LOW FREQUENCY SCATTERING BY A PLANAR CRACK

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INTRODUCTION

The detection of cracks with the aid of ultrasonics is an important nondestructive evaluation technique. The corresponding theoretical problem of the scattering of elastic waves by cracks has attracted considerable attention. Scattering of time harmonic plane wave by an isolated two dimensional Griffith, or an penny-shaped crack in an unbounded elastic medium has been studied extensively. However, studies of the scattering problem by a three dimensional crack other than circular shape have been rather limited. Few studies of scattering from an elliptical crack in an elastic body of infinite extent can be found in the literature. Datta[1] studied the problem using the method of matched asymptotic expansion. Gubernatis et al. [2] and Budiansky and O'Connell [3] have used the elastostatic approximation to determine the scattered field. The backscattered field from an elliptical crack has been obtained by Kino [4] in the low frequency limit by a formula derived from elastodynamic reciprocity theorem. An integro-differential equation technique was employed by Roy [5]-[6] to study the same problem.

In this paper, a method to obtain the low frequency asymptotic solution to the problem of scattering of elastic waves by a planar crack of arbitrary shape is described in the spirit of the low frequency scattering theory [7]. In the low frequency regime, a formal series solution in the power of the non-dimensionalized wavenumber is proposed. Each coefficient of this series is the solution to the problem of a crack subject to static surface loading. Thus the scattering problem is reduced to a sequence of static equilibrium problems, which are somewhat easier to deal with. Furthermore, this method has the potential to be applied to cracks in anisotropic solids.

The general formulation is developed in section 2. In section 3, an elliptical crack under normal incidence of a

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longitudinal time harmonic plane wave is considered. The low-frequency expansions of the crack opening displacement and the dynamic stress intensity factor are obtained. Explicit expressions are given for the first two non-zero terms in these expansions. Plots of the dynamic stress intensity factor versus the polar angle for various incident frequencies are also presented.

GENERAL FORMULATION

Consider a planar crack in an elastic solid of infinite extent. Without loss of generality, we can assume that the crack occupies a finite region S in the plane $x_3 = 0$. A time-harmonic plane wave is incident from infinity. The equation governing the total displacement field (incident field plus scattered field) u_i is given, omitting the time factor $\exp(-i\omega t)$, by

$$C_{ijkl}u_{k',lj} + \varepsilon^2 u_i = 0 \quad , \tag{1}$$

where $\varepsilon = \omega\rho^{1/2}$, ω is the angular frequency of the incident wave and ρ the mass density of the solid. In (1) C_{ijkl} is the elastic constant tensor of the solid. The traction free condition on the crack faces implies

$$C_{i3kl}u_{k',l} = 0, \qquad \underset{\sim}{x} \in S \tag{2}$$

where $\underset{\sim}{x} = (x_1, x_2, x_3)$.

Using Green's formula (or the Betti-Rayleigh reciprocal theorem) yields (see [8], p.34)

$$u_i(\underset{\sim}{x}) = v_i(\underset{\sim}{x}) - \iint_S T_{im3}(\underset{\sim}{x}-\underset{\sim}{x}_o)\Delta u_m(\underset{\sim}{x}_o)ds_o \quad , \tag{3}$$

where

$$\Delta u_m = u_m \big|_{S^+} - u_m \big|_{S^-} \quad , \tag{4}$$

$$T_{mij} = C_{ijkl}g_{mk',l} \tag{5}$$

in which g_{mk} is the three dimensional steady state elastodynamic Green's tensor and v_i is the displacement field of the incident wave.

Now let's assume that the solution to the boundary value problem (1)-(2), considered as a function of ε, is analytic in a neighborhood of $\varepsilon = 0$, so that it can be expanded into a convergent power series of ε:

$$u_i(\underset{\sim}{x}) = \sum_{n=o}^{\infty} u_i^{(n)}(\underset{\sim}{x})\varepsilon^n/n! \quad . \tag{6}$$

Similarly, one can also expand the incident displacement v_i and the traction Green's tensor T_{mij} to give

$$v_i(\underset{\sim}{x}) = \sum_{n=o}^{\infty} v_i^{(n)}(\underset{\sim}{x})\varepsilon^n/n! \quad , \tag{7}$$

$$T_{mij}(\underset{\sim}{x}) = \sum_{n=o}^{\infty} T_{mij}^{(n)}(\underset{\sim}{x})\varepsilon^n/n! \quad . \tag{8}$$

