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Coupled conduction and convection heat transfer occurs in soil when a significant amount of water is moving continuously through soil. Prime examples are rainfall and irrigation. We developed an analytical solution for the heat conduction-convection equation. The solution for the upper boundary of the first type is obtained by Fourier transformation. Results from the analytical solution are compared with data from a field infiltration experiment with natural temperature variations. The predicted temperature values are very similar to the observed values. Temperature changes with time for different soil depths are predicted from conduction-convection theory and from conduction theory alone. During infiltration, convective heat transfer contributed significantly to the temperature changes at all soil depths monitored. The theory also quite accurately predicts temperature effects on surface infiltration.

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Analytical Solution for One-Dimensional Heat Conduction–Convection Equation

Mingan Shao, Robert Horton,* and D. B. Jaynes

ABSTRACT

Coupled conduction and convection heat transfer occurs in soil when a significant amount of water is moving continuously through soil. Prime examples are rainfall and irrigation. We developed an analytical solution for the heat conduction–convection equation. The solution for the upper boundary of the first type is obtained by Fourier transformation. Results from the analytical solution are compared with data from a field infiltration experiment with natural temperature variations. The predicted temperature values are very similar to the observed values. Temperature changes with time for different soil depths are predicted from conduction–convection theory and from conduction theory alone. During infiltration, convective heat transfer contributed significantly to the temperature changes at all soil depths monitored. The theory also quite accurately predicts temperature effects on surface infiltration.

In recent decades, efforts have been made to understand the effects of temperature on soil physical and chemical properties. Recent efforts have focused on the modeling of water and heat transfer in soils, together with studying temperature effects on the physical and chemical properties of soils (Nassar and Horton, 1992a,b). Soil hydraulic properties are temperature-dependent in part because of the temperature effect on water viscosity. While the effect of temperature on hydraulic properties of a soil has been studied under laboratory conditions (Constantz, 1982), little information on the same topic can be found for field conditions because either the effect may be too small to be worthy of consideration (Jaynes, 1990), or the conditions are too complicated to be handled. Some observations of temperature effects on infiltration have been made, however (Musgrave, 1955; Bouwer et al., 1974). Increases in seepage or infiltration rate were observed in response to temperature increases (e.g., Constantz et al., 1994). Additional mathematical and physical studies may lead to the development of methods for estimating seepage rates based on soil temperature changes.

Water and heat transfer in soils can be modeled either numerically or analytically. Most research on the modeling of water and heat movement in soils has been made by numerical techniques (Jaynes, 1990; Horton and Chung, 1991; Nassar and Horton, 1992a,b). Few analytical solutions are available for isothermal water flow in soil (Knight and Philip, 1974; Parlange and Fleming, 1984; Sander et al., 1988, 1991; Barry and Sposito, 1989; Barry and Sander, 1991). Even fewer are available for coupled heat and water transport (Bredehoeft and Papadopulos, 1965; Milly, 1984). Nevertheless, it is possible to analytically solve the simultaneous transfer problem of water and heat in soils under certain conditions. New analytical solutions will improve our understanding of coupled heat- and water-flow problems because the analytical solutions themselves contain more explicit information of process descriptions, model parameters, and initial and boundary conditions than do numerical methods. Analytical solutions will also provide standards for comparison with the numerical solutions. The objective of this study was to derive an analytical solution to water and heat transfer during infiltration under field conditions. The analytical solution will be compared with field-measured data.

MODEL

The partial differential equation for one-dimensional simultaneous nonsteady heat and water transfer through an isotropic, homogeneous porous medium is (Bredehoeft and Papadopulos, 1965):

$$\frac{c_s\rho_s}{\kappa}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} - \frac{c_l\rho_l}{\kappa}\frac{\partial (qT)}{\partial x} \quad [1]$$

where T is temperature at any point and at any time (°C), t is time (s), x is the depth (m, positive downward), κ is soil thermal conductivity (W m^{-1} °C^{-1}), c_l and c_s, and ρ_l and ρ_s are specific heats of the liquid and solid (J kg^{-1} °C^{-1}), and the liquid and bulk densities (kg m^{-3}), respectively, and q is the liquid infiltration rate or volume flux density (m^3 s^{-1} m^{-2}). The infiltration rate is a function of time. Therefore, for one-dimensioinal infiltration into soils, the equation can be reduced to:

$$\frac{\partial T}{\partial t} = \frac{\kappa}{c_s\rho_s}\frac{\partial^2 T}{\partial x^2} - q(t)\,\frac{c_l\rho_l}{c_s\rho_s}\frac{\partial T}{\partial x} \quad [2]$$

