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**Abstract**

Dual-depth subsurface drainage is considered to be more effective in removing excess water from soil than single-depth drainage, but this problem has not been analyzed in detail. Therefore, assuming that uniform, water-saturated soil covered by ponded water and overlying an impervious barrier is drained by equally spaced, alternating deep circular drain tubes, existing potential flow theory for a single-depth drainage system was extended. Sample calculations with the newly derived equations show that a dual-depth subsurface drainage system can be highly effective to remove excess water from soil. For example, a relative drain discharge of 160% is calculated when new drain tubes, added at the 0.60 m depth, are placed midway between the original drain tubes, which are 25 m apart and at the 1.20 m depth. In this calculation we have assumed that the impervious layer is at the 3.0 m depth, the radius of the tubes is 0.05 m, the soil hydraulic conductivity is 1 m/d, and the thickness of the ponded water is 0.0 m. For the same conditions, but with the additional tubes at the 1.20 m depth (same depth as original tubes), the relative drain discharge becomes nearly 200%, and with the additional tubes at the 2.40 depth (1.20 m below original tubes) it is more than 250%. When the impervious layer is at a greater depth and when the original drain spacing is more than 25.0 m, the relative drain discharge becomes even larger. The effectiveness of the dual-depth tube system becomes particularly large, if the second tube system is placed below the level of the first one.

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# Potential theory for dual-depth subsurface drainage of ponded land

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**Abstract.** Dual-depth subsurface drainage is considered to be more effective in removing excess water from soil than single-depth drainage, but this problem has not been analyzed in detail. Therefore, assuming that uniform, water-saturated soil covered by ponded water and overlying an impervious barrier is drained by equally spaced, alternating deep circular drain tubes, existing potential flow theory for a single-depth drainage system was extended. Sample calculations with the newly derived equations show that a dual-depth subsurface drainage system can be highly effective to remove excess water from soil. For example, a relative drain discharge of 160% is calculated when new drain tubes, added at the 0.60 m depth, are placed midway between the original drain tubes, which are 25 m apart and at the 1.20 m depth. In this calculation we have assumed that the impervious layer is at the 3.0 m depth, the radius of the tubes is 0.05 m, the soil hydraulic conductivity is 1 m/d, and the thickness of the ponded water is 0.0 m. For the same conditions, but with the additional tubes at the 1.20 m depth (same depth as original tubes), the relative drain discharge becomes nearly 200%, and with the additional tubes at the 2.40 depth (1.20 m below original tubes) it is more than 250%. When the impervious layer is at a greater depth and when the original drain spacing is more than 25.0 m, the relative drain discharge becomes even larger. The effectiveness of the dual-depth tube system becomes particularly large, if the second tube system is placed below the level of the first one.

## 1. Introduction

Occasionally, one might wish to increase the capacity of an existing single-depth tube drainage system. Instead of removing the existing system and replacing it by a more narrowly spaced one, it might be just as effective and more economical to install additional tubes midway between the ones already present. The additional tubes do not need to be in the same depth as the existing ones, but they can be either shallower or deeper. Besides being more effective in removing excess water, a dual-depth drainage system also enables separation of drain water of different quality. Higher quality drain water may be collected and reused for irrigation. Another advantage of a dual-depth drainage system is that it offers more flexibility in groundwater table management.

For the design of subsurface drainage systems theory that frequently is used assumes steady state flow conditions [see, e.g., *van Beers*, 1976; *Eggelsmann*, 1981; *Schwab et al.*, 1981]. Either potential flow concepts or Dupuit-Forchheimer assumptions are used to derive drain spacing equations that relate the relevant field parameters. A review of steady state flow theory to drains and wells is given by *Lovell and Youngs* [1984] and R. R. van der Ploeg et al. (manuscript in preparation, 1997). For the derivation of steady state subsurface drain spacing equations it is usually assumed that at some finite depth below the soil surface there is an impervious barrier that is parallel to the soil surface. It is further assumed that the soil is drained by parallel, equally spaced, drain tubes. Examples of such derivations, pertinent to the present paper, are given by, among others, *Kirkham* [1940, 1945, 1949, 1958]. In these publications, either a free water table between the drain tubes at some depth below the soil surface is considered, or a soil that is completely saturated, for example, because of ponded surface water.

In the aforementioned publications it is assumed that the soil is drained by a single-depth system of (subsurface) tubes. For soils drained by ditches, theory has been derived for unequal water level heights (e.g., by *Kirkham* [1965] or by *Powers et al.* [1967]). Such theory, however, has, to our knowledge, not yet been derived for a system of subsurface drains. It is the objective of the present paper to derive drain spacing equations for the case of a dual-depth subsurface system for water-ponded land. To this end, previous work of *Kirkham* [1940, 1945, 1949] will be extended. In these early papers, Kirkham used complex variables and the method of multiple drain images to derive equations for the hydraulic head, the velocity potential, the stream function, and the drain discharge rate. The use of these methods in other areas of subsurface hydrol-



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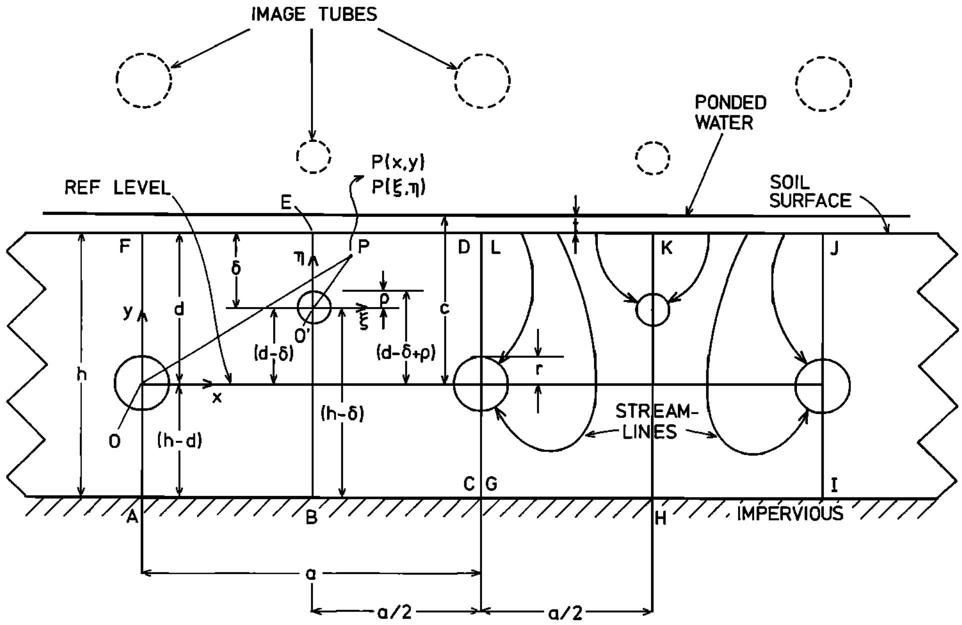


**Figure 1.** Geometry for dual-depth tube drainage of ponded water; only a few of the infinite number of arrays of image tubes that are implicitly used in the analysis [see *Kirkham*, 1949] are shown.

ogy has been described by *Muskat* [1946]. Another objective of the present paper is to prepare, for engineering purposes, some simple nomographs that allow a user to compare the performance of a dual-depth subsurface drainage system with that of a single-depth one for a wide range of field parameters. For information about the drainage of agricultural land in general the reader is referred to *van Schilfgaarde* [1974].

## 2. Analysis

The problem is two dimensional. That is, we use an $(x, y)$ system of coordinates with a unit length (1 m) of flow medium taken in the direction perpendicular to the $(x, y)$ plane. The flow medium is taken to be homogeneous and isotropic with constant hydraulic conductivity $k$ (meter per day). Darcy's law is assumed. Therefore Laplace's equation for the hydraulic head $\phi$ (units in meters) is valid as in the work of *Kirkham and Powers* [1984, pp. 46–52] (hereinafter referred to as KP), where we now use $\phi$ (instead of $h$ as in the work of KP), given by

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad (1)$$

The head $\phi$ in (1) can be determined if we know the boundary conditions; that is, we must know the head $\phi$, or normal derivative of the head $\partial\phi/\partial n$ (meter per meter), for each boundary segment of the flow region. Before we can establish the boundary conditions, we must know the flow region geometry.

### 2.1. Flow Region Geometry

Figure 1, which includes some fictitious image tubes (for a discussion of the image tube method, see *Kirkham* [1949]) above the ponded water, shows at the left of the figure that the thickness of the flow medium is $h$ (meters), and at the center top right of the figure that the ponded water is of the thickness $t$. Deep drain tubes are shown by large circles with depth $d$ to the tube centers. Shallow drains are shown by small circles with depth $\delta$ to the tube centers. The horizontal distance between centers of adjacent large and shallow drains is $a/2$. The radius of the large drain, as the one in the middle of the figure, is $r$, and that of a shallow drain, as at the left center top of the figure, is $\rho$.

