Extraction methods for multidirectional driving point accelerance and transfer point accelerance matrices

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Extraction methods for multidirectional driving point accelerance
and transfer point accelerance matrices

by

Jianrong Dong

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

Department: Aerospace Engineering and Engineering Mechanics
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Graduate College

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For the Major Program

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For the Graduate College
Dedication

This dissertation is dedicated to

Liming, Richard and ???

for our wonderful lives in Ames
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CHAPTER 1 INTRODUCTION TO SUBSTRUCTURE TESTING

Introduction

The introduction of foreign competition to the U.S. auto market in the early 1970's has had a major impact on the automotive industry. The competitiveness of the market has challenged both manufacturers and their component suppliers to continually improve the quality of their products. An increasing measure of vehicle quality is how a vehicle vibrates and sounds to the passengers riding inside. The vibration and noise level experienced when the car is started, running and braked often determines whether a customer is willing to buy the car. NVH, the abbreviation of Noise, Vibration and Harshness, has become a hot area of research in the auto industry today.

In order to reduce the cost, the auto giants like Ford, GM and Chrysler now require their part suppliers to undertake more research and testing efforts of their products than ever before. The brand manufacturers themselves only make the body in white, such as the chassis, and have become more assembly oriented, focusing on the synthesized behavior of the completed car.

In the design stage, the finite element method (FEM) and the boundary element method (BEM) have provided analytical tools that can be used to model vibration and structure-borne noise in automotive applications. However, the accuracy of these models has been limited to several hundred Hz or so in most cases. The cost required to extend the accuracy of FEM and BEM models to higher frequencies is prohibitive. Testing methods offer an approach that is potentially less costly and more accurate than the modeling methods [46]. Traditionally the verification of a new design has been done by trial and error. After
all the parts from the suppliers are assembled, various vibration and sound quality tests must be performed to check the quality of the vehicle. When problems are identified, some parts need to be redesigned and likely other related parts need to be adjusted as well. This final stage do-it-again method is clearly inefficient for both time and cost. From the supplier's point of view, its engineers have considerable experience on the dynamic characteristics of the part, such as the range of natural frequencies, damping etc., for a series of products. Besides they know what variations in design can be made in order to achieve certain design goals when the part stands alone. But they do not know the part's in-situ behavior until it is assembled onto a specific vehicle and the whole vehicle test is done. If the in-situ behavior of the part could be predicted based on its finite element model in its design stage, or based on its testing data after the part is manufactured, they would be able to modify its design well in advance.

From the assembler's point of view, their engineers have accumulated a lot of data on the behavior of various white bodies or bare vehicles. But they do not know the performance of a fully loaded car until the whole car test is done. If the in-situ behavior of the fully loaded car could be predicted, given the necessary parts information in the form of a finite element model or some test data on the part, they would be able to do the modification on the body part before the car is finally assembled.

Moreover, after either party made modifications, they need to assemble the car and go through another round of the trial and error process. The question is, can they evaluate the design changes on individual components without having to assemble the overall system again and again? With the great amount of time and money saved, the importance of substructure testing is apparent.
Another important problem is the structural modification problem. The current test processing software like ME-Scope [54] provides limited structural modification capability provided the raw test data is given. The change is limited to add a linear spring or a linear damper to a point on the structure. However in reality, structural modification can be more general. A bar can be bolted, welded, or riveted to a point. If the test data for the original system is available, and we know the dynamics of the added bar when it is not mounted, can we predict the performance of the modified system? Or do we have to mount it again and again?

From the academic point of view, these problems all lead to a substructure testing and analysis method. Among various substructure formulations, Varoto and McConnell proposed READI (the Rules for the Exchange and Analysis of Dynamic Information) [42][43][53]. READI is a general-purpose substructure testing and analysis method formulated in frequency domain. With the finite element formulation, it aims to obtain the combined system response of the test item based on test item and the vehicle test/analysis information. It indicates the parameters to be measured or calculated, and provides a computation algorithm to obtain the dynamic characteristics of the combined system.

But is the current testing capability sufficient for READI or other substructure testing methods to be applied to sophisticated industrial applications? Are they getting the same success as their finite element counterparts where the interface motion continuity and action balance can be taken for granted?

Several years ago, Dr. Mark French, Dr. McConnell and the author worked on a substructure test project with the car seat and the body in white. The aim was to set up the test standard, take the data on the car seat and the bare vehicle respectively, apply READI to
predict the in-situ behavior of the car seat, and check the prediction with in-situ testing data on the combined structure. Due to the lack of certain instruments to measure angular acceleration and interface moments, only the normal accelerance of the attachment interface for the car seat and the chassis were measured. The accelerance is defined as the ratio of acceleration divided by force in the frequency domain. It is one of the frequency response functions (FRF). The accelerance normal to the interface is used to form the intermediate matrices required by READI. It turned out the prediction deviated from the result of the in-situ test on the combined structure. The main reason was that there were four interface attachment surfaces that were all bolted together. These interface connections have significant moments and shear forces acting on these connection surfaces. Therefore, any calculation model that neglects the degrees of freedom (DOF) corresponding to the shear force and moments is inaccurate.

In fact our inability to obtain the accelerances that correspond to the tangential and rotational degrees of freedom is the bottleneck for the application of READI in the real world. Unfortunately the rotational degrees of freedom accelerances are a notoriously difficult problem [20]. For a small contacting surface, there are three linear accelerations and three rotational accelerations. Correspondingly, there are three linear forces and three rotational moments. Thus, 50% of all coordinates are rotations (as opposed to translations) and 75% of all frequency response functions involve rotational information. However, it is extremely rare to find any reference to methods for the measurement of rotational accelerations since almost none are made. This situation arises since it is extremely difficult to measure either rotational responses or excitations such as moments [20].
During a trip to Detroit, the author learned from Dr. Glen Steyer, technical director of the Noise and Vibration Division of MTS Systems, that the current substructure testing practice in the automobile industry is still limited to those special components having linear springs at connection points. In other words, the complexity involved in dealing with rotational degrees of freedom is carefully avoided in practice. There appears to be approximately 30-year lag between experimental mechanics and the finite element method in fully implementing the substructure method.