If we let $D = \kappa/(c_s\rho_s)$ and $r = c_l\rho_l/(c_s\rho_s)$, then Eq. [2] is reduced to

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} - rq(t)\frac{\partial T}{\partial x} \quad [3]$$

where D is the thermal diffusivity (m^2 s^{-1}). The initial and typical boundary conditions for [3] are

$$T(x,0) = f(x) \quad [4]$$

$$T(\infty,t) = T_1 \quad [5]$$

$$T(0,t) = T_0 + A\sin(\omega t + \phi) \quad [6]$$

where in Eq. [5] and [6], T_0 is the average temperature (°C) of the soil surface; A (°C) is the amplitude of surface temperature oscillations of angular frequency ω (rad s^{-1}); the term T_1 is defined as a constant temperature at infinite depth, but is usually approximated by the temperature at a relatively large depth; and $f(x)$ (°C) is the initial temperature distribution in

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Abbreviations: K_{sr}, saturated hydraulic conductivity at the reference temperature; RMSE, root mean square error.

the soil profile. Also in Eq. [6], the symbol ϕ is for an initial phase angle (rad).

For the case of ponded surface infiltration (Jaynes, 1990), the water flux density $q(t)$ of Eq. [3] may also be expressed by a periodic function of time. We demonstrate this as follows. First, we express $q(t)$ as a function of h, the pressure head of soil water:

$$q(t) = -\frac{\eta_r}{\eta_T} K_r(h) \left(\frac{\partial h}{\partial x} - 1\right) \qquad [7]$$

where η_r and η_T are the liquid viscosities for a reference temperature and for the current temperature (kg m^{-1} s^{-1}), h is the pressure head of soil water (m), and $K_r(h)$ is the relative hydraulic conductivity (m s^{-1}) determined at the reference temperature.

Second, the relative hydraulic conductivity $K_r(h)$ is expressed by Campbell's equation (Campbell, 1974):

$$K_r(h) = K_{sr}\left|\frac{h_a}{h}\right|^n \qquad h < h_a \qquad [8]$$

where K_{sr} is the saturated hydraulic conductivity at the reference temperature (m s^{-1}), h_a is the bubbling pressure head (m), and n is a dimensionless constant.

For the nearly saturated case of downward seepage, $h \geq h_a$, described by Jaynes (1990), we may assume that h is constant throughout the profile (unit gradient of soil water potential for the soil profile); then $\partial h/\partial = 0$, $K_r(h) = K_{sr}$, and $q(t)$ is now expressed as

$$q(t) = \frac{\eta_r}{\eta_T} K_{sr} \qquad [9]$$

The ratio η_r/η_T in Eq. [9] can be approximated by a linear function for the range of temperature variation in field conditions, i.e., we take in Eq. [9]:

$$\frac{\eta_r}{\eta_T} = V_0 + V_1 T \qquad [10]$$

where V_0 (dimensionless) and V_1 (°C^{-1}) are constants, and T is the periodic time-dependent surface temperature. For Eq. [10] we further take T as

$$T = T_0 + A\sin(\omega t) \qquad [11]$$

where T, T_0, and A have dimensions of temperature (°C). Combining Eq. [9], [10], and [11], then gives $q(t)$ as

$$q(t) = a_1 + b_1 \sin(\omega t) \qquad [12]$$

in which $a_1 = K_{sr}(V_0 + V_1 T_0)$ and $b_1 = K_{sr}V_1A$. If we let $a = ra_1$ and $b = rb_1$, then from Eq. [12] the term $rq(t)$, in [3] becomes:

$$rq(T) = a + b\sin(\omega t) \qquad [13]$$

where from Eq. [12] the dimensions of a_1 and b_1 are m s^{-1}. This completes the model development.