Tacitly required, to solve (1) for head $\phi$, are the location of an origin for the $(x, y)$ coordinates and a reference level for the head $\phi$. The origin of the $(x, y)$ coordinates is taken at the center of the deep left tube and the REF LEVEL is taken as the level of the same deep drain tube.

In Figure 1, some distances, namely, $(h - d)$, $(d - \delta)$, $(d - \delta + \rho)$, and $(h - \delta)$ are shown for later use, and also for later use we define a distance $c$ as

$$c = d + t \qquad (2)$$

Our Figure 1 indicates by zigzag marks at either side of the figure that there exist additional drain tubes. We assume that there are enough side drain tubes so that no edge effects occur. Thus (in Figure 1) the zigzag marks indicate that flow in the left-half rectangle ABCDEFA is the same as in the right-half rectangle GHIJKLG and thus that either or both of these rectangles could be used for the flow region of (1).

Figure 1 shows an $(x, y)$ system origin of coordinates at the left deep drain tube center and a $(\xi, \eta)$ system of coordinates for the shallow drain which has its origin at the left shallow drain center. The $(x, y)$ and $(\xi, \eta)$ coordinate systems will enable us to write down presently a needed expression for the head $\phi$ of (1) for a certain point $P(x, y)$ and for the same point also labeled $P(\xi, \eta)$ as shown in Figure 1.

Before we formulate boundary conditions, we specify ranges of variables and parameters and give two expressions for drain tube radii $r$ and $\rho$ when they are small.

Parameter ranges

$$\infty > (a/2) > (r + \rho)$$

$$d > \delta > \rho > 0$$

$$\infty > h > d + r > d > r \geq \rho > 0$$
$$t \geq 0$$
$$c \geq d$$
$$x^2 + y^2 = r^2 \qquad \text{for } r \to 0$$
$$\xi^2 + \eta^2 = \rho^2 \qquad \text{for } \rho \to 0$$

### 2.2. Boundary Conditions, Heads, and Related Equations

We next introduce Figure 2, a boundary condition figure. Figure 2 represents the left half of Figure 1. Because of flow symmetry (see the streamlines in Figure 1), we deduce that the flow is periodic and that no flow in Figures 1 and 2 passes through vertical lines (planes) that go through deep and shallow drain tube centers. That is, we formulate the $\partial\phi/\partial n$-type boundary conditions that we noted under (1) for Figure 2 by first reading upward at the left-hand side of Figure 2, then reading upward at the middle of Figure 2 as

along AA′ and F′F:

$$\partial \phi_{\text{KDB}}/\partial n = 0 \qquad (3)$$

along BB′ and E′E:

$$\partial \phi_{\text{KDB}}/\partial n = 0 \qquad (4)$$

where in (3) and (4) the normal $n$ is in the $x$ direction and the subscript KDB refers to the use of both deep and shallow tubes. The subscript KDB needs a short explanation. The K of KDB refers to *Kirkham* [1949], the article in which the problem of water-ponded land drained by a single-depth subsurface system was solved. The subscript DB refers to Darrell DeBoer from North Dakota State University, who asked in a personal communication if the same problem for a dual-depth drain system could be solved.

The expressions (3) and (4) have been written down from physical considerations. Analytically, (3) and (4) will also be found to be correct, as will appear later. For now, we see that in addition to the stretches AA′, ⋯, E′E of (3) and (4), Figure 2 indicates four other boundary stretches. They are labeled I at the top, II, III, and IV, and of these the stretch IV (that pertains to the bottom impermeable barrier) yields (as stems from an image procedure in the work of *Kirkham* [1949]) the boundary condition BC for stretch IV,

BC IV, for stretch IV:

$$\partial \phi_{KDB}/\partial n = 0 \qquad (5)$$

There remains to write boundary condition equations for stretches I, II, and III. From piezometer readings seen or implied (see Figure 2) we obtain

BC I, for stretch I, $0 \leq x \leq a/2$, $y = d$:

$$\phi_{\text{KDB}} = d + t \qquad (6)$$

BC II, for stretch II, $x^2 + y^2 = r^2$, $r^2 \to 0$:

$$\phi_{\text{KDB}} = r \qquad (7)$$

BC III, for stretch III, $\xi^2 + \eta^2 = \rho^2$, $\rho^2 \to 0$:

$$\phi_{\text{KDB}} = d - \delta + \rho \qquad (8)$$

where for stretches II and III (on the circumference of the tubes) the tubes are assumed to run full with zero back pressure, as the piezometers indicate, and it is assumed that $r$ and $\rho$ are small compared to $d$, $\delta$, $h$, and $a$.

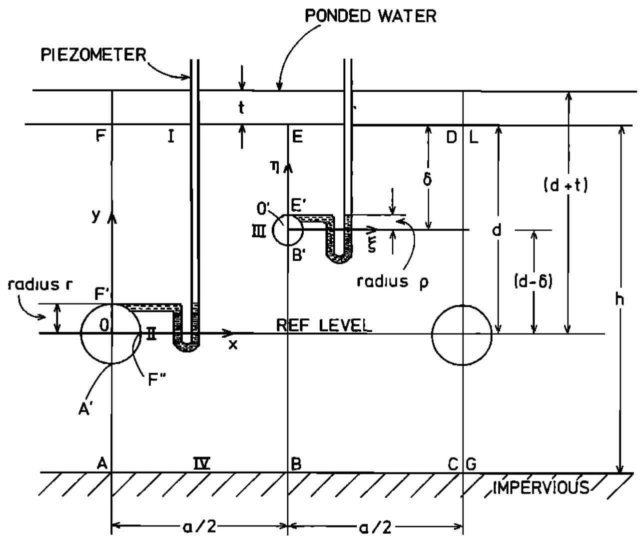


**Figure 2.** Boundary conditions for dual-depth drainage of ponded water.

We need to apply the boundary conditions of (6), (7), and (8) to our as yet unknown head function which we denote by $\phi_{\text{KDB}}$. We assume that the function $\phi_{\text{KDB}}$ that we seek is made up of three parts; a part $\phi_{\text{K}}$ associated in particular (but not only) with the deep tubes, a part $\phi_{\text{DB}}$ associated in particular (but not only) with the shallow tubes, and a part that is a constant. Consequently, we need to get $\phi_{\text{KDB}}$, and to get $\phi_{\text{KDB}}$, we need in turn an auxiliary function, for example, $\phi_{\text{K}}$ which we obtain from *Kirkham* [1949] as follows.

**2.2.1. Hydraulic head $\phi_{\text{K}}$.** We refer to *Kirkham* [1949], and we see that his equation (13) is an expression for the hydraulic head, which is periodic and applies to our present problem for the special case in which the shallow tubes in our Figures 1 and 2 are omitted. We copy *Kirkham* [1949, equation (13)] just as it stands, except that we use the hydraulic head symbol $\phi_{\text{K}}$ instead of his $\phi_a$ (meters) and use hydraulic head coefficient $q_{\text{K}}$ (meters) for his $q_a$. Making these changes, we obtain from *Kirkham* [1949, equation (13)] the head function $\phi_{\text{K}}$ for the deep tubes as

$$\phi_{\text{K}} = q_{\text{K}} \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh \dfrac{2\pi(y - 2nh)}{a} - \cos \dfrac{2\pi x}{a}}{\cosh \dfrac{2\pi(y - 2d - 2nh)}{a} - \cos \dfrac{2\pi x}{a}} + C_3, \qquad n = 0, \text{ incl.} \qquad (9)$$

where we take the origin of $(x, y)$ coordinates as in the work of *Kirkham* [1949, equation (13)], namely, at the center of the deep tube at the left of Figures 1 and 2. The constant $C_3$ in (9) will depend on a reference level for head that up to equation (13) in the work of *Kirkham* [1949] is not used. In (9) and hereinafter it is to be understood that $n = 0$ has to be included in the summation.

**2.2.2. Hydraulic head $\phi_{\text{DB}}$.** We now return to (9) and observe that (9) gives the hydraulic head function (defined as a solution of Laplace's equation) for the deep tubes for an origin of $(x, y)$ coordinates (as in the work of *Kirkham* [1949]) taken at the center of the left-hand deep tube. Equation (9) has the head coefficient $q_K$ before the summation sign. We now define a head coefficient $q_{DB}$, similar to $q_K$ but which we now use in a head function $\phi_{DB}$ for a $(\xi, \eta)$ system of rectangular coordinates as at point $P(\xi, \eta)$ in Figure 1 for the shallow tubes with origin of coordinates taken at the center of the first left shallow tube.