In this dissertation, the author investigates the importance of the ignored DOF terms, develops an original method to obtain the full interface accelerance FRF matrices, and provides a prototype design of a new concept force-moment transducer together with its corresponding analysis software. This approach shows a promising avenue for solving the problem of substructure testing with rotational degrees of freedom in the near future.

Literature Review

Before plunging into details, let us define some technical terms since a lot of terms used daily in vibration testing were originally borrowed from electronics and acoustics. Consequently, the same word can have several completely different meanings dependent on the sentence’s context. In the frequency domain, accelerance is defined as the acceleration divided by force; mobility is defined as the velocity divided by force; receptance is defined as the displacement divided by force. Their reciprocals are termed as dynamic force, impedance, and dynamic stiffness [43].

Accelerance is classified according to the relative location of the input and output points. If the input and output are measured at different locations, the measurement is called
a transfer accelerance. If the input and output are measured at the same point and in the same direction, the measurement is termed a driving point accelerance. If both the input and the output are obtained at the same point but in different directions, the measurement is termed driving point cross accelerance.

Component Modal Synthesis

The substructure method itself is not new. Historically, shortly after the birth of finite element methods (FEM) in the 1960s, many substructure methods were proposed to solve the limited computer capacity problem. Instead of solving a huge model, they tried to solve a series of smaller problems [4][9][10][11][12][13][31]. Such developments led to a series of analysis methods often referred to as “Component modal synthesis methods” which sought to reduce the overall system model to the most compact form possible – the Modal Model. A modal model is a mathematical representation of a structure or system based on natural frequency and model shape information. Modal models can be developed from lumped parameter or analytical models as well as from modal test data [20][35]. The component modal synthesis methods couple the structures by enforcing constraint relationships using modal coordinates.

Component modal synthesis was pioneered by Hurty in the 1960's [31]. Advances by other investigators have resulted in techniques that can be used to couple structures, including techniques that utilize experimental data [12][36]. The accuracy of component models based on modal coordinate techniques will depend on the accuracy of the modal data. When the modal data is accurate, convergence is achieved by increasing the number of modes included in the model. Out of range higher frequency modes, called residuals, are
generally important to the overall model. Although some of the formulations are suitable for
finite element analysis, experimentally they are very difficult to implement because some of
the more refined methods employ models of the substructures with different boundary
conditions, many of which are not practical to use [20].

**Mobility Techniques**

Another group of substructure formulation is to use physical coordinates. These
physical coordinate methods are based on principles similar to electrical circuit analysis; and
hence, are called Mobility Techniques. Velocity and force in a mechanical mobility analysis
are analogous to voltage and current, respectively, in an electrical circuit analysis. However
in an electrical circuit each node has only one degree of freedom (DOF) while a structural
node can have up to six DOF's, in the most general case. In 1960s, O'Hara developed a
substructure test scheme based on mobility techniques [45]. The basis for this technique is to
model the individual components with matrices of frequency response function (FRF).
These FRF’s model the input/output relationships between the various input, output and
connection terminals of each component. Force equilibrium and motion continuity at the
interface is used to couple the two structures. The term “mobility” refers to a frequency
response function (FRF) that describes the magnitude and phase relationship between the
input and output of a linear system. It is a vague concept that can refer to accelerance,
mobility, receptance etc.

The mobility technique has potential benefits over the more traditional methods based
on modal coupling. An FRF can always be obtained where classical normal mode techniques
are impractical. For example, it may be difficult to obtain a modal model of a component
with high modal density or with modes that are well damped and closely coupled, especially with experimental data. Another advantage of using measured FRF directly is that the contribution from all participating modes is taken into account by the response measurement and important residual mode information is automatically included in the model. These FRF measurements have become more attractive with the availability of multi-channel data acquisition systems.

Numerous substructure coupling methods with the physical coordinates have been proposed. Ren and Beard proposed some criteria to justify an algorithm and based on them, developed a new coupling method to satisfy both the physical and mathematical generality, and to retain the computation efficiency [48]. If no optimization such as the least-squares algorithm is involved in the coupling process, the final results from different receptance coupling methods are mathematically exactly the same. Therefore in reality, the coupling methods can be different in computation errors, computation speed, and memory requirement. Accuracy (sensitivity to noise), efficiency (computer time and human time), simplicity (same formula for different problems), and generality (no severe restrictions) are the four criteria they proposed. The new coupling method is called the Generalized Receptance Coupling method (GRC). GRC is a computationally efficient and a physically generalized method.

Recently Varoto and McConnell formulated a frequency substructure method called READI (Rules for Exchanging Analysis of the Dynamic Information [42][43][53]. The contribution of this work is not limited to a detailed formulation of a frequency domain substructure method. It also provides a framework to guide substructure testing in the lab to mimic the field environment. The work contributes to the various methods that can be used
to identify the interface forces for the test item under field environment in order to control the 
exciter to reproduce the force in the lab. The numerical examples of the work were 
performed on a mass-spring-dash pot model. Although it pointed out the importance of 
moments and rotational degrees of freedom (RDOF) at the contacting surfaces on the general 
structure, it did not provide a way to find the driving point and transfer point accelerance 
matrices involving RDOF.

**Bending Moment and Rotational Motion Measurement**

To find accelerance relating the rotational degrees of freedom (RDOF), it is useful to 
investigate the state-of-the-art techniques for moment and rotational motion measurement if 
we want to find the accelerance from its definition, i.e., rotational acceleration divided by 
moment.

