_2$. Stay there for time $T$, then jump to $p_2$, next jump is in $B_2 \cap B_3$. Hence we form our $\epsilon - T$ chain (finite number of times...
\( T_1, \ldots, T_{2n+1} \geq T \) as it is a finite open cover). Using symmetry we can retrace our path from \( q \) to \( p \). Thus \( E_1 \) is a chain recurrent set.

It can be shown in a similar manner that \( E_2 \) is a chain recurrent set.

Next we want to show that the chain recurrent sets \( E_1 \) and \( E_2 \) are disjoint. To show this, we show that once close enough to \( E_1 \) there is no way for a chain to go back near \( E_2 \).

Let \( \eta = \text{dist}(E_1, E_2) = \min_{x,y \in F} d(\{x\} \times IP_1(x), \{y\} \times IP_2(y)) \) \( [E_1, E_2 \text{ are compact}.] \)

Consider a neighborhood of \( E_1 : N(E_1, \frac{\eta}{3}) \) and of \( E_2 : N(E_2, \frac{\eta}{3}) \). Recall that all points on \( E_1 \) are attracting fixed points for the flow in the projective space and all points on \( E_2 \) are the repelling ones.

Then, for all \( p \in N(E_1, \frac{\eta}{3}) \), all \( (\epsilon, T) \) chains starting at \( p \) with \( \epsilon < \frac{\eta}{3} \) can never enter (go back to) \( N(E_2, \frac{\eta}{3}) \) around \( E_2 \).

Thus \( E_1 \) and \( E_2 \) are disjoint, connected, chain recurrent sets and we have two chain recurrent components for the flow in the projective space.

Each of the two components \( E_1 \) and \( E_2 \) comes with a set \( I_1 = \{\lambda_1(x), x \in F\}, I_2 = \{\lambda_2(x), x \in F\} \), \( F \) is closed and connected.

As \( F \) is a closed subset of a compact set it is compact and \( \lambda_1 \) is a continuous function from \( F \) to \( \mathbb{R} \). Hence \( \lambda_1[F] \) must be compact, thus it is closed and bounded in \( \mathbb{R} \). Also it is connected. Thus \( I_1 \) is actually a closed interval. Similarly, \( I_2 \) is a closed interval.

**Theorem 4.7.** The Morse spectrum over \( E_1 \) is \( I_1 \), i.e. \( \sum_{Mo}(E_1) = I_1 \).

**Proof:** By Theorem (5.3.5) in [8], for a linear flow on a vector bundle, Morse spectrum contains the Lyapunov spectrum. As eigenvalues (real in this case) belong to the Lyapunov spectrum, they belong to the Morse spectrum. Thus \( I_1 \subset \sum_{Mo}(E_1) \).

To show the other direction of inclusion we use a result by Lars Grune [13], in particular, Theorem 2.6.
In our case $E_1$ (⊂ the projective bundle) is a compact invariant set (as the set of eigenfunctions is an invariant set) for the projected flow $\mathbb{P}\Phi$ and we proved the chain transitive part.

Pick $e_n \in E_1$ with $e_n = (x_n, p_n)$ where $x_n \in M$ and $p_n \in \mathbb{P}p^{-1}(x_n)$. Hence $\sum_{\mathcal{M}_0}(E_1) = \sum_{U \in E}(E_1) = \{ \lambda \in \mathbb{R}, \text{such that } \exists t_n \to \infty \text{ and points } e_n \in E_1 \text{ such that } \lim_{n \to \infty} \frac{1}{t_n} \log \frac{|T\Phi(t_n, e_n)|}{|e_n|} = \lambda \}.$

$T\Phi(t_n, e_n)$ has two components, one in the base space and the other one in the tangent space over $x_n$. As $x_n$ is a fixed point, flow in the base space just remains there, whereas the linear component follows the second component of the linearized equation and has the form $e^{\lambda_1(p_n)t_n}p_n$. Thus

$$\lambda_n = \frac{1}{t_n} \log \frac{|T\Phi(t_n, e_n)|}{|e_n|} = \frac{1}{t_n} \log \frac{|e^{\lambda_1(p_n)t_n}p_n|}{|p_n|} = \frac{1}{t_n} \lambda_1(p_n)t_n = \lambda_1(p_n)$$

and, $\lim_{n \to \infty} \lambda_n = \lim_{n \to \infty} \lambda_1(p_n)$.

Since $I_1$ is closed $\lim_{n \to \infty} \lambda_1(p_n) \in I_1$. Hence $\sum_{\mathcal{M}_0}(E_1) \subseteq I_1$ and the proof is complete.

Similarly, $\sum_{\mathcal{M}_0}(E_2) = I_2$ and the Morse spectrum for Case(1) is $I_1 \cup I_2$.

4.3.2 Morse Spectrum when $A_{x^*}$ has Both Eigenvalues Zero

Here we consider case (2), i.e. we assume that for at least one $x^* \in F$, $A_{x^*}$ has both eigenvalues zero. We first show that for a fixed point $x^*$, when $A_{x^*}$ has both eigenvalues the same (zero here), the entire projective bundle over $x^*$, $\{x^*\} \times \mathbb{P}_{(x^*)}$, is chain recurrent.

**Lemma 4.2.** Let $A_{x^*}$ have both eigenvalues zero. Then the entire projective bundle $\{x^*\} \times \mathbb{P}_{(x^*)}$ is chain recurrent.
**Proof:** Consider the matrix $A_x$. There exists a real nonsingular constant matrix $U$ such that if $y = Uv$ then the transformed system $\dot{y} = (UA_xU^{-1})y$ has a real coefficient matrix $J = UA_xU^{-1}$ which has one of the following real canonical forms:

$$
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}.
$$

The matrix $A_x$ has a nontrivial null-space. Any vector in the null-space of $A_x$ is an equilibrium point for the system; i.e. the system has an equilibrium subspace rather than an equilibrium point. The dimension of the null-space can be one or two. If it is two, $A_x$ will be the zero matrix. This is a case where every point in the plane is an equilibrium point.

When the dimension of the null space is one and both the eigenvalues are zero, the change of variables $y = Uv$ results in

$$
\begin{align*}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= 0
\end{align*}
$$

whose solution is

$$
\begin{align*}
y_1(t) &= c_1 + c_2t \\
y_2(t) &= c_2
\end{align*}
$$

The term $c_2t$ will increase or decrease, depending on the sign of $c_2$. The $y_1$-axis is equilibrium subspace [see Figure 4.3].

In the first case the projection of the flow on the projective space gives a flow which cannot move, i.e. all points on the projective bundle over $x^*$ are fixed points for the projective flow.

For the second case the flow on the projective space will have a single fixed point corresponding to the single eigendirection for the linear component of $T^\Phi$. To show that
Figure 4.3 Phase portrait in the $v_1 - v_2$ plane: The dashed line is the equilibrium subspace. Trajectories starting off the equilibrium subspace move parallel to it.
\{x^*\} \times \mathbb{P}_{\{x^*\}} is chain recurrent we have to show that for all \(\epsilon, T > 0\) we can form a periodic \(\epsilon - T\) chain from any point of \(\{x^*\} \times \mathbb{P}_{\{x^*\}}\). If the point is the fixed point the chain is trivial. If it is any other point \(P\) on \(\{x^*\} \times \mathbb{P}_{\{x^*\}}\), consider an \(\epsilon\)-ball around the fixed point. Choose \(T_1 > T\) large enough so that \(\mathbb{P}\Phi(T_1, P)\) ends up in that ball. Choose that point the next point in the chain. Now consider the backward flow from \(P\). There exists a time \(T_2 > T\) such that the backward flow starting from \(P\) ends up in the \(\epsilon\)-neighborhood of the fixed point. Choose that point to be the next point in the chain, i.e. make a jump over the fixed point. From this point the forward flow will end up at \(P\) after time \(T_2\). Hence we form a periodic \(\epsilon - T\) chain and \(\{x^*\} \times \mathbb{P}_{\{x^*\}}\) is chain recurrent.

Thus, when we consider a continua of fixed points, for each \(x^*\), \(\{x^*\} \times \mathbb{P}_{\{x^*\}}\) is chain recurrent. We define \(E_1\) and \(E_2\) as in case (1), but as the eigenvalue 0 of \(A_{x^*}\) is not simple, eigenvectors may not be continuous, i.e. \(E_1\) and \(E_2\) may not be continuous in this case.

**Theorem 4.8.** Let \(F\) be a connected set of fixed points. For each \(x \in F\) let \(A_x\) be the Jacobian matrix of partial derivatives for the linearized component. If for at least one \(x^* \in F\), \(A_{x^*}\) has both eigenvalues zero, then the entire projective bundle over \(F\) is a single chain recurrent component.

**Proof:** From all \(x^* \in F\) for which \(A_{x^*}\) has both eigenvalues zero, pick one, say \(x_0\). As discussed above, \(\{x_0\} \times \mathbb{P}_{\{x_0\}}\) is chain recurrent. Let \(a\) and \(b\) be any two points in the projective bundle. Given \(\epsilon\) and \(T > 0\) we form a chain from \(a\) to \(b\) in the following way:

Let \(p_a\) and \(p_b\) be the corresponding base points in \(F\) for \(a\) and \(b\). Let \(\gamma : [0, 1] \rightarrow F\) be a parametrization of a portion of \(F\) with \(\gamma(0) = p_a, \gamma(1) = x_0\), i.e. \(\gamma\) is a continuous path from \(p_a\) to \(x_0\) on \(F\). Being a closed subset of a compact set, \(\gamma\) is compact. Thus we can cover \(\gamma\) with a finite number of open \(\epsilon/2\) balls. We order these balls from \(p_a\) to \(x_0\) as \(B_1(p_a, \epsilon/2), B_2(p_2, \epsilon/2), B_3(p_3, \epsilon/2), \ldots, B_n(x_0, \epsilon/2)\).
Similarly we can consider a continuous path from \(p_0\) to \(x_0\) and cover it by open \(\epsilon/2\) balls \(\bar{B}_1(p_0, \epsilon/2), \bar{B}_2(p_2, \epsilon/2), \bar{B}_3(p_3, \epsilon/2), \ldots, \bar{B}_m(x_0, \epsilon/2)\).

As the projective bundle is compact we can cover the projective bundle by a finite number of open coverings. In particular, we can choose the inverse projection images of the balls \(B_i\) to cover the closed subset of the projective bundle consisting of inverse projection images of all the points from \(p_a\) to \(x_0\) in the base.

Start at \(a\). Follow the flow for time \(T_1 = T\). If \(a\) is a fixed point this means staying at \(a\) for time \(T_1\). \(IP(T_1, a)\) will belong to the inverse projection image of \(p_a\). Choose the next point \(a_2\) of the chain from the inverse projection image of \(p_2\) so that \(d(IP(T_1, a), a_2) \leq \epsilon\). Again follow the flow for time \(T_2 = T\). The next point \(a_3\) in the chain will be on the inverse projection image of \(p_3\) such that \(d(IP(T_2, a_2), a_3) \leq \epsilon\). We continue choosing points this way, so that ultimately we choose a point on \(\{x_0\} \times IP(z_0)\). Call this point \(\hat{A}\).

It takes a finite number of steps from \(a\) to \(\hat{A}\).

Next start from \(b\). Follow the flow in backward time \(T\). Choose a point \(b_2\) from the inverse projection image of \(p_2\) so that \(d(IP(-T, b), b_2) \leq \epsilon\). From \(b_2\), follow the flow for time \(-T\) and choose the next point \(b_3\) from the inverse projection image of \(p_3\) so that \(d(IP(-T, b_2), b_3) \leq \epsilon\). Ultimately, in a finite number of steps we reach a point on \(\{x_0\} \times IP(z_0)\). Call this point \(\hat{B}\).

As \(\{x_0\} \times IP(z_0)\) is chain recurrent we can join \(\hat{A}\) and \(\hat{B}\) by an \(\epsilon - T\) chain.

Concatenate the chains from \(a\) to \(\hat{A}\), \(\hat{A}\) to \(\hat{B}\), and \(\hat{B}\) to \(b\) (i.e. \(b\) to \(\hat{B}\) backwards).

This gives an \(\epsilon - T\) chain from \(a\) to \(b\).

Thus the entire projective bundle becomes a single chain recurrent component. \(\Box\)

**Conclusion:** In this case the Morse spectrum is the single interval \(I_1\) consisting of the eigenvalues of \(A_x\). \(I_1\) contains zero. There will not be anything else in the Morse spectrum as the boundaries of the Morse intervals are Lyapunov exponents [by Theorem 4.2 and Theorem 2.4] and \(I_1\) contains all the Lyapunov exponents.

There cannot be any complex eigenvalue of \(A_x\) as one eigenvalue of \(A_x\) is always zero
and complex eigenvalues appear in complex conjugate pairs.

For all the cases above we found that the Lyapunov spectrum and the Morse spectrum coincide.

4.4 Case 3: Morse Spectrum for a Periodic Orbit

The system in base $M$ is given by

\[ \dot{x} = f(x) \tag{4.12} \]

Let $\gamma = \{\Phi(t, p), \ t \in [0, T]\}$ be a $T$-periodic orbit, i.e. a nonconstant $T$-periodic solution of the above equation. This periodic orbit is assumed to be an isolated chain recurrent component.

Linearizing the system (4.12) over $\gamma$ we get

\[ \dot{x} = f(x) \tag{4.13} \]

\[ \dot{v} = D_xf(\Phi(t, p))v(t) = A(t)v(t) \tag{4.14} \]

where $A(t)$ is a periodic matrix with the same period $T$, i.e. $A(t + T) = A(t)$.

The linear flow is given by $T\Phi$ which locally is $(\Phi(t, p), D\Phi(t, (p, v)))$.

As we project this linear flow to the projective space we get an induced system in the projective bundle given by

\[ \dot{x} = f(x) \tag{4.15} \]

\[ \dot{s}(t) = (A(t) - s(t)^T A(t) s(t) Id)s(t) \tag{4.16} \]

where $s = \frac{v}{||v||}$.

The flow for this system in the projective bundle is denoted by $\mathbb{P}\Phi$ and we locally denote this flow by $(\Phi, \mathbb{P}D\Phi)$.

It is known [for reference see [26] pp.113-116] that for the second component of linearization (4.14) if $\Xi(t)$ is a fundamental matrix, then so is $\Xi(t + T), \ \forall t \in \mathbb{R}$. 
Moreover, corresponding to every $\Xi$, there exists a nonsingular matrix $P$ which is also periodic with period $T$ and a constant matrix $R$, such that

$$\Xi(t) = P(t)e^{tR}$$

(4.17)

The real parts of the eigenvalues of $R$ can be uniquely determined.

Also as shown in Chapter 3, Theorem 3.1, Theorem 3.2, one eigenvalue of the matrix $R$ is zero, i.e. one Lyapunov exponent is always zero.

The other Lyapunov exponent is negative or positive according to whether the periodic orbit is attracting or repelling. The second Lyapunov exponent can be zero too.

