INTRODUCTION

Using Newton's recently formulated laws of motion, Brook Taylor (1685-1721) discovered the wave equation by means of physical insight alone [1]. Daniel Bernouli (1700-1782) showed that an infinite summation of sinusoids can represent the general solution of the wave equation with given initial conditions [2]. Finally Jean Baptiste Joseph Fourier (1768-1830) showed that such an infinite sum, a Fourier series, can represent any discontinuous function under general conditions [3]. From this early work connecting the wave equation and the Fourier transform, much of engineering mathematics of wave motion and transformations has been developed.

Seismic data consist of recording of seismic waves (i.e., sound waves in the earth). The task of seismic analysis is to process the received data so as to obtain an image of the subsurface structure of the earth [4, 5]. In exploration geophysics manmade signals are sent downward from the surface of the earth where they encounter interfaces between subsurface rock strata. The signals are reflected at these discontinuities, and are returned to the surface where they are recorded. This received data are then computer processed in order to produce the final image. The basic mathematical technique makes use of the wave equation, but solves it in a reverse-time sense in order to downward extrapolate the surface wave-motion back into the subsurface. If the problem is solved in the space-time domain, then either the Kirchhoff integral solution of the wave equation [6], or the finite-difference approximation to the wave equation [7], can be used. However a more computationally efficient method makes use of Fourier techniques by transforming the whole problem to the frequency-wavenumber domain [8, 9]. In this paper we describe this Fourier method of imaging seismic waves to produce a picture of the geological structure of the Earth [10].

SEISMIC IMAGING BY MIGRATION

The purpose of reflection seismology is to determine the structure of the subsurface from seismic traces recorded at the surface. The recorded seismic data are subjected to various processing operations by means of digital computers in order to transform the data into a valid image of cross sections of the earth which can
be interpreted in geological terms. The operation that produces the final subsurface image in seismic data processing is called migration. The word *migration* refers to the movement, or migration, of the observed events on the stacked section to their true spatial positions.

The data processing operations prior to the migration process produce a seismic section known as the stacked section. The stacked section may be considered as a wave-field measured at the surface of the earth. Given the approximate velocity variations within the earth as given by the velocity function, the migration process downward continues this wavefield into the subsurface and thereby elucidates the sources of the reflected and diffracted seismic events. Therefore, migration is the inverse process in which the recorded seismic waves are depropagated (with time running backward) to the corresponding reflector locations. In the process of seismic data acquisition, the upward-travelling waves are recorded at the surface of the earth. In migration, these recorded waves, in the form of the stacked section, are used either as boundary conditions or initial conditions for a wavefield governed by the wave equation. Migration is the inverse propagation (or depropagation) process, which pushes these upgoing waves back into the earth in reverse time in order to arrive at the reflector locations.

**AN EXAMPLE**

Let us now give a simple example that explains what the mathematics is doing. Ocean waves have wavelengths comparable to the seismic waves used in petroleum exploration, but the velocity of ocean waves is much smaller so that they can be easily observed. Let us imagine a long straight beach which we take as the x-axis. We let the z-axis point directly seaward, with z = 0 corresponding to the beach line. For this simple example we still suppose that the ocean waves are sinusoidal with frequency \( w \), velocity \( v \), and direction of travel \( q \), all fixed. The angle \( q \) is measured with respect to the z-axis. We now suppose that someone on a ship at sea radios us on the beach that a large-crested wave passed his ship at \( t = 0 \). We observe that the same wave hitting the beach at \( t = t_0 \). The question is: What is the location (range and bearing) of his ship? The part of the question as to the range is easy. The range is \( R = vt_0 \), so the ship can be anywhere on a circle of radius \( R \) and center at our position on the beach. Let us now make our analogy with oil exploration. We think of the beach as the surface of the ground, and think of the ocean as the subsurface geological rock structure. The boat with unknown position corresponds to an unknown oil reservoir for which we are exploring. The ocean waves correspond to the seismic waves. By auxiliary means we can find the seismic wave velocity \( v \), and we can measure the arrival time \( t_0 \) (in this case, oneway time from depth to the surface of the earth) of the seismic wavelet due to the oil reservoir. Thus we can immediately determine the range \( R = vt_0 \) of the oil reservoir.