Static bending moment measurement is widely used in all industries. In the medical 
instrument industry, pedicle screws are commonly used in spinal reconstruction, and failure 
of pedicle screws due to bending is a significant clinical problem. Strain gauges are mounted 
inside the screw to measure moments. Smith designed some special screws to measure 
flexion-extension moments at a single cross-section as dictated by strain gauge placement 
[49]. It is possible to measure moments of up to 12 Nm at location along the length of the 
screw by constructing transducers with varying strain gauge placements. In another 
application, a device capable of simultaneously measuring the isometric moments generated 
about the joints of all fingers was developed to utilize a four-bar linkage to transmit 
moments, but not forces to the device. Strain gauges mounted to the aluminum bars were 
used to measure the bending moment [34].
Another field that uses the multiple DOF control is robotics and automation. For the execution of 6-DOF tasks, both the end effector position and orientation of the robotic manipulator must be handled. Impedance control is a well-established framework to manage the interaction of the end effector of a robot manipulator with the environment. In order to perform 6-DOF tasks, a suitable representation of end-effector linear and angular displacement with respect to contact forces and moments should be sought. Caccavale presented a new approach to 6-DOF impedance control based on angle/axis representations of the orientation displacement [8]. In his experimental verifications, a six-axis force/torque sensor ATI-FT 130 with force range of ±130 N and torque range of ±10 Nm was mounted at the wrist of the robot manipulator. Contact forces and moments were measured. However, the speed of the force and moment change is too small to be considered a dynamics problem. This is a typical quasi-static case.

Transducers measuring moments are usually implemented by the strain gauge methods [14]. The shape of the mechanical element in a force transducer can be link, beam, ring, or shear web for force transducers and Wheatstone bridges are used to obtain the output. For torque measurement, the torque cells contain a mechanical element (usually a shaft with a circular cross section) and a sensor (usually electrical resistance stain gages mounted 45 degrees with respect to the cross section). For combined force-moments measurement, the mechanical element is a link and different wiring is used in Wheatstone bridges to obtain the moment. For combined force-torque measurement, the strain gages are placed on different angles to the shaft to measure force and torque separately.

Although the response of the gage, largely controlled by its inertia is sufficient to permit recording of dynamic strains with frequency components exceeding 100 KHz, the
inherent low sensitivity of the resistance strain gage prevents its wide use in measuring acceleration. Since the sensitivity of the gage is low, the deformation has to be large to generate signals. Therefore the stiffness for the base where the stain gage is attached is small. The small stiffness causes the transducer to have a low natural frequency. When inserted into the force path to measure forces and moments, the dynamic characteristics of the transducer often changes the dynamic properties of the original dynamic system.

The piezoelectric sensor is ideal for the dynamic application because it serves as a very stiff spring with good sensitivity. Consequently, these seismic transducers are small and lightweight with a high natural frequency. These lightweight transducers have little effect on most vibrating structures.

However the dynamic moment measurement transducers have a cross-axis sensitivity problem. When a pure force is applied, the moment channel senses a small percentage of the force signal. When the structure is experiencing a vertical resonance, the cross-axis leakage is large enough to cause strange peaks and notches in the moment channel FRF. The author performed a study on the cross axis sensitivity of a piezoelectric force-moment transducer over the last several years. A method was developed to identify the cross axis sensitivity errors within the transducer from vibration tests. Then, a correction matrix was formed to remove the cross axis signal contamination from measurements made with this transducer [16][17][18][44].

Strain gages are not totally ruled out for dynamic applications, however. Hillary and Ewins investigated the force identification problem for both deterministic (periodic and transient) and random forces [28][29]. In the case of deterministic signals, they found that by employing strain gages in the structure's response measurements an improvement is obtained
in the conditioning of the inversion process at the low frequencies. Later we will show this current work suffers from the noise contamination most severely at the lower frequencies in our test scenarios. Therefore, similar strain gage measurements might improve the signal quality over that from piezoelectric transducers at lower frequencies by providing some redundancy.

**Basic Research on Rotational FRF Measurement**

If we try to find the rotational FRF using its definition, there are two problems to tackle. The first one is how to measure rotational accelerations and the second one is how to generate and measure the rotational excitations or dynamic moments. Moment measurement and angular acceleration measurement schemes pose notoriously difficult problems. Over the last 30 years, contrary to the rapid development and application of substructure methods in the numerical simulation field, mostly finite element, the corresponding experimental work is way behind. A number of methods have been tried, with limited success, but these are still in a development stage. However, it is believed that these FRF terms will be of increasing importance [20]. In fact most of the basic research work on rotational FRF measurement has been done by the group lead by Prof. D. J. Ewins in the Imperial College at University of London, UK. Early investigations in building substructure models have focused on coupling beam type structures with rotational degrees of freedom.

Ewins and Sainsbury were among the first to investigate modeling the structural coupling problem using experimental FRF's [23]. They studied the problem of modeling two simple structures with experimental FRF data. The test items were a massive block and a simple beam. The interface between the test items was modeled with two translational
DOF's and one rotational DOF. The frequency range for this analysis was from 10 Hz to 1000 Hz. The results showed that only when the component mobility data is complete and accurate was the mobility approach reliable. Therefore, how to obtain high quality rotational mobility data became the key issue. Methods to more accurately determine the rotational mobility were proposed.

Ewins and Gleeson advanced with the goal of more accurately determining rotational mobility information [22]. The literature review showed the information conveyed by this paper was rarely surmounted in two decades thereafter. The objective of this investigation was to predict the mobility at the connection point between two steel beams connected end to end. The coupling between the beams was modeled using one translational and one rotational degree of freedom. A rigid block was added to the test item. Two off center impacts were used to input both forces and moments into the system. The mass and mass moment of inertia of the block was used to model the block. Substructure formulation was used to find the multidirectional driving point accelerance matrix of the test item. Two accelerometers were used and their sum and difference were related to the linear and rotational accelerations respectively. The frequency range for this analysis was 30 - 1000 Hz. The rotational mobility information was derived from translational accelerance data using a finite difference technique. The data showed good prediction was achieved for the force related FRF on single beams. The quality of moment related FRF was poor, especially the $\theta/M$ type of FRF. The smoothed FRF for single beam are much closer to the theoretical value. But the result showed no improvement when using the curve-fitted FRF of single beams to predict the FRF for the combined beam. An alternative method was given to derive $\theta$ related FRF from the smoothed measurements of $x$ related FRF using the modal
identification techniques, without measuring the rotations directly. The prediction was still not satisfactory.