ANALYTICAL SOLUTION

The solution to Eq. [3] satisfying Eq. [4], [5], and [6] may be obtained by transforming Eq. [3] into the classical heat equation (Cannon, 1984). The model thus reformulated is a moving-boundary problem. Two methods can be employed in the analytical solution. One is Fourier transformation, if movement of the boundary is so small that it can be ignored (Powers, 1987). The other is the moving-boundary approach by use of known mathematical solutions (Cannon, 1984). In this study, the Fourier transformation method was used because it produces an explicit analytical solution to the problem. We did not use the moving-boundary approach because it produces an implicit solution which requires additional intensive numerical integrations. Fourier transforms (Fourier integral), as we use them, are fully detailed in Powers (1987).

Transformation to the Classical Heat Equation

Returning to Eq. [3], we can first make a homogeneous boundary condition by means of transformation $T^* = T(x,t) - T_1$. After this transformation and the combination with Eq. [13], Eq. [3] through [6] become

$$\frac{\partial T^*}{\partial t} = D\frac{\partial^2 T^*}{\partial x^2} - a\frac{\partial T^*}{\partial x} - b\sin(\omega t)\frac{\partial T^*}{\partial x} \qquad [3a]$$

$$T^*(0,t) = (T_0 - T_1) + A\sin(\omega t + \phi) \qquad [4a]$$

$$T^*(\infty,t) = 0 \qquad [5a]$$

$$T^*(x,0) = f(x) - T_1 = F(x) \qquad [6a]$$

Then the term $a\partial T^*/\partial x$ needs to be eliminated. This can be done by the substitution of $U(x,t) = T^* \exp[a^2t/(4D) - ax/(2D)]$. By using this substitution, Eq. [3a] through [6a] become, respectively,

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2} - [b\sin(\omega t)]\frac{\partial U}{\partial x} - \left[\frac{ab}{2D}\sin(\omega t)\right]U(x,t) \qquad [3b]$$

$$U(0,t) = \exp\left(\frac{a^2t}{4D}\right)[(T_0 - T_1) + A\sin(\omega t + \phi)] \qquad [4b]$$

$$U(\infty,t) = 0 \qquad [5b]$$

$$U(x,0) = F(x)\exp\left(\frac{-ax}{2D}\right) \qquad [6b]$$

The next step is to remove the term $b\sin(\omega t)\partial U/\partial x$ in Eq. [3b]. This can be done by introducing a parameter $\lambda_1(t)$, in m, which is defined by

$$\lambda_1(t) = \int_0^t b\sin(\omega t)\,dt = \frac{b}{\omega}[1 - \cos(\omega t)] \qquad [14]$$

Let $z = x - \lambda_1(t)$. Then, for function U of Eq. [3b], we have $U(x,t) = U[z + \lambda_1(t),t] = V(z,t)$. The differential relationships with respect to time and depth between U and V are given by

$$\frac{\partial U}{\partial t} = \frac{\partial V}{\partial t} - b\sin(\omega t)\frac{\partial V}{\partial z} \qquad [15]$$

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial z} \qquad [16]$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 V}{\partial z^2} \qquad [17]$$

If we combine Eq. [15], [16], and [17] with [3b], then we have

$$\frac{\partial V}{\partial t} = D\frac{\partial^2 V}{\partial z^2} - \left[\frac{ab}{2D}\sin(\omega t)\right]V \quad [18]$$

$$V[-\lambda_1(t),t] = \exp\left(\frac{a^2 t}{4D}\right)[(T_0 - T_1) + A\sin(\omega t + \phi)] \quad [19]$$

$$V(\infty,t) = 0 \quad [20]$$

$$V(z,0) = F(z)\exp\left(\frac{-az}{2D}\right) \quad [21]$$

We note that $\lambda_1(t)$ is often small compared with the depth of soil profile concerned. From the field experimental data (Jaynes, 1990), λ_1 is from 0 to 0.04 m (the depth of the profile concerned was 0.6 m).