It follows from substituting (6) into (1) and (2) that

$$C_{ijkl}u^{(n)}_{k,lj} = -n(n-1)u^{(n-2)}_i \quad , \tag{9}$$

$$C_{i3kl}u^{(n)}_{k,l} = 0, \qquad \underset{\sim}{x} \in S \quad . \tag{10}$$

The integral relations among the coefficients are obtained by substituting (6)-(8) into (3)

$$u^{(n)}_i(\underset{\sim}{x}) = v^{(n)}_i(\underset{\sim}{x}) - \sum_{m=o}^{n} \frac{n!}{m!(n-m)!} \iint_S T^{(n-m)}_{ij3}(\underset{\sim}{x}-\underset{\sim}{x}_o)\Delta u^{(m)}_j(\underset{\sim}{x}_o)ds_o \quad . \tag{11}$$

It is easy to show, by direct substitution, that

$$P^{(o)}_i(\underset{\sim}{x}) = 0 \quad , \quad P^{(1)}_i(\underset{\sim}{x}) = v^{(1)}_i(\underset{\sim}{x}) \quad , \tag{12a,b}$$

$$P^{(n)}_i(\underset{\sim}{x}) = v^{(n)}_i(\underset{\sim}{x}) - \sum_{m=o}^{n-1} \frac{n!}{m!(n-m)!} \iint_S T^{(n-m)}_{ij3}(\underset{\sim}{x}-\underset{\sim}{x}_o)\Delta u^{(m)}_j(\underset{\sim}{x}_o)ds_o$$

$$n = 2,3,\ldots \tag{13}$$

are the particular solutions to (9), i.e.

$$C_{ijkl}P^{(n)}_{k,lj} = -n(n-1)u^{(n-2)}_i \quad . \tag{14}$$

Thus, the solutions to the boundary value problem (1)-(2) can be written as

$$u^{(n)}_i = P^{(n)}_i + U^{(n)}_i \quad , \tag{15}$$

where $U^{(n)}_i$ satisfies the following boundary value problem

$$C_{ijkl}U^{(n)}_{k,lj} = 0 \quad , \tag{16}$$

$$C_{i3kl}U^{(n)}_{k,l} = -C_{i3kl}P^{(n)}_{k,l} \quad , \qquad \underset{\sim}{x} \in S \quad . \tag{17}$$

Equations (12)-(17) build up a hierarchy of equations to compute $u^{(n)}_i$. In fact once (16)-(17) have been solved for the n'th step in terms of $P^{(n)}_i$ (assuming $P^{(n)}_i$ is known), $P^{(n+1)}_i$ can be calculated from (13). Since the initial terms $P^{(o)}_i$ and $P^{(1)}_i$ are both known from the incident field, one can then find all the coefficients $u^{(n)}_i$ by repeatedly solving (16)-(17).

Clearly, the feasibility of the above iteration process depends on the solvability of the boundary value problem (16)-(17). However, it is very helpful to recognize that the solution to (16)-(17) is nothing but the one for a planar crack subjected to static surface traction of magnitude $-C_{i3kl}P^{(n)}_{k,l}$. To illustrate the above formulation, an example will be given in the next section.

As a final remark, we notice that the above formulation was developed for materials that obey the general Hooke's law. Therefore, it is valid for to both isotropic and anisotropic solids.

ELLIPTICAL CRACK

Consider an elliptical crack S, where

$$S=\{ \underset{\sim}{x} \mid (x_1/a)^2 + (x_2/b)^2 \leq 1,\ x_3=0\},\quad 0 < b \leq a < \infty .$$