Brassard and Massoud also investigated the coupling between beam-type structures [7]. Their aim was to predict the full mobility matrix with emphasis on transfer accelerance. The example they used was two aluminum beams joined by a lap weld at the end. The coupling between the beams was modeled with a single transverse degree of freedom. Although Brassard and Massoud considered the results encouraging, there were many mismatched peaks and valleys in the predicted and measured FRF for the combined system. Since the weld was very narrow, it was suspicious that the coupling was weak so that the dominant factor was the force balance and linear motion compatibility. This effect might be used to explain the general trend that the FRF looked similar while there were still many discrepancies. It is likely that a more complete model of the coupling between the joints would have significantly improved the results.

Imregum and Ewins formulated the structure modification equations based on the FRF matrices of the original structure and the added structure [32]. His numerical case studies included three cases, a beam added by another beam (2 DOF involved), a frame added by a beam at one point (3 DOF involved), and a frame added by a link between two points on the beam (6 DOF involved). The simulation frequency is over the 100-1100 Hz frequency range. The effect of errors or inconsistencies in the component FRF measurements was investigated by polluting the FRF matrices with 5% noise. It shows the errors incurred in the coupled structure prediction are much greater than those implanted into the individual component matrices. There is evidence to suggest that the ill conditioning is most pronounced in vicinity of natural frequencies of the separate components; and that, it is
greatly diminished when subsystems are damped, even moderately. A modal fit for each of the raw, measured curves respectively provided some improvement in the overall model but the result still shows signs of sensitivity to the accuracy of the component data since systematic errors between one FRF and the next (i.e., values of the natural frequency, damping etc.) are still inherently present in the analysis. Global curve fitting to all FRF curves yields the best prediction, as all the regenerated FRF data are consistent. The error introduced by neglecting rotational DOF was briefly shown.

Williams and Green [55] proposed a spatial curve fitting technique for estimating rotational degrees of freedom. The prime advantage of the approach is that existing modal displacement data can be used, obviating the need for additional measurements, the use of special rotational transducers or the development of a complementary finite element model. Since it is applied as a post-measurement processing technique, it also has the advantage that it does not require previous knowledge of the modification sites at the time that the displacement measurements are taken. The key here is to fit cubic polynomials to measured translational modal vectors to provide local approximations to the mode shape function. The polynomial functions may then be differentiated to give the required rotations. Examples showed that this method works better for lower modes than for higher modes since the accuracy of the estimates depends on the density of the original displacement measurements. Tests suggest that an error of about 15% will be incurred if there are 2 measurement points between nodes, but that this drops to less than 5% if the number of measurement points between nodes is increased to 3 points.

Ashory studied the problem of mass cancellation in modal testing and proposes a general solution based upon a direct substructure technique, Structural Modification Using
experimental frequency Response Functions (SMURF) [2]. It is shown that for the non-

driving points, the FRF can be corrected if the measurement is repeated with an 
accelerometer of different mass. It is also shown that the driving point FRF of the 
response point can be obtained by the same measurement. A similar procedure is used to 
correct the suspension effects on the test structure. For the case that the substructure is 
suspended with one spring, it is proved that for all of the points of the structure, the FRF can 
be corrected if the measurement is repeated with two other springs with different stiffness. It 
shows the method is exact but in practical situations may be vulnerable to noise. Noise has 
been considered in the measured FRF and a way to prevent error is discussed.

Liu and Ewins studied the extent of errors resulting in coupling analysis without 
using RDOF data [38]. They investigated the consequence of omitting RDOF-related FRF 
from FRF coupling analysis method in a systematic study. The importance of RDOF-related 
FRF was quantitatively described by explicit error functions for both weakly coupled TDOF-
RDOF systems and more general cases. These error functions revealed the composition of 
the error caused by the absence of RDOF-related FRF. In the error function for general 
cases, the error was decomposed and the contributions of both TDOF and RDOF related FRF 
to the error are fully discussed. Mathematically this paper gave a form of inverse so that 
each element of the inverse matrix can be related to some elements of the original matrix 
without computing the determinant.

Approaches and Difficulties

Ewins discussed the approaches used by the researchers and their difficulties [20]. 
The first difficulty is how to measure rotational accelerations. Typically a pair of matched
Accelerometers are placed a short distance apart on the structure or on a fixture attached to the structure. Figure 1.1 shows both configurations and also introduces the coordinates $x_0$ and $\theta_0$. Both accelerometer signals are measured. The responses can be deduced by taking the mean and difference of $x_A$ and $x_B$. Since we know the force from the impact hammer readings as well as with $x$ and $\theta$, we can determine all FRF's of the $x/F$ or $\theta/F$ type. The main difficulty is the measurement of the $x/M$ and $\theta/M$ type of FRF's since they require both rotational acceleration and moments be measured. The dynamic moment can be applied through the concept of force and couple.

Figure 1.2 shows a way to simultaneously apply a force and a moment. A second test hits at position 2 gives another simultaneous force and moment. By adding and differentiating the responses produced by these two separate excitation conditions, we can deduce the translational and rotational responses to the force and the moment separately. Then all four types of accelerations, that is, $x/F$, $\theta/F$, $x/M$ and $\theta/M$ can be determined.

Two traditional test setups used by other researchers are based on two closely spaced linear accelerometers.

Figure 1.1 Measurement of Rotational Response
A hidden assumption here is that the block should be very rigid so its own natural frequency is very high compared with the excitation frequency. Therefore under the excitation the block can be treated as a rigid body and its own dynamic characteristics is not reflected into the response. That is why the block is designed to be small and the end that connects to the interface is very short. Later, we show that our method described in this dissertation does not have this rigid body restriction and therefore gives more flexibility for design optimization.

The same principle can be extended to more directions by the use of multidimensional excitation fixture until the full 6 x 6 accelerance matrix at any given point can be measured. However, the procedures involved are quite demanding, because they require the acquisition and processing of many different measurements that are made at different times. Therefore software is developed for the analysis of a large quantity of data.