If $\xi_0$ is an eigenvector of $R$ belonging to an eigenvalue $\lambda$, so that $e^{Rt}\xi_0 = \xi_0 e^{\lambda t}$, then the solution $v = \Xi(t)\xi_0$ of (4.14) is of the form $z_1(t)e^{\lambda t}$ where the vector $z_1(t) = P(t)\xi_0$ has the period $T$. Thus after time $T$, it turns out to be

$$e^{\lambda(t+T)}P(t+T)\xi_0 = e^{\lambda T}e^{\lambda t}P(t)\xi_0 = e^{\lambda T}e^{\lambda t}z_1(t) = e^{\lambda T}\Xi(t)\xi_0$$

Hence, for the linear flow, if the initial direction is an eigendirection, after time $T$ the only change is in the magnitude of the solution. When we project the solution on $\mathbb{P}^d$ we get back the same vector again after time $T$. Hence for $\mathbb{P}\Phi$, if the flow starts at a projection of an eigendirection, after time $T$ it comes back to the same point. For $\mathbb{P}\Phi$, thus we would have a periodic trajectory of period $T$ which passes through the projection of an eigendirection of $R$.

First consider the case when the periodic orbit is attracting, i.e. one Lyapunov exponent is zero, the other one, $-\beta$, is negative ($\beta > 0$). As these are two distinct eigenvalues of $R$, $R$ is diagonalizable. Let $\xi_1$ and $\xi_2$ be the eigenvectors of $R$ corresponding to $-\beta$ and 0 respectively. With respect to the basis vectors $\xi_1$ and $\xi_2$, $R$ is diagonal and $e^{Rt}$ can be expressed as

$$\begin{pmatrix} e^{-\beta t} & 0 \\ 0 & 1 \end{pmatrix}.$$
As explained above, the two eigendirections will give two periodic trajectories for the flow in the projective space. As these periodic solutions are closed solutions, these two trajectories will be parts of chain recurrent components.

Consider the projective bundle over $\gamma$ at $t = 0$, denote this by $\gamma(0) \times I\mathbb{P}(0)$. Denote the projections of $\xi_1$, $\xi_2$, the two eigendirections of $R$, as $I\mathbb{P}_{\xi_1}$ and $I\mathbb{P}_{\xi_2}$. These two points are on two periodic trajectories of $I\mathbb{P}(0)$. Call these two periodic trajectories $A^*$ and $A$ respectively [see Figure 4.4].

![Figure 4.4 A and A* are periodic trajectories of I\mathbb{P}(0)](image)

Over $\gamma(0)$, next consider a trajectory in the linear space starting at a point $\zeta_0$ which is not an eigendirection of $R$. Let $\zeta_0$ have the representation $\langle \zeta_{01}, \zeta_{02} \rangle^T$ with respect to the basis $\xi_1$, $\xi_2$. Trajectory in the linear space is given by

$$P(t)e^{tR} \begin{pmatrix} \zeta_{01} \\ \zeta_{02} \end{pmatrix} = P(t) \begin{pmatrix} e^{-\beta t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta_{01} \\ \zeta_{02} \end{pmatrix} = P(t) \begin{pmatrix} e^{-\beta t} \zeta_{01} \\ \zeta_{02} \end{pmatrix}$$

Choosing $\Xi(0) = Id$ we have $P(0) = P(T) = Id$.

That we can choose $\Xi(0) = Id$ can be seen as follows:
Suppose in some basis \( \Xi(0) = Id \). Then any solution for a linear system is given by,

\[
\begin{bmatrix}
  x_1(t) \\
  x_2(t)
\end{bmatrix} = \Xi(t) \begin{bmatrix}
  x_1(0) \\
  x_2(0)
\end{bmatrix}
\]

As we change from this basis to the basis \((\xi_1, \xi_2)\) we make a linear transformation,

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix} \rightarrow T \begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
\]

i.e.

\[
T \begin{pmatrix}
  x_1(t) \\
  x_2(t)
\end{pmatrix} = T\Xi(t)T^{-1}T \begin{pmatrix}
  x_1(0) \\
  x_2(0)
\end{pmatrix}
\]

Hence if \( \Xi(0) = Id \) in the first choice of basis, it can still be chosen to be identity with respect to the new basis.

At \( t = 0 \), the projection of \( \zeta_0 \) on \( IP_M \) is given by

\[
IP\zeta_0 = \frac{\zeta_0}{\|\zeta_0\|} = \frac{\langle \zeta_{01}, \zeta_{02} \rangle}{\sqrt{\zeta_{01}^2 + \zeta_{02}^2}}
\]

At \( t = nT \), where \( n \) is an integer, the position of this trajectory in the linear space is

\[
P(nT)e^{nTR} \begin{pmatrix}
  \zeta_{01} \\
  \zeta_{02}
\end{pmatrix} = e^{nTR} \begin{pmatrix}
  \zeta_{01} \\
  \zeta_{02}
\end{pmatrix} = \begin{pmatrix}
  e^{-n\beta T} 0 \\
  0 1
\end{pmatrix} \begin{pmatrix}
  \zeta_{01} \\
  \zeta_{02}
\end{pmatrix} = \begin{pmatrix}
  e^{-n\beta T}\zeta_{01} \\
  \zeta_{02}
\end{pmatrix} = \zeta_n
\]

Projection to \( IP^d \) gives

\[
IP\zeta_n = \frac{\zeta_n}{\|\zeta_n\|} = \frac{\langle e^{-n\beta T}\zeta_{01}, \zeta_{02} \rangle}{\sqrt{e^{-2n\beta T}\zeta_{01}^2 + \zeta_{02}^2}}
\]

At time \( t = (n + 1)T \),

\[
\zeta_{n+1} = P((n + 1)T)e^{(n+1)TR} \begin{pmatrix}
  \zeta_{01} \\
  \zeta_{02}
\end{pmatrix} = \begin{pmatrix}
  e^{-(n+1)\beta T}\zeta_{01} \\
  \zeta_{02}
\end{pmatrix}
\]

\[
IP\zeta_{n+1} = \frac{\langle e^{-(n+1)\beta T}\zeta_{01}, \zeta_{02} \rangle}{\sqrt{e^{-2(n+1)\beta T}\zeta_{01}^2 + \zeta_{02}^2}}
\]
Thus after each time interval $T$ we get a sequence of points $P_0, P_1, \ldots, P_n, P_{n+1}, \ldots$ on the projective space over $\gamma(0)$.

We define the angle between two vectors $X, Y \in \gamma(0) \times P_\gamma(0)$ to be the unique $\theta \in [0, \pi]$ satisfying $\cos \theta = \frac{(X, Y)}{|X||Y|}$, [ref [24] pp.23,24].

**Theorem 4.9.** All the trajectories of $P_\Phi$, other than the periodic ones passing through $P_{\xi_1}$ and $P_{\xi_2}$, converge monotonically and uniformly to the periodic trajectory $A$ through $P_{\xi_2}$, and all of them diverge monotonically and uniformly from the periodic trajectory $A^*$ through $P_{\xi_1}$.

**Proof:** Consider $\gamma(0) \times P_\gamma(0)$. After each time interval $T$ all the trajectories of $P_\Phi$ intersects $\gamma(0) \times P_\gamma(0)$. At $t = 0$, let $P_{\xi_0}$ be a point on $\gamma(0) \times P_\gamma(0)$. With respect to the basis $(\xi_1, \xi_2)$, $P_{\xi_0}$ has the representation $\frac{(\xi_0, \xi_2)}{\sqrt{\xi_0^2 + \xi_2^2}}$.

As mentioned above, at $t = nT$, with $n$ being an integer, the position of the trajectory starting at $P_{\xi_0}$ on $\gamma(0) \times P_\gamma(0)$ is

$$P_{\xi_n} = \frac{(e^{-nT}\zeta_1, \zeta_2)}{\sqrt{e^{-2nT}\zeta_1^2 + \zeta_2^2}}.$$

Taking limit as $n \to \infty$, we get $P_{\xi_n} \to \frac{0}{1}$. Thus the sequence $P_{\xi_n} \to P_{\xi_2}$. Taking limit as $n \to -\infty$, we get $P_{\xi_n} \to P_{\xi_1}$. Thus the sequence $P_{\xi_n} \to P_{\xi_1}$.

Hence the trajectory of $P_\Phi$ starting at $P_{\xi_0}$ diverges from the periodic trajectory through $P_{\xi_1}$ and converges to the periodic trajectory through $P_{\xi_2}$. To see that this convergence (or divergence) is monotonic consider the angular distance between $P_{\xi_n}$ and $P_{\xi_2}$ (or $P_{\xi_1}$).

$$\cos \theta_n = \langle P_{\xi_n}, P_{\xi_2} \rangle = \frac{\zeta_2}{\sqrt{e^{-2nT}\zeta_1^2 + \zeta_2^2}}.$$

$$\cos \theta_{n+1} = \langle P_{\xi_{n+1}}, P_{\xi_2} \rangle = \frac{\zeta_2}{\sqrt{e^{-2(n+1)T}\zeta_1^2 + \zeta_2^2}}.$$ 

i.e. $\cos \theta_{n+1} > \cos \theta_n \quad \theta_n, \theta_{n+1} \in [0, \pi] \implies \theta_{n+1} < \theta_n$. 
Also
\[
\cos \phi_n = \langle \Pi \xi_n, \Pi \xi_1 \rangle = \frac{e^{-n\beta T \zeta_{01}}}{\sqrt{e^{-2\beta T \zeta_{01}^2} + \zeta_{02}^2}}.
\]
\[
\cos \phi_{n+1} = \langle \Pi \xi_{n+1}, \Pi \xi_1 \rangle = \frac{e^{-(n+1)\beta T \zeta_{01}}}{\sqrt{e^{-2(n+1)\beta T \zeta_{01}^2} + \zeta_{02}^2}}.
\]
i.e. \(\cos \phi_{n+1} < \cos \phi_n \Rightarrow \phi_{n+1} > \phi_n\).

Hence the angular distance between \(\Pi \xi_n\) and \(\Pi \xi_2\) decreases monotonically and between \(\Pi \xi_n\) and \(\Pi \xi_1\) increases monotonically. The convergence (or divergence) of the trajectories passing through \(\Pi \xi_n\) is thus monotonic.

To show uniform convergence consider any point \(\Pi \rho\) on \(\gamma(0) \times \Pi \gamma(0)\) between \(\Pi \xi_0\) and \(\Pi \xi_1\). Let the trajectory passing through \(\Pi \rho\) intersect \(\gamma(0) \times \Pi \gamma(0)\) after time \(T\) at \(\Pi \rho_1\). From uniqueness, \(\Pi \rho_1\) cannot lie between \(\Pi \xi_0\) and \(\Pi \xi_1\), it must lie between \(\Pi \xi_1\) and \(\Pi \xi_2\). This is true for all \(\Pi \rho\) in between \(\Pi \xi_0\) and \(\Pi \xi_1\). Thus all the trajectories of \(\Pi \Phi\) through points in between \(\Pi \xi_0\) and \(\Pi \xi_1\) move together. Thus for every \(\epsilon > 0\) there is an integer \(N\) such that \(n \geq N\) implies \(d(\Pi \xi_n, \Pi \xi_2) < \epsilon\), no matter where \(\Pi \xi_0\) is on \(\gamma(0) \times \Pi \gamma(0)\), other than at \(\Pi \xi_1\). The convergence of the trajectories of \(\Pi \Phi\) (other than \(A^*\)) to the periodic trajectory passing through \(\Pi \xi_2\) is thus uniform. Similarly, the divergence from the periodic trajectory through \(\Pi \xi_1\) is uniform.

Next consider the case when the periodic orbit is repelling, i.e. one Lyapunov exponent is zero, the other one, \(\beta\), is positive. Let \(\xi_1\) and \(\xi_2\) be the eigenvectors of \(R\) corresponding to \(\beta\) and zero respectively. Let \(\Pi \xi_1\) and \(\Pi \xi_2\) be the projections of \(\xi_1\) and \(\xi_2\) on \(\Pi M\). As before, trajectories passing through \(\Pi \xi_1\) and \(\Pi \xi_2\) will give two periodic trajectories of \(\Pi \Phi\). Denote them by \(A\) and \(A^*\) respectively.

This time, on \(\gamma(0) \times \Pi \gamma(0)\), we would get a sequence \(\Pi \xi_n\) given by (in terms of the basis \(\xi_1, \xi_2\))
\[
\Pi \xi_n = \frac{(e^{n\beta T \zeta_{01}}, \zeta_{02})}{\sqrt{e^{2\beta T \zeta_{01}^2} + \zeta_{02}^2}}.
\]
Taking limit as \(n \to \infty\), we get \(\Pi \xi_n \to (1, 0)\), i.e. \(\Pi \xi_n \to_{n \to \infty} \Pi \xi_1\). Taking limit as \(n \to -\infty\), we get \(\Pi \xi_n \to (0, 1)\), i.e. \(\Pi \xi_n \to_{n \to -\infty} \Pi \xi_2\).
Hence for this case we summarize our result as:

**Theorem 4.10.** All the trajectories of $\mathbb{P}\Phi$ (other than the periodic ones passing through $\mathbb{P}_{\xi_1}$ and $\mathbb{P}_{\xi_2}$) converge monotonically and uniformly to the periodic trajectory $A$ passing through $\mathbb{P}_{\xi_1}$, and all of them diverge monotonically and uniformly from the periodic trajectory $A^*$ passing through $\mathbb{P}_{\xi_2}$.

**Proof:** The details of the proof can be done exactly as before in Theorem (4.9). □

From what is stated in Theorem (4.9) and in Theorem (4.10) we get the following conclusion.

**Theorem 4.11.** Consider a $T$-periodic solution $\gamma$ of system (4.12) in $M$. Assume that $\gamma$ is an isolated chain recurrent component. Linearization over $\gamma$ gives the linearized system (4.13,4.14). Corresponding to every fundamental matrix of (4.14) there exists a constant matrix $R$ and the real parts of the eigenvalues of $R$ can be uniquely determined. As one eigenvalue of $R$ is zero the other one must be real. If $0$ and $3 (\neq 0)$ are the two eigenvalues of $R$ we get two distinct eigendirections. Projections of these give two periodic trajectories for the flow $\mathbb{P}\Phi$ on the projective bundle $\mathbb{P}M$. Denote the periodic trajectory corresponding to the larger eigenvalue as $A$, the other one as $A^*$. Then $A$ and $A^*$ form two distinct chain recurrent components.

**Proof:** First we show that $(A, A^*)$ is an attractor-repeller pair.

$A$ and $A^*$, being periodic trajectories of the flow $\mathbb{P}\Phi$ on compact metric space $\mathbb{P}M$, are compact and invariant. All the trajectories of $\mathbb{P}\Phi$, other than $A$ and $A^*$, converge to $A$ and diverge from $A^*$.

Let $\eta$ be the distance between $A$ and $A^*$. This means

$$\eta = \min_{\mathbb{P}v \in A, \mathbb{P}v' \in A^*} d(\mathbb{P}v, \mathbb{P}v')$$

where the distance function is as defined in [8] pg 530 and pg 527.
Cover $A$ with open balls of radius $\eta / 3$. The union of all these balls gives a neighborhood $N$ of $A$. Similarly we form $N^*$, an $\eta / 3$ neighborhood of $A^*$. From Theorem (4.9) and Theorem (4.10), $\omega(N) = A$ and $\omega^*(N^*) = A^*$. Thus $A$ and $A^*$ are attractor and repeller respectively.