Analytical Solution of Equation [18] Subject to Equations [19] through [21]

The analytical solution of Eq. [18] subject to Eq. [19] through [21] may be found by using the Fourier sine transformation, given by

$$V(\lambda,t) = \int_0^\infty V(z,t)\sin(\lambda z)\,dz \quad [22]$$

where λ, in m, is the parameter of the Fourier transformation. By using this transformation, the problem becomes the following initial value problem of an ordinary differential equation:

$$\frac{dV(\lambda,t)}{dt} = -\left[D\lambda^2 + \frac{ab}{2D}\sin(\omega t)\right]V(\lambda,t) + D\lambda\exp\left(\frac{a^2 t}{4D}\right)[(T_0 - T_1) + A\sin(\omega t + \phi)] \quad [23]$$

$$V(\lambda,0) = \int_0^\infty F(z)\exp\left(\frac{-az}{2D}\right)\sin(\lambda z)dz \quad [24]$$

Integrating Eq. [23] and using the initial condition Eq. [24], the explicit analytical solution is expressed as

$$V(\lambda,t) = [I_1 + I_3 - I_2 - I_4 + C]\exp[A_3\cos(\omega t) - D\lambda^2 t] \quad [25]$$

in which

$$I_1 = A_1\exp(A_2 t)\frac{A_2\sin(\omega t + \phi) - \omega\cos(\omega t + \phi)}{A_2^2 + \omega^2} \quad [26]$$

$$I_2 = \frac{I_2^*[(A_2^2 + 4\omega^2)\sin(\phi) + A_2^2\sin(2\omega t + \phi) - 2A_2\,\omega\cos(2\omega t + \phi)]}{2A_2(A_2^2 + \omega^2)} \quad [27]$$

$$I_2^* = A_1 A_3\exp(A_2 t) \quad [28]$$

$$I_3 = (A_4/A_2)\exp(A_2 t) \quad [29]$$

$$I_4 = \frac{A_3 A_4\exp(A_2 t)[A_2\cos(\omega t) + \omega\sin(\omega t)]}{A_2^2 + \omega^2} \quad [30]$$

in which A_1, A_2, A_3, A_4, and C are constants given by

$$A_1 = D\lambda A \quad [31]$$

$$A_2 = \left(\frac{a^2}{4D} + D\lambda^2\right) \quad [32]$$

$$A_3 = \frac{ab}{2D\omega} \quad [33]$$

$$A_4 = D\lambda(T_0 - T_1) \quad [34]$$

$$C = \exp(-A_3)V(\lambda,0) + I_2(0) + I_4(0) - I_1(0) - I_3(0) \quad [35]$$

in which

$$I_1(0) = \frac{A_1[A_2\sin(\phi) - \omega\cos(\phi)]}{A_2^2 + \omega^2} \quad [36]$$

$$I_2(0) = \frac{A_1 A_3[(A_2^2 + 2\omega^2)\sin(\phi) - A_2\,\omega\cos(\phi)]}{A_2(A_2^2 + 4\omega^2)} \quad [37]$$

$$I_3(0) = \frac{A_4}{A_2} \quad [38]$$

$$I_4(0) = \frac{A_2 A_3 A_4}{A_2^2 + \omega^2} \quad [39]$$

From Eq. [25], the solution to $V(z,t)$ is expressed by

$$V(z,t) = \frac{2}{\pi}\int_0^\infty V(\lambda,t)\sin(\lambda z)\,d\lambda \quad [40]$$

Analytical Solution to the Original Problem

By the substitutions used, we can obtain the solution to the original problem. From $V(z,t)$ of Eq. [40], we can obtain the $U(x,t)$ as

$$U(x,t) = \frac{2}{\pi}\int_0^\infty V(\lambda,t)\sin\{\lambda[x - \lambda_1(t)]\}\,d\lambda \quad [41]$$

Then, $T^*(x,t)$ is given by

$$T^*(x,t) = \exp\left(\frac{ax}{2D} - \frac{a^2 t}{4D}\right)U(x,t) \quad [42]$$

The solution to the original problem, Eq. [2], is given by

$$T(x,t) = T_1 + T^*(x,t) \quad [43]$$

where $T^*(x,t)$ is given by Eq. [42] and T_1 by Eq. [5]. Because Eq. [41] is explicit, the final solution (Eq. [43]) is explicit rather than implicit. This is one of the advantages of using Fourier transformation rather than using the moving-boundary theory, which can only give an implicit solution for this problem.