![Figure 1.2 Application of Moment Excitation](image-url)
However, there is a major problem on using two linear accelerometers in the above method. The prevailing levels of output signal generated by the translational components of the structure's movement often overshadow those due to the rotational motions, a fact which makes the differencing operations liable to serious errors. For example, $x_A - x_B$ is often of the order to 1-2% of either the two individual values. When the transducers have a cross-axis sensitivity of the order of 1-2%, the errors in the rotations can be enormous. Therefore in our research work, a specially developed rotational accelerometer called a TAP, is employed to improve quality of the rotational acceleration signal.

**Industrial Application of Substructure Testing**

For real structures encountered in industry, the attachments are usually connected through a surface contact instead of through a point contact. The connection can be bolted, welded, riveted, etc. [20]. Substructure testing techniques are based on enforcing compatibility and equilibrium conditions at the connection points between structures. So the force equilibrium and motion compatibility equations apply to each of the six DOF at each attachment.

However, due to the immaturity of the rotational DOF test techniques, practitioners in industry have to find ways to avoid the problem. One way is to study simple surface contact where only normal forces exist. Another way is to study slender connecting parts where vertical forces are dominant so the error related to RDOF is small. For complicated structures, a straightforward thinking is to separate the structure at pinned-type joints instead of the bolted or welded joints as rotational DOF at a ball or pinned joint are generally not existent. Ball or pinned joints are easier to model than rigidly bolted or welded connections.
where rotational motions and moments cannot be ignored. For example, the steering knuckle and strut of the test vehicle, the car seat and the chassis, were bolted together. As it is difficult to accurately model rotational accelerance, leaving these two components attached together as a single component improves the accuracy of the overall model by eliminating a difficult-to-model joint. The price paid for this simplification is that a design change on either of the bolted components cannot be modeled by measuring the newly designed component alone and thus a complete test must be conducted.

A literature review reveals very few modern applications of the mobility method on practical engineering systems. Ochsner provides a comprehensive review of this topic in his work up to 1991 [46].

Hemingway indicates how vibration responses of a vehicle body produced by inputs at the rear wheels were modeled using a mobility matrices coupling method [30]. The model simulated the multiple coupling of the sub-assemblies comprising the rear axle, rear suspension and vehicle body in the lower audio-frequency range. When modeling the interface, the author used some simplifications. Although for a vehicle on the road, inputs were applied to the wheels in both the vertical and longitudinal, and to a smaller extent in the lateral directions, the responses in the study were limited to a vertical input at the rear wheels. Therefore only forces transmitted along the vertical coordinates were considered at the interface. An angular rubber bush existed at every point of interaction between components that possessed relative angular motion, and this rubber bush possessed a rotational stiffness that was very low when compared with its radial stiffness. Therefore it was incapable of transmitting any significant couples. Because of this consideration, moments were omitted and consequently each connection point was represented by a single
coordinate in the vertical direction. The components were tested using swept sine testing over the 25 Hz - 250 Hz frequency range. Although the major peaks and general trends of the measured FRF were predicted, the details of the simulation and experimental curves were quite different. It indicated that the mobility technique had promise for structure-borne road noise applications, but in order to improve the results, other degrees of freedom should be included into the interface model.

Klosterman used a building block approach to model vehicle systems, i.e., the truck frame and the cab [35][36]. In his simulation, only the characteristics in the three translational directions were used because in this special case, the torques applied by the cab mounts were not significant. The results appeared very good. Klosterman referred to this method as a general impedance method. Kienholz and Smith also developed similar methods [33]. Their test article was a simple two-bay truss constructed of steel angle. Three legs of the upper bay bolt to the corners of the lower bay. Though bolted, only three linear DOF were considered. The lower bay was bolted at its feet to a rigid base. Burst random excitation was used for all horizontal DOF of the lower bay and impact test was used for upward direction. Upper bay was hung from soft springs and only the impact test was used exclusively. The combined structure was also tested to verify the predictions. Unfortunately without the consideration of rotational DOF at the connections, the predicted curve had distinct peaks and valleys with the measurement curve.

Ochsner and Bernhard applied substructure testing to the analysis of noise that is structure-borne from the tire spindle, through the suspension, into the passenger compartment of an automobile [46][47]. The model was developed with experimental FRF data that is measured on isolated suspension and body structure components of a midsize automobile.
The most significant contribution of this paper is that several important modeling issues were resolved. Multiple degrees-of-freedom are required to model the coupling at joints between the front-wheel suspension and body structure. Slender suspension bushings were modeled as simple stiffness elements to improve the quality of the mobility-component measurements. The choice of stiffness elements for the bushings enhanced the accuracy and increased the value of the model because the effects of stiffness changes can be easily predicted. For each connecting point, three force components are present. Therefore the driving point mobility matrix is 3 by 3. Also experimentally investigated was the importance of including transfer mobility information in the driving point mobility matrix if multiple bushings were present.

Lim applied the bushing models in similar applications [37]. Lim discussed a successful application of FEM and an experimental modeling approach to automotive structure-borne noise and vibration control problems when the excitation sources are the powertrain and tire-road interaction forces in the 100 Hz - 1000 Hz frequency range. In this range, the system dynamics are significantly influenced by higher-order modes of the car body and its interior acoustics. The sound-generating subframe is modeled as a finite element substructure that is coupled to the body at several bushing locations. Force transmissibility and dynamic stiffness matrices at the subframe side of the boundary are computed from this model. Compliance functions at the boundary and response points are measured on the vehicle body in the free-free mode. Predicted responses correlate well with measurements.

Steyer and Lim [50] proposed a hybrid simulation method to use a finite element model of a suspension component and the experimental model of an automotive body to simulate the automotive interior noise. Two different interfaces are developed. First, when
the impedance of the body is considerably higher than the impedance of the suspension or powertrain, the vehicle components are weakly coupled. The interface reaction forces are then equal to the rigid constraint forces. Practically this situation happens when flexible bushings are used to attach these components to the body. A method called the Complex Vector method is proposed. Second, in a strongly coupled system, the interface dynamic forces are influenced by the impedance matching between vehicle components. This class of problem requires that the dynamic stiffness (or impedance) of the body be incorporated into the model of the suspension or powertrain to accurately predict the dynamic forces transmitted to the body. The rigid constraint reaction forces for the suspension or powertrain will over estimate the magnitude of the forces transmitted to the body. An algorithm that is referred to as System Modeling and Analysis using the Response Technique (SMART) is proposed.