Also in this case the complementary repeller for $A$, i.e. the set $\{x \in \Pi M, \omega(x) \cap A = \phi\}$ is nothing but $A^*$. Thus by Lemma B.2.11, [8], pg 535, $(A, A^*)$ forms an attractor-repeller pair. There is no other attractor-repeller pair in $\Pi M$. Therefore, by Theorem B.2.15, [8], pg 536, $M_1 = A^*$ and $M_2 = A$ defines a Morse decomposition. According to Theorem B.2.25, [8], pg 541, the chain recurrent set $\mathcal{R}$ satisfies

$$\mathcal{R} = \{A \cup A^*, A \text{ is an attractor}\}.$$ 

In our case the chain recurrent set $\mathcal{R}$ has only two connected components, viz., $A$ and $A^*$. The Morse sets coincide with the chain recurrent components of $\mathcal{R}$. 

Our inference about the Morse spectrum in this case is:

**Theorem 4.12.** Under the conditions stated in Theorem (4.11) the Morse spectrum coincides with the Lyapunov spectrum.

**Proof:** Two chain recurrent components indicate that there will be two Morse intervals. As there are only two Lyapunov exponents over the periodic orbit, viz. 0 and $\beta$, and these Lyapunov exponents form the boundaries of the Morse intervals, (Theorem 2.4 and Theorem 4.2), the Morse intervals are simply $\{0\}$ and $\{\beta\}$. 

We considered the case where the matrix $R$, defined in (4.17) above, has two distinct eigenvalues. Next, suppose $R$ has both eigenvalues 0. Possible canonical forms for $R$ are

$$
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}.
$$

**Theorem 4.13.** When $R$ is the zero matrix, the Morse spectrum is $\{0\}$. 

Proof: Here $e^{tR}$ is the identity matrix. Thus all the trajectories in the linear flow are periodic; i.e., the flow $(\Phi, D\Phi)$ is periodic. Hence the local flow $(\Phi, \text{IPD}\Phi)$ is periodic. From uniqueness of the solutions the projective flow $\text{IP}\Phi$ is periodic. As all the trajectories of $\text{IP}\Phi$ are periodic we can form a periodic $e^{-T}$ chain for any point on $\text{IPM}$. Thus the entire projective bundle is a single chain recurrent component. This means that there will be a single Morse interval. As 0 is the only Lyapunov exponent here the Morse spectrum is $\{0\}$.

Theorem 4.14. If the canonical form of $R$ is

\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]

the Morse spectrum is $\{0\}$.

Proof: Consider $\gamma(0) \times \text{IP}\gamma(0)$. The canonical form of $R$ suggests that there is one eigendirection $\xi$ and one generalized eigendirection $\xi$ of $R$. Expressed in the basis of these $e^{tR}$ appears to be

\[
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix}
\]

As explained earlier, the trajectory of $\text{IP}\Phi$ passing through $\text{IP}\xi$ is a periodic one. The question now is how do the other trajectories behave.

Let $\text{IP}\xi(\neq \text{IP}\xi) \in \gamma(0) \times \text{IP}\gamma(0)$. Let representation of $\zeta_0$ in terms of the basis $($$\xi$, $\bar{\xi})$ be $(\zeta_{01}, \zeta_{02})$. Over $\gamma(0)$, trajectory in the linear space is given by

\[
\Xi(t)\zeta_0 = P(t)e^{tR} \begin{pmatrix}
\zeta_{01} \\
\zeta_{02}
\end{pmatrix} = P(t) \begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\zeta_{01} \\
\zeta_{02}
\end{pmatrix} = P(t) \begin{pmatrix}
\zeta_{01} + t\zeta_{02} \\
\zeta_{02}
\end{pmatrix}
\]

Choosing $\Xi(0) = \text{Id}$ we have $P(0) = P(T) = \text{Id}$.

At $t = 0$, the projection of $\zeta_0$ on $\text{IPM}$ is given by

\[
\text{IP}\zeta_0 = \frac{\zeta_0}{\|\zeta_0\|} = \frac{\langle \zeta_{01}, \zeta_{02} \rangle}{\sqrt{\zeta_{01}^2 + \zeta_{02}^2}}.
\]
At $t = nT$, where $n$ is an integer,

$$\zeta_n = P(nT)e^{nTR} \begin{pmatrix} \zeta_{01} \\ \zeta_{02} \end{pmatrix} = e^{nTR} \begin{pmatrix} \zeta_{01} \\ \zeta_{02} \end{pmatrix} = \begin{pmatrix} 1 & nT \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta_{01} \\ \zeta_{02} \end{pmatrix} = \begin{pmatrix} \zeta_{01} + nT \zeta_{02} \\ \zeta_{02} \end{pmatrix}$$

Projection to $\mathbb{P}^1$ gives

$$\Pi\zeta_n = \frac{\zeta_n}{||\zeta_n||} = \frac{\langle\zeta_{01} + nT\zeta_{02}, \zeta_{02}\rangle}{\sqrt{\langle\zeta_{01} + nT\zeta_{02}\rangle^2 + \zeta_{02}^2}}.$$

At time $t = (n + 1)T$,

$$\zeta_{n+1} = P((n + 1)T)e^{(n+1)TR} = \begin{pmatrix} \zeta_{01} + (n + 1)T \zeta_{02} \\ \zeta_{02} \end{pmatrix}$$

$$\Pi\zeta_{n+1} = \frac{\langle\zeta_{01} + (n + 1)T\zeta_{02}, \zeta_{02}\rangle}{\sqrt{\langle\zeta_{01} + (n + 1)T\zeta_{02}\rangle^2 + \zeta_{02}^2}}.$$

Thus after each time interval $T$ we get a sequence of points $\Pi\zeta_0, \Pi\zeta_1, \ldots, \Pi\zeta_n, \Pi\zeta_{n+1}, \ldots$ on the projective bundle over $\gamma(0)$. After each time interval $T$ all the trajectories of $\Pi\Phi$ intersect $\gamma(0) \times \Pi\gamma(0)$ and the trajectory starting at $\Pi\zeta_0$ at $t = 0$ intersects $\gamma(0) \times \Pi\gamma(0)$ at $\Pi\zeta_0, \Pi\zeta_1, \ldots, \Pi\zeta_n, \Pi\zeta_{n+1}, \ldots$.

Taking limit as $n \to \infty$, we get $\Pi\zeta_n \to_{n \to \infty} (1, 0)$. Taking limit as $n \to -\infty$, we get $\Pi\zeta_n \to_{n \to -\infty} (1, 0)$. Thus the sequence $\Pi\zeta_n \to_{n \to \pm\infty} \Pi \xi$. If $\theta_n$ be the angle between $\Pi\zeta_n$ and $\Pi \xi$,

$$\cos \theta_n = \langle \Pi\zeta_n, \Pi \xi \rangle = \frac{\zeta_{01} + nT \zeta_{02}}{\sqrt{\langle\zeta_{01} + nT\zeta_{02}\rangle^2 + \zeta_{02}^2}}.$$

$$\cos \theta_{n+1} = \langle \Pi\zeta_{n+1}, \Pi \xi \rangle = \frac{\zeta_{01} + (n + 1)T \zeta_{02}}{\sqrt{\langle\zeta_{01} + (n + 1)T\zeta_{02}\rangle^2 + \zeta_{02}^2}}.$$

$$\cos \theta_{n+1} > \cos \theta_n$$

$$\theta_{n+1} < \theta_n \text{ when } \theta_n, \theta_{n+1} \in [0, \pi]$$
\[ \theta_{n+1} > \theta_n \quad \text{when} \quad \theta_n, \theta_{n+1} \in [-\pi, 0] \]

Hence the convergence and divergence of the sequence \( \{\Gamma \} \) is monotonic.

As before, owing to uniqueness, all the trajectories of \( \Phi \) passing through points of \( \gamma(0) \times \gamma(t(0)) \) between \( \Gamma \) and \( \Gamma \) move together as a band.

Next we show that it is possible to form an \( \epsilon - T \) chain between any two points of \( \Gamma \). If the two points lie on the periodic trajectory we can simply follow the flow for suitable times \( T_i > T \) to form an \( \epsilon - T \) chain. For positions other than the periodic trajectory, starting from the initial point \( A \) we can follow the trajectory passing through \( A \). As this trajectory converges to the periodic trajectory, for every \( \epsilon \), there is a time \( \hat{T} \) when this trajectory is \( \epsilon \)-close to the periodic one. Choose \( T_1 = \max \{T, \hat{T}\} \) and follow the trajectory through \( A \) for time \( T_1 \). Choose the next point \( P_1 \) of the chain on the periodic trajectory \( \epsilon \)-close to \( \Phi(T, A) \). Also, the trajectory passing through the final point \( B \) diverges from the periodic trajectory. Thus \( \exists \) a time \( \hat{T} \) such that \( \Phi(-\hat{T}, B) \) is \( \epsilon \)-close to the periodic trajectory. Let \( T_2 = \max \{\hat{T}, T\} \). Choose \( \Phi(-T_2, B) \) a point of the chain, so that from this point, following the trajectory passing through \( B \) for time \( T_2 \), we can reach \( B \). Choose a point \( P_2 \) on the periodic trajectory which is \( \epsilon \)-close to \( \Phi(-T_2, B) \). On the periodic trajectory form an \( \epsilon - T \) chain from \( P_1 \) to \( P_2 \). Concatenate the chains from \( A \) to \( P_1 \), from \( P_1 \) to \( P_2 \), and from \( P_2 \) to \( B \).

This makes the entire projective bundle a single chain recurrent component. Hence there will be a single Morse interval whose boundary points are Lyapunov exponents. As the Lyapunov spectrum is \( \{0\} \), the Morse spectrum will be \( \{0\} \) as well.

We summarize our findings in the following theorem:

**Theorem 4.15.** If a periodic orbit is an isolated chain recurrent component for the system \( \dot{x} = f(x) \), then the Morse spectrum coincides with the Lyapunov spectrum.

**Proof:** Theorems (4.12),(4.13) and (4.14) prove our assertion.
4.5 Case 4: Morse Spectrum for a Cycle

The system in base $M$ is given by

$$\dot{x} = f(x)$$

(4.18)

Let $\Gamma : \{x_1, \gamma_1, x_2, \gamma_2, x_3, \gamma_3, \cdots \}$ be a cycle on $M$ where $x_i$ are the fixed points and $\gamma_i$ are the connecting orbits. Consider linearization over this cycle.

Over the fixed point $x_1$, $\Phi(t, x_1) = x_1$ and the linearized system is

$$\dot{x} = f(x)$$

(4.19)

$$\dot{v} = A_1(\Phi(t, x_1))v = A_1(x_1)v = A_1v = A_{x_1}v$$

(4.20)

where $A_1$ is a constant matrix.

Over the trajectory $\gamma_1$, if $y_1 \in \gamma_1$ the trajectory is $\Phi(t, y_1)$ and the linearized system is

$$\dot{x} = f(x)$$

(4.21)

$$\dot{v} = B_1(\Phi(t, y_1))v$$

(4.22)

Over the fixed point $x_2$, $\Phi(t, x_2) = x_2$ and the linearized system is

$$\dot{x} = f(x)$$

(4.23)

$$\dot{v} = A_2(\Phi(t, x_2))v = A_2(x_2)v = A_2v = A_{x_2}v$$

(4.24)

To calculate the Morse spectrum over a cycle we first have to find out the chain recurrent components for the flow in the projective bundle. First consider the linearizations over the fixed points $x_i$.

Lemma 4.3. $A_{x_i}$ cannot have a complex pair of eigenvalues.

Proof: Suppose for the linear system at $x_i$, $A_{x_i}$ has a complex pair of eigenvalues. Then the linear system will have either a center or a focus. If the linear system has a
center at the origin then by Theorem 5, [28], pg 143, the nonlinear system will have a
center, a center-focus or a focus at \( x_i \). But as the nonlinear system is a cycle, according
to the definitions in [28], pg 139, \( x_i \) cannot be a center, center-focus or focus. □

So there are two real eigenvalues for \( A_{x_i} \). The possibilities are either they are the
same or different.

**Lemma 4.4. If the two eigenvalues of \( A_{x_i} \) are the same, then they must be zero.**

**Proof:** Suppose there are two nonzero equal eigenvalues of \( A_{x_i} \). If the eigenvalues are
negative (or positive) then the linearized system over \( x_i \) will have a stable (or unstable)
node. Then by Theorem 4, [28], pg 143, the fixed point for the nonlinear system, \( x_i \), will
be a stable (or unstable) node. This means each trajectory in a deleted neighborhood
of \( x_i \) will approach (move away) the fixed point (\( x_i \)) along a well defined tangent line as
t \( \to \infty \) [Definition 4, [28], pg 139]. But as \( x_i \) is a fixed point in a cycle there is at least
one trajectory which moves away (approaches) from \( x_i \). Thus it is not possible for \( x_i \) to
be a stable (or unstable) node. Hence the only possibility for equal eigenvalues for \( A_{x_i} \)
is when both of them are zero. □

The above proof also suggests that for the hyperbolic case, i.e. when \( A_{x_i} \) has two unequal
nonzero real eigenvalues, one of them must be positive and the other one negative.
The proof is exactly the same as above.

So the possibilities are:

Hyperbolic case: \( A_{x_i} \) will have one eigenvalue positive, the other one negative.

Nonhyperbolic case:

(a) Both eigenvalues of \( A_{x_i} \) are zero.

(b) One eigenvalue of \( A_{x_i} \) is zero, the other one positive.

(c) One eigenvalue of \( A_{x_i} \) is zero, the other one negative.
For simplicity, we first consider a cycle with two fixed points and trajectories connecting them. Also we first consider the hyperbolic situation, i.e. both \( A_{x_i} \) \( (i = 1, 2) \) will have two real and distinct eigenvalues, one positive, one negative.

### 4.5.1 Chain Recurrent Components for the Hyperbolic Case

Call \( \mathbb{P}_{\{x_i\}} \) and \( \mathbb{P}_{\{x_2\}} \) the projective spaces over \( x_1 \) and \( x_2 \) respectively. Projection of the two eigendirections of \( A_{x_1} \) gives the two fixed points for the flow on \( \{x_1\} \times \mathbb{P}_{\{x_1\}} \), we call them \( (x_1, e_{11}) \) and \( (x_1, e_{12}) \) or sometimes, in short form as \( e_{11} \) and \( e_{12} \), where \( e_{11} \) is the repelling fixed point and \( e_{12} \) is the attracting one.

Similarly, for \( A_{x_2} \) we get two fixed points \( (x_2, e_{21}) \) or \( e_{21} \) and \( (x_2, e_{22}) \) or \( e_{22} \) for the flow on \( \{x_2\} \times \mathbb{P}_{\{x_2\}} \) where \( e_{21} \) is the repelling one and \( e_{22} \) is the attracting one [see Figure 4.5].

![Figure 4.5 Flow in the projective bundles over \( X_1 \) and \( X_2 \)](image)

Consider the projective bundle over \( x_1, \gamma, x_2 \). The projective bundle is compact and the projective flow is continuous. Thus each trajectory of \( \mathbb{P} \Phi \) must have at least one limit point in this projective bundle. [For reference see [30], pp.193,194].

As \( e_{11}, e_{12}, e_{21}, e_{22} \) are the limit points for the projective flows over \( \{x_1\} \times \mathbb{P}_{\{x_1\}} \) and \( \{x_2\} \times \mathbb{P}_{\{x_2\}} \), they might be possible limit points for \( \mathbb{P} \Phi \).
Theorem 4.16. The trajectories of \( IP \) over \( \gamma_1 \) will converge to \((x_2, e_{21})\) or to \((x_2, e_{22})\).