The crack is contained in an infinite isotropic and linearly elastic solid, characterized by the Lamé constants λ and μ, and the mass density ρ. The incident wave is assumed to be a longitudinal time harmonic plane wave whose propagation vector is normal to the crack faces:

$$v_i(\underset{\sim}{x}) = \delta_{i3}\exp(ik_L x_3) \quad , \tag{18}$$

where δ_{ij} is the Kronecker delta function and the longitudinal wave number k_L is given by

$$k_L = \omega/c_L \ , \quad c_L^2 = (\lambda+2\mu)/\rho \quad . \tag{19}$$

The Green's tensor for an isotropic solid can be found in a variety of sources (see [9]-[10])

$$g_{ij} = (4\pi\varepsilon^2 x)^{-1}\{\beta^2\varepsilon^3\delta_{ij}e^{i\beta x} - x[e^{i\alpha x}/x - e^{i\beta x}/x]_{,ij}\} \quad , \tag{20}$$

where

$$x^2 = (x_i x_j \delta_{ij}) \quad , \ \alpha^2 = 1/(\lambda+2\mu) \quad , \ \beta^2 = 1/\mu \quad . \tag{21}$$

It follows from (7)-(8) that

$$v_j^{(n)} = \delta_{j3}(ix_3/c_L\rho^{1/2})^n \ , \tag{22}$$

and

$$\begin{aligned} T_{mij}^{(n)} = {} & [i^n\beta^n(n-1)/4\pi]\{[(\lambda/\mu)\tau^{n+2} \\ & + 2(\tau^{n+2} - 1)/(n+2)]x_m x^{n-3}\delta_{ij} + \\ & + [n+2\tau^{n+2}/(n+2)][x_i\delta_{mj} + x_j\delta_{mi}]x^{n-3} \\ & + [2(n-3)/(n+2)][\tau^{n+2}-1)x_i x_j x_m x^{-5}\} \quad , \end{aligned} \tag{23}$$

where

$$\tau = \alpha/\beta < 1 \quad . \tag{24}$$

According to (12)

$$P_j^{(o)} = 0 \ , \quad P_j^{(1)} = ix_3\alpha\delta_{j3} \quad . \tag{25}$$

By substituting (25) into (16)-(17) for n = 0 and n = 1, respectively, we obtained $U_i^{(o)} = 0$ and

$$C_{ijkl}U_{k'lj}^{(1)} = 0 \quad , \tag{26}$$

$$C_{i3kl}U^{(1)}_{k'l} = -i\alpha\delta_{i3} \quad , \qquad \underset{\sim}{x} \in S \quad . \tag{27}$$

This is the boundary value problem that corresponds to the solution of an elliptical crack under uniform loading over the crack face. Equations similar to (26)-(27) have been solved by Green and Sneddon [11]

$$\Delta U^{(1)} = i\alpha b\delta_{i3}[(1-\tau^2)\mu E(\sigma)]^{-1}\{1-(x_1/a)^2-(x_2/b)^2\}^{1/2} \quad , \tag{28}$$

where $E(\sigma)$ is the complete elliptical integral of the second kind with modulus $\sigma = [1-(b/a)^2]^{1/2}$.

Using (13) and (16)-(17), we find for n = 2

$$P^{(1)}_j = -\delta_{j3}x_3^2/\rho c_L \quad , \quad U^{(2)}_j = 0 \quad . \tag{29}$$

For n = 3, (13) gives

$$P^{(3)}_j = V^{(3)}_j - 3\iint_S T^{(3)}_{jm3}\Delta u^{(1)}_m ds_o \quad . \tag{30}$$

Substituting (30) into (17) yields the following boundary value problem for $U^{(3)}_i$

$$C_{ijkl}U^{(1)}_{k'lj} = 0 \quad , \tag{31}$$

$$C_{i3kl}U^{(1)}_{k'l} = -i\alpha D\delta_{i3}I(x_1,x_2) \quad , \qquad \underset{\sim}{x} \in S \quad . \tag{32}$$

where

$$D = b[3(3\tau^4-4\tau^2+3)][\mu(1-\tau^2)E(\sigma)]/4\pi \quad , \tag{33}$$

$$I(x_1,x_2) = \iint_S [1-(\frac{x}{a})^2-(\frac{y}{b})^2]^{1/2}[(x-x_1)^2+(y-x_2)^2]^{-1/2}dxdy$$

$$= (b\pi^2/2)[s_1-s_2(x_1/a)^2-s_3(x_2/b)^2] \quad . \tag{34}$$

The definitions of s_j (j=1,2,3) are given in the Appendix and the details of the integration in (34) can be found in [12].

