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Errors in the approximations of functions by the use of functionals

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ERRORS IN THE APPROXIMATIONS OF FUNCTIONS

BY THE USE OF FUNCTIONALS

by

Roy Herbert Cook

A Thesis Submitted to the Graduate Faculty
for the Degree of

DOCTOR OF PHILOSOPHY

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Iowa State College

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I. INTRODUCTION

It is the purpose of this thesis to present remainder operators for certain infinite series. The series discussed are of a special nature, and are based upon the notion of fundamental sets of functionals for a given set of functions. That is, one may select any linearly independent set of functions, form a set of functionals according to certain definitions and develop series expansions for which the remainder is given. An advantage of this scheme is that one has a rather wide choice in his selection of functionals, and thus need not be bothered for example with the evaluation of complicated integrals.

Fundamental sets of functionals of both finite and infinite order are considered and it is shown that the set of infinite order is in general the set ordinarily associated with the given set of functions.
II. REVIEW OF PREVIOUS INVESTIGATIONS

A. General Work

The evaluation of remainders has received surprisingly little attention. However, much has been done on convergence of series and also on rates of convergence. A very substantial bibliography and review of such work, as well as some material on remainders, is contained in an article by Shohat (4), which discusses only orthogonal polynomials. To the above bibliography should be added an article by Peebles (3), and one by Walsh and Wiener (5). Jackson (2) has contributed much material of a related nature, material which is certainly an important part of the literature on series.

To attempt a discussion in any detail of one or several of the above would require space and time not justified in this work. However, in part B of this section is a fairly complete statement of a very recent and interesting piece of work.

B. Generalized Taylor Expansions

In an unpublished paper Atanasoff (1) has shown that if one were given an operator \( \mathcal{O} \), an operator \( \mathcal{J} \), a function \( \phi \), and a functional \( \mathcal{L} \) such that
\[ D^0 = 1, \]  

and

\[ D^i = 1 - \varphi L, \]  

then he could form a biorthogonal system \( \{ \varphi_i \} , \{ \ell_i \} \) through the following definitions:

\[ \varphi_0 = \varphi, \]  

\[ \varphi_k = D^k \varphi, \]  

and

\[ \ell_k = L D^k. \]

Further, if a function, \( f \), were expanded to \( n \) terms by this system as follows:

\[ f = \sum_{i=0}^{n-1} \alpha_i \varphi_i, \]  

where

\[ \alpha_i = \ell_i \varphi, \]

then the remainder would be given by \( D^n f \).

In the case of Taylor's expansion one has

\[ D = \ell_k, \quad \varphi = \int_0^x \frac{d^m}{d k^m} ( \cdot ) \, d x, \quad \varphi = 1; \]

with a remainder operator

\[ \int_{x_0}^{x_1} \cdots \int_{x_0}^{x_1} \frac{d^m}{d k^m} ( \cdot ) \, d x. \]
Thus, one might choose $\mathcal{D}$, $\mathcal{I}$, $\mathcal{P}$, $\mathcal{F}$ in such a way that (1) and (2) are satisfied and develop a type of expansion for which the remainder would be known. A more interesting problem, however, is that of finding $\mathcal{D}$, $\mathcal{I}$, and $\mathcal{F}$ if given a biorthogonal system $\{\Phi_i\}$, $\{\mathcal{Q}_i\}$ where $\Phi_i$ is a function and $\mathcal{Q}_i$ a functional. Atanasoff indicated the formal possibility of the above by showing that

\begin{align}
\mathcal{D} &= \sum_{i=0}^{\infty} \Phi_i \mathcal{Q}_i \\
\mathcal{I} &= \sum_{i=0}^{\infty} \Phi_{i+1} \mathcal{Q}_i \\
\mathcal{P} &= \Phi_0, \\
\mathcal{F} &= \mathcal{Q}_0,
\end{align}

satisfy equations (1) to (5) inclusive. Thus a formal procedure is defined for the construction of a remainder operator. The difficulty here is that $\mathcal{D}$ and $\mathcal{I}$ are not closed forms and hence practical limitations arise.