Proof: The flow \( IP \) has two components of which \( \Phi \) is on the base and \( \Phi_t(y) \rightarrow t \rightarrow \infty \) for all \( y \in \gamma_1 \). Consider \( (U, \varphi) \) a chart for \( M \) with \( x_2 \in U = \gamma \). Then \( \varphi[U] \subset \mathbb{R}^2 \) and for \( \Pi_{\gamma} : \Pi M \rightarrow M, \varphi[\Pi_{\gamma}[U]] \) can be considered as \( U \times \mathbb{P}^1 \). Here \( \varphi \) is the induced diffeomorphism in \( \Pi M \) and \( U \times \mathbb{P}^1 \) is a neighborhood of \( \{x_2\} \times \Pi \{x_2\} \).

Then, as \( t \rightarrow \infty \), trajectories of \( IP \) enter \( U \times \mathbb{P}^1 \). As closure of \( \gamma \times \mathbb{P}^1 \) is compact and each trajectory of \( IP \) is continuous, each trajectory of \( IP \) must have at least one accumulation point. As the base flow converges to \( x_2 \), these accumulation points must be on \( \{x_2\} \times \Pi \{x_2\} \). We claim that only \((x_2, e_{21})\) and \((x_2, e_{22})\) can be the possible accumulation points. On \( \{x_2\} \times \mathbb{P}^1 \) denote one section between \( e_{21} \) and \( e_{22} \) by \( A \) and the other one by \( B \). Let \( \alpha \) be a point on \( A \).

On \( \{x_2\} \times \mathbb{P}^1 \), as \( e_{22} \) is an attracting fixed point, for all \( \epsilon > 0 \) and \( \alpha \in A \), there exists \( \delta > 0 \) and \( T > 0 \) such that \( IP(t, y) \in B(e_{22}, \epsilon) \) for all \( t \geq T \) and \( y \in B(\alpha, \delta) \). On \( U \times \mathbb{P}^1 \), continuity with respect to initial conditions states that for all \( \epsilon > 0 \) and \( t > 0 \) there exists \( \eta(\epsilon, t) > 0 \) such that \( IP(t, z) \in B(IP(t, \alpha), \frac{\epsilon}{2}) \) whenever \( z \in B(\alpha, \eta(\epsilon, t)) \).

Let \( \epsilon \) be given, then there exists a \( T \) such that \( IP(t, \alpha) \in B(e_{22}, \epsilon/2) \) \( \forall t \geq T \).

Then
\[
d(IP(t, z), e_{22}) < d(IP(t, z), IP(t, \alpha)) + d(IP(t, \alpha), e_{22})
\]
\[
\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
whenever \( z \in B(\alpha, \eta(\epsilon, t)) \) and \( t \geq T \). This means, on \( U \times \mathbb{P}^1 \) for all \( \epsilon > 0 \) we would find an \( N_1 \) so that for all \( n \geq N_1 \) we can find \( \eta(\epsilon, n) > 0 \) such that
\[
IP(s_n, z) \in B(e_{22}, \epsilon) \text{ for all } z \in B(\alpha, \eta(\epsilon, n)) \tag{4.25}
\]

Suppose \( \alpha \) is an accumulation point for a trajectory of \( IP \). Then there exists a sequence \( \{t_m\} \) such that \( IP(t_m, y) \rightarrow \alpha \) as \( m \rightarrow \infty \). For each \( n \geq N_1 \) choose \( \epsilon_n = \eta(\epsilon, n) \). Then we could find an \( N_2 \) such that \( IP(t_m, y) \in B(\alpha, \epsilon_n) = B(\alpha, \eta(\epsilon, n)) \) for all \( m \geq N_2 \). By
(4.25) for each \( n \geq N_1 \) we have

\[
P(s_n, \Phi(t_m, y)) \in B(e_{22}, \epsilon)
\]

for all \( m \geq N_2 \). This implies, for all \( \epsilon > 0 \)

\[
P(s_n + t_m, y) \in B(e_{22}, \epsilon)
\]

for all \( n \geq N_1 \) and all \( m \geq N_2 \). This means we can get a sequence of times \( \{r_k\} \) so that

\[
P(r_k, y) \to e_{22} \text{ as } k \to \infty.
\]

As \( PM \) is Hausdorff we can choose \( \epsilon \) so that \( B(e_{22}, \epsilon) \) and \( B(\alpha, \epsilon) \) are disjoint. Then the trajectory through \( y \) actually oscillates between \( B(\alpha, \epsilon) \) and \( B(e_{22}, \epsilon) \).

Let \( \beta \) be the midpoint of the stretch of \( \{x_2\} \times \Phi \) lying between \( \alpha \) and \( e_{22} \). Then \( \beta \) is a nonequilibrium point of \( \Phi \). We would construct a "flow box" in a neighborhood of \( \beta \). For description of flow box we would rely mostly on \cite{2} pp. 24–26 and \cite{18} pp. 242–243.

A flow box gives a complete description of the flow \( \Phi \) in a neighborhood of \( \beta \) by means of special (nonlinear) coordinates. The description is that points move in parallel straight lines at constant speed [see Figure 4.6].

Consider a local (cross) section at \( \beta \), i.e., an open set \( S \) containing \( \beta \) in a hyperplane \( H \subseteq \mathbb{R}^2 \) which is transverse to the vector field at \( \beta \). For convenience, we will assume that \( H \) has normal vector field at \( \beta \) in the following discussion. There is a neighborhood \( V \) of \( \beta \) such that any point \( x \in V \) can be written as \( x = \Phi_u(y) \) where \( y \in S \). In other words, we can use the trajectories of the flow \( \Phi \) to define new coordinates on \( V \).

These new coordinates are best related to local coordinates at \( \beta \), therefore, let \( x \mapsto x - \beta \) so that \( \beta \) is at the origin of both sets of coordinates. Now suppose we choose a basis in \( \mathbb{R}^3 \) which has \( X(0) \), the vector field at \( 0 \) as its first vector. Then the first coordinate of every point \( y \in S \) is zero and \( S \) defines a neighborhood \( \hat{S} \) of the origin in \( \mathbb{R}^2 \) (see 4.6). Each point of \( \hat{S} \) can be specified by \( \xi \in \mathbb{R}^2 \) and every point \( x \) of \( V \) can be written as \( x = \Phi_u((0, \xi)) = \Phi(u, (0, \xi)) = h(u, \xi) \).
Figure 4.6 Flow-box containing the ordinary point $\beta$:
(a) in the original coordinates; (b) in local coordinates at $\beta$ and (c) using local coordinates defined by the flow lines.
As explained in [2] pg 24 \( h \) is a diffeomorphism. In the new coordinates, the trajectories of the flow are simply lines of constant \( \xi \) (see 4.6), i.e., \( \psi_\xi(u, \xi) = (t + u, \xi) \). Also it is shown in [2] pg 26 that \( \psi_t \) and \( \text{IP}_t \) are conjugate. Based on these arguments we mention the "Flow-box" Theorem.

**Theorem 4.17 (Flow-box Theorem).** Let \( \beta \) be an ordinary point of the flow \( \text{IP}_\Phi \). Then in every sufficiently small neighborhood of \( \beta \), \( \text{IP}_\Phi \) is \( C^\infty \)-conjugate to the flow \( \psi(t, x) = x + te_1 \) where \( e_1 \) is a unit vector parallel to the \( x_1 \)-axis.

The flow box theorem guarantees the existence of a "box"-like neighborhood of any nonequilibrium point such that the orbits of the system enter at one end of the box and flow out through the other. Moreover, no orbit leaves through the sides of the box and if a point belongs to the flow box then the orbit through that point will intersect the local section for a unique \( t \in (-\sigma, \sigma) \).

We come back to our discussion about the cycle. As \( n \to \infty \), we found that the trajectory passing through \( y \) keeps oscillating between the two neighborhoods of \( e_{21} \) and \( \alpha \). Thus this trajectory enters and exits the flow-box around \( \beta \) in opposing directions. This type of oscillation is a contradiction to the flow-box theorem. Hence, every point \( \alpha \) on \( A \) or \( B \) cannot be an accumulation point for the trajectories of \( \text{IP}_\Phi \). The only possible accumulation points are \( e_{21} \) and \( e_{22} \).

Also, there cannot be more than one accumulation point for a trajectory of \( \text{IP}_\Phi \) because if both \( e_{21} \) and \( e_{22} \) are accumulation points then again the trajectory will oscillate, violating the flow box theorem.

The next result describes the behavior of the trajectories of \( \text{IP}_\Phi \) as \( t \) tends to infinity.

**Theorem 4.18.** The \( \omega \)-limit set for the projective flow \( \text{IP}_\Phi \) over \( \gamma_1 \) is \( e_{22} \), the attracting fixed point on \( \{x_2\} \times \text{IP}_\{x_2\} \), for all the trajectories except one for which it is \( e_{21} \), the repelling fixed point on \( \{x_2\} \times \text{IP}_\{x_2\} \). In other words: if \( y \in \gamma_1 \) \( \text{IP}_\Phi(t, y, v) \to (x_2, e_{22}) \) for all \( v \in \text{IP}_\{v\}(y_1) \) except for one point \( v_1 \) for which \( \text{IP}_\Phi(t, y, v_1) \to_{t \to \infty} (x_2, e_{21}) \).
Proof: In Theorem 4.16 above we proved that all trajectories of $\mathbb{P}\Phi$ over $\gamma_1$ have either $e_{21}$ or $e_{22}$ as their $\omega$-limit sets. As we did above in (4.16), let $U \times \mathbb{P}^d$ be an invariant neighborhood of $\{x_2\} \times \mathbb{P}(x_2)$. Let $y \in U$ be a point on $\gamma_1$. Denote the projective bundle over $y$ as $\{y\} \times \mathbb{P}^d$.

Let $S \subset \{y\} \times \mathbb{P}^d$ be the set consisting of all the points of $\{y\} \times \mathbb{P}^d$ such that the trajectories of $\mathbb{P}\Phi$ through these points have $e_{22}$ as their $\omega$-limit set.

Lemma 4.5. The set $S$ is nonempty.

Proof: Assume $S$ is empty, i.e., all the trajectories passing through $\{y\} \times \mathbb{P}^d$ have $e_{21}$ as their $\omega$-limit set. This means that for all $\epsilon > 0$, there exists $T > 0$ such that $\mathbb{P}\Phi(t, g) \in B(e_{21}, \epsilon)$ for all $t \geq T$ and $g \in \{y\} \times \mathbb{P}^d$.

Let $\beta$ be a point on $\{x_2\} \times \mathbb{P}^d$ lying between $e_{21}$ and $e_{22}$. As $e_{21}$ is a repelling fixed point and $e_{22}$ is an attracting fixed point for the flow on $\{x_2\} \times \mathbb{P}^d$, the trajectory passing through $\beta$ has a specific direction, viz., from $e_{21}$ to $e_{22}$. As in Theorem 4.16 consider a flow box around the ordinary point $\beta$.

From the discussion about flow box in Theorem 4.16 the flow $\mathbb{P}\Phi$ is conjugate to a flow where each trajectory in a flow box neighborhood moves in the specific direction as the vector field through $\beta$. Consider a point $y_1$ in the flow box. The trajectory through $y_1$ has direction from $e_{21}$ towards $e_{22}$. By uniqueness there is a point $g \in \{y\} \times \mathbb{P}^d$ such that $y_1 = \mathbb{P}\Phi(t, g)$ for some $t$. Hence the trajectory through $y_1$ has $e_{21}$ as its limit set. Then, for all $\epsilon > 0$ there exists an $N$ such that $\mathbb{P}\Phi(t_n, y_1) \in B(e_{21}, \epsilon)$ for all $n \geq N$. Choose $\epsilon > 0$ so that $B(e_{21}, \epsilon)$ and the flow box around $\beta$ are disjoint. Then this trajectory must reenter the flow box in an opposite direction to reach $B(e_{21}, \epsilon)$, which is a contradiction to the flow box theorem. Hence all the trajectories through $\{y\} \times \mathbb{P}^d$ cannot have $e_{21}$ as their $\omega$-limit set, i.e., $S$ is non-empty.

Lemma 4.6. The set $S$ is open.
Proof: We will show that every \( g \) in \( S \) is an interior point of \( S \), i.e., we find a neighborhood of \( g \) such that the trajectories passing through the points in that neighborhood will have \( e_{22} \) as their \( \omega \)-limit set.

As before, we name the two portions of \( \{x_2\} \times \mathbb{T} \) between \( e_{21} \) and \( e_{22} \) as \( A \) and \( B \). The flow on \( \{x_2\} \times \mathbb{T} \) moves from \( e_{21} \) towards \( e_{22} \) on \( A \) and \( B \). For every \( \beta \) in \( A \) and \( B \) we can construct a flow box neighborhoods \( V_\beta \). All the trajectories in \( V_\beta \) move parallel to each other in the direction of the trajectories through \( \beta \), i.e., from \( e_{21} \) towards \( e_{22} \). Define \( \text{rad} V_\beta \), radius of \( V_\beta \), to be the maximum distance from the boundary of \( V_\beta \) to \( \{x_2\} \times \mathbb{T} \). Choose \( \epsilon < \min\{\text{rad} V_\beta\} \), \( \beta \) in \( A \) or \( B \), and consider \( B(e_{22}, \epsilon) \). As \( g \in S \), trajectory of \( \Phi \) through \( g \) will have \( e_{22} \) as its \( \omega \)-limit set. Thus there exists a \( T > 0 \) such that for all \( t \geq T \), \( \Phi(t, g) \in B(e_{22}, \epsilon) \). Let \( t_1 \geq T \). Choose \( \epsilon_1 \) so that the \( \epsilon_1 \)-neighborhood of \( \Phi(t_1, g) \) is contained in \( B(e_{22}, \epsilon) \). By continuous dependence on initial conditions, there exists \( \eta(\epsilon_1, t_1) > 0 \) such that \( \Phi(t_1, z) \in B(\Phi(t_1, g), \epsilon_1) \) whenever \( z \in B(g, \eta(\epsilon_1, t_1)) \). This \( B(g, \eta(\epsilon_1, t_1)) \) gives us a neighborhood of \( g \) such that all the trajectories passing through that neighborhood will be in the \( \epsilon_1 \) neighborhood of \( \Phi(t_1, g) \) after time \( t_1 \). We have to prove that all these trajectories actually converge to \( e_{22} \). As \( \epsilon_1 \) neighborhood of \( \Phi(t_1, g) \) is contained in \( B(e_{22}, \epsilon) \), for every \( z \in B(g, \eta(\epsilon_1, t_1)) \) the trajectory \( \Phi(t_1, z) \in B(e_{22}, \epsilon) \). If possible, let for some \( z_1 \in B(g, \eta(\epsilon_1, t_1)) \) the trajectory through \( z \) have \( e_{21} \) as its \( \omega \)-limit set. As \( e_{22} \) is not an \( \omega \)-limit set for the trajectory through \( z_1 \), eventually, for some time \( t > t_1 \) the trajectory \( \Phi(t, z_1) \) has to leave \( B(e_{22}, \epsilon) \). As this trajectory has to move towards \( e_{21} \) and it is in a neighborhood of \( e_{22} \), there will be some time \( t > t_1 \) so that at that time it has to enter at least one of the \( V_\beta \)'s from the opposite direction of the flow in that \( V_\beta \). This is a contradiction to the flow box theorem. Thus, all the trajectories through \( B(g, \eta(\epsilon_1, t_1)) \) actually converge to \( e_{22} \), hence \( B(g, \eta(\epsilon, t))_{\{\mathbb{T}\} \times \mathbb{T}} \subset S \). This makes \( g \) an interior point of \( S \) and \( S \) is open.