Observing now in Figure 1 that the geometry of the shallow tube system is similar to the geometry of the large tube system, we can write, with (9) and Figure 1 in view, the equation for $\phi_{DB}$ (change $x$ to $\xi$, $y$ to $\eta$, $d$ to $\delta$, and leave $h$ and $a$ unchanged; also change $\phi_K$ to $\phi_{DB}(\xi, \eta)$) as

$$\phi_{DB}(\xi, \eta) = q_{DB} \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh \dfrac{2\pi(\eta - 2nh)}{a} - \cos \dfrac{2\pi\xi}{a}}{\cosh \dfrac{2\pi(\eta - 2\delta - 2nh)}{a} - \cos \dfrac{2\pi\xi}{a}} + C_4, \; n = 0, \text{ incl.} \tag{10}$$

where $C_4$ is a constant analogous to $C_3$ in (9), and it is to be understood again, unless otherwise indicated, that the summation is to be done with $n = 0$ included.

We wish now to add the potentials of (9) and (10) for point P in Figure 1. However, because we have presented Laplace's equation (1) in $(x, y)$ coordinates, with origin at the center of the left deep tube of Figure 1, we must, before doing this adding, change (10) which is in $(\xi, \eta)$ coordinates to $(x, y)$ coordinates. To do this, we see from the geometry of Figure 1 that the following substitutions are needed (the lengths $h$ and $a$ do not change):

$$\xi = x - (a/2) \tag{11}$$

$$\eta = y - (d - \delta) \tag{12}$$

$$-\cos\left(\frac{2\pi\xi}{a}\right) = -\cos\left(\frac{2\pi(x - a/2)}{a}\right) = \cos(2\pi x/a) \tag{13}$$

We put values from (11)–(13) in (10), and after we simplify and also change the left-hand side (hereinafter LHS) of (10) to $\phi_{DB}(x, y)$, we get (10) as

$$\phi_{DB}(x, y) = q_{DB} \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh \dfrac{2\pi(y - d + \delta - 2nh)}{a} + \cos \dfrac{2\pi x}{a}}{\cosh \dfrac{2\pi(y - d - \delta - 2nh)}{a} + \cos \dfrac{2\pi x}{a}} + C_4, \cdots. \tag{14}$$

**2.2.3. Hydraulic head $\phi_{KDB}$.** We now add (9) and (14) and after letting $\phi_{KDB}(x, y)$ denote the sum of (9) and (14) and after putting $C_3 + C_4$ at the end, we get $\phi_{KDB}(x, y)$ as

$$\phi_{KDB}(x, y) = q_K \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh \dfrac{2\pi(y - 2nh)}{a} - \cos \dfrac{2\pi x}{a}}{\cosh \dfrac{2\pi(y - 2d - 2nh)}{a} - \cos \dfrac{2\pi x}{a}} + q_{DB} \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh \dfrac{2\pi(y - d + \delta - 2nh)}{a} + \cos \dfrac{2\pi x}{a}}{\cosh \dfrac{2\pi(y - d - \delta - 2nh)}{a} + \cos \dfrac{2\pi x}{a}} + C_3 + C_4, \cdots. \tag{15}$$

and (15) may be written, in view of (9) and (14), as

$$\phi_{KDB}(x, y) = \phi_K(x, y) + \phi_{DB}(x, y) + C_3 + C_4 \tag{16}$$

where the two $\phi$ terms on the right-hand side (hereinafter RHS) of (16) are the respective sum terms on the RHS of (15).

### 2.3. Applying the Boundary Conditions

In (15) we need to determine the head coefficient $q_K$ and $q_{DB}$ and also the combined constant $(C_3 + C_4)$. To do this, we apply the boundary condition equations (6)–(8) to (15) as follows. We put $(d + t)$ of (6) in place of the LHS of (15) and put $x = x$ and $y = d$ into the RHS of (15) to get (17) as

$$d + t = q_K \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh \dfrac{2\pi(d - 2nh)}{a} - \cos \dfrac{2\pi x}{a}}{\cosh \dfrac{2\pi(d - 2d - 2nh)}{a} - \cos \dfrac{2\pi x}{a}} + q_{DB} \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh \dfrac{2\pi(d - d + \delta - 2nh)}{a} + \cos \dfrac{2\pi x}{a}}{\cosh \dfrac{2\pi(d - d - \delta - 2nh)}{a} + \cos \dfrac{2\pi x}{a}} + (C_3 + C_4), \cdots. \tag{17}$$

We simplify (17) whereby in the first summand in the denominator we use, after changing $(d - 2d)$ to $-d$, the identity

$$\cosh(-d) = \cosh(d)$$

to get (17) as

$$d + t = q_K \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh \dfrac{2\pi(d - 2nh)}{a} - \cos \dfrac{2\pi x}{a}}{\cosh \dfrac{2\pi(d + 2nh)}{a} - \cos \dfrac{2\pi x}{a}} + q_{DB} \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh \dfrac{2\pi(\delta - 2nh)}{a} + \cos \dfrac{2\pi x}{a}}{\cosh \dfrac{2\pi(\delta + 2nh)}{a} + \cos \dfrac{2\pi x}{a}} + (C_3 + C_4), \cdots. \tag{18}$$

**2.3.1. Constants $S_1$, $S_2$, and $(C_3 + C_4)$.** We define $S_1$ and $S_2$ to be the sums in (18), namely,

$$S_1 = \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh \dfrac{2\pi(d - 2nh)}{a} - \cos \dfrac{2\pi x}{a}}{\cosh \dfrac{2\pi(d + 2nh)}{a} - \cos \dfrac{2\pi x}{a}} \tag{19}$$

$$S_2 = \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh \dfrac{2\pi(\delta - 2nh)}{a} + \cos \dfrac{2\pi x}{a}}{\cosh \dfrac{2\pi(\delta + 2nh)}{a} + \cos \dfrac{2\pi x}{a}} \tag{20}$$

so that (18)–(20) give the constant $(C_3 + C_4)$ as

$$C_3 + C_4 = d + t - (q_K S_1 + q_{DB} S_2) \quad (21)$$

We now see in (21) (by expanding the sums in equations (19) and (20) and rearranging) that $S_1$ and $S_2$ each vanish; that is, we get (because the term for $n = 0$ is equal to zero and the terms for $n = +1$ and $n = -1$, for $n = +2$ and $n = -2$, etc. cancel each other out)

$$S_1 = 0 \quad (22)$$

and similarly we obtain

$$S_2 = 0 \quad (23)$$

so that (21)–(23) give

$$(C_3 + C_4) = d + t \quad (24)$$

**2.3.2. Constants $S_3$ and $S_4$.** We now apply boundary condition equation (7) to (15), which for $x = 0$ and $y = r$, together with (21), gives

$$r = q_K S_3 + q_{DB} S_4 + d + t \quad (25)$$

where we now designate the sums in (15) as $S_3$ and $S_4$ and write

$$S_3 = \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh\left[\dfrac{2\pi(r - 2nh)}{a}\right] - 1}{\cosh\left[\dfrac{2\pi(r - 2d - 2nh)}{a}\right] - 1} \quad (26)$$

and

$$S_4 = \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh\left[\dfrac{2\pi(r - d + \delta - 2nh)}{a}\right] + 1}{\cosh\left[\dfrac{2\pi(r - d - \delta - 2nh)}{a}\right] + 1} \quad (27)$$

**2.3.3. Constants $S_5$ and $S_6$.** We apply boundary condition III (see equation (8)) to (15), which, with Figure 2 in view and with (24), gives (put $\phi_{KDB} = d - \delta + \rho$, $x = a/2$, $y = d - \delta + \rho$ into equation (15)) the relation

$$d - \delta + \rho = q_K S_5 + q_{DB} S_6 + d + t \quad (28)$$

where with some simplification we define $S_5$ and $S_6$ as

$$S_5 = \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh\left[\dfrac{2\pi(d - \delta + \rho - 2nh)}{a}\right] + 1}{\cosh\left[\dfrac{2\pi(-\delta + \rho - d - 2nh)}{a}\right] + 1} \quad (29)$$

and

$$S_6 = \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh\left[\dfrac{2\pi(\rho - 2nh)}{a}\right] - 1}{\cosh\left[\dfrac{2\pi(\rho - 2\delta - 2nh)}{a}\right] - 1} \quad (30)$$

**2.3.4. Head coefficients $q_K$ and $q_{DB}$.** From (25) and (28) we find, by algebra, $q_{DB}$ and $q_K$ as

$$q_{DB} = \frac{1}{S_4 S_5 - S_3 S_6} [S_5(r - c) - S_3(d - \delta + \rho - c)] \quad (31)$$

and

$$q_K = (r - c - q_{DB} S_4)/S_3 \quad (32)$$

where $S_3$, $S_4$, $S_5$, and $S_6$ are given by (26), (27), (29), and (30).