Clearly, however, in case closed forms for $\mathcal{D}$ and $\mathcal{I}$ are known the problem of remainder is solved.
III. THE INVESTIGATION

A. Notation

In this thesis we shall sum over each index that appears more than once in any term except in the case of capitalized indices. Limits of summation will be written as follows:

\[ \sum_{i}^{m} \sum_{j}^{n} A_{ij} A_{ji} \frac{L_{j}}{L_{i}} \]

This is to be interpreted as a sum in which \( i \) goes from zero to \( m \) and \( j \) from zero to \( n \). If no limits are given for an index then zero and infinity are to be understood.

A symbol which appears in the position of an exponent will be an ordinary index unless underlined, in which case it will be an exponent.

The symbol \( \delta_{ij} \) has the value zero for \( i \neq j \) and one for \( i = j \).

B. Fundamental Sets of Functionals

Let us suppose that a set of linearly independent functions, \( \{ \Phi_{i} \} \),
is given. This set may be either finite or infinite. Consider a biorthogonal system \( \{ \varphi_i \} \), \( \{ \xi_i \} \) in terms of which the \( \Phi \)'s may be expanded. In this system \( \varphi_i \) is a function and \( \xi_i \) a functional.

We shall discuss the problem under two cases.

Case I. \( \{ \Phi_i \} \) an infinite set.

Expand each \( \Phi \) as follows:

\[
\Phi_k = A_{ki} \varphi_i \xi_k \quad (k = 0, 1, 2, \ldots) \quad \text{(12)}
\]

where

\[
A_{ki} = \xi_k \Phi_k. \quad \text{(13)}
\]

In general equations (12) can be solved for the \( \varphi \)'s to give

\[
\varphi_k = A_{kj} \xi_j \varphi \xi_k \quad (k = 0, 1, 2, \ldots) \quad \text{(14)}
\]

where \( A_{kj} \) is the quotient of the cofactor of \( A_{jk} \) by the determinant of the \( A \)'s. We now define a set of functionals

\[
\xi_k = A_{jk} \xi_j \quad (k = 0, 1, 2, \ldots) \quad \text{(15)}
\]

We shall call this set "the fundamental set of functionals of order infinity relative to the \( \varphi \)'s."
We shall define another set of functionals in this case as follows:

\[ \Phi_k^M = \sum_{i=-m+1}^{m-1} A_{k'i} \Phi_{i}^M, \quad (k=0, 1, \ldots, m-1), \quad (16) \]

where

\[ A_{k'i} = \mathcal{L}_i \Phi_k. \]

In general equations (16) can be solved for the \( \Phi_k^M \) to give

\[ \Phi_k^M = \frac{A_{k'j}}{\frac{1}{f_{-m+1} \cdot \cdots \cdot f_{-m+1}}} \cdot (k=0, 1, \ldots, m-1). \quad (17) \]

In which \( A_{k'j} \) is the quotient of the cofactor of \( A_{k'j} \) by the determinant of the \( A_{{k'}j} \). We define a set of functionals

\[ \Phi_k^M = \frac{A_{k'j}}{\frac{1}{f_{-m+1} \cdot \cdots \cdot f_{-m+1}}} \cdot (k=0, 1, \ldots, m-1). \quad (18) \]

We shall call this set "the fundamental set of functionals of order \( m \) relative to the \( \Phi_k^M \)."

Case II. \( \{ \Phi_i \} \) a finite set of order \( n \).