We thus know the range of the ship or the oil reservoir. With no other information, we cannot determine its bearing, so the bearing angle could be anywhere from -90° to 90°. The average value could be zero, so we could guess that the ship or oil reservoir made an angle of zero with the z-axis; that is, the ship or reservoir was at right angles to the beach or earth's surface from our observation position. An unmigrated seismic record makes this assumption; it puts the cause of each event directly under the position on the earth's surface where this event was observed. In other words, an unmigrated seismic record always draws each bearing as straight down into the earth.
The reasoning for performing the data processing operation of migration is to compute the true bearing for each event, and then put the cause of the event at the computed range in the direction of the computed bearing. Of course, if all the geologic rock layers are flat and horizontal, then indeed the seismic waves from an exploding reflector model go straight up (each with a bearing angle of zero degrees), and in such a case an unmigrated seismic section serves perfectly well. It also serves well provided the dips of the layers are all small and random. However, when there are many small dips all going in the same direction, or when there are some large dips as in the overthrust belt of the Rocky Mountains, or a combination of both, then an unmigrated seismic record section does not serve exploration well.

In order to find the bearing of the ship or the oil reservoir we must measure some additional quantities. On the beach we need a time piece and a measuring stick, and we must take measurements at two or more stations on shore. With the time piece we measure the time between two consecutive crests of the wave at a given station. This time measurement is the wave period $T$. With the meter stick we measure the distance between two stations where adjacent crests hit the beach at the same instant. This distance measurement is horizontal ($x$ coordinate) wavelength $\lambda_x$. With these two measurements we can determine the bearing $\theta$ as follows. First we must determine the wavelength of the wave. The wavelength $\lambda$ is given by $\lambda = vT$; that is, the wavelength is equal to the distance a crest travels during the elapsed time of one period. In Fig. 1, we see the necessary relationships. The bearing angle $\theta$ is given by,

$$\sin \theta = \frac{\lambda}{\lambda_x}$$  \hspace{1cm} (1)

This same principle applies to all migration schemes. In effect, we find the bearing angle $\theta$ and then backtrack along this bearing by letting time run backwards, from the arrival time $t_0$, to the source time $0$. When we reach time $0$, we know we have...
reached the source, and the total distance that we have backtracked is equal to the range $R = vt_0$. Thus we locate the source of the event seen on the seismic record section. This process is seismic migration, and it involves the depropagation of the seismic waves observed at the surface of the earth.

**COMPUTER IMPLEMENTATION**

Of course, migration must be done by a computer, and it involves many seismic waves coming from different directions and various arrival times. In order to make things mathematically tractable, we appeal to the power of the Fourier transform. When we do things in the frequency domain instead of the time domain, we use spatial frequencies (also called wavenumbers) instead of wavelengths, and we use temporal frequencies (simply called frequencies) instead of wave periods. The well-known relationships are

\[
\begin{align*}
    k &= \frac{2\pi}{\lambda} \\
    k_x &= \frac{2\pi}{\lambda_x} \\
    \omega &= \frac{2\pi}{T}
\end{align*}
\]

where $k$ is called the wavenumber, $k_x$ the horizontal wavenumber, and $\omega$ the frequency. We can also define the vertical wavenumber $k_z$ by the equation

\[k_z^2 = k^2 - k_x^2\]  

This equation says that $k_x$ and $k_y$ are the sides of a right triangle with hypotenuse $k$. The wavenumber $k$ is equal to

\[
k = \frac{2\pi}{\lambda} = \frac{2\pi}{vT} = \frac{\omega}{v}
\]

We define the propagation vector $\mathbf{k}$ pointing in the direction of the wave (i.e., $\mathbf{k}$ makes an angle $\theta$ with the $z$-axis) where $\mathbf{k}$ has length $k$ and components

\[
k_x = k \sin \theta \quad k_z = k \cos \theta.
\]

The seismic disturbance (wave motion) at any point $(x, z)$ at any time $t$ may be denoted by the symbol $u(x, z, t)$. The surface of the earth is given by depth $z = 0$, so the wave motion which we measure at the receivers on the ground is $u(x, 0, t)$. We can compute the two-dimensional Fourier transform of the observed wave motion $u(x, 0, t)$ with respect to $x$ and $t$ to obtain the surface wavefield spectrum

\[
U(k_x, 0, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, 0, t) \exp \left[-j(\omega t - k_x x)\right] \, dx \, dt.
\]

Our purpose is to take the wave motion associated with the sinusoidal wave characterized by $k_x$, $\omega$, and then depropagate this sinusoidal wave in the direction $\theta$ as determined by

\[\sin \theta = \frac{k_x}{k} = \frac{k_x v}{\omega} \]
The depropagation terminates when we reach the range distance $R = v t_0$.