**2.3.5. Terms $\sigma_K$ and $\sigma_{DB}$.** We return to (15) and for brevity we define sums $\sigma_K$ and $\sigma_{DB}$ of (15), respectively, as

$$\sigma_K = \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh \dfrac{2\pi(y - 2nh)}{a} - \cos \dfrac{2\pi x}{a}}{\cosh \dfrac{2\pi(y - 2d - 2nh)}{a} - \cos \dfrac{2\pi x}{a}} \quad (33)$$

and

$$\sigma_{DB} = \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh \dfrac{2\pi(y - d + \delta - 2nh)}{a} + \cos \dfrac{2\pi x}{a}}{\cosh \dfrac{2\pi(y - d - \delta - 2nh)}{a} + \cos \dfrac{2\pi x}{a}} \quad (34)$$

so that returning to (15), we may write the two sums there as

$$\phi_K = q_K \sigma_K \quad (35)$$

and

$$\phi_{DB} = q_{DB} \sigma_{DB} \quad (36)$$

It is noted that the functions $\sigma_K$ and $\sigma_{DB}$ are independent of $r$ and $\rho$.

### 2.4. Some General Hydraulic Head Equations

From foregoing work we now write down some general results for heads. From (15) and (24) we can write $\phi_{KDB}$ as

$$\phi_{KDB} = \phi_K + \phi_{DB} + d + t \quad (37)$$

We put the value of (33) into (35) as

$$\phi_K = q_K[\text{RHS of (33)}] \quad (38)$$

We put the value of (34) into (36) as

$$\phi_{DB} = q_{DB}[\text{RHS of (34)}] \quad (39)$$

As an alternative for $\phi_{KDB}$ of (37), we replace $(C_3 + C_4)$ in (15) by $(d + t)$ of (24) to get (15) as

$$\phi_{KDB}(x, y) = q_K \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh \dfrac{2\pi(y - 2nh)}{a} - \cos \dfrac{2\pi x}{a}}{\cosh \dfrac{2\pi(y - 2d - 2nh)}{a} - \cos \dfrac{2\pi x}{a}} + q_{DB} \sum_{n=-\infty}^{\infty} (-1)^n \ln \frac{\cosh \dfrac{2\pi(y - d + \delta - 2nh)}{a} + \cos \dfrac{2\pi x}{a}}{\cosh \dfrac{2\pi(y - d - \delta - 2nh)}{a} + \cos \dfrac{2\pi x}{a}} + (d + t), \quad (40)$$

which is an essential expression in our analysis.

### 2.5. Tube Inflows $Q_K$, $Q_{DB}$, and $Q_{KDB}$

We let $Q_K$ ($m^3$/m/d) denote the inflow into both sides of a deep tube per unit length of tube and similarly let $Q_{DB}$ denote the inflow into both sides of a shallow tube. Then, in view of Figure 2, we see that the total inflow $Q_{KDB}$ for a distance "$a$"

between two adjacent deep tube centers per unit length of tube will be given by

$$Q_{KDB} = Q_K + Q_{DB} \tag{41}$$

We will first derive an expression for $Q_K$.

**2.5.1. Expression for $Q_K$.** To compute the quantity of water $Q_K$ entering a unit length (1 m) of tube, we shall return to the head relation (9) where we assume, as we have tacitly done up to this point, that hydrologically an equipotential surface can be considered as a thin perforated rigid tube surrounded by a layer of thin infinitely conductive "gravel"; and similarly for tube drains. It is also now assumed that the radius of the tube is small compared to distances $h$, $h - d$, $d$, and $a$. With these assumptions we may write, provided $y$ and $x$ are taken of the same small order of size as $r$, the head $\phi_K$ of (9) (whereby in equation (9) we single out the $n = 0$ term of the sum and take $n = -1$, $n = +1$, etc. terms as approximately constant in the neighborhood of the tube as in *Kirkham* [1949]) to get (9) as

$$\phi_K = \{q_K \ln [\cosh(2\pi y/a) - \cos (2\pi x/a)]\} + \text{const approx.} \tag{42}$$

In (42) the constant $C_3$ of (9) is included in const approx.

We may simplify $\phi_K$ of (42). To do so, we first write down the expansions [*Dwight*, 1961, number 657.2, 415.02]

$$\cosh A = 1 + A^2/2! + \cdots \tag{43}$$

$$\cos A = 1 - A^2/2! + \cdots \tag{44}$$

Then we write (42) as

$$\phi_K = q_K \ln [(2\pi y/a)^2 + (2\pi x/a)^2] + \text{const approx.} \tag{45}$$

In (45), cosh and cos are approximated by their series expansions.

Now the inflow per unit area for a unit length of deep drain tube at $r = r_1$ near the tube is by Darcy's law given by the expression $-k\partial\phi_K/\partial r_1$, so that the inflow $Q_K$ for the corresponding whole outside surface area of deep tube per unit length is ($2\pi r$ is the tube area per unit length) given by

$$Q_K = (2\pi r_1)(-k\partial\phi_K/\partial r_1) \tag{46}$$

To get the ratio which is needed, $\partial\phi_K/\partial r_1$ for (46), we return to (45) and revise (45) as

$$\phi_K = q_K \ln \left[ (2\pi)^2\left(\frac{y_1^2}{a^2} + \frac{x_1^2}{a^2}\right) \right] + \text{const approx.} \tag{47}$$

that is, with $r_1$ taken as the radial distance from the tube axis to a point in the soil near the deep tube of which the cartesian coordinates are $(x_1, y_1)$ we next write (48), with $r_1^2 = x_1^2 + y_1^2$, as

$$\phi_K = q_K \ln (r_1/a)^2 + \text{const approx.}$$

or

$$\phi_K = 2q_K \ln (r_1/a) + \text{const approx.} \tag{48}$$

where "const approx." now includes the term $q_K \ln (2\pi)^2$.

For small enough $r_1$ in (48) we may drop the "approx." in (48) and then differentiate (48) ([*Dwight*, 1961, number 82.1] and his Figure 82.1, third quadrant, which shows that a negative sign is needed) as

$$\frac{\partial\phi_K}{\partial r_1} = -2q_K \frac{1}{r_1} \tag{49}$$

We next replace the quantity $\partial\phi_K/\partial r_1$ in (46) by the RHS of (49) to get (46) as

$$Q_K = (2\pi r_1)(-k)\left(-2q_K \frac{1}{r_1}\right) \tag{50}$$

and (50) simplifies as

$$Q_K = 4\pi k q_K \tag{51}$$

**2.5.2. Expression for $Q_{DB}$.** To obtain an expression for $Q_{DB}$, the shallow tube inflow rate per unit length of tube, we work as we did in getting the deep tube inflow rate $Q_K$, except that now we start with $\phi_{DB}$ of (10) with its $(\xi, \eta)$ coordinates, instead of with (9), with the $(x, y)$ coordinates. Thus we get the shallow tube head $\phi_{DB}$, similar to $\phi_K$ of (42), as

$$\phi_{DB} = q_{DB} \ln\left(\cosh\frac{2\pi\eta}{a} - \cos\frac{2\pi\xi}{a}\right) + \text{const approx.} \tag{52}$$

Next, in (52), after we have changed $(\xi, \eta)$ to $(\xi_1, \eta_1)$ to indicate that (52) applies to a point $P(\xi_1, \eta_1)$ near a shallow tube, and after again we have used (43) and (44), we get from (52) the head $\phi_{DB}$ as

$$\phi_{DB} = q_{DB} \ln [(2\pi\eta/a)^2 + (2\pi\xi/a)^2] + \text{const approx.} \tag{53}$$

an expression which is similar to (45).

We next get, similarly to the development of $Q_K$ of (46), $Q_{DB}$ as

$$Q_{DB} = (2\pi\rho_1)(-k\partial\phi_{DB}/\partial\rho_1)$$

and similar now to the development of (47) we get $\phi_{DB}$ as

$$\phi_{DB} = q_{DB} \ln \left[ (2\pi)^2\left(\frac{\eta_1^2}{a^2} + \frac{\xi_1^2}{a^2}\right) \right] + \text{const approx.}$$

and similar to (48), with now $\rho_1^2 = \xi_1^2 + \eta_1^2$, we get

$$\phi_{DB} = 2q_{DB} \ln (\rho_1/a) + \text{const approx.} \tag{54}$$

and instead of (49),

$$\frac{\partial\phi_{DB}}{\partial\rho_1} = -2q_{DB} \frac{1}{\rho_1}$$

and instead of (50),

$$Q_{DB} = (2\pi\rho_1)(-k)\left(-2q_{DB} \frac{1}{\rho_1}\right) \tag{55}$$

which can be simplified as

$$Q_{DB} = 4\pi k q_{DB} \tag{56}$$

where $Q_{DB}$ is the inflow (or discharge) of both sides of a single shallow drain tube per unit length.