FIELD EXPERIMENT

A detailed description of the field experiment can be found in Jaynes (1990). A brief summary is provided here. Ponded infiltration rates were observed near Phoenix, AZ. The soil is an Avondale clay loam (fine-loamy, mixed, hyperthermic Torrifluventic Haplustoll). The leaching-basin method was used in the field-infiltration experiment. A 6.1 by 6.1 m area was isolated by driving a sheet metal strip, 0.4 m wide, 0.2 m into the ground. The center 3.66 by 3.66 m was divided into four

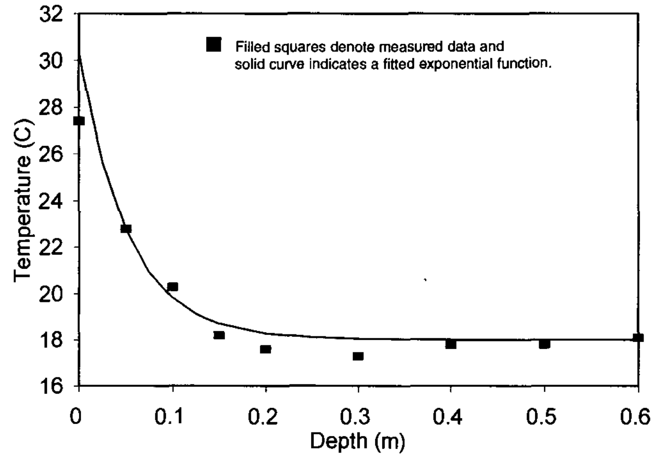

Fig. 1. The initial temperature distribution of the soil profile.

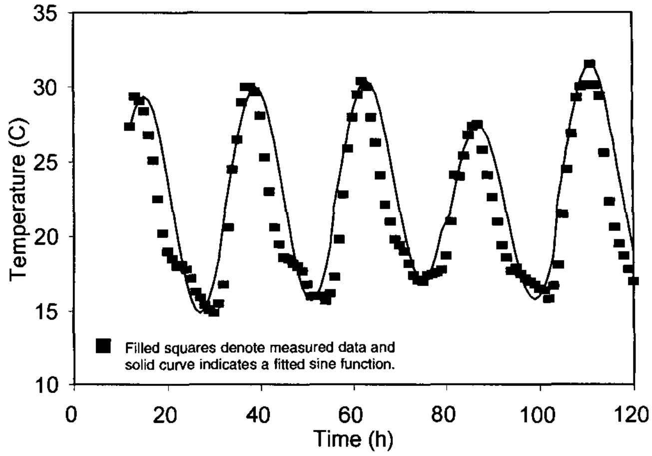

Fig. 2. The change of the soil surface temperature with time.

sub-basins, 1.83 m on each side, with similar metal borders.

Soil temperatures were measured by Cu-constant thermocouples. Temperature measurements were observed hourly at depths of 0.0, 0.05, 0.1, 0.15, 0.2, 0.3, 0.4, 0.5, and 0.6 m. Infiltration rates were measured by flow meters and corrected for changes in measured ponding depth. All of the measurements were continued for a period of 120 h.