Expand each \( \Phi_k \) as follows:

\[ \Phi_k = \sum_{i=-n+1}^{n-1} A_{k'i} \Phi_{i}^M, \quad (k=0, 1, \ldots, n-1), \quad (19) \]

where

\[ A_{k'i} = \mathcal{L}_i \Phi_k. \quad (20) \]
In general equations (19) can be solved for the \( \varphi^i \) to give

\[
\varphi^i_j = \frac{A_{jk}^N \Phi^i_k}{k - n - 1}, \quad (j = 0, 1, \ldots, n - 1).
\] (21)

In which \( A_{jk}^N \) is a quotient as above. We then define a set of functionals

\[
\psi^N_k = \frac{A_{jk}^N \psi^j}{j - n - 1}, \quad (k = 0, 1, \ldots, n - 1).
\] (22)

Thus a "fundamental set of functionals of order \( n \) relative to the \( \varphi^i \) is formed.

C. Theorems

**Theorem I.** The system \( \{ \phi_i \}, \{ \psi_i \} \) is biorthogonal.

**Proof:**

\[
\psi^j_k \phi^i_j = A_{ik} \Delta \phi^i_j \psi^j_k = A_{ik} \delta_{ij} = A_{ij} \Delta \phi^i_j
\]

so

\[
\psi^j_k \phi^i_j = \delta_{kj}.
\] (24)
Theorem II. The system \( \{ \Phi_i^M \} \), \( \{ \Phi_i^M \} \) is biorthogonal.

Proof: Let \( \Phi_k^M \) operate on \( \Phi_j^M \).

\[
\sum_k \Phi_k^M \Phi_j^M = \frac{A_{ik} A_{jr} \Phi_k \Phi_r}{c - m - 1} \quad \text{for } r = m - 1
\]

so

\[
\sum_k \Phi_k^M \Phi_j^M = \delta_{kj} \quad \text{for } j = 0, 1, \ldots, m - 1.
\]

Theorem III. The system \( \{ \Phi_i \} \), \( \{ \Phi_i^M \} \) is biorthogonal.

Proof: Let \( \Phi_k^M \) operate on \( \Phi_j \).

\[
\sum_k \Phi_k^M \Phi_j = \frac{A_{ik} A_{jr} \Phi_k \Phi_r}{c - m - 1} \quad \text{for } r = m - 1
\]

so

\[
\sum_k \Phi_k^M \Phi_j = \delta_{kj} \quad (j = 0, 1, \ldots, m - 1).
\]
Theorem IV. If in the complete biorthogonal system \( \{ \Phi_i \}, \{ \psi_i \} \),
where \( \Phi_i \) is a function and \( \psi_i \) a functional, \( \psi_i \) is expressible
as a linear sum of the \( \Phi_i \) in the biorthogonal system \( \{ \phi_i \}, \{ \chi_i \} \)
then \( \{ \psi_i \} \) is identical with "the fundamental set of order infinity
relative to the \( \phi_i 's. " \)

Proof: Let
\[
L_k = \alpha_{ki} \phi_i,
\]
where \( \alpha_{ki} \) is a constant. Let \( L_k \) operate on \( \phi_j \).

\[
L_k \phi_j = \alpha_{ki} \phi_i \phi_j = \alpha_{kj}.
\]

But since the expansion of \( \phi_j \) in terms of the \( \phi_i 's \) is unique we have

\[
L_k \phi_j = A_{jk},
\]

thus
\[
A_{jk} = \alpha_{kj}.
\]

and
\[
L_k = \frac{\psi_j}{\phi_k}. \quad (26)
\]
Theorem V. If $E_{j}$ can be exactly represented by a sum of

not more than $j$ of the $\Phi_{s}$ then the $A_{jk}$ are stable, that is

$$A_{jk}^{N} = A_{jk}^{M}$$

Proof: Consider the fundamental sets \{ $\Phi_{j}^{M}$ \} and \{ $\Phi_{j}^{N}$ \}

where $n > m$. Let $E_{k}^{N}$ operate on $\Phi_{j}$ where

$j < m$

$$E_{k}^{N} \Phi_{j} = A_{jk}^{M} E_{k}^{N} \Phi_{i}^{M}$$

but $E_{k}^{M} = E_{k}^{N}$, so

$$E_{k}^{N} \Phi_{i}^{M} = S_{k} \Phi_{i}$$

and thus

$$A_{jk}^{N} = A_{jk}^{M} \quad (j < m) \quad \text{(27)}$$
D. Expansions and Remainders