About This Work

From the above discussion we can clearly see that the substructure method has great potential for industrial application. Academically, it is also one of the last fortresses to be conquered in doing substructure testing. However, due to the unique difficulty in measuring angular acceleration, shear forces, and moments, we have not gone very far in finding a general method that is effective and efficient in obtaining the rotational accelerances. The engineers in industry avoid the rotational DOF problem by dealing with special interfaces that only have linear degrees of freedom.

In order to progress at this stage, new technologies and strategies are required. Since it is extremely difficult to measure and apply dynamic shear forces and moments, an
approach that uses the definition of the rotational accelerance is experimentally unfeasible. We need to use an indirect method that measures a related quantity, and then use theory of elastics to compute the actual FRF's. A similar case in the history of experimental mechanics is to use an arbitrarily oriented rosette of strain gages to measure strains and compute the principle stresses. The work described in this dissertation has a similar flavor. However this time we have something new and more powerful, that is, the finite element method and a new accelerometer that can measure both linear and rotational accelerations.

Let us review some of the advantages we have today over the pioneers.

**Advantage 1: Finite element with rotational degrees of freedom**

In order to model the interface with both linear degrees of freedom and rotational freedom, the 3D spatial beam is an ideal candidate, as it has three translational and three rotational degrees of freedom for each point. If the interface is modeled by one end of a 3D beam element, then the other end needs to be coupled with some other elements that also have six degrees of freedom. Otherwise the moment cannot be transmitted to the rest of the structure. Yunus et al introduced a set of new hybrid elements with rotational degrees of freedom [56][57]. The solid, 8 point, hexahedron element is developed for solving general three-dimensional elasticity problems. This element has three translational and three rotational DOF at each point and is based on a 42 parameter, 3D stress field in a natural coordinate system. The middle point translational DOF are expressed in terms of the corner nodal translations and rotations using appropriate transforms. The stiffness matrix is derived based on the Hellinger-Reissner variational principle. As this element can have curved boundaries, it can be used to model the real structure in any shape.
Advantage 2: Experiment and Finite Element Correlation and Updating Techniques

The development of experimental and finite element correlation and model updating techniques show promise in creating high quality finite element model that mimics the physical model. The 1990's saw a rapid development in finite element model updating techniques [26].

Advantage 3: TAP

In his review of the next twenty years of development of experimental modal analysis in 1987, Allemang predicted that the next twenty years in transducer hardware would see even more significant changes. Out of them all, reliable rotational transducers will be available with reasonable low noise, physical size, and cost characteristics [1]. Now 10 years have passed and progress is continuously being made towards this goal. It is difficult to design angular seismic accelerometers with both very high sensitivity for the desired angular acceleration and very low sensitivity for linear acceleration. However, recent developments in micromachining and fabrication of synthetic piezoelectric materials have resulted in a new transducer -- TAP that allows both linear and angular acceleration to be measured simultaneously. The key design here is two identical cantilever beams fabricated from piezoelectric ceramic. The beams are mounted on a center post that is mounted on the base. Figure 1.3 shows a schematic graph of the inside construction of a piezobeam.

With proper polarity in the piezobeam, the entire top surface of each beam becomes positively charged while the entire bottom surface is negatively charged because of the acceleration induced stresses in the beams. Similarly, when the beam accelerates downward, the polarity of the electrical charge is reversed. Under angular acceleration, the top of one
piezobeam is negative while the bottom is positive and the top of the other beam is positive while the bottom is negative. Through the wiring and sum and difference amplifiers, we can separate the voltage caused by linear motion and the voltage caused by angular motion [42].

The piezobeam design exhibits a high sensitivity of about 1000 mV/g with +/−10g range for linear acceleration and either 0.5, 5, or $50\text{mV/(rad/s}^2\text{)}$ for angular acceleration. The usable frequency range is up to 2000 Hz and the natural frequency is around 8000 Hz. The major advantage is that both linear and angular acceleration can be measured simultaneously for one point, instead of going through the difference procedure that is used in previous studies. Besides when three accelerometers are placed in a tri-axial mounting, all six DOF of motion at a single location can be measured simultaneously. The major disadvantages are its limited range and the requirement that the power unit and transducer must be used together as a single unit.
Based on Advantages 1 and 2, it is reasonable to believe that given a simple structure, we can develop its FE model with sufficient accuracy so that all linear and rotational FRF's are available. This simple conceptual structure is named as an "Instrument Cluster". The "Instrument Cluster" has two features. It is a "white box" structurally, and it also houses some measurement instruments so that when installed to the test item, it can provide the response measurement of the combined structure. If we attach this "white box" to the unknown test structure, the "black box", the combined structure is a "gray box". If we can measure some signals in both linear and rotational DOF using the TAP (Advantage 3) from the "gray box", with the knowledge of the "white box", it is possible for us to derive some features of the "black box". This idea is further developed in Chapters 5 and 6 and two specific implementations of the conceptual "Instrument Cluster", a T-bar and a C-bar are proposed.

Layout of the Dissertation

Chapter 1 introduces the background of the substructure testing followed by a literature review. Instrument and method advantages of this work over the previous work are discussed.

Chapter 2 reviews some basic concepts on modal analysis and substructure testing. It serves the theoretical basis of this work.

Chapter 3 exploits various errors in substructure testing. Based on the derived motion transmissibility from the ground to a point of interest on the test item, various errors involved in substructure testing such as DOF deficiency, bias, noise, rocking table motion and measurement point inconsistency errors are investigated.
Chapter 4 investigates the composition of the DOF deficiency error in detail. Both 2D and 3D cases are studied. It shows that DOF deficiency is a very complicated problem.