RESULTS AND DISCUSSION

Initial Profile Temperature Distribution

The observed initial soil temperature vs. depth can be approximated by the following exponential function:

$$f(x) = T_1 + Be^{-kx} \quad [44]$$

where T_1 (18.02°C) is the constant temperature when x approaches infinity, and B (12.3°C) and k (19.07 m^{-1}) are coefficients. The result of the best fit for the initial temperature distribution is shown in Fig. 1. From Fig. 1, we can see that the initial temperature may be approximated by the exponential function. With this initial temperature, the needed $V(\lambda,0)$ of Eq. [24] can be expressed as

$$V(\lambda,0) = \frac{B\lambda}{\lambda^2 + (k + a/2D)^2} \quad [45]$$

Required Parameters

The parameters required in the analytical solution were obtained either from the experiment (Jaynes, 1990) or from calculations. The heat capacity and thermal conductivity of the soil were calculated (Campbell, 1985). Their values are 2090 J kg^{-1} K^{-1} and 1.434 W m^{-1} K^{-1}, respectively. The T_0 (21.5°C) was calculated using the measured surface temperature. The K_{sr} (at 21.5°C) was found to be 0.022 m h^{-1}. The c_l and ρ_l are assumed to be 4180 J kg^{-1} K^{-1} and 1000 kg m^{-3}, respectively. The ρ_s was 1.50 Mg m^{-3} by measurement. Daily amplitude of the surface temperature was obtained by fitting a sine function to the observed temperature, for which the amplitude ranged from 5.85 to 7.25°C. The ω (angular frequency) was assumed to be $2\pi/24$ (rad h^{-1}). The values of V_0 (0.46, dimensionless) and V_1 (0.02, K^{-1}) were obtained by a linear regression. Viscosity data were taken from Weast (1986). With these parameters, the analytical solution Eq. [43] can be obtained by integration. Equation [41] was integrated numerically using a trapezoidal method.

Temperature Changes

Measured and fitted surface temperature are shown in Fig. 2. The surface temperature oscillations can be estimated by a simple sine function with amplitude varying from day to day. In general, higher order harmonics (especially the second and third harmonics) are used to describe the surface temperature. For the observed data of surface temperature under these specific field conditions and for simplicity, however, the fundamental harmonic alone is appropriate. Allowing the amplitude to vary from day to day is more important than using higher order harmonics. The comparisons of measured temperature with temperature predicted analytically are shown in Fig. 3. Two objective quantitative measures, R^2 and root mean square error (RMSE) (Willmott et al., 1985), are used to estimate the accuracy of prediction. At a depth of 0.1 m (Fig. 3A), R^2 is 0.84 and RMSE is 1.64 (°C). At a depth of 0.2 m (Fig. 3B), R^2 is 0.68 and RMSE is 1.72 (°C). At a depth of 0.6 m (Fig. 3C), R^2 is 0.65 and RMSE is 1.19 (°C). Therefore, all RMSE values are within 2°C of the observations. For most of the time, the analytical solution predicts the temperatures within 2°C of the corresponding field-observed temperatures. Reasons for the discrepancies between the observed and simulated temperatures may be (i) the assumption of strictly one-dimensional heat transfer and (ii) parameter estimations. In reality, heat transfer under field conditions may be three dimensional; lateral heat transfer may occur. Water moving laterally carries heat laterally, resulting in decreased vertical heat transfer. This may explain why the analytical solution tends to overestimate soil temperatures for all depths. Actual measurement of the parameters in the coupled heat and water transfer should increase the accuracy of temperature prediction of the analytical solution. This does not affect the analytical approach for understanding the problem itself, however.

The analytical solution is sensitive to the K_{sr} of the soil. The K_{sr} affects the amplitude of the soil temperature