Lemma 4.7. Trajectory through a boundary point of \( S \) does not have \( e_{22} \) as its \( \omega \)-limit
Proof: If these trajectories have $e_{22}$ as their $\omega$-limit set, then following the same method of proof as above it can be shown that there exists a neighborhood of a boundary point everything in which will have $e_{22}$ as $\omega$-limit set. This will prevent the point from being a boundary point of $S$ [for definition of boundary point see [1] pg 64].

This shows that not all the trajectories through $\{y\} \times \mathbb{P}^d$ will have $e_{22}$ as their $\omega$-limit set. There exists at least one trajectory which has $e_{21}$ as its $\omega$-limit set.

Lemma 4.8. A unique trajectory through $\{y\} \times \mathbb{P}^d$ has $e_{21}$ as its $\omega$-limit set.

Proof: Let $\hat{S}$ be the set of points on $\{y\} \times \mathbb{P}^d$ trajectories through which have $e_{21}$ as their $\omega$-limit set. We already proved above that $\hat{S}$ is nonempty.

If possible, let $\hat{S}$ have at least two points $y_1$ and $y_2$. Uniqueness implies that an entire portion of $\{y\} \times \mathbb{P}^d$ lying between $y_1$ and $y_2$ will be in $\hat{S}$. We can think of this portion as an interval $[y_1, y_2]$, a closed subset of $\hat{S}$.

Consider $q \times \mathbb{P}^d$ sufficiently close to $\{x_2\} \times \mathbb{P}^d$. Uniqueness guarantees that $\mathbb{P} \Phi(t, z_i)$ for $z_i \in [y_1, y_2]$ will be a closed interval $[q_1, q_2]$ on $q \times \mathbb{P}^d$. We can construct flow-box neighborhoods for all ordinary points $\beta$ on $\{x_2\} \times \mathbb{P}(x_2)$. As the direction of flow on $\{x_2\} \times \mathbb{P}^d$ is away from $e_{21}$, the direction of flow in these flow boxes will be away from $e_{21}$ and reentering the box is not possible. As all the trajectories through $[q_1, q_2]$ converge to $e_{21}$, none of them can enter any of these flow boxes around $\beta$. Also, we can construct flow box neighborhoods for trajectories through $[q_1, q_2]$, the flow in these boxes will be towards $e_{21}$, hence they cannot be along the same direction as flow in boxes around $\beta \in A \cup B$.

Consider a neighborhood $B$ of $e_{21}$. In this neighborhood we would have at least one $\beta \in A \cup B$ and points of trajectories through $[q_1, q_2]$. In $B$, construct flow boxes around all the $\beta$'s and around points of trajectories converging to $e_{21}$. Then in an intersection of these two different direction flow boxes the flow will have two different directions, which
is not possible. Hence \( \mathcal{S} \) cannot have two points, but as it is nonempty, there is exactly one point in \( \mathcal{S} \). Uniqueness implies exactly one trajectory of \( \mathcal{I} \mathbf{P}_\Phi \) converges to \( e_{21} \). \( \square \)

The above theorem describes the trajectories of the flow \( \mathcal{I} \mathbf{P}_\Phi \) as time tends to infinity. To get a complete picture we need to describe the situation when time tends to negative infinity; i.e. we describe the \( \alpha \)-limit sets next.

For the \( \alpha \)-limit sets, we consider the backward time. Over \( x_1 \), the projective bundle is \( \{x_1\} \times \mathbf{P}(x_i) \) where \((x_1, e_{11})\) is the repelling fixed point and \((x_1, e_{12})\) is the attracting fixed point. So, in the reversed time sense, \((x_1, e_{11})\) becomes the attracting and \((x_1, e_{12})\) becomes the repelling fixed point [see Figure 4.7].

![Figure 4.7](image)

Figure 4.7 In backward time direction of flow changes on the projective bundle over \( x_1 \)

In the reversed time sense, all the trajectories, except one, will be attracted to \((x_1, e_{11})\) and exactly one trajectory will be attracted to \((x_1, e_{12})\). In other words, \((x_1, e_{11})\) will be the \( \alpha \)-limit set for all but one of the trajectories of \( \mathcal{I} \mathbf{P}_\Phi \) and \((x_1, e_{12})\) will be the \( \alpha \)-limit set of exactly one trajectory. We summarize this in the following theorem [see Figure 4.8].

**Theorem 4.19.** \( \mathcal{I} \mathbf{P}_\Phi(t, y, v) \rightarrow_{t \rightarrow -\infty} (x_1, e_{11}) \quad \forall \ v \in \mathbf{P}(\gamma_1) \) except \( v_2 \), for which

\[
\mathcal{I} \mathbf{P}_\Phi(t, y, v_2) \rightarrow_{t \rightarrow -\infty} (x_1, e_{12}).
\]
Figure 4.8 $e_{11}$ and $e_{12}$ are the $\alpha$—limit sets

Now that we have analyzed the long term behavior of the trajectories we use the attractor theory to see what happens for times in between.

**Theorem 4.20.** The single trajectory $\mathbb{P}\Phi(t,y,v)$ which has $(x_2, e_{21})$ as its $\omega$—limit set will have $(x_1, e_{11})$ as its $\alpha$—limit set and the single trajectory $\mathbb{P}\Phi(t,y,v)$ which has $(x_1, e_{12})$ as its $\alpha$—limit set will have $(x_2, e_{22})$ as its $\omega$—limit set.

**Proof:** Consider the trajectory $\eta_2$ which has $(x_1, e_{12})$ as its $\alpha$—limit set. Suppose the $\omega$—limit set for $\eta_2$ is $(x_2, e_{21})$.

As we are considering a flow on a compact space (the projective bundle) there will be at least one attractor for the entire dynamics. As $(x_1, e_{12})$ is an attracting fixed point on $\{x_1\} \times \mathbb{P}(x_1)$ it must belong to an attractor for the entire flow on the projective bundle. Hence $\eta_2$ must also belong to the attractor since otherwise $(x_1, e_{12})$ cannot be an attracting point being the $\alpha$—limit set of $\eta_2$.

On $\{x_2\} \times \mathbb{P}(x_2)$, $(x_2, e_{21})$ is a repelling fixed point. Thus if $(x_2, e_{21})$ be the $\omega$—limit
set for $\eta_2$, then $(x_1, e_{12}) \cup \eta_2$, attracting sets, end up being a repeller. More precisely, whatever neighborhood we choose of $(x_1, e_{12}) \cup \eta_2 \cup (x_2, e_{21})$, there is always at least one trajectory, e.g. the one leaving $(x_2, e_{21})$ on $\{x_2\} \times IP_{(x_2)}$, which leaves the neighborhood. Also we cannot choose simply $(x_1, e_{12}) \cup \eta_2$ as an attractor for the same reason. As $(x_2, e_{21})$ is the $\omega$—limit set for $\eta_2$, we have to include that point no matter what neighborhood we choose for $(x_1, e_{12}) \cup \eta_2$, and immediately the two trajectories, as mentioned above, leave that neighborhood. Hence $(x_2, e_{21})$ cannot be the $\omega$—limit set for $\eta_2$. The only other possibility for an $\omega$—limit set is the point $(x_2, e_{22})$ as we proved above in Theorem 4.16. Thus $\omega$—limit set for $\eta_2$ must be $(x_2, e_{22})$. [See Figure 4.9].

![Figure 4.9 Projective flow over $\{x_1\} \cup \gamma_1 \cup \{x_2\}$](image)

In a similar manner it can be shown that the $\alpha$—limit set for the single trajectory $\eta_1$ with $\omega$—limit set $(x_2, e_{21})$ will be $(x_1, e_{11})$. [See Figure 4.9].

**Theorem 4.21.** For the flow on the projective bundle over $\{x_1\} \cup \gamma_1 \cup \{x_2\}$ the attractor is $\{(x_1, e_{12})\} \cup \eta_2 \cup \{(x_2, e_{22})\}$ and the complementary repeller is $\{(x_1, e_{11})\} \cup \eta_1 \cup$
\{(x_2, e_{21})\}.

**Proof:** Let \(A_1 = (x_1, e_{12}) \cup \eta_2 \cup (x_2, e_{22})\). \(A_1\) is compact and invariant. To show that \(A_1\) is an attractor we need to find a neighborhood \(\mathcal{N}\) such that \(\omega(\mathcal{N}) = A_1\).

First we consider a distance on the projective bundle. Note that \(\mathbb{P}M\) is a compact metric space under the metric \(d(\mathbb{P}v, \mathbb{P}v') = \min\{d\left(\frac{v}{||v||}, \frac{v'}{||v'||}\right), d\left(\frac{v}{||v||}, -\frac{v'}{||v'||}\right)\}\) for \(\frac{v}{||v||}, \frac{v'}{||v'||} \in TM \setminus \{0\}\) [[8] pg 583].

Define the distance between \(\eta_1\) and \(\eta_2\) as

\[
\xi = d(\eta_1, \eta_2) = \min_{\mathbb{P}v_1 \in \eta_1, \mathbb{P}v_2 \in \eta_2} [d(\mathbb{P}v_1, \mathbb{P}v_2)]
\]

[The minimum exists as the distance function is a continuous function on a compact space, the projective bundle.]

Cover \(A_1\) with open balls of radius \(\xi/2\). The union of all these balls makes a neighborhood \(\mathcal{N}\) of \(A_1\) such that any trajectory in that neighborhood will have \(\omega\)-limit set in \(A_1\). More precisely, \(\omega(x_1, e_{12}) = (x_1, e_{12})\); for any other trajectory in \(\mathcal{N}\), the \(\omega\)-limit set is \((x_2, e_{22})\). Hence \(A_1 = \{(x_1, e_{12})\} \cup \eta_2 \cup \{(x_2, e_{22})\}\) is an attractor.

Next we want to show that given \(A_1\) as an attractor, \(A_1^* = \{(x_1, e_{11})\} \cup \eta_1 \cup \{(x_2, e_{21})\}\) is the complementary repeller and \((A_1, A_1^*)\) forms an attractor-repeller pair.

We know that

\[
\omega(x_1, e_{11}) = (x_1, e_{11})
\]
\[
\omega(\eta_1) = (x_2, e_{21})
\]
\[
\omega(x_2, e_{21}) = (x_2, e_{21})
\]

(4.26)

Consider the set \(\hat{A}^* = \{(x, p) \in \mathbb{P}M | \omega(x, p) \cap A_1 = \emptyset\}\). We proved above that all the trajectories but \(\eta_1\) in \(\mathbb{P}\{\eta_1\}\) have \((x_2, e_{22})\) as their \(\omega\)-limit set. Thus if \((x, p) \in \hat{A}^*\) then \((x, p)\) must be either \((x_1, e_{11})\) or \((x_2, e_{21})\) or \((x, p)\) should lie on \(\eta_1\). Thus \(\hat{A}^* \subset A_1^*\).

By (4.26) above, \(A_1^* \subset \hat{A}^*\).
Therefore $A^*_1 = \tilde{A}^*$. and by Lemma B.2.11 [8], pg 535, $A^*_1 = \{(x_1, e_{11})\} \cup \eta_1 \cup \{(x_2, e_{21})\}$ is a complementary repeller and $(A, A^*)$ forms an attractor-repeller pair.

Also it is true that if $(x, p) \notin A_1 \cup A^*_1$ then $\omega^*(x, p) \subset A^*_1$ and $\omega(x, p) \subset A_1$. Thus the criteria cited in Proposition B.2.12, [8], pg 535, is satisfied.

So far we considered the linearization, and then the projection, over only one trajectory of the cycle, viz. $\gamma_1$, and the two fixed points. Next we consider the entire cycle, i.e. the two fixed points $x_1, x_2$ and both the trajectories $\gamma_1$ and $\gamma_2$. [See Figure 4.10].

![Figure 4.10 Projective flow over the entire cycle](image)

**Lemma 4.9.** Over $\gamma_2$, all but one of the trajectories will have $\omega$–limit set at $(x_1, e_{12})$, the attracting fixed point on $\{x_1\} \times \Pi_{\{x_1\}}$. Exactly one trajectory will have $(x_1, e_{11})$ as its $\omega$–limit set. Also, all but one of the trajectories will have $(x_2, e_{21})$ as their $\alpha$–limit set and one will have $(x_2, e_{22})$ as its $\alpha$–limit set.

**Proof:** These can be shown exactly in the same manner as over $\gamma_1$ in Theorem 4.16, 4.18, 4.20. □
We will use the following nomenclature in the later discussions.

Call the trajectory having \((x_1, \epsilon_{12})\) as its \(\alpha\)-limit set and \((x_2, \epsilon_{22})\) as its \(\omega\)-limit set \(\eta_{12}\); the trajectory having \((x_2, \epsilon_{22})\) as its \(\alpha\)-limit set and \((x_1, \epsilon_{12})\) as its \(\omega\)-limit set \(\eta_{22}\); the trajectory having \((x_1, \epsilon_{11})\) as its \(\alpha\)-limit set and \((x_2, \epsilon_{21})\) as its \(\omega\)-limit set \(\eta_{11}\); the trajectory having \((x_2, \epsilon_{21})\) as its \(\alpha\)-limit set and \((x_1, \epsilon_{11})\) as its \(\omega\)-limit set \(\eta_{12}\).

**Lemma 4.10.** For the flow in the projective bundle over the cycle \(\Gamma : \{x_1\} \cup \gamma_1 \cup \{x_2\} \cup \gamma_2\) the attractor is \(A = (x_1, \epsilon_{12}) \cup \eta_{12} \cup \{(x_2, \epsilon_{22}) \cup \eta_{22}\) and the complementary repeller is \(A^* = (x_1, \epsilon_{11}) \cup \eta_{11} \cup \{(x_2, \epsilon_{21}) \cup \eta_{12}\).

**Proof:** Just as we did over \(\gamma_1\), it can be shown over \(\gamma_2\) that the attractor is \(A_2 = (x_2, \epsilon_{22}) \cup \eta_{22} \cup (x_1, \epsilon_{12})\) and the complementary repeller is \(A_2^* = (x_2, \epsilon_{21}) \cup \eta_{12} \cup (x_1, \epsilon_{11})\). Hence, over \(\gamma_1\), \(A_1\) admits a neighborhood \(\mathcal{N}_1\) such that \(\omega(\mathcal{N}_1) = A_1\) and over \(\gamma_2\), \(A_2\) admits a neighborhood \(\mathcal{N}_2\) such that \(\omega(\mathcal{N}_2) = A_2\). Let \(\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2\) which is a neighborhood of \(A = A_1 \cup A_2\) and \(\omega(\mathcal{N}) = A\). Also \(A\) is compact and invariant. Thus \(A\) is an attractor for the projective flow over \(\Gamma\).

In a similar manner it can be shown that \(A^* = (x_1, \epsilon_{11}) \cup \eta_{11} \cup \{(x_2, \epsilon_{21}) \cup \eta_{12}\) is a repeller for the projective flow over \(\Gamma\).