In view of Theorems I to III one is led to attempt expansions according to the following operators:

\[ \sum_{M} = \sum_{k=1}^{M} \frac{\Phi_k \Phi^*_{k-1}}{k^{\alpha-1}} \]  \hspace{1cm} (28)

\[ \sum_{M}^{M} = \sum_{k=1}^{M} \frac{\Phi_k \Phi_k^{*}}{k^{\alpha-1}} \]  \hspace{1cm} (29)

\[ M \sum_{M}^{M} = \sum_{k=1}^{M} \frac{\Phi_k \Phi_k^{*}}{k^{\alpha-1}} \]  \hspace{1cm} (30)

We shall call

\[ \sum_{j=1}^{M} \frac{\Phi_j \Phi_j^{*}}{j^{\alpha-1}} = S_{M} \]  \hspace{1cm} (31)

and

\[ 1 - S_{M} = R_{M} \]  \hspace{1cm} (32)

Further call

\[ 1 - \sum_{M} = R_{M} \]  \hspace{1cm} (33)

\[ 1 - \sum_{M}^{M} = R_{M}^{M} \]  \hspace{1cm} (34)

and

\[ 1 - M \sum_{M}^{M} = M \sum_{M}^{M} \]  \hspace{1cm} (35)
From Theorem IV we see that in general the operator given by (28) will produce the usual expansion in terms of the $\Phi^s$ and their regularly associated functionals. Since $\varphi_k$, as defined, is not a closed form one would naturally make use of those regularly associated functionals. This, however, may prove impractical if the evaluation of the numbers $L_i F$ becomes complicated. In fact this is the stumbling block most frequently met in applied mathematics. In such an event one could make an expansion according to the operator of (29). That is one could select $\{\varphi_i\}$, $\{f_i\}$ in such a way that the functional set $\{q_i^M\}$ was of simple form. Let us examine this latter expansion more critically. Consider

$$\sum_{M}^{N} = \sum_{k=m}^{N} \phi_k \varphi_i^M$$

$$= \alpha_{k_i} A_j \frac{\varphi_i}{j} \frac{f_j}{k}$$

$$= \sum_{m}^{N} \sigma_M + \sum_{m}^{N} \sum_{m}^{M}$$

(38)

so that

$$R_M^M = \sum_{M}^{N} \sum_{m}^{N}$$

(37)

Thus we are able to give a remainder operator which requires that only a finite number of operations be applied. There are two advantages then
for this scheme: (1) The numbers \( \mathcal{G}_k \) need not be difficult to
evaluate, (2) the remainder operator contains but a finite number of
terms. Clearly this remainder operator is of no use unless we know
the \( \mathcal{G} \) and \( \mathcal{J} \) for the \( \varphi \) series, and know them in closed forms.

In all that follows we shall assume that \( \sum_{M} \) is known in
closed form.

Let us now consider the operator

\[
\sum_{M}^{M} = \sum_{k=0}^{M} \mathcal{G}_k \mathcal{J}_k
\]

\[
= \sum_{j=0}^{M} \sum_{k=0}^{M} \mathcal{G}_j \mathcal{J}_k
\]

\[
= \sum_{j=0}^{M} \sum_{k=0}^{M} \mathcal{G}_j \mathcal{J}_k
\]

Thus the expansion given by \( \sum_{M}^{M} \) will be identical with that given by
the \( \varphi \)'s. Hence the remainder

\[
\sum_{M}^{M} \mathcal{R}_M = \sum_{M}^{M} \mathcal{R}_M
\]