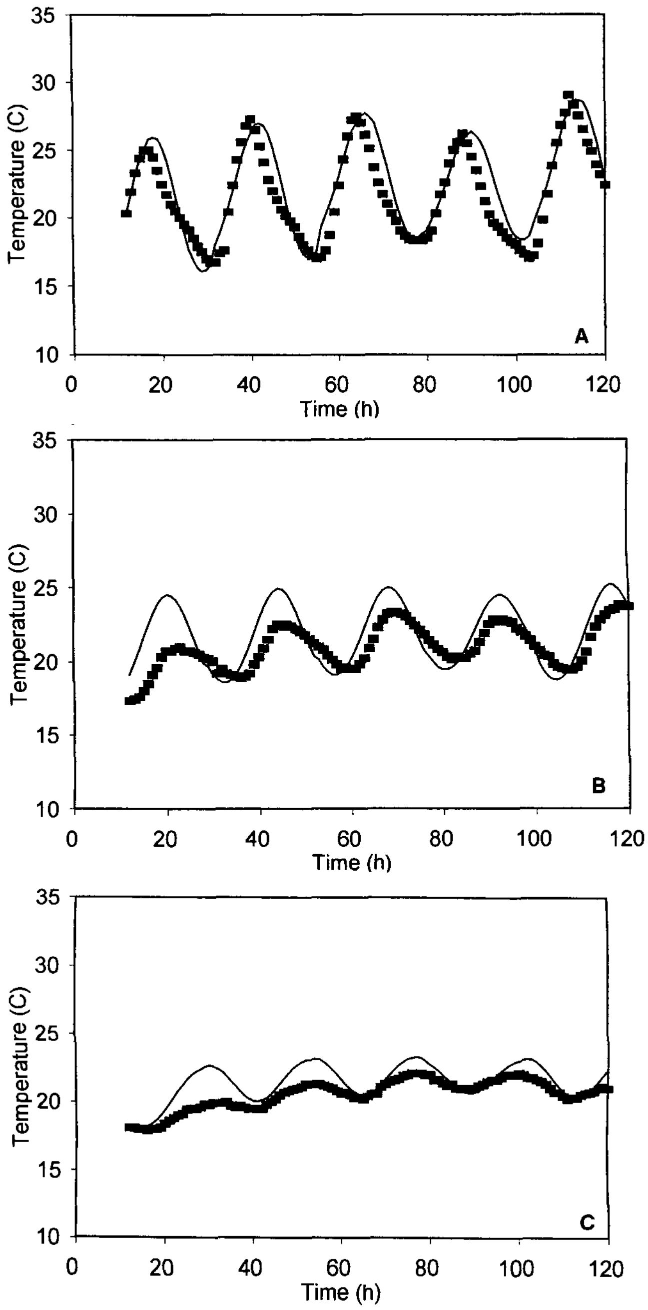

Fig. 3. Comparison of the analytical solution of the soil temperature (solid line) with the observed temperatures (filled squares) at (A) 0.1 m, (B) 0.3 m, and (C) 0.6 m.

at different depths, because percolating water carries heat down the soil profile. This effect is particularly important for the deeper depths. This can be shown by comparing the temperature profiles for conduction–convection (percolating water) vs. conduction (no water flow) alone. The results of the comparisons are given in Fig. 4A, 4B, and 4C. The temperature difference between the two mechanisms persists with depth. The convection affects not only the amplitude but also the mean temperature at the deeper depths.

Oscillating Surface Infiltration

The measured and predicted surface infiltration rates are shown in Fig. 5. The flux changed with time somewhat like a sine function. The reason is that surface temperature oscillates with time, causing the water viscosity to oscillate with time. As viscosity fluctuates, the K_{sr} fluctuates also. Specifically, in the daytime, the temperature of water on the soil surface increases, then water viscosity decreases; therefore, K_{sr} increases and so does the infiltration rate. A similar argument can be applied to the night.