Next we show that \(A^*\) is the complementary repeller to \(A\). Consider the set \(B = \{(x, p) \in \Pi \mathcal{M}_{\Gamma}, \, \omega(x, p) \cap A = \phi\}\).

\[
\begin{align*}
\omega(x_1, \epsilon_{11}) &= (x_1, \epsilon_{11}) \\
\omega(\eta_{11}) &= (x_2, \epsilon_{21}) \\
\omega(x_2, \epsilon_{21}) &= (x_2, \epsilon_{21}) \\
\omega(\eta_{12}) &= (x_1, \epsilon_{11}) \\
\end{align*}
\]

Thus \(\omega(A^*) \cap A = \phi\). Hence \(A^* \subset B\).

Next consider any \((x, p) \in \Pi \mathcal{M}_{\Gamma}\). If \((x, p) \notin A^*\) then we proved above in Theorem (4.16) that \(\omega(x, p)\) will be either \((x_2, \epsilon_{22})\) or \((x_1, \epsilon_{12})\). Thus \(\omega(x, p) \cap A \neq \phi \forall (x, p) \notin A^*\).
Hence $B \subset A^\ast$. Thus $A^\ast = B$ and $(A, A^\ast)$ forms an attractor-repeller pair.

To prove our next assertion we would use two theorems from [8]. Let us mention the theorems first.

For reference see B.2.15, [8], pg 536:

**Theorem 4.22.** For a flow on a compact metric space $S$ a finite collection of subsets \{\mathcal{M}_1, \ldots, \mathcal{M}_n\} defines a Morse decomposition if and only if there is a strictly increasing sequence of attractors

$$\phi = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n = X,$$

such that $\mathcal{M}_{n-i} = A_{i+1} \cap A_i^\ast$ for $0 \leq i \leq n - 1$.

For reference see B.2.25, [8], pg 541:

**Theorem 4.23.** The chain recurrent set $\mathcal{R}$ satisfies $\mathcal{R} = \bigcap \{A \cup A^\ast. \ A \text{ is an attractor}\}$.

In particular, there exists a finest Morse decomposition \{\mathcal{M}_1, \ldots, \mathcal{M}_n\} if and only if the chain recurrent set $\mathcal{R}$ has only finitely many connected components. In this case, the Morse sets coincide with the chain recurrent components of $\mathcal{R}$ and the flow restricted to every Morse set is chain transitive and chain recurrent.

Based on these two theorems we prove the following theorem.

**Theorem 4.24.** There are two chain recurrent components for the projective flow over the cycle $\Gamma$. The chain recurrent components are $A$ and $A^\ast$.

**Proof:** For the projective flow over $\Gamma$ we have one trivial attractor $\phi = A_0$ and the other attractor $A = A_1$. Thus according to Theorem 4.22 we have a strictly increasing sequence of attractors

$$\phi \subset A \subset \text{IPM}|_{\Gamma}$$
\[ \phi = A_0 \subset A_1 (= A) \subset A_2 = \mathbb{I} \mathbb{P} M |_\Gamma \]

Then

\[ \mathcal{M}_2 = A_1 \cap A_0^* = A_1 \cap \mathbb{I} \mathbb{P} M |_\Gamma = A_1 = A \]

\[ \mathcal{M}_1 = A_2 \cap A_1^* = \mathbb{I} \mathbb{P} M |_\Gamma \cap A^* = A^* \]

By Theorem 4.22 \{\mathcal{M}_1, \mathcal{M}_2\} or \{A^*, A\} defines a Morse decomposition.

According to Theorem 4.23, the chain recurrent set \( \mathcal{R} = A \cup A^* \) and as there exists a Morse decomposition \( \mathcal{R} \) has only finitely many components. Also, the Morse sets, \( A \) and \( A^* \) coincide with the chain recurrent components of \( \mathcal{R} \). Hence the chain recurrent components for the flow in the projective space over the cycle \( \Gamma \) are \( A \) and \( A^* \).

Let us analyze the situation in a little more detail. We want to check whether \( A \) and \( A^* \) are actually chain recurrent components. Both of them are connected. Consider \( A \). We want to show that given \( \epsilon > 0 \), \( T > 0 \) we can form a periodic \( \epsilon - T \) chain from any point on \( A \).

Choose any point \( P_0 \) on \( A \) restricted over \( \gamma_1 \). If \( P_0 \) is \((x_1, e_{12})\) stay there for time \( T_1 = T \), then choose the next point of the chain within an \( \epsilon \)-neighborhood of \((x_1, e_{12})\) on \( \eta_{21} \).

If \( P_0 \) is on the trajectory \( \eta_{21} \), choose \( T_1 \geq T \) such that following the flow for time \( T_1 \) ends up in an \( \epsilon/2 \)-neighborhood of \((x_2, e_{22})\). Choose the next point \( P_1 \) of the chain within that neighborhood on \( \eta_{22} \). Choose time \( T_2 \geq T \) such that it takes \( T_2 \) from \( P_1 \) to reach an \( \epsilon \)-neighborhood of \((x_1, e_{12})\). If \( P_0 \) was \((x_1, e_{12})\) the next point in the chain would be \( P_0 \) and the chain will be complete. If not, then consider the backward flow from \( P_0 \) on \( \eta_{21} \). As \((x_1, e_{12})\) is the \( \alpha \)-limit set of \( \eta_{21} \), there exists a point \( P_2 \) in the \( \epsilon \)-neighborhood of \((x_1, e_{12})\) and a time \( T_3 \geq T \) such that \( \mathbb{I} \Phi(T_3, P_2) = P_0 \). Choose \( P_2 \) to be the next point in the chain. The last point of the chain of course will be \( P_0 \).
If the initial point $P_0$ is on $A$ restricted over $\gamma_2$ then following exactly the same procedure we can form a periodic $\epsilon - T$ chain. Thus $A$ is a maximal chain transitive subset of $\mathcal{R}$. Hence $A$ is a chain recurrent component.

For $A^*$, similar procedure works. We follow the trajectories $\eta_{11}$ and $\eta_{12}$ and can jump over the two fixed points to form a chain in $A^*$.

Also we can show that nothing else belongs to the chain recurrent components other than $A$ and $A^*$. Consider any other point $(y, p)$ in $\operatorname{IPM}|\mathcal{R}$. As $A$ is an attractor, any trajectory starting at $(y, p)$ will converge either to $(x_2, e_{22})$ or to $(x_1, e_{12})$. Jumping over the fixed point will end up on the trajectories $\eta_{22}$ or $\eta_{21}$, the single trajectories going out of $(x_1, e_{12})$ or to $(x_2, e_{22})$ respectively. $A$ being an attractor, following the flow, it is not possible to go back to the point $(y, p)$ to complete a periodic $\epsilon - T$ chain. Thus $A$ and $A^*$ are the only chain recurrent components.

Next we generalize the cycle to more than two fixed points and two trajectories. We still assume that all the fixed points are hyperbolic. So, now the case is that there is a finite number of hyperbolic fixed points $x_1, x_2, \ldots, x_n = x_1$ and trajectories $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ connecting them. For each fixed point $x_i$, we will get two fixed points for the projective flow – one attracting, one repelling. The attracting fixed point will be the $\omega$–limit set for all trajectories except one for the projective flow over $\gamma_{i-1}$ and will be the $\alpha$–limit set for a single trajectory for the projective flow over $\gamma_i$.

The repelling point will be the $\alpha$–limit set for all but one trajectories of the projective flow over $\gamma_i$ and the $\omega$–limit set for a single trajectory for the projective flow over $\gamma_{i-1}$. Using the same arguments as above we can say that the attractor $A$ is the union of all the attracting fixed points in the projective bundle and the trajectories to which they are the $\alpha$–limit sets. The complementary repeller $A^*$ is the union of all the repelling fixed points for the flow in the projective space and the trajectories for the projective flow to which the fixed points are the $\omega$–limit sets. $A$ and $A^*$ are the two chain recurrent components.
Conclusion: For a finite number of hyperbolic fixed points and trajectories connecting them there are two chain recurrent components.

4.5.2 Chain Recurrent Components for the Nonhyperbolic Case

Next we would consider the nonhyperbolic case. Suppose a fixed point $x_i$ is not hyperbolic. If $A_{x_i}$ has one eigenvalue zero and one nonzero, in the projective bundle over $x_i$, i.e. in $\{x_i\} \times \mathbb{P}(x_i)$, we still get two fixed points corresponding to the two distinct eigendirections of $A_{x_i}$. Of these two fixed points, one is attracting and the other one is repelling. So, in the projective bundle, the situation remains exactly the same as in the previous hyperbolic case and we get two chain recurrent components.

Theorem 4.25. Let for at least one fixed point $x_i$ in a cycle, $A_{x_i}$ have both eigenvalues zero. Then the entire projective bundle over the cycle is a single chain recurrent component.

Proof: We showed in Lemma 4.2 that under the given conditions in this theorem, for the projective flow on $\{x_i\} \times \mathbb{P}(x_i)$, the projective bundle $\{x_i\} \times \mathbb{P}(x_i)$ is chain recurrent.

For simplicity, let us first consider two fixed points $x_1$ and $x_2$ with trajectories $\gamma_1$ and $\gamma_2$ connecting them. Let us assume that $A_{x_1}$ has two distinct eigenvalues and $A_{x_2}$ has both eigenvalues zero. Projection of the two distinct eigenvectors of $A_{x_1}$ gives two fixed points $(x_1, e_{11})$ and $(x_1, e_{12})$ in the projective bundle. Let $(x_1, e_{12})$ be the attracting one and $(x_1, e_{11})$ be the repelling one. From what we proved above $(x_1, e_{12})$ is the $\alpha$—limit set for a single trajectory $\gamma_2$ and $(x_1, e_{11})$ is the $\alpha$—limit set for all the other trajectories for the projective flow over $\gamma_1$. Also $(x_1, e_{11})$ is the $\omega$—limit set for a single trajectory $\eta_1$ and $(x_1, e_{12})$ is the $\omega$—limit set for all the other trajectories for the projective flow over $\gamma_2$.

The projective space over $x_2$, $\{x_2\} \times \mathbb{P}(x_2)$ is chain recurrent. [See Figure 4.11].
We want to show that given $\epsilon$, $T > 0$ we can form a periodic $\epsilon - T$ chain from any point of $\mathbb{P}M$. Let a point $P_1$ be over $\gamma_1$ in $\mathbb{P}M$ but not on $\eta_2$. There will be a trajectory of $\mathbb{P}\Phi$ through $P_1$ with $\alpha-$limit set $(x_1, e_1)$. Consider an $\epsilon-$neighborhood of $\{x_2\} \times \mathbb{P}(x_2)$. As $x_2$ is the $\omega-$limit set for $\gamma_1$ in the base, there will be a time $T'$ such that $\forall t > T', d(\mathbb{P}\Phi(t, P_1), \{x_2\} \times \mathbb{P}(x_2)) < \epsilon$. Also, as $x_2$ is the $\alpha-$limit set for $\gamma_2$ in the base, there will be a time $T''$ such that $\forall t > T'' d(\mathbb{P}\Phi(-t, P_{n_1}), \{x_2\} \times \mathbb{P}(x_2)) < \epsilon$ where $P_{n_1}$ is a point on $\eta_1$. We form the $\epsilon - T$ chain as follows:

Start from $P_1 = z_0$ and follow the flow for time $T_1 > \max(T, T')$. The trajectory will end up within the $\epsilon-$neighborhood of $\{x_2\} \times \mathbb{P}(x_2)$. Choose the next point $z_1$ of the chain on $\{x_2\} \times \mathbb{P}(x_2)$.

Next consider a point on the trajectory $\eta_1$ which is within the $\epsilon-$distance of a point $B$ of $\{x_2\} \times \mathbb{P}(x_2)$. As $\{x_2\} \times \mathbb{P}(x_2)$ is chain transitive we can form an $\epsilon - T$ chain from $z_1$ to $B$. The next point $z_2$ in the chain will be the point on $\eta_1$ within the $\epsilon-$distance to $B$.

Consider an $\epsilon-$ball around $(x_1, e_1)$. Choose time $T_2 > T$ such that $\mathbb{P}\Phi(T_2, z_2)$ ends
up within this $\epsilon-$ball at $z_3$, the next point in the chain. Let $z_4 = (x_1, e_{11})$. As $(x_1, e_{11})$ is the $\alpha-$limit set for the trajectory through $P_1$, there exists a $T''$ such that $\forall \ t > T''$, $d(e_{11}, \mathcal{I} \Phi(-t, P_1)) < \epsilon$. In other words, in the $\epsilon-$ball around $e_{11}$, we can find a point of the trajectory through $P_1$. Let $T_3 > \max(T, T'')$. Choose the point $\mathcal{I} \Phi(-T_3, P_1) = z_5$ the next point in the chain. Then following the trajectory for time $T_3$ we would end up in $P_1$ and thus the $\epsilon - T$ chain would be complete.

If a point $P_2$ is somewhere over $\gamma_2$ (other than $\eta_1$) we would follow the trajectory through $P_2$, end up in an $\epsilon-$neighborhood of $(x_1, e_{12})$, choose the next point within that $\epsilon-$ball on $\eta_2$, follow the trajectory $\eta_2$ to enter an $\epsilon-$neighborhood of $\{x_2\} \times \mathcal{I} P\{x_2\}$. Choose the next point on $\{x_2\} \times \mathcal{I} P\{x_2\}$. Consider the backward flow from $P_2$. As $\gamma_2 \to_{t \to -\infty} x_2$, we can find a $\hat{T}$ such that for all $t > \max(\hat{T}, T)$, $\mathcal{I} \Phi(-t, P_2)$ lies within the $\epsilon-$neighborhood of $\{x_2\} \times \mathcal{I} P\{x_2\}$. As $\{x_2\} \times \mathcal{I} P\{x_2\}$ is chain transitive we can connect this point to our point of the chain on $\{x_2\} \times \mathcal{I} P\{x_2\}$ by an $\epsilon - T$ chain.

If a point $P_3$ lies on $\eta_1$ we would follow the trajectory, as we come near $\{x_1\} \times \mathcal{I} P\{x_1\}$, we can make a jump over $e_{11}$, follow the flow on $\{x_1\} \times \mathcal{I} P\{x_1\}$, make a jump over $e_{12}$ and continue along $\eta_2$. As we come near $\{x_2\} \times \mathcal{I} P\{x_2\}$ we can form chain on it to reach $\eta_1$. Considering the backward flow from $P_3$ we can complete the chain. The same idea works for $P_3$ on $\eta_2$.

Thus, for any point $P$ on $\mathcal{I} P M$ we can form a periodic $\epsilon - T$ chain. Hence the entire projective bundle is a single chain recurrent component.

The same idea of forming chains can be extended for a finite number of fixed points, where, for at least one $x_i$, $A_{x_i}$ has both eigenvalues zero making $\{x_i\} \times \mathcal{I} P\{x_i\}$ chain recurrent. In this situation consider the following cases [see Figure 4.12]:

1. $P$ lies on the projective bundle over $\gamma_i$ where none of $\{x_i\} \times \mathcal{I} P\{x_i\}$, or $\{x_{i+1}\} \times \mathcal{I} P\{x_{i+1}\}$ are chain recurrent.

2. $P$ lies on the projective bundle over $\gamma_i$ where $\{x_i\} \times \mathcal{I} P\{x_i\}$ is not chain recurrent
Figure 4.12 Illustration of a possible combination of different situations for the projective flow over a cycle

but \( \{x_{i+1}\} \times \mathbb{P}(x_{i+1}) \) is.