The solution to (31)-(32) is given by Kassir and Sih [13]

$$\Delta U^{(3)} = \frac{i\alpha b\delta_{j3}D\pi^2}{1-\tau^2}\left(1-\frac{x_1^2}{a^2}-\frac{x_2^2}{b^2}\right)^{1/2}\left\{\frac{bs_1}{2\mu E(\sigma)}-\frac{4A_o}{ab}\right.$$

$$\left.-\frac{32}{a^3b}\left(1-\frac{4x_1^2}{a^2}-\frac{x_2^2}{b^2}\right)A_1+\frac{32}{ab^3}\left(1-\frac{x_1^2}{a^2}-\frac{4x_2^2}{b^2}\right)A_2\right\} \quad , \tag{35}$$

were A_i (i=0,1,2) are constants involving the material properties and the geometry of the crack (see the Appendix).

Now, by (6) and (15), the crack opening displacement can be approximated by the first two non-zero terms in the low frequency expansion

$$\Delta u_i = \{\Delta U^{(1)}_i\varepsilon + (1/6)\Delta U^{(3)}_i\varepsilon^3\}\delta_{i3} + o(\varepsilon^3) \tag{36}$$

with $\Delta U_i^{(1)}$ and $\Delta U_i^{(3)}$ given by (28) and (35), respectively.

The stress intensity factors associated with $u_i^{(1)}$ and $u_i^{(3)}$ can be found in [14]

$$K_I^{(1)} = [i\alpha/E(\sigma)](b/a)^{1/2}(a^2\sin^2\theta + b^2\cos^2\theta)^{1/4} \quad , \qquad (37)$$

$$K_I^{(3)} = K_I^{(1)} Db\{[240\mu E(\sigma)/ab^2][A_1(\cos\theta/a)^2 + A_2(\sin\theta/b)^2] + \pi^2 s_1/2\} \quad , \qquad (38)$$

where θ is the polar coordinate defined by

$$x_1 = ar\cos\theta \quad , \quad x_2 = br\sin\theta \quad .$$

It follows from (36) that the dynamic stress intensity factor is given approximately by

$$K_I = K_I^{(1)}\varepsilon + K_I^{(3)}\varepsilon^3/6 + o(\varepsilon^3) \quad . \qquad (39)$$

The amplitude of K_I given in (39), normalized by the value of corresponding static stress intensity factor at $\theta = \pi/2$, is plotted in Fig. 1 for an elliptical crack of b/a = 0.7.

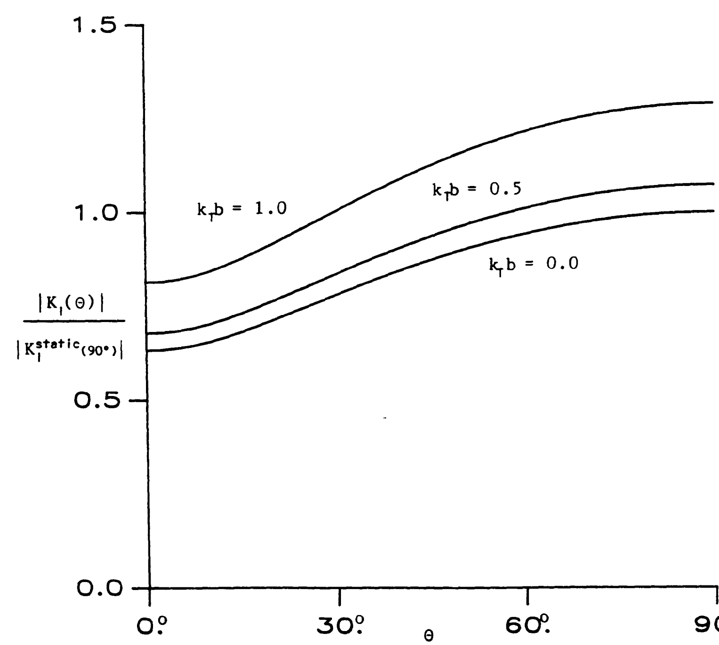

Fig. 1 Normalized amplitude of the stress intensity factor of an elliptical crack (b/a=0.7) versus polar angle θ for normal incidence of a longitudinal wave.