Chapter 5 studies the six extraction scenarios for the interface multidirectional driving point or transfer point accelerance matrices in 2D space. A special design of the "Instrument Cluster" in the 2D space, an elastic T-bar, is proposed. Robustness of various scenarios is studied with polluted signals.

Chapter 6 describes an experiment that implements Scenario 5 of Chapter 5 in order to extract the multidirectional driving point FRF. Experimental results showed that some of the extracted FRF's have nearly perfect matches with the theoretical predictions while some other FRF's are not as good. It shows the cross axis sensitivity of the TAP is a significant source of error. Correction methods are illustrated.

Chapter 7 studies four extraction scenarios for the interface multidirectional driving point and transfer point accelerance matrices in 3D space. A specific design of the "Instrument Cluster" in 3D space, a C-bar, is proposed. Robustness of various methods is studied by using polluted signals.

Chapter 8 is the general conclusion of this work.
CHAPTER 2 CONCEPTS OF EXPERIMENTAL MODAL ANALYSIS AND SUBSTRUCTURE TESTING

This chapter reviews some fundamental experimental modal analysis and substructure testing concepts as related to this dissertation. To begin with, the basics of mechanical vibration theory will be presented. This is to serve as both a review of the theory and an introduction of the notation.

Three Different Models

There are three different types of mathematical model used for dynamic analysis, [20]. They are

- The SPATIAL MODEL, consisting of mass, stiffness, and damping matrices;
- The MODAL MODEL, comprising natural frequencies and mode shapes;
- The RESPONSE MODEL, expressed as a set of frequency response functions (FRF).

For the theoretical route, generally we start with a description of the structure’s physical characteristics, usually in terms of its mass, stiffness, and damping properties. This is referred to as the SPATIAL MODEL. This model is most intuitively related to the physical nature of the system and thus widely employed in numerical simulations such as the finite element method. However, when applied to large complex structural assemblies such models tend to be extremely large. With millions of degrees of freedom and very sparse matrices, it is inefficient to capture the nature of the system. To reduce the data quantity while capturing the essence of the structural dynamics properties, it is customary to perform an analytical modal analysis of the spatial model that leads to a set of natural frequencies,
vibration modes and modal damping ratios: the MODAL MODEL. Natural frequencies and mode shapes describe the various configurations in which the structure is capable of vibrating naturally. The modal model has a clear physical meaning.

However, in most applications, it is required to predict the vibration response levels of some points of interest when the structure is excited at one or more points. What is desired is the structural response in terms of amplitude and phase due to a given excitation condition. This model is referred to as the RESPONSE MODEL. It is a bridge to relate the input and output. The standard excitation is a unit-amplitude sinusoidal force.

Hence the theoretical analysis starts from the spatial model, and then leads to the modal model, and then to the response model.

After the prototype of the product is manufactured, it needs to be tested to verify whether it has met the design goals. The experimental route of modal analysis starts from measuring the raw data in the form of FRF, which is the content of the response model. A lot of the study in experimental modal analysis has been focused on how to obtain FRF data of high quality using different combinations of transducers, excitation, and data acquisition systems. Unfortunately in reality some of the data is difficult to obtain when compared to their theoretical counterparts, such as moments and angular accelerations. This inherent restriction makes the development of experimental mechanics lagged behind that of the numerical simulations.

We can deduce modal properties by using various curve-fitting methods to extract the response properties from the measured frequency response functions. During the last thirty years, the greatest impact of technology on those who work in the area of modal analysis is in the area of modal parameter estimation. Single degrees of freedom, multiple degrees of
freedom, multiple measurement, multiple reference methods were developed in a natural progression [1].

With the inevitable inaccuracy and incompleteness in the test data, the spatial model cannot be constructed from the test. It is also found that the modal model approach is hampered by difficulties in ascertaining in advance how many modes must be included in its model and, in many cases, by practical obstacles to obtaining sufficient number of these modal data from tests. From the substructure testing point of view, it is very difficult to predict how many modes are involved in the combined structure with some FRF for the substructures only. Therefore in experiment it seems the response models, that is, dealing with measured FRF directly is a reliable analysis procedure. This approach uses the FRF of each individual component to predict FRF of the complete structure without determining the modes of the combined structure as an intermediate stage [20].

Natural Frequencies and Normal Modes of Vibration

The governing equations of an \( N \) degrees of freedom (DOF) linear system with viscous damping can be written as [4][53]:

\[
[M]{\ddot{x}} + [C]{\dot{x}} + [K]{x} = \{f\} 
\] (2.1)

where the symmetric \( N \times N \) matrices \([M]\), \([C]\), and \([K]\) are the structural mass, damping, and stiffness matrices. The \( N \times 1 \) vectors \( \{f\} \) and \( \{x\} \) contain the input force applied to the structure and the output response displacement. Both of them are functions of time \( t \). The dot denotes differentiation with respect to time.
The natural frequencies and normal modes of the structure are obtained by solving the undamped free vibration equation which can be obtained by ignoring the damping matrix $[C]$ and setting $\{f\}$ equal to zero in Equation 2.1.

$$[M]\ddot{x} + [K][x] = \{0\} \tag{2.2}$$

Suppose the solution of Equation 2.2 is

$$\{x\} = \{\phi\}e^{s't} \tag{2.3}$$

where $\{\phi\}$ is a $N \times 1$ vector of real or complex entries and $\lambda$ is a complex number. Substitution of Equation 2.3 in Equation 2.2 gives

$$\lambda^2 [M] + [K][\phi] = \{0\} \tag{2.4}$$

For Equation 2.4, a non-trivial solution exists if and only if the following relationship holds

$$\det(\lambda^2 [M] + [K]) = 0 \tag{2.5}$$

The solution of Equation 2.5 is composed of $N$ pairs of eigenvalues and the structure's $r^{th}$ natural frequency $\omega_r$ is obtained from the $r^{th}$ eigenvalue $\lambda_r$ through the following expression

$$\lambda_r = \pm j \omega_r \tag{2.6}$$

where $j = \sqrt{-1}$.