**2.5.3. Expression for $Q_{KDB}$.** We add (51) and (56) to get their sum $Q_{KDB}$ (defined in equation (41)) as

$$Q_{KDB} = 4\pi k q_K + 4\pi k q_{DB}$$

or

$$Q_{KDB} = 4\pi k(q_K + q_{DB}) \tag{57}$$

For later use we refer to *Kirkham* [1949] and to his expression for the inflow rate per unit length of tube for the case of a single-depth drain, which we here denote as $Q_S$. For $Q_S$, *Kirkham* [1949] derived the following relation (in our notation)

$$Q_S = 4\pi k q_S \tag{58}$$

where $q_S$ is a head coefficient, similar to $q_K$ in our present (51) or $q_{DB}$ in (56).

For $q_S$ in (58), *Kirkham* [1949] provided the following expression (his equation (9)):

$$q_S = -(d + t - r)/F \tag{59}$$

with $F$ (his equation (10)) given as

$$F = -2\Bigg( \ln\,[\tan \pi(2d - r)/4h]/(\tan \pi r/4h) + \sum_{n=1}^{\infty} \ln \{[\cosh \pi na/2h + \cos \pi r/2h] /[\cosh \pi na/2h - \cos \pi r/2h]\} \cdot \{[\cosh \pi na/2h - \cos \pi(2d - r)/2h] /[\cosh \pi na/2h + \cos \pi(2d - r)/2h]\}\Bigg) \tag{60}$$

The quantity $Q_S$ of (58) for a single-depth drainage system will be used later to evaluate the performance of the dual-depth drainage system. We shall next work on stream functions.

### 2.6. Stream Functions

We let $\psi_K$ (square meter per day) be a stream function component strongly associated with the deep tubes, and $\psi_{DB}$ similarly, for the shallow tubes, and such that the complete stream function $\psi_{KDB}$ is given by

$$\psi_{KDB} = \psi_K + \psi_{DB} + C_5 \tag{61}$$

where $C_5$ will be an arbitrary constant. We need to work on the RHS of (61). Before we start working on the RHS of (61), we introduce the velocity potential $\Phi$ (square meter per day), which is related to the hydraulic head $\phi$ (meter) of (1) by the equation

$$\Phi = k\phi \tag{62}$$

in which expression $k$ (meter per day) is the hydraulic conductivity as before. In terms of velocity potential, (37) can thus be written as

$$\Phi_{KDB} = k\phi_{KDB} = k\phi_K + k\phi_{DB} + k(d + t), \tag{63}$$

where $\phi_K$, $\phi_{DB}$, and $\phi_{KDB}$ are given by (38), (39), and (40).

**2.6.1. Stream function $\psi_K$.** In view of *Kirkham* [1949, equation (14)] and our (9) and (63), the stream function $\psi_K$ for deep tubes can be written as

$$\psi_K = 2kq_K \sum_{n=-\infty}^{\infty} (-1)^n \left\{ \tan^{-1}\left[\tanh \frac{\pi(y - 2nh)}{a} \cot \frac{\pi x}{a}\right] - \tan^{-1}\left[\tanh \frac{\pi(y - 2d - 2nh)}{a} \cot \frac{\pi x}{a}\right]\right\} + C_6, \qquad n = 0, \text{ incl.} \tag{64}$$

in which expression $q_K$ is given by (32). In (64), $C_6$ is an arbitrary constant, and in (64) the summation index $n$ goes from $-\infty$ to $+\infty$ with $n = 0$ included. In (32) it is seen that $q_K$ is a function of $q_{DB}$. Consequently, $\psi_K$ is a function of parameters of both deep and shallow tubes.

**2.6.2. Stream function $\psi_{DB}$.** Working as we did with (9) and (10)–(14), to change $\phi_K$ of (9) for deep drain tubes to $\phi_{DB}$ for shallow tubes, we proceed in two steps. In the first step we change in (64) $x$ to $\xi$, $y$ to $\eta$, $d$ to $\delta$, and $q_K$ to $q_{DB}$ (and leave the symbols $h$ and $a$ unchanged), and also change $\psi_K$ of (64) to $\psi_{DB}$ and change $C_6$ of (64) to an arbitrary constant $C_7$; to get $\psi_K$ of (64) converted to $\psi_{DB}$ as

$$\psi_{DB} = 2kq_{DB} \sum_{n=-\infty}^{\infty} (-1)^n \left\{ \tan^{-1}\left[\tanh \frac{\pi(\eta - 2nh)}{a} \cot \frac{\pi \xi}{a}\right] - \tan^{-1}\left[\tanh \frac{\pi(\eta - 2\delta - 2nh)}{a} \cot \frac{\pi \xi}{a}\right]\right\} + C_7, \cdots. \tag{65}$$

We now do the second step whereby we change the $(\xi, \eta)$ coordinates of (65) to the $(x, y)$ coordinates of our present Figure 1. That is, with (11), (12), and (65) in view, we get $\psi_{DB}$ of (65) converted to and partly simplified as the relation

$$\psi_{DB} = 2kq_{DB} \sum_{n=-\infty}^{\infty} (-1)^n \left\{ \tan^{-1}\left[\tanh \frac{\pi(y - d + \delta - 2nh)}{a} \cdot \cot \frac{\pi(x - a/2)}{a}\right] - \tan^{-1}\left[\tanh \frac{\pi(y - d - \delta - 2nh)}{a} \cot \frac{\pi(x - a/2)}{a}\right]\right\} + C_7,. \tag{66}$$

To simplify (66) further, we need a relation obtained from *Dwight* [1961] (formulas 400.05, 401.02, and 401.04), which we find as

$$\cot \frac{\pi(x - a/2)}{a} = -\tan \frac{\pi x}{a}, \tag{67}$$

$$x \neq a/2 \qquad \text{otherwise } 0 < x < a$$

We put (67) in (66) to get $\psi_{DB}$ of (66) as

$$\psi_{DB} = 2kq_{DB} \sum_{n=-\infty}^{\infty} (-1)^n \left\{ \tan^{-1}\left[\tanh \frac{\pi(y - d + \delta - 2nh)}{a}\right] \cdot \left(-\tan \frac{\pi x}{a}\right) - \tan^{-1}\left[\tanh \frac{\pi(y - d - \delta - 2nh)}{a}\right] \cdot \left(-\tan \frac{\pi x}{a}\right)\right\} + C_7, \qquad x \neq (a/2), \cdots. \tag{68}$$

**2.6.3. Stream function $\psi_{KDB}$.** We now get the complete stream function $\psi_{KDB}$ of (61) by addition of (64) and (68) as

$$\psi_{KDB} = \text{RHS of (64)} + \text{RHS of (68)} \tag{69}$$

and (69), after we drop the arbitrary constants $C_6$ and $C_7$ of (64) and (68), as we may, gives (69) as

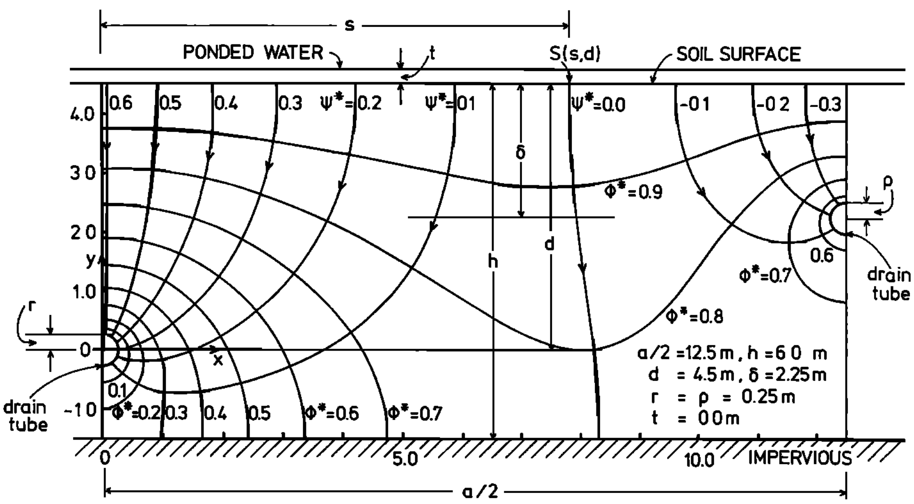


**Figure 3.** Flow net for a ponded-water soil that is drained by a dual-depth tube system.

$$\psi_{KDB} = 2kq_K \sum_{n=-\infty}^{\infty} (-1)^n \left\{ \tan^{-1}\left[\tanh\left(\frac{\pi(y-2nh)}{a}\right)\cot\frac{\pi x}{a}\right] - \tan^{-1}\left[\tanh\left(\frac{\pi(y-2d-2nh)}{a}\right)\cot\frac{\pi x}{a}\right]\right\} + 2kq_{DB}$$
$$\cdot \sum_{n=-\infty}^{\infty} (-1)^n \left\{ \tan^{-1}\left[\tanh\left(\frac{\pi(y-d+\delta-2nh)}{a}\right)\left(-\tan\frac{\pi x}{a}\right)\right] - \tan^{-1}\left[\tanh\left(\frac{\pi(y-d-\delta-2nh)}{a}\right)\left(-\tan\frac{\pi x}{a}\right)\right]\right\} \quad (70)$$