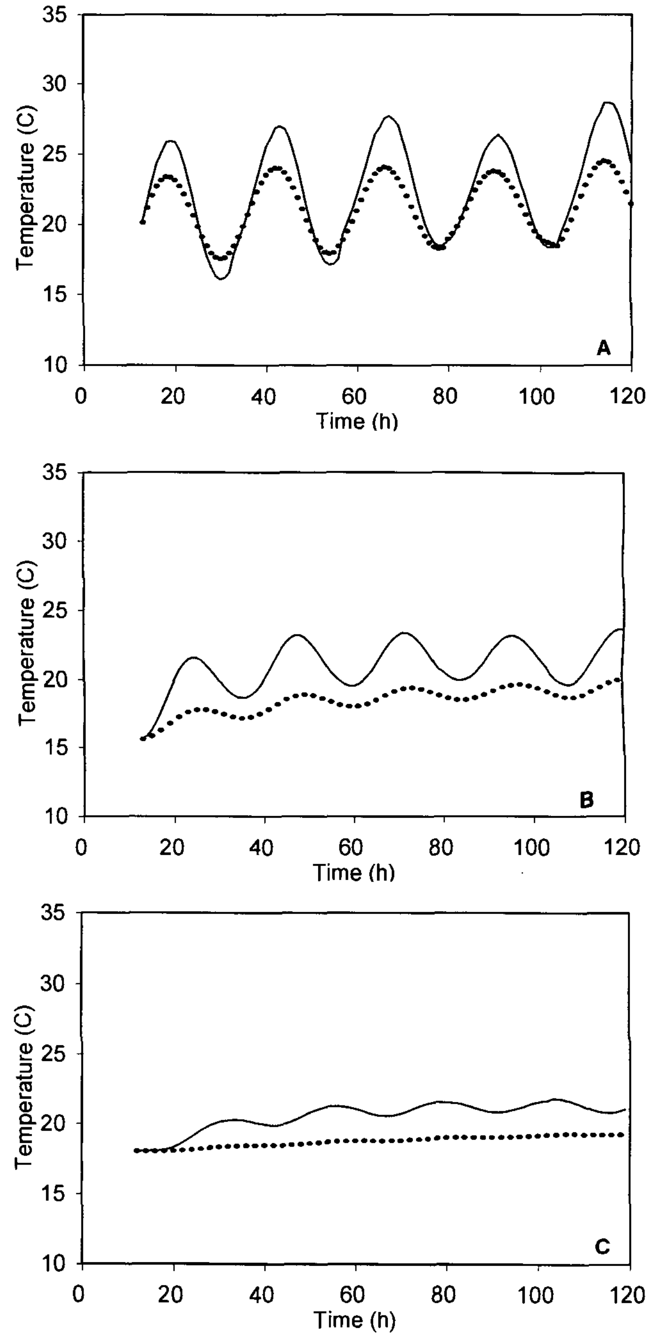

Fig. 4. Comparison of soil temperatures by conduction–convection (solid curve) with those by conduction alone (filled circles) at (A) 0.1 m, (B) 0.3 m, and (C) 0.6 m.

CONCLUSIONS

The analytical solution for coupled heat and water transfer under typical field initial and boundary conditions can be obtained by using some variable substitu-

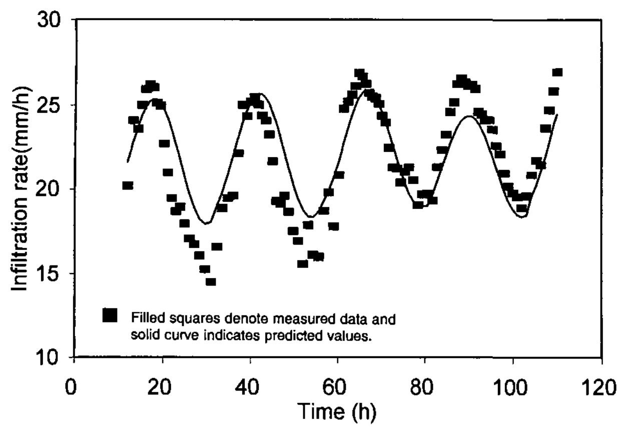

Fig. 5. **The change of infiltration rate (surface flux) with time.**

tions and Fourier transformation. The analytical procedure for the solution of the heat conduction–convection equation is straightforward and may be useful in checking coupled water and heat numerical procedures. The analytical solution improves our understanding of the coupled heat- and water-transfer problem. One example is the analysis of the relative importance between conductive heat transfer and convective heat transfer. Furthermore, the analytical solution itself may provide useful water and heat flux predictions for field conditions when significant water and heat transfer is occurring. For instance, we can use temperature profiles as an indicator for percolation rates of a streambed or seepage rates for a canal. Other appropriate field conditions for application are rainfall infiltration and flood irrigation.

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