The implementation of this depropagation scheme in the frequency domain is done as follows. The travelt ime $t$ has differential

$$dt = \frac{\partial t}{\partial x} \, dx + \frac{\partial t}{\partial z} \, dz.$$

The derivative $dx/dt$ is the horizontal apparent velocity, so

$$\lambda_x = \frac{dx}{dt} = T.$$

As a result we have

$$\frac{dt}{dx} = \frac{T}{\lambda_x} = \frac{2\pi}{\lambda_x} \frac{T}{2\pi} = \frac{k_x}{\omega}.$$  

Likewise

$$\frac{dt}{dz} = \frac{k_z}{\omega}.$$  

Thus the time differential is

$$dt = \frac{k_x}{\omega} \, dx + \frac{k_z}{\omega} \, dz. \quad (7)$$

When we depropagate by a time span of $t_0$, we let time run backward. This means in engineering terms that we must introduce a time advance of $t_0$. A time-advance operator is the pure phase-shift system given in the frequency domain by $\exp [j \omega t_0]$. Let the depropagation path be from the receiver point $(x, z = 0)$ on the surface of the earth to the source point at depth given by the point $(x = 0, z)$. Here we assume that time $t_0$ is the one-way time from source to receiver. Thus the time advance is equal to

$$\int_0^{t_0} dt = \frac{1}{\omega} \int_x^0 k_x \, dx + \frac{1}{\omega} \int_0^z k_z \, dz$$

which (for the constant-velocity medium treated here) is

$$\omega t_0 = -k_x x + k_x z.$$  

The observed seismic wave motion has sinusoidal component

$$U(k_x, 0, \omega) e^{j \omega t}.$$  

We multiply this component by the phase-shift (pure advance) filter $\exp [j \omega t_0]$ to obtain

$$U(k_x, 0, \omega) \exp [j \omega (t + t_0)] = U(k_x, 0, \omega) \exp [j (\omega t - k_x x + k_x z)].$$  

This expression gives the depropagating sinusoidal wave. We thus integrate this
expression over \( k_x \) and \( w \) to obtain the depropagating wave. We remember that \( k_z \) is not an independent variable, but is given by the positive square root.

\[
k_z = + \left[ k^2 - k_x^2 \right]^{1/2} = + \left[ (\omega/v)^2 - k_x^2 \right]^{1/2}.
\]

In fact, it is this link of \( k_z \) to \( k_x \) and \( w \) that makes the operation of depropagation possible. Thus the required integral is

\[
u(x, z, t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(k_x, 0, \omega) \exp \left\{ j \left[ (\omega/v)^2 - k_x^2 \right]^{1/2} z \right\} \cdot \exp \left\{ j(\omega t - k_x x) \right\} dk_x \, d\omega.
\]

This integral is the inverse Fourier transform of

\[
U(k_x, 0, \omega) \exp \left\{ j \left[ (\omega/v)^2 - k_x^2 \right]^{1/2} z \right\}.
\]

Thus depropagation is achieved by multiplying the surface wavefield spectrum by the filter

\[
\exp \left\{ j \left[ (\omega/v)^2 - k_x^2 \right]^{1/2} z \right\}.
\]

It is for this reason that this filter is called the depropagation (or migration) filter.

We have thus found the wave field \( u(x, z, t) \) at an arbitrary space-time point \((x, z, t)\) by depropagation. In other words, we have found the correct bearings. This first step of the depropagation process is called wavefield reconstruction. The second step of the depropagation process involves stopping at the correct range. We recall that the signal originated at the source at time \( t = 0 \). Thus we set \( t = 0 \) in (9), and thus obtain the final answer.

\[
u(x, z, 0) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(k_x, 0, \omega) \exp \left\{ j \left[ (\omega/v)^2 - k_x^2 \right]^{1/2} z \right\} \cdot \exp \left\{ -(jk_x x) \right\} dk_x \, d\omega.
\]

This second step is called imaging, as it gives the required sources.

In our discussion up to this point we have assumed that the velocity \( v \) is constant. A usual geophysical assumption, however, is the stratified earth assumption in which we assume that \( v \) varies in the depth direction but not in the horizontal direction, so we write \( v(z) \) indicating that the velocity is a function of depth. In the stratified case, the depropagation operator (10) becomes

\[
\exp \left[ j \int_0^z k_z(z) \, dz \right] = \exp \left[ j \int_0^z \left\{ \frac{\omega}{v(z)} \right\}^2 - k_x^2 \right]^{1/2} \, dz.
\]
CONCLUSION

The geophysical processing operation known as migration represents a method of depropagating (or backtracking) seismic waves. The paper contains a simplified example and then gives the mathematics describing the migration process. This mathematics is based on the use of Fourier methods to solve the wave equation, which originally was put forth in the classical work of Taylor, Bernoulli, and Fourier.

REFERENCES

2. D. Bernoulli, Hydrodynamics (Basel, Switzerland, 1738)
3. J. Fourier, Théorie Analytique de la Chaleur (Didot, Paris, 1822)