A partial check on the low frequency solutions (36) and (39) is possible by taking the limit when the ellipse degenerates into a circle, i.e., when b -> a. The limiting values are

$$\sigma = 0 \quad , \quad E(\sigma) = \pi/2 \; ,$$

$$A_o = a^3/20\mu\pi \quad , \quad A_1 = A_2 = a^5/720\mu\pi \; .$$

The corresponding crack opening displacement reduces to

$$\Delta u_3 = \frac{2i\alpha\varepsilon}{\pi\mu(1-\tau^2)}(a^2-r^2)^{1/2}\{1+ \frac{\varepsilon^2 a^2[3(\tau^2-1) + 2\tau^2]}{36\mu(1-\tau^2)}(4- \frac{r^2}{a^2})\} \qquad (40)$$

and the stress intensity factor becomes

$$K_I = \frac{2i\alpha\varepsilon a^{1/2}}{\pi}\{1 + \frac{a^2\varepsilon^2}{12\mu(1-\tau^2)}(3\tau^4 + 4\tau^2 + 3)\} \; . \qquad (41)$$

Equations (40) and (41) are in agreement with similar expressions given by Robertson [15] and Mal [16], respectively.

SUMMARY

The problem of elastic wave scattering by a planar crack has been considered in the low frequency limit. The total displacement field is expanded into a power series of the wave number. Each coefficient of the expansion is the solution to the same crack under prescribed static loading. Thus, the scattering problem is transformed into a series of static problems. In the case of an elliptical crack, solutions for static loading of polynomial distribution are available in the literature. Therefore, the low frequency solutions have been easily constructed without the need of solving any new equations.

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APPENDIX

$$s_1 = (2/\pi)K(\sigma) \quad , \quad s_2 = (2/\pi\sigma^2)[K(\sigma)-E(\sigma)]$$

$$s_3 = (2/\pi\sigma^2)[E(\sigma)-(b/a)^2K(\sigma)]$$

$$A_1 = [s_2(J_{12} + 5J_{o3}) - (a/b)^2 s_3(J_{21} + J_{12})]a^5/288\mu\Delta$$

$$A_2 = [-s_2(J_{12} + J_{21}) + (a/b)^2 s_3(J_{21} + 5J_{3o})]a^5/288\mu\Delta$$

$$A_3 = 12[A_1(J_{11} + 3J_{2o}) + A_2(J_{11} + 3J_{o2})]a^2/(J_{o1} + J_{1o})$$

$$\Delta = (J_{21} + 5J_{3o})(J_{12} + 5J_{o3}) - (J_{12} + J_{21})^2$$

$$J_{o1} = [(\sigma^2 - 1)K(\sigma) + E(\sigma)]a^2/b^2\sigma^2$$

$$J_{1o} = [K(\sigma) - E(\sigma)]/\sigma^2$$

$$J_{12} = \{(8-9\sigma^2)K(\sigma) - [10-3\sigma^2-(2a^2/b^2)]E(\sigma)\}a^2/3b^2\sigma^6$$

$$J_{21} = \{-(8 + \sigma^2)K(\sigma) + [5 + 2\sigma^2 + (3a^2/b^2)]E(\sigma)\}/3\sigma^6$$

$$J_{o2} = \{-2(1 - 2\sigma^2)E(\sigma) + (2 - 3\sigma^2)(2a^2/b^2)K(\sigma)\}a^4/3b^4\sigma^4$$

$$J_{2o} = [(2 + \sigma^2)K(\sigma) - 2(1 + \sigma^2)E(\sigma)]/3\sigma^4$$

$$J_{o3} = \{-(8-19\sigma^2+15\sigma^4)K(\sigma)-[23\sigma^2-(8a^2/b^2)]E(\sigma)\}a^2/15b^2\sigma^6$$

$$J_{3o} = \{(8 + 3\sigma^2+ 4\sigma^4)K(\sigma) - [8 + 7\sigma^2 + 8\sigma^4]E(\sigma)\}a^2/15\sigma^6$$

where $K(\sigma)$ and $E(\sigma)$ are the complete elliptical integrals of the first and second kind, respectively.