with $(x/a) \neq 1/2$

In (70) we note that we have also dropped the constant $C_5$ and that the parameters $r$, $\rho$, and $t$ do not occur in the summands but do occur through the head coefficients $q_K$ and $q_{DB}$, as is seen in (31), (32), (25), and (28). We also note that our (70) as well as our earlier (40) for $\phi_{KDB}$ are valid for $\delta = 0$. That is, we find, by putting $\delta/a = 0$ in (40) and (70), that after algebraic reduction, we get $\phi_{KDB}$ and $\psi_{KDB}$ as (see also equation (24))

$$\phi_{KDB} = d + t \qquad \text{for } \delta/a = 0 \qquad (71a)$$

and

$$\psi_{KDB} = \psi_K \qquad \text{for } \delta/a = 0 \qquad (71b)$$

where for (71b) we have to remember that the symbol $\psi_{DB}$ was defined by (65) where in (65) a constant $C_7$ was later dropped, that is, taken as zero. Also, we note that for $\delta = 0$, we must put $\rho = 0$ as per range limitations on parameters, given in subsection 2.1. If $d$ and $\delta$ are both zero so that $r$ and $\rho$ are both zero, then we find (40) as

$$\phi_{KDB} = d + t \qquad \text{for } d, \delta, r, \rho \text{ all zero} \qquad (71c)$$

and find

$$\psi_{KDB} = 0 \qquad \text{for } d, \delta, r, \rho \text{ all zero} \qquad (71d)$$

where in (71a)–(71d) we still have $k$, $t$, $h$, and $a$ arbitrary.

To check the validity of the expressions for $\phi$, $\Phi$, and $\psi$, that is (40), (63), and (70), it is noted that (40) has to satisfy the Laplace equation (1), and (63) and (70) the Cauchy-Riemann relations, which can be written [see *Kirkham and Powers*, 1984] as

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad (72)$$

and

$$\frac{\partial \Phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad (73)$$

Although somewhat tedious, it is a straightforward matter to show that (40) indeed is a solution of the Laplace equation (1) and that (63) and (70) satisfy the Cauchy-Riemann relations.

## 3. Results and Discussion

Principal analytical results up to now are algebraic formulas for the head $\phi_{KDB}$ (40), for the velocity potential $\Phi_{KDB}$ (63), for the stream function $\psi_{KDB}$ (70), deep tube intake $Q_K$ (51), shallow tube intake $Q_{DB}$ (56), and combined tube intake (or drain discharge) $Q_{KDB}$ (57). With these formulas, numerical calculations are possible, and the results of some computational work will now be presented and discussed.

Figure 3 is the flow net for a drain configuration as depicted in Figures 1 and 2. The parameters chosen for the calculation of Figure 3 are $a/2 = 12.5$ m, $h = 6.0$ m, $d = 4.5$ m, $\delta = 2.25$ m, $r = \rho = 0.25$ m, and $t = 0.0$ m. The somewhat unusual values for $d$ (equal to 4.5 m), $\delta$ (equal to 2.25 m), $r$, and $\rho$ (equal to 0.25 m) are chosen for illustration purposes. The figure shows, among others, a set of equiheads and streamlines. The equiheads were calculated with (40) and the streamlines with (70). However, the equiheads as well as the streamlines were normalized. The star superscript indicates normalization. Shown in Figure 3 are values for $\phi^*$ and $\psi^*$, where $\phi^*$ was calculated as

$$\phi^* = \frac{\phi - r}{(d + t) - r}, \qquad (74)$$

and $\psi^*$ as

$$\psi^* = \frac{\psi - \psi(s, d)}{0.5(Q_K + Q_{DB})} \qquad (75)$$

where $\phi = (d + t)$ is the maximum value for the head in the flow region, $\phi = r$ is the smallest value, $\psi(s, d)$ is the value for $\psi$ at $S(s, d)$, where $S(s, d)$ denotes the location of the water divide between the deep and the shallow drain tube at the soil surface, and $0.5(Q_K + Q_{DB}) = 0.5Q_{KDB}$ is the total amount of water removed from the flow region per unit length of drain per unit time. This implies, for example, that the amount of water that flows to the deep drain (in the lower left corner of Figure 3) between the streamlines $\psi^* = 0.1$ and $\psi^* = 0.2$ is one tenth of the total discharge of the flow region ($= 0.05Q_{KDB}$). This also means that the flow net shown in Figure 3 is independent of the soil hydraulic conductivity $k$ because $k$ does not appear in the $\phi^*$ and $\psi^*$ equations.

Figure 4 shows the drain discharge ($Q_K$, $Q_{DB}$, and $Q_{KDB}$) as function of the drain spacing $a$ for the same set of parameters as in Figure 3. Also shown is the single-depth drain discharge $Q_S$ (our equation (58)) of *Kirkham* [1949]. Unless the drain spacing $a$ is very small ("$a$" less than 5 m), $Q_{KDB}$ is always larger than $Q_S$. For example, when $a = 25.0$ m (see Figure 4), $Q_S = 6.327$ m³/m/d, and $Q_{KDB} = 10.154$ m³/m/d. This means that for the chosen set of parameters (see Figure 4) the dual-depth drain system is always more effective in removing water than the single-depth drain system. The figure also shows that for large drain spacings ("$a$" larger than 30 m) the ratio $Q_{KDB}/Q_S$ becomes a constant. Finally, the figure indicates that for not too large values of the drain spacing $a$ (e.g., "$a$" less than 40 m), $Q_K$ is smaller than $Q_S$, but that for larger values of "$a$" $Q_K$ approaches $Q_S$, as expected.

Some further results can be seen in Figure 5. In this figure the drain discharge $Q_{KDB}$ is expressed as a function of the shallow drain depth $\delta$, again for the set of parameters used for

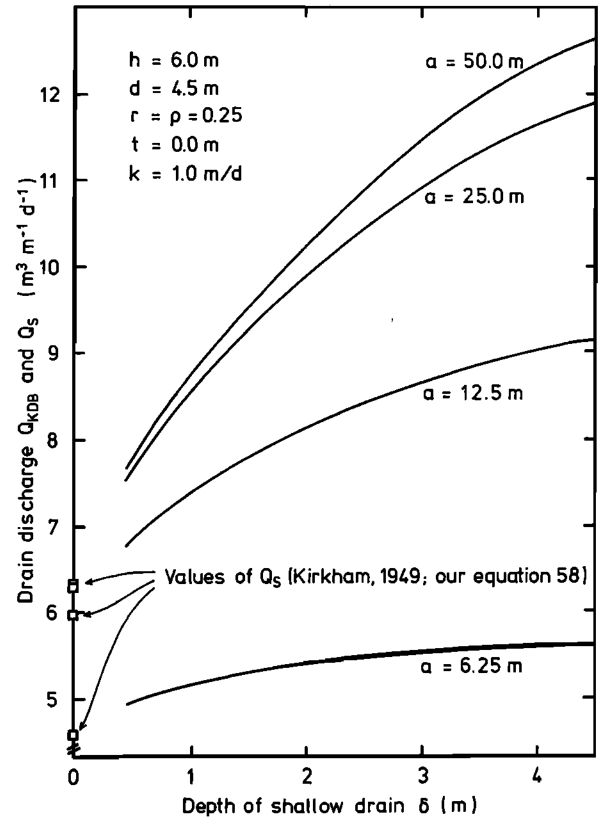


**Figure 5.** Dual-depth drain discharge $Q_{KDB}$ as a function of the shallow drain depth $\delta$ for different values of the drain spacing $a$; for comparison, also the single-depth drain discharge $Q_S$ of *Kirkham* [1949] is shown.

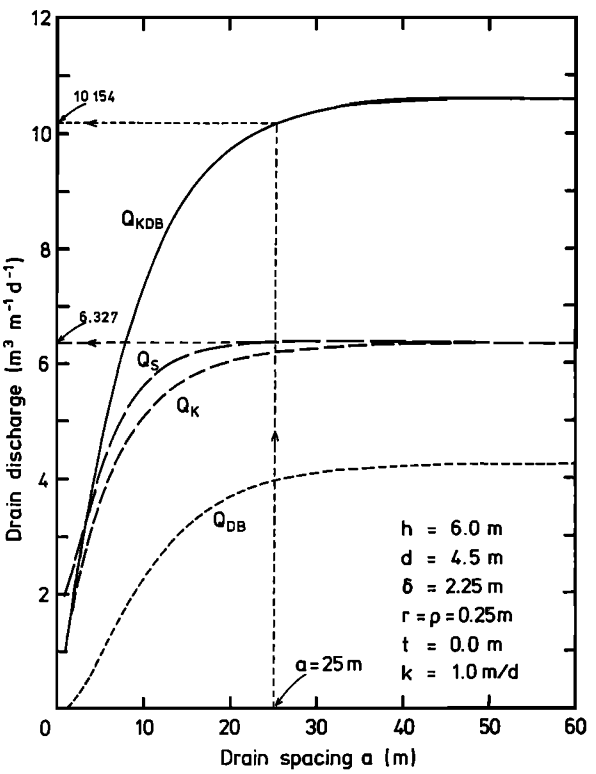


**Figure 4.** Dual-depth drain discharge $Q_K$, $Q_{DB}$, and $Q_{KDB}$ as functions of the drain spacing $a$; for comparison, also the single-depth drain discharge $Q_S$ of *Kirkham* [1949] is shown.