3. \( P \) lies on the projective bundle over \( \gamma_i \) where none of \( \{x_i\} \times \mathbb{P}(x_i) \) is chain recurrent but \( \{x_{i+1}\} \times \mathbb{P}(x_{i+1}) \) is not.

4. \( P \) lies on the projective bundle over \( \gamma_i \) where both \( \{x_i\} \times \mathbb{P}(x_i) \) and \( \{x_{i+1}\} \times \mathbb{P}(x_{i+1}) \) are chain recurrent.

We want to show that no matter where \( P \) lies, we can form a periodic \( \epsilon - T \) chain starting from \( P \). In each segment we would define sections of \( \epsilon - T \) chain starting from \( P \) and ending at \( P \), then continue forming chains over all the other segments. Whenever necessary we would refer to Fig 4.12.

If \( P \) lies over a segment like case (1):

The flows on the projective spaces over \( x_i \) and \( x_{i+1} \) have two fixed points, one attracting, one repelling. Over \( x_{i+1} \), the attracting fixed point will be the \( \omega \)-limit set for all but one of the trajectories for the projective flow over \( \gamma_i \), the repelling fixed point will be the \( \omega \)-limit set for a single trajectory. Over \( x_i \), the attracting fixed point will be the \( \alpha \)-limit set for a single trajectory and the repelling fixed point will be the \( \alpha \)-limit
set for all the other trajectories. The single trajectory converging to the repelling fixed point over \( x_{i+1} \) will have the repelling fixed point over \( x_i \) as its \( \alpha \)-limit set. The single trajectory originating from the attracting fixed point over \( x_i \) will have the attracting fixed point over \( x_{i+1} \) as its \( \omega \)-limit set.

If \( P \) lies anywhere in the projective bundle over \( \gamma_i \) other than the two "special" single trajectories \( \eta_1 \) and \( \eta_2 \), as shown over \( \gamma_1 \) in Fig 4.12, we start from \( P \) and follow the trajectory through \( P \) for some time \( T_1 > T \) so that after time \( T_1 \) it ends up within the \( \epsilon \)-ball of \( C \), the attracting fixed point. Choose this point, \( \mathbb{P}\Phi(T_1, P) \) as a point in the chain. Choose \( C \) to be the next point where the flow can stay for time \( T \).

To get the other end of the chain, i.e. the end of the periodic \( \epsilon - T \) chain that comes back to \( P \), consider the backward flow starting from \( P \). We can find a time \( T_2 > T \) such that the backward flow starting from \( P \) will end up in the \( \epsilon \)-neighborhood of \( B \). Choose this point as the last but one point \( \hat{P} \) for the chain so that \( \mathbb{P}\Phi(T_2, \hat{P}) = P \). Let the point previous to this be \( B \).

If \( P \) lies on one of the single trajectories \( \eta_1 \) or \( \eta_2 \):

If \( P \) lies on \( \eta_1 \) then the forward chain will be same as above. For the flow backward for some time \( T_2 > T \) we would end up in an \( \epsilon \)-neighborhood of the point \( A \). Choose this point as the point \( \hat{P} \), previous to \( P \) in the chain. Let the point \( A \) be the point previous to \( \hat{P} \). Consider the backward flow from \( A \) on \( \{x_1\} \times \mathbb{P}_{\{x_1\}} \). There exists a time \( T_3 > T \) such that the backward flow starting from \( A \) ends up within the \( \epsilon \)-neighborhood of \( B \). Let this point be the point previous to \( A \) in the chain and the point previous to this one will be \( B \).

If \( P \) lies on \( \eta_2 \) then the backward chain will be a point in the \( \epsilon \)-neighborhood of \( B \). The forward flow from \( P \), after some time \( T_1 > T \) will enter in the \( \epsilon \)-neighborhood of \( D \), the repelling point in \( \{x_2\} \times \mathbb{P}_{\{x_2\}} \). Let this point be the point next to \( P \) in the chain, and \( D \) the following point. If necessary, we can continue the chain from \( D \) to \( C \). There will be a time \( T_3 > T \) such that the trajectory starting near \( D \) will enter the
$\epsilon$–neighborhood of C in time $T_3$. Let this be a point in the chain and C the next point where the flow can stay for time $T$.

Thus for case (1), no matter where P lies, we can form a chain starting from B which will end in P and from P we can form a chain up to C.

Case (2) is when P lies over $\gamma_i$ where $\{x_i\} \times IP(x_i)$ is not chain recurrent but $\{x_{i+1}\} \times IP(x_{i+1})$ is. In Fig. 4.12 this case is similar to segment 2 over $\gamma_2$. The projective flow in $\{x_2\} \times IP(x_2)$ has two fixed points, one attracting, C, in Fig 4.12, one repelling, D, in Fig. 4.12. A single trajectory $\eta_3$ in Fig 4.12 has C as its $\alpha$–limit set, all the rest of the trajectories have D as their $\alpha$–limit set. The projective space $\{x_3\} \times IP(x_3)$ over $x_3$ is chain recurrent. The trajectory $\gamma_2$ converges to $x_3$ in the base. Thus, given $\epsilon > 0$, there exists a time $T_1 > T$ such that all the trajectories for the projective flow over $\gamma_2$ enters the $\epsilon$–neighborhood of $\{x_3\} \times IP(x_3)$.

If P is on any trajectory other than $\eta_3$, follow the flow for time $T_1$ to end up within an $\epsilon$–neighborhood of $\{x_3\} \times IP(x_3)$. As $\{x_3\} \times IP(x_3)$ is chain transitive we can construct a chain on it between any two points. We will choose a suitable point to form this chain depending on the next stage of the chain that needs to be joined to this one.

From P, following the backward flow for some time $T_2 > T$ we enter a neighborhood of D. Choose D to be the previous point in the chain.

If P lies on $\eta_3$ the forward chain is just as above. The backward flow, after a time $T_1 > T$ will end up in an $\epsilon$–neighborhood of the point C. Choose C to be the previous point in the chain. Choose the point previous to C on $\{x_2\} \times IP(x_2)$, inside the $\epsilon$–neighborhood of C. If we follow the backward flow from this point, after time $T_4 > T$ would reach in the $\epsilon$–neighborhood of D. Thus, if necessary we can connect D and C by a chain.

Case (3) is like segment (3) over $\gamma_3$ in Fig 4.12. If P lies on any trajectory other than $\eta_4$ the forward flow will lead us to construct a chain including the fixed point E. The backward flow will end up in the $\epsilon$–neighborhood of $\{x_3\} \times IP(x_3)$. 
If $P$ lies on $\eta_4$, the forward flow will include the fixed point $F$ in the chain. If necessary $E$ and $F$ can be connected by a chain. The backward flow will end up in an $\epsilon$—neighborhood of $\{x_3\} \times \mathbb{P}(x_3)$.

Case (4) is displayed in segment 5 over $\gamma_5$ in Fig 4.12. Both $\{x_5\} \times \mathbb{P}(x_5)$ and $\{x_6\} \times \mathbb{P}(x_6)$ are chain recurrent. No matter where $P$ is, there is a time $T_1 > T$ such that the flow starting from $P$ will enter an $\epsilon$—neighborhood of $\{x_6\} \times \mathbb{P}(x_6)$ in time $T_1$ and $\exists$ a time $T_2 > T$ such that the backward flow from $P$ will enter the $\epsilon$—neighborhood of $\{x_5\} \times \mathbb{P}(x_5)$ in time $T_2$.

As an illustration of a specific situation let $P$ belong to segment 2 over $\gamma_2$ in Fig 4.12. We want to form a periodic $\epsilon - T$ chain at $P$.

There exists a time $T_1 > T$ such that $\Phi(T_1, P)$ enters an $\epsilon$—neighborhood of $\{x_3\} \times \mathbb{P}(x_3)$. Choose $p_1 = \Phi(T_1, P)$ and $p_2$ = a point on $\{x_3\} \times \mathbb{P}(x_3)$ within an $\epsilon$—distance to $p_1$. As $\{x_3\} \times \mathbb{P}(x_3)$ is chain transitive we can form a chain between any two points on it. Form a chain from $p_3$ to $G$. Form an $\epsilon$—ball around $G$ and choose a point $p_{i_1}$ in that ball on $\eta_4$. Form an $\epsilon$—ball around $F$. There exists a time $T_{p_{i_1}} > T$ such that $p_{i_2} = \Phi(T_{p_{i_1}}, p_{i_1})$ lies in the $\epsilon$—ball around $F$. Choose $F$ to be the next point in the chain. $F$ is the $\alpha$—limit set for all but one trajectories in segment 4 over $\gamma_4$. We can choose $p_{i_3}$ on any one of them within an $\epsilon$—distance to $F$. Starting from $p_{i_3}$ follow the flow for a time $T_{p_{i_3}} > T$, the trajectory will reach an $\epsilon$—neighborhood of $\{x_5\} \times \mathbb{P}(x_5)$. Choose the next point $p_{i_4}$ on $\{x_5\} \times \mathbb{P}(x_5)$. All the trajectories of $\Phi$ over $\gamma_6$ will enter an $\epsilon$—neighborhood of $\{x_5\} \times \mathbb{P}(x_5)$ as $t \to -\infty$. Choose $p_{i_5}$ on any one of them, follow the flow for time $T_{p_{i_5}} > T$ so that $\Phi(T_{p_{i_5}}, p_{i_5}) = p_{i_6}$ lies in the $\epsilon$—neighborhood of $\{x_6\} \times \mathbb{P}(x_6)$. As $t \to -\infty$ the trajectory $\eta_6$ over $\gamma_6$ (which is the single trajectory having the point $B$ as its $\omega$—limit set) enters the $\epsilon$—neighborhood of $\{x_6\} \times \mathbb{P}(x_6)$. Choose $p_{i_7}$ on $\{x_6\} \times \mathbb{P}(x_6)$. Let the point on $\eta_6$ in the neighborhood of $\{x_6\} \times \mathbb{P}(x_6)$ be $p_{j_1}$. Let $p_{j_0}$ be a point on $\{x_6\} \times \mathbb{P}(x_6)$ within an $\epsilon$—distance of $p_{j_1}$. As $\{x_6\} \times \mathbb{P}(x_6)$ is chain transitive, form an $\epsilon - T$ chain from $p_{i_7}$ to $p_{j_0}$. Choose the next point $p_{j_1}$ for
the chain on \( \eta_5 \). Following the flow on \( \eta_5 \) after some time \( T_{p_{j_1}} > T \) we would reach the \( \epsilon \)-neighborhood of \( B \). Let \( p_{j_2} = \Phi(T_{p_{j_1}}, p_{j_1}) \). Choose \( B \) to the next point in the chain.

Now, \( B \) is the \( \alpha \)-limit set for all but one trajectories of \( \Phi \) over \( \gamma_1 \). Among them is \( \eta_2 \), the single trajectory that has the repelling point \( D \) of \( \{x_2\} \times \mathbb{P}(x_2) \) as its \( \omega \)-limit set. Let \( p_{j_2} \) be a point on \( \eta_2 \) lying in the \( \epsilon \)-neighborhood of \( B \). Choose \( p_{j_2} \) the next in the chain, follow the flow for some time \( T_{p_{j_1}} > T \) to reach \( p_{j_2} \) in the \( \epsilon \)-neighborhood of \( D \). Choose \( D \) as the next point. For \( \Phi \) over \( \gamma_2 \), \( D \) is the \( \alpha \)-limit set for all but one of the trajectories. Among them is the trajectory passing through \( P \). Starting from \( P \) we consider the backward flow. There exists some time \( T_{p_{j_4}} > T \) such that the backward flow trajectory enters the \( \epsilon \)-neighborhood of \( D \). Let \( p_{j_5} \) be a point on this trajectory in this neighborhood. Choose \( p_{j_5} \) as the next point in the chain. Then starting from \( p_{j_5} \) after time \( T_{p_{j_5}} > T \) the chain will end at \( P \). \( \square \)

4.5.3 Morse Intervals for a Cycle

In the above we found the possible chain recurrent components for different situations that can occur in a cycle. Next we compute the Morse spectrum.

**Theorem 4.26.** Let \( \{x_1, \gamma_1, x_2, \gamma_2, \ldots, x_i, \gamma_i, \ldots, x_n, \gamma_n\} \) be a cycle for a flow \( \dot{x} = f(x) \). Here \( \{x_1, x_2, \ldots, x_n\} \) are fixed points and \( \gamma_1, \gamma_2, \ldots, \gamma_n \) are trajectories connecting them.

We assume that the cycle is an isolated chain recurrent component. Linearization over a fixed point gives the linearized system

\[
\begin{align*}
\dot{x} &= f(x) \\
\dot{v} &= A_x v
\end{align*}
\]

We assume that all the matrices \( A_{x_1}, A_{x_2}, \ldots, A_{x_n} \) have two distinct Lyapunov exponents and one of them might be zero. Let the Lyapunov exponents over \( x_i \) be given by \( \lambda_{i_1}, \lambda_{i_2} \) where \( \lambda_{i_1} < \lambda_{i_2} \) (one of them might be zero). Then the Morse intervals are given by \( [\min_{1 \leq i \leq n} \lambda_{i_1}, \max_{1 \leq i \leq n} \lambda_{i_1}] \) and \( [\min_{1 \leq i \leq n} \lambda_{i_2}, \max_{1 \leq i \leq n} \lambda_{i_2}] \).
Proof: Let us analyze the situation for a cycle with two fixed points \( x_1 \) and \( x_2 \) with trajectories \( \gamma_1 \) and \( \gamma_2 \) connecting them. Over \( x_1 \) the Lyapunov exponents are \( \lambda_{11}, \lambda_{12} \) with \( \lambda_{11} < \lambda_{12} \). Over \( x_2 \) the Lyapunov exponents are \( \lambda_{21}, \lambda_{22} \) with \( \lambda_{21} < \lambda_{22} \).

It is known by Theorem 2.4 that the Lyapunov exponents form the boundaries of the Morse intervals. Also, as there are two chain recurrent components here, there will be two Morse intervals. Let \( e_{11}, e_{12} \) be the eigendirections for \( A_{x_1} \) and \( e_{21}, e_{22} \) be the eigendirections for \( A_{x_2} \). Projections of these give four fixed points for the projective flow. Denote them by \( \text{Pe}_{11}, \text{Pe}_{12}, \text{Pe}_{21}, \text{Pe}_{22} \) with \( \text{Pe}_{12} \) and \( \text{Pe}_{22} \) attracting and the other two repelling. Let the single trajectory of \( \text{IP} \Phi \) over \( \gamma_1 \) converging to \( \text{Pe}_{21} \) and emerging from \( \text{Pe}_{11} \) be \( \eta_{11} \) and the single trajectory emerging from \( \text{Pe}_{12} \) and converging to \( \text{Pe}_{22} \) be \( \eta_{12} \). [See Fig 4.13] Similarly, for the flow \( \text{IP} \Phi \) over \( \gamma_2 \), the single trajectory converging to \( \text{Pe}_{11} \) is \( \eta_{21} \) and the single trajectory emerging from \( \text{Pe}_{21} \) is \( \eta_{22} \). \( \eta_{21} \) has \( \text{Pe}_{21} \) as its \( \alpha \)-limit set and \( \eta_{22} \) has \( \text{Pe}_{12} \) as its \( \omega \)-limit set. Denote the chain recurrent components by \( \mathcal{FK}_1 \) and \( \mathcal{FK}_2 \) where \( \mathcal{FK}_1 \) is \( \{\text{Pe}_{12}\} \cup \eta_{12} \cup \text{Pe}_{22} \cup \eta_{22} \), the attractor, and \( \mathcal{FK}_2 \) is \( \{\text{Pe}_{11}\} \cup \eta_{11} \cup \text{Pe}_{21} \cup \eta_{21} \), the repeller. [See Figure 4.13].