In this thesis we shall not study \( \sum_{M}^{M} \) in the most general case,
but in that special case for which Theorem V is valid, i.e. each \( \mathcal{E}_j \)
can be represented by a sum of not more than \( j \) of the \( \varphi \)'s. This
may at first seem to be a serious restriction, however, a little
consideration shows that this is not true. For example, if the \( \phi \)'s
were the Maclaurin functions, \( \phi_k = \frac{x^k}{k!} \) and the \( \Phi \)'s the
Legendre, or Laguerre polynomials the conditions for this case are
satisfied. Let us examine \( \sum_M \) under these conditions. We have

\[
\sum_M = \frac{\Phi_k \phi_k}{k ^{m-1}} \]

\[
= \frac{a_{ki} A\phi_i \phi_j}{k ^{m-1}} \]

But in this case the index \( i \) cannot exceed the index \( k \), so

\[
\sum_M = \frac{a_{ki} A\phi_i \phi_j}{k ^{m-1}} \]

Since \( A_{jk} = A_{jk}^M \) \( \text{(Theorem V)} \)

\[
\sum_M = \frac{a_{ki} A\phi_i \phi_j}{k ^{m-1}} \\
+ \frac{a_{ki} A_{j+k} \phi_i \phi_{j+k}}{k ^{m-1}} \\
\sum_M = \sum_M + \sum_M \cap \sum_M \] (40)
And the remainder operator is given by

\[ R_M = r_M - \sum_M r_M. \]  \hspace{1cm} (41)

Since the Legendre polynomials, for example, are of the type just discussed, it is conceivable that one transform them by \( \chi = \cos \theta \)

and thus pass on to trigonometric series of the cosine type. Other generalizations of a fundamental nature suggest themselves also, but cannot be considered in the present paper.
IN many problems of physics and engineering the set of functions used for approximation is dictated by the workers experience or intuition as well as by the auxiliary conditions involved. Once a set of functions is selected the choice of a functional set follows one of two lines: (1) The functionals, regularly associated with selected functions, are used. (2) An arbitrary set of functionals is chosen usually in such a way as to simplify the evaluation of the expansion constants.

The scheme presented in this thesis makes it possible in many cases to evaluate the remainder under choice (1). In choice (2) the present scheme reduces somewhat the degree of arbitrariness with usually no serious effect upon the simplicity desired. At the same time evaluation of the remainder is made possible.

In regard to non-homogeneous linear differential equations and similar problems a method of obtaining a particular solution suggests itself immediately. Consider

\[ L \cdot u = f \]

where \( L \) is a linear operator; \( u \), an unknown function; and \( f \), a known function. Assume that \( u \) may be expanded by the complete biorthogonal system \( \{ \varphi_i \} \) and \( \{ f_i \} \) and write
\[ L a_i \varphi_i = f \]

or

\[ a_i L \varphi_i = f \]

We may now take, in general, \( \Phi_i = L \varphi_i \) and determine a fundamental set of functionals \( \{ \varphi^M_k \} \), say, relative to the \( \varphi \)'s.

This means essentially that we may approximate \( f \) as closely as we please and provide a \( U \) for that approximation.
Certain operators in applied problems.

Such information would aid in the selection of suitable sets of operators. The remainder of this work is that of estimating the problem indirectly. By the present work the most efficient use of the operators is obtained.

Remedial operators have been determined for these approximations. Examples are shown in terms of operators. Thus a means to present the problems is one large.

Theorems of operators have been developed for these approximations. Theorem operators are related to the chosen functions. At the same time, these operators are related to the condition, for a given set of conditions. Thus it may estimate the action that it permits some choice in the selection of the set of approximations. It is necessary to based upon the fact that a scheme of approximation, which should prove useful in applied

V. SOMEHOW

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VI. LITERATURE CITED


VII. ACKNOWLEDGMENTS

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