Figures 3 and 4. Shown is $Q_{KDB}$ for different values of the drain spacing $a$. The figure indicates (as can be expected) that the drain discharge $Q_{KDB}$ increases when the drain spacing $a$ becomes larger. However, it appears that with increasing drain spacing $a$ the relative increase in drain discharge becomes smaller. Also shown in Figure 5 (on the ordinate) are values for the single-depth drain discharge $Q_S$ from *Kirkham* [1949], as calculated with our (58). The figure suggests that with decreasing shallow drain depth $\delta$ the values of $Q_{KDB}$ approach the $Q_S$ value of *Kirkham* [1949], as it should.

The dual-depth drain discharge $Q_{KDB}$ can also be considered as a function of the depth $h$ of the impervious barrier (Figure 6). However, the ratio $(Q_{KDB}/Q_S) \times 100$ is plotted on the ordinate rather than $Q_{KDB}$. This ratio represents the improvement in drain discharge due to the additional drain tubes midway between the original ones. In Figure 6 this ratio $Q_{KDB}/Q_S$ is calculated for two values of the drain spacing $a$ ($a = 6.25$ m and $a = 25.0$ m) and for four values of the tube radii $r$ and $\rho$($r = \rho = 0.04$, 0.06, 0.08, or 0.10 m). These values of tube radii represent the size of drain tubes that are commonly used in the field [e.g., *Schwab et al.*, 1981]. The other parameters needed to prepare Figure 6 were the same as in Figures 3, 4, and 5 (i.e., $d = 4.5$ m, $\delta = 2.25$ m, and $t = 0.0$ m; see Figures 1 and 2).

Figure 6 shows that the relative drain discharge $Q_{KDB}/Q_S$ depends strongly on the depth of the impervious barrier $h$ when the deep drain tubes are located near this barrier but is almost independent of $h$ when the impervious barrier is at greater depth. The figure also shows that for the chosen set of

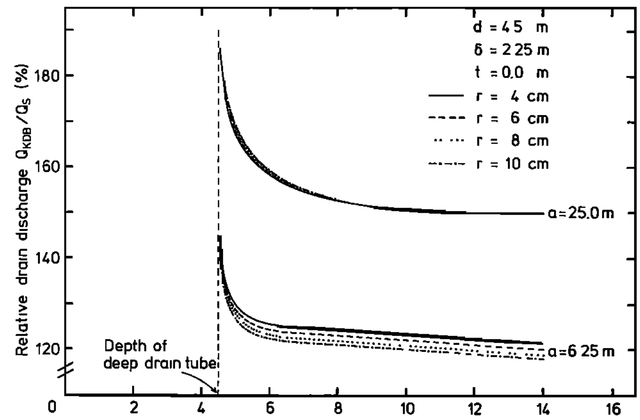


**Figure 6.** Relative drain discharge $Q_{\mathrm{KDB}}/Q_S$ (in percent) as a function of the depth of the impervious barrier $h$, for different values of the drain tube radii ($r = \rho = 0.04$, 0.06, 0.08, or 0.10 m) for two values of the drain spacing ($a = 6.25$ or 25.0 m).

parameters the ratio $Q_{\mathrm{KDB}}/Q_S$ depends on the drain spacing $a$; for larger values of $a$ the ratio $Q_{\mathrm{KDB}}/Q_S$ also appears to increase. Furthermore, Figure 6 indicates that the radii of the drain tubes have some influence on $Q_{\mathrm{KDB}}/Q_S$ but that this influence is small. Finally, Figure 6 illustrates that for the chosen set of parameters the dual-depth system is always more efficient in removing soil water than the single-depth system. Roughly speaking, the improvement is 120% when the original drain spacing $a$ is already narrow ($a = 6.25$ m) and 150% for a wider original spacing ($a = 25.0$ m). It is remarked that for any other combination of field parameters, such estimates could also be derived.

For engineering purposes, an attempt can be made to generalize the obtained results. In general, drain tubes are installed at about one meter depth; minimum depth for drain tubes is about 0.60 m, and the depth of subsurface tubes seldom is more than 2.50 m [see *Schwab et al.*, 1981]. The diameter of drain tubes commonly is about 0.10 m, and this diameter affects the drain discharge only slightly (see Figure 6). Therefore a standard single-depth drainage system with drain tubes at 1.20 m depth and a radius of 5 cm was taken as the reference system, with which the performance of a dual-depth system can be compared. The ratio $Q_{\mathrm{KDB}}/Q_S$ can be taken as the performance index to evaluate the dual-depth system, with $Q_S$ being the drain discharge of the standard single-depth system, as calculated according to *Kirkham* [1949] (our equation (58)), and $Q_{\mathrm{KDB}}$ (equal to $Q_{\mathrm{K}} + Q_{\mathrm{DB}}$) being the combined drain discharge of the dual-depth system, see (41), as calculated with (57).

With the drain discharge $Q_S$ of this single-depth system as a reference value, simple nomographs can be prepared to estimate the performance of an arbitrary dual-depth drainage system. By taking the depth $\delta$ of the additional drain tubes midway between the ones of the single-depth system either as $\delta = 0.60$ m, $\delta = 1.20$ m, or as $\delta = 2.40$ m, a wide range of possible flow configurations is covered (see Figures 1 and 2). The last two values for $\delta$ ($\delta = 1.20$ m and $\delta = 2.40$ m) seem to violate the range limitations, discussed in section 2.1. However, for $d = \delta = 1.20$ m, the dual-depth drainage system reduces to a single-depth system as treated by *Kirkham* [1949], with the new drain spacing being half the original drain spacing. The case with $\delta = 2.40$ m and $d = 1.20$ m can be handled by interchanging the symbols, that is, by taking $d = 2.40$ m and $\delta = 1.20$ m. However, for the calculation of $Q_S$, one should still take $d = 1.20$ m. The range of values for the nomographs can be extended, if additionally different values for the single-depth drain spacing $a$ are considered. To this end, the following values for $a$ were taken for the calculation of the nomographs: $a = 6.25$ m, $a = 12.50$ m, $a = 25.0$ m, $a = 50.0$ m, and $a = 100.0$ m.

In Figures 7a, 7b, and 7c the ratio $Q_{\mathrm{KDB}}/Q_S$ is depicted as a function of the depth $h$ of the impervious barrier (see Figures 1 and 2). Figure 7a is for $\delta = 0.60$ m, Figure 7b is for $\delta = 1.20$ m (the same depth as for the existing drain tubes), and Figure 7c is for $\delta = 2.40$ m. For all three figures the radii of the tubes were taken as $r = \rho = 0.05$ m, and for the depth of the ponded water, $t = 0.0$ m was used. As mentioned before, the shown results are independent of the soil hydraulic conductivity $k$. For Figures 7a and 7b the minimum value for the depth $h$ of the impervious barrier was 1.25 m and for Figure 7c it was 2.45 m.

The figures show that for the set of considered parameters the dual-depth system is always more effective in removing soil water than the single-depth system. The effect of the additional system is particularly pronounced for small values of the depth $h$ of the impervious barrier. For large values of $h$, however, the ratio $Q_{\mathrm{KDB}}/Q_S$ becomes independent of $h$, as the figures show. For large values of the (original) drain spacing $a$ the ratio $Q_{\mathrm{KDB}}/Q_S$ also reaches a constant value; for $\delta = 0.60$ m this value (in percent) is about 160; for $\delta = 1.20$ m it is 200, and for $\delta = 2.40$ m (when the additional tubes are twice as deep as the original ones) it is about 270. In case the impervious barrier is at a relatively shallow depth, these values for $\delta = 0.60$ m and $\delta = 1.20$ m are even larger; for $\delta = 2.40$ m, however, they are somewhat smaller, as the figures show.

The following example demonstrates how to use the nomographs. Suppose that for a water-ponded soil with a single-depth subsurface drainage system at 1.20 m depth it is desired to increase the drainage capacity. Suppose further that the existing drain spacing is 25 m ($a = 25$ m), that the depth of the impervious barrier is 3 m ($h = 3.0$ m), that the soil hydraulic conductivity is 0.75 m/d ($k = 0.75$ m/d), and that the thickness of the ponded water is zero ($t = 0.0$ m). Although a value for $k$ is given, the nomographs do not depend on $k$. The question is, How large is the relative discharge when midway between the existing drain tubes additional drain tubes are installed at either 0.60, 1.20, or 2.40 m depth? From Figures 7a, 7b, and 7c it can be seen that a dual-depth system would result in a relative drain discharge of about 160, 200, and 250%, respectively. For intermediate values of $\delta$ and $a$ an estimate of the benefits of a dual-depth system may be obtained by interpolation.