![Figure 4.13 Projective flow over a two points two trajectories cycle](image-url)
Corollary 4.6 in Lars Grüne’s paper [13] states:

Let $\mathcal{P}K \subseteq \mathcal{P}M$ be a compact invariant set for the projected flow $\mathcal{P}\Phi$ such that $\mathcal{P}\Phi$ is chain transitive on $\mathcal{P}K$. Then the Morse spectrum $\Sigma_{M_o}(\mathcal{P}K)$ is a closed interval whose extremal points are actually Lyapunov exponents for some points $\mathcal{P}e^*$ and $\mathcal{P}e \in \mathcal{P}M$. For these points the Lyapunov exponents are actually limits.

In our case, both $\mathcal{P}K_1$ and $\mathcal{P}K_2$ are connected compact invariant sets for $\mathcal{P}\Phi$ and $\mathcal{P}\Phi$ is chain transitive on them.

Theorem 4.4 in [13] states:

Let $\mathcal{P}K \subseteq \mathcal{P}M$ be a connected compact invariant set for the projected flow $\mathcal{P}\Phi$. Then $\Sigma_{L_y}(\mathcal{P}K) \subseteq \Sigma_{U_e}(\mathcal{P}K)$ and there exist points $\mathcal{P}e^*$ and $\mathcal{P}e \in \mathcal{P}K$ such that

$$\lambda(\mathcal{P}e^*) = \min_{\Sigma_{U_e}(\mathcal{P}K)} \lambda(\mathcal{P}K) \quad \text{and} \quad \lambda(\mathcal{P}e) = \max_{\Sigma_{U_e}(\mathcal{P}K)} \lambda(\mathcal{P}K).$$

For these points the Lyapunov exponents are actually limits.

Theorem 4.5 in [13] shows the equivalence of the Morse spectrum and the uniform exponential spectrum. We cite the theorem here.

**Theorem 4.27.** Let $\mathcal{P}K \subseteq \mathcal{P}M$ be a compact invariant set for the projected flow $\mathcal{P}\Phi$ such that $\mathcal{P}\Phi|_{\mathcal{P}K}$ is chain transitive. Then

$$\Sigma_{M_o}(\mathcal{P}K) = \Sigma_{U_e}(\mathcal{P}K).$$

Combination of the last two results from [13] gives existence of $\mathcal{P}e^*$ and $\mathcal{P}e$ in $\mathcal{P}K_1$ corresponding to which we get the Lyapunov exponents which form the boundaries of the Morse intervals.

It was proved before in Theorem 4.2 that the end points for the Morse intervals are Lyapunov exponents coming from the chain-recurrent component, i.e., the cycle in this case. Also, in Chapter 3 it was proved that the Lyapunov Exponents for the cycle comes from the fixed points only. So $\mathcal{P}e^*$ and $\mathcal{P}e \in \mathcal{P}K_1$ must come from the linearization over the fixed points.
For the case of \( \{x_1, x_2, \ldots, x_n\} \), i.e., two fixed points, two trajectories, there are two such points, \( P_1 \) and \( P_2 \), corresponding to \( pK_1 \). They correspond to the larger eigenvalues of \( A_{x_1} \) and \( A_{x_2} \) respectively. Hence those two eigenvalues (or Lyapunov exponents) will form the boundary of a Morse interval.

For \( pK_2 \), the smaller eigenvalues of \( A_{x_1} \) and \( A_{x_2} \) will form the interval.

That everything inside these intervals are Morse exponents (UE spectrum) is proved in [13] in Theorem 3.3. Also, this is shown in [8] page 162, Theorem 5.3.4.

For more than two fixed points, \( pK_1 \) has points \( P_{12}, P_{22}, P_{32}, \ldots \) all of which correspond to the larger eigenvalues \( \lambda_{12}, \lambda_{22}, \lambda_{32}, \ldots \) of \( A_{x_1}, A_{x_2}, A_{x_3}, \ldots \) respectively; and \( pK_2 \) has points \( P_{11}, P_{21}, P_{31}, \ldots \) all of which correspond to the smaller eigenvalues \( \lambda_{11}, \lambda_{21}, \lambda_{31}, \ldots \) of \( A_{x_1}, A_{x_2}, A_{x_3}, \ldots \) respectively. Hence the Morse intervals in this case would be \( [\min \lambda_{ij}, \max \lambda_{ij}] \) and \( [\min \lambda_{ij}, \max \lambda_{ij}] \).

**Theorem 4.28.** Let \( \{x_1, x_2, \ldots, x_n\} \) be a cycle for a flow \( \dot{x} = f(x) \). Here \( \{x_1, x_2, \ldots, x_n\} \) are fixed points and \( \gamma_1, \gamma_2, \ldots, \gamma_n \) are trajectories connecting them and we assume that the cycle is an isolated chain recurrent component. Linearization over a fixed point \( x_i \) gives the linearized system

\[
\dot{x} = f(x) \\
\dot{v} = A_{x_i} v
\]

We assume that at least one matrix \( A_{x_i} \) has both eigenvalues zero. Let the Lyapunov exponent over each \( x_i \) be given by \( \lambda_{i1}, \lambda_{i2} \). Then there is a single Morse interval given by

\[
[\min_{1 \leq i \leq n, 1 \leq j \leq 2} \lambda_{ij}, \max_{1 \leq i \leq n, 1 \leq j \leq 2} \lambda_{ij}].
\]

**Proof:** It is proved above in Theorem 4.25 that for the situation stated in this theorem there is a single chain recurrent component. This means there would be a single Morse interval whose boundary points are Lyapunov exponents. Hence

\[
[\min_{1 \leq i \leq n, 1 \leq j \leq 2} \lambda_{ij}, \max_{1 \leq i \leq n, 1 \leq j \leq 2} \lambda_{ij}]
\]
gives the Morse interval.

Now that we have identified the intervals that give us the Morse spectrum, let us analyze these intervals in a little more detail.

For the sake of convenience, consider a two-fixed points, two-trajectories cycle:

$$\Gamma\{x_1, \gamma_1; x_2, \gamma_2\}$$

Assume that both the matrices of linearization, $A_{x_1}$ and $A_{x_2}$ over $x_1$ and $x_2$, have two distinct eigenvalues. Projection of the eigendirections give four fixed points for the projective flow. Denote them by $e_{11}, e_{12}, e_{21}$ and $e_{22}$. Then the chain recurrent components are $pK_1 = e_{11} \cup \eta_{11} \cup e_{21} \cup \eta_{12}$ and $pK_2 = e_{12} \cup \eta_{21} \cup e_{22} \cup \eta_{22}$. [See Figure 4.14].

![Figure 4.14 Attractor and Repeller for the projective flow](image)

Let $[\lambda_{11}, \lambda_{21}]$ denote the Morse interval corresponding to $pK_1$. We want to show that every point in $[\lambda_{11}, \lambda_{21}]$ gives a Morse exponent. First consider the (rational) convex combinations of $\lambda_{11}$ and $\lambda_{21}$; i.e., $\frac{\lambda_{11} + m \lambda_{21}}{l + m} \in [\lambda_{11}, \lambda_{21}]$, where $l$ and $m$ are integers. For $\epsilon_n, T_n > 0$, denote an $(\epsilon_n, T_n)$-chain of $p\Phi$ by $\zeta_n$. Given $\epsilon_n, T_n > 0$, we want to construct
\(\zeta_n\) such that for \(\epsilon_n \to 0\) and \(T_n \to \infty\), we have \(\lambda(\zeta_n) \to \frac{\lambda_{l_1} + m \lambda_{l_2}}{l + m} \) as \(n \to \infty\). Choose \(\epsilon_n = \frac{1}{n}\). We form a chain as follows.

Consider the neighborhood \(B(e_{11}, e_n)\) and let \(e_{1n} \in B(e_{11}, e_n) \cap \eta_{11}\). Also, consider the neighborhood \(B(e_{21}, e_n)\) and let \(e_{2n} \in B(e_{21}, e_n) \cap \eta_{11}\). Let \(T_{1n}\) be the time taken for a trajectory of the flow \(\Phi\) to move from \(e_{1n}\) to \(e_{2n}\).

Next, consider the two neighborhoods \(B(e_{21}, e_n)\) and \(B(e_{11}, e_n)\) and let \(e_{2n} \in B(e_{21}, e_n) \cap \eta_{12}\) and \(e_{1n} \in B(e_{11}, e_n) \cap \eta_{12}\). Let \(T_{2n}\) be the time taken for the flow \(\Phi\) to move from \(e_{2n}\) to \(e_{1n}\).

Let \(T_n = \max\{T_{1n}, T_{2n}\}\). This makes sure that starting from \(e_{1n}\) or \(e_{2n}\) after time \(T_n\) the flow will be in the \(\epsilon\)-neighborhood of \(e_{21}\) or \(e_{11}\) respectively.

Let the flow stay at \(e_{11}\) for time \(nT_n\), then choose the next point \(e_{1n}\) which is within distance \(\epsilon_n\) from \(e_{11}\). Starting from \(e_{1n}\) follow the flow for time \(T_n\). This ends up at a point inside \(B(e_{21}, e_n)\). Choose \(e_{21}\) to be the next point in the chain and stay there for time \(mnT_n\). The following point in the chain is \(e_{2n}\). Starting from \(e_{2n}\) follow the flow for time \(T_n\) to end up in \(B(e_{11}, e_n)\). Choose the final point in the chain as \(e_{11}\). Thus a periodic \((e_n, T_n)\) chain is formed from \(e_{11}\). Let us see what \(\lambda(\zeta_n)\) is in this case.

\[
\lambda(\zeta_n) = \frac{\log \epsilon_n \lambda_{l_1} T_n + \log |T\Phi(T_n, e_{1n})| + \log \epsilon_n \lambda_{l_2} T_n + \log |T\Phi(T_n, e_{2n})|}{nT_n + mnT_n + T_n} = \frac{nl_{11} T_n + n\lambda_{l_2} T_n}{(2 + nl + mn)T_n} + \frac{log |T\Phi(T_n, e_{1n})| + log |T\Phi(T_n, e_{2n})|}{(2 + nl + mn)T_n}
\]

As \(n \to \infty\) (i.e., \(\epsilon_n \to 0\)) the first term on the right hand side

\[
\frac{nl_{11} T_n + n\lambda_{l_2} T_n}{(2 + nl + mn)T_n} \to \frac{\lambda_{l_1} + m\lambda_{l_2}}{l + m}
\]

As \(n \to \infty\), the terms \(\log |T\Phi(T_n, e_{1n})|\) and \(\log |T\Phi(T_n, e_{2n})|\) will tend to finite limits. This is so because the finite numbers \(\lambda_{l_1}\) and \(\lambda_{l_2}\) are the infimum and supremum of the Morse exponents (limits of the exponential growth rates) over \(K_1\), and the two terms are exponential growth rates. Let \(\log |T\Phi(T_n, e_{1n})| \to_{n \to \infty} l_1\) and \(\log |T\Phi(T_n, e_{2n})| \to_{n \to \infty} l_2\).

Let us reformulate the problem like this: there are two sequences \(\alpha_n\) and \(\beta_n\) such that \(\alpha_n \to l_1\) and \(\beta_n \to l_2\) as \(n \to \infty\). The question is what happens to \(\frac{\alpha_n + \beta_n}{2 + (l_1 + l_2)n}\). Given
\( \epsilon > 0 \) we can find \( N_1 \) and \( N_2 \) such that \( |a_n - l_1| < \epsilon \) for all \( n \geq N_1 \), and \( |\beta_n - l_2| < \epsilon \) for all \( n \geq N_2 \). Choose \( N = \max\{N_1, N_2\} \). Then for \( n \geq N \) we have both \( |a_n - l_1| < \epsilon \) and \( |\beta_n - l_2| < \epsilon \). Then for all \( n \geq N \)

\[
\left| \frac{\alpha_n + \beta_n}{2 + (l + m)n} \right| \leq \frac{|\alpha_n|}{2 + (l + m)n} + \frac{|\beta_n|}{2 + (l + m)n} < \frac{|l_1| + \epsilon}{2 + (l + m)n} + \frac{|l_2| + \epsilon}{2 + (l + m)n}
\]

which is less than \( \epsilon \) if \( n > \frac{|l_1| + |l_2|}{(l + m)\epsilon} \). Given \( \epsilon > 0 \), if we choose \( M \geq \max\{N, \frac{|l_1| + |l_2|}{(l + m)\epsilon}\} \), then \( |\frac{\alpha_n + \beta_n}{2 + (l + m)n}| < \epsilon \) for all \( n \geq M \), or, as \( n \to \infty \), the term \( |\frac{\alpha_n + \beta_n}{2 + (l + m)n}| \to 0 \). Thus, as \( n \to \infty \),

\[
\frac{\log |T(\hat{\mathcal{P}}(\hat{T}_n, \hat{e}_{1n}))| + \log |T(\hat{\mathcal{P}}(\hat{T}_n, \hat{e}_{2n})|}{(2 + nl + mn)\hat{T}_n} \to 0
\]

Therefore,

\[
\lambda(\zeta_n) \to n \to \infty \frac{\lambda_{11} + m\lambda_{21}}{l + m}
\]

Now that we are done with the convex (rational) combinations of \( \lambda_{11} \) and \( \lambda_{21} \) consider a \( \lambda \in [\lambda_{11}, \lambda_{21}] \) which is an irrational combination of the boundary points \( \lambda_{11} \) and \( \lambda_{21} \). As the set of rational numbers is dense in \( \mathbb{R} \), \( \lambda \) is a limit point of the rational numbers in \( [\lambda_{11}, \lambda_{21}] \). As each of these rational numbers is a Morse exponent and the Morse spectrum is closed, \( \lambda \) must be a Morse exponent.

The argument for a cycle with \( n > 2 \) fixed points and connecting trajectories is similar. \( \square \)

We summarize our results in the following theorem.

**Theorem 4.29.** The Morse spectrum and the Lyapunov spectrum do not coincide for the case of a cycle where the cycle is an isolated chain recurrent component.

**Proof.** From what we discussed above, in this case, the Morse spectrum consists of intervals where boundary points are the Lyapunov spectrum. \( \square \)
5 CONCLUDING REMARKS

In this paper we analyzed the relationship between the Lyapunov and the Morse spectra. Our goal was to check whether the two spectra were identical and if they were not then under what conditions they could become identical. We considered three specific situations in two dimensions. By direct computation of the two spectra we found a situation where the two spectra were different, giving an answer for our first question. With the current techniques that we were aware of, we could not proceed any further. The second question remains open.

An extension of our analysis might be to consider control/perturbation systems instead of just dynamical systems. This will be more appropriate for the purpose of application.

Our result implies that in general, for a perturbation analysis, using Morse spectrum for a perturbed system and applying its nice continuity and persistency properties, we can only conclude about the Morse spectrum of the original system, not about the Lyapunov spectrum. As Lyapunov spectrum is very important for stability analysis, this difficulty again leads to our second question.

Here we did analysis in two dimensions. Another extension might be to consider higher dimensions to get more general results. But there, as far as we are aware of, a case by case direct study like this cannot be done.