## 4. Conclusions

With a procedure based on multiple drain images and complex variables it was possible to extend existing soil water flow theory for the case of water-ponded soil drained by a single-depth subsurface drain tube system to soil water flow for a dual-depth drain tube system. For a homogeneous, water-ponded soil underlain by an impervious barrier, analytic expressions for the hydraulic head in the flow region, as well as for the velocity potential, for the stream function, and for the drain tube discharge, either for the deep tube or for the shal-

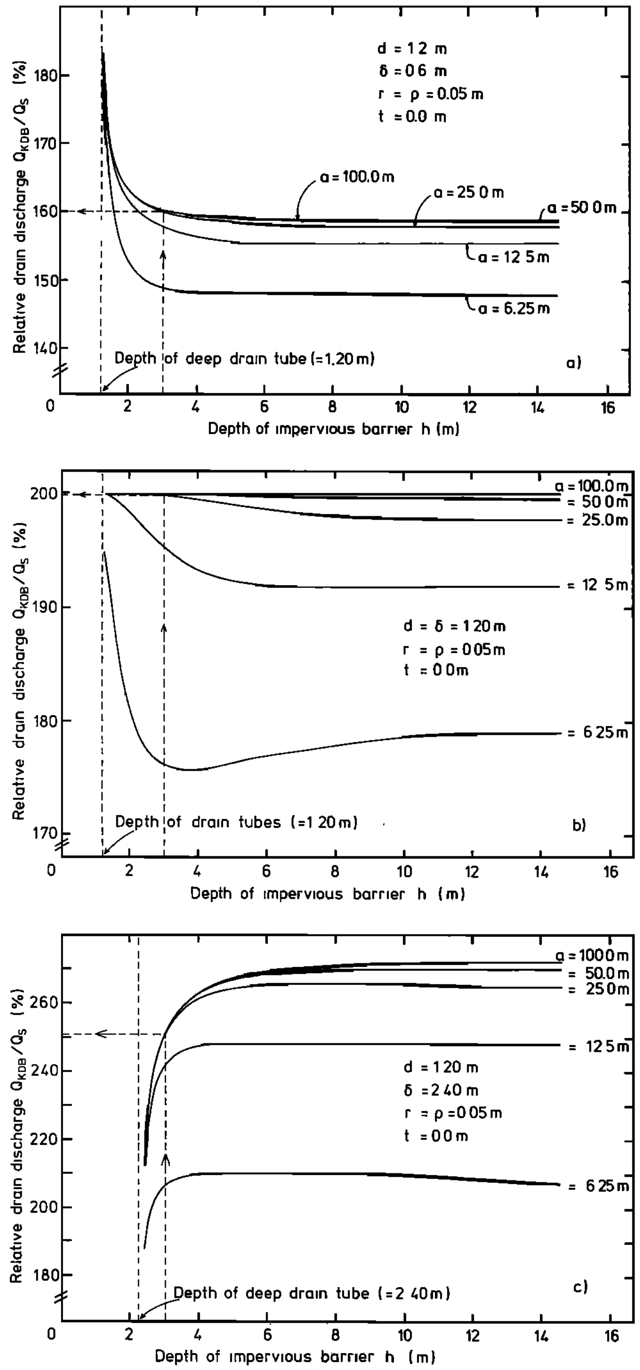


**Figure 7.** Simple nomographs to estimate the relative drain discharge $Q_{KDB}/Q_S$ (in percent) for a dual-depth subsurface drainage system, when the depth of the single-depth tube is 1.20 m and the additional tube is (a) at 0.60 m, (b) at 1.20 m, and (c) at 2.40 m depth; other parameters are shown in the figures.

low tube, were derived. Sample calculations with the newly derived expressions show that the effectiveness of a single-depth subsurface drainage system in removing soil water can be increased by 50% and more if midway between the existing tubes additional tubes are installed. Sample calculations also show that the depth of the impervious barrier strongly affects the drain discharge of the dual-depth system but that the radii of the tubes have only a minor influence on the drain discharge. For design purposes, simple nomographs can be used to evaluate the performance of a dual-depth subsurface drainage system in comparison with an existing single-depth system.

## Notation

- $a$ drain spacing of the single-depth drain system, m (Figure 1 or 2).
- $c$ height of the ponded water surface above the reference level, m (Figure 1 or 2).
- $C_3$, $C_4$ constants in the expressions for the hydraulic head functions $\phi_K$ and $\phi_{DB}$, m, m.
- $C_5$, $C_6$, $C_7$ constants in the expressions for the stream functions $\psi_{KDB}$, $\psi_K$, and $\psi_{DB}$, respectively, m²/d.
- $d$ height of the soil surface above the reference level, m (Figure 1 or 2).
- $F$ auxiliary function of *Kirkham* [1949, equation (10)], dimensionless.
- $h$ depth of the impervious barrier below the soil surface, m (Figure 1 or 2).
- $k$ hydraulic conductivity of the soil, m/d.
- $n$ summation integer; $n$ goes from $-\infty$ to $+\infty$, inclusive $n$ equal to 0.
- $q_{DB}$ hydraulic head coefficient, associated with the shallow drain tube, m.
- $q_K$ hydraulic head coefficient, associated with the deep drain tube, m.
- $q_S$ hydraulic head coefficient of *Kirkham* [1949] for a single-depth drain system, m.
- $Q_{DB}$ total inflow into a shallow drain tube per unit length of tube per unit time, m³/m/d.
- $Q_K$ total inflow into a deep drain tube per unit length of tube per unit time, m³/m/d.
- $Q_{KDB}$ combined inflow of a deep and a shallow drain tube per unit length of tube per unit time, equal to $Q_K + Q_{DB}$, m³/m/d.
- $Q_S$ total inflow per unit length of drain tube per unit time, for the single-depth drain system of *Kirkham* [1949], m³/m/d.
- $r$ radius of the deep drain tube, m.
- $r_1$ (short) radial distance, measured from the center O of the deep tube, m.
- $S_1$, $S_2$ summation coefficients, both equal to zero (see equations (19) and (20)), dimensionless.
- $S_3$, $S_4$ summation coefficients, as defined by (26) and (27), dimensionless.
- $S_5$, $S_6$ summation coefficients, as defined by (29) and (30), dimensionless.
- $t$ height of the ponded water surface above the soil surface, m (Figure 1 or 2).
- $x$, $y$ rectangular coordinates measured from the center O of the deep tube in the lower left corner of the flow region, m, m (Figure 1 or 2).
- $x_1$, $y_1$ rectangular coordinates of a point P near a deep tube, m, m.
- $\delta$ height of the soil surface above the center of the shallow drain tube, m (Figure 1 or 2).
- $\phi$ hydraulic head, m.
- $\phi^*$ normalized combined hydraulic head in the flow region (see equation (74)), dimensionless.
- $\phi_{DB}$ hydraulic head component, associated (but not only) with the shallow tube in the flow region, m.

$\phi_K$ hydraulic head component, associated (but not only) with the deep tube in the flow region, m.
$\phi_{KDB}$ total hydraulic head in the flow region, equal to $\phi_K + \phi_{DB}$, m.
$\Phi$ velocity potential, equal to $k\phi$, $m^2/d$.
$\Phi_{KDB}$ combined velocity potential in the flow region, equal to $k\phi_{KDB} = k(\phi_K + \phi_{DB})$, $m^2/d$.
$\xi$, $\eta$ rectangular coordinates measured from the center O′ of the shallow tube in the right upper corner of the flow region under consideration, m, m (Figure 1 or 2).
$\xi_1$, $\eta_1$ rectangular coordinates of a point P near a shallow tube, m, m.
$\pi$ a constant, approximately 3.14159.
$\rho$ radius of the shallow drain tube, m.
$\rho_1$ (short) radial distance, measured from the center O′ of the shallow tube, m.
$\sigma_K$, $\sigma_{DB}$ summation coefficients, as defined by (33) and (34), dimensionless.
$\psi^*$ normalized combined stream function in the flow region (see equation (75)), dimensionless.
$\psi_K$ stream function component, associated (but not only) with the deep tube in the flow region, $m^2/d$.
$\psi_{DB}$ stream function component, associated (but not only) with the shallow tube in the flow region, $m^2/d$.
$\psi_{KDB}$ combined stream function in the flow region, $\psi_{KDB} = \psi_K + \psi_{DB}$, $m^2/d$.
$\psi(s, d)$ value of the stream function $\psi$ at $x = s$ and $y = d$ (see Figure 3), $m^2/d$.

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