The solution for the real eigenvectors, i.e., the structure's undamped vibration modes, requires that Equation 2.4 be solved for each value of $\omega_r$. Since Equation 2.4 is homogeneous, there are an infinite number of eigenvectors satisfying this equation. Therefore the amplitude of the structure's modes is indeterminate but the shape of each mode of vibration can be obtained by assuming arbitrarily one entry of vector to be unity and solving for the remaining coordinates of that mode.
The set containing all natural frequencies and mode shapes constitutes the structure's undamped modal model and can be written in terms of a \( N \times N \) diagonal matrix containing the square of the structure's natural frequencies.

\[
[\Omega_r] = \begin{bmatrix}
\omega_1^2 & 0 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \omega_r^2 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \omega_N^2
\end{bmatrix}
\]  

(2.7)

And the real \( N \times N \) matrix \([\Phi]\) contains the mode shapes

\[
[\Phi] = [\phi_1 \ldots \phi_r \ldots \phi_N]
\]  

(2.8)

Consider any two mode shape vectors \( \{\phi_r\} \) and \( \{\phi_s\} \) corresponding to distinct natural frequencies \( \omega_r \) and \( \omega_s \), the orthogonality conditions of the two mode shapes with respect to the mass and stiffness matrices states that

\[
\{\phi_r\}^T [M] \{\phi_s\} = 0
\]  

(2.9)

\[
\{\phi_r\}^T [K] \{\phi_s\} = 0
\]  

(2.10)

For the case where \( r = s \), we have

\[
\{\phi_r\}^T [M] \{\phi_r\} = m_r
\]  

(2.11)

\[
\{\phi_r\}^T [K] \{\phi_r\} = k_r
\]  

(2.12)

The constants \( m_r \) and \( k_r \) are called the modal mass and modal stiffness and \( \{\phi_r\} \) is the \( r^{th} \) mode shape. For all the modes, we have the diagonal matrices for the structural modal mass and modal stiffness matrices.

\[
[\Phi]^T [M] [\Phi] = \text{diag}[m_r]
\]  

(2.13)
The $r^{th}$ natural frequency can be expressed in terms of the $r^{th}$ modal mass and stiffness coefficients as

$$\omega_r = \sqrt{\frac{k_r}{m_r}}$$  \hspace{1cm} (2.15)

The $r^{th}$ mode shape vector can be normalized by the modal mass values by using the following relationship

$$\{\psi_r\} = (m_r)^{-1/2} \{\phi_r\}$$  \hspace{1cm} (2.16)

where $\{\psi_r\}$ represents the $r^{th}$ mass normalized mode shape.

Using the mass normalized mode shape matrix we can simplify Equation 2.13 and Equation 2.14 to be

$$[\Psi]^T [M][\Psi] = [I]$$  \hspace{1cm} (2.17)

$$[\Psi]^T [K][\Psi] = \text{diag}[\Omega_r]$$  \hspace{1cm} (2.18)

where $[I]$ is the $N \times N$ identity matrix, $[\Psi]$ is the $N \times N$ mass normalized mode shape matrix, and $[\Omega_r]$ is the $N \times N$ diagonal matrix whose diagonals are the squares of the structural natural frequencies.

**Frequency Response Function - FRF**

To find the particular solution that relates to the excitation, the external excitation vector $\{f\}$ in Equation 2.1 has a special form, i.e.,

$$\{f\} = \{f_0\} e^{j\omega t}$$  \hspace{1cm} (2.19)

where
\[
\{f_q\} = \begin{bmatrix} 0 & \cdots & f_q & \cdots & 0 \end{bmatrix}^T \tag{2.20}
\]

where the element \(f_q\) is the magnitude of the sinusoidal force applied at the \(q^{th}\) coordinate and \(\omega\) is the excitation circular frequency.

Now, let us consider the linear transformation

\[
\{x\} = \Phi \{y\} \tag{2.21}
\]

that relates the structure’s displacements in the physical domain \(\{x\}\) to the structure’s displacements in the modal domain \(\{y\}\).

Then, substitution of Equations 2.19 and 2.21 into Equation 2.1 and pre-multiplication of both sides of the resulting equation by \(\Phi^T\) gives

\[
[\Phi^T [M \Phi] \{y\} + [\Phi^T [C \Phi] \{y\} + [\Phi^T [K \Phi] \{y\}] = [\Phi^T \{f_0\}] \text{e}^{j\omega t} \tag{2.22}
\]

Equation 2.22 reduces to a set of \(N\) uncoupled single degree of freedom equations of motion in the modal coordinates if and only if the damping matrix \([C]\) obeys the following orthogonality requirement

\[
[\Phi^T [C \Phi] = [c_r] \tag{2.23}
\]

where the \(N \times N\) \([c_r]\) is a diagonal matrix that contains modal damping coefficients. This orthogonality requirement can be met if a proportional damping distribution is assumed, i.e., the damping matrix \([C]\) is proportional to the mass matrix \([M]\) and the stiffness matrix \([K]\).

A widely used proportional damping distribution is the Rayleigh damping, i.e.,

\[
[C] = a_0[M] + a_1[K] \tag{2.24}
\]

Pre and post multiplication of both sides of Equation 2.24 by \(\Phi^T\) and \(\Phi\) respectively, gives the following result of \([c_r]\)

\[
[c_r] = [\Phi^T [C \Phi] = a_0[m_r] + a_1[k_r] \tag{2.25}
\]
Here \([c_r]\) is a combination of diagonal matrices \([m_r]\) and \([k_r]\) too.

If proportional damping is assumed, Equation 2.22 reduces to

\[
[m_r]\ddot{y} + [c_r]\dot{y} + [k_r]y = \Phi^T \{f_0\} e^{j\omega t} \tag{2.26}
\]

That represents a set of \(N\) uncoupled equations of motion in the modal domain. The equation of motion for the \(r^{th}\) degree of freedom is

\[
m_r\ddot{y}_r + c_r\dot{y}_r + k_r y_r = \phi_r^T \{f_0\} e^{j\omega t} \tag{2.27}
\]

Division of both sides of Equation 2.27 by the modal mass \(m_r\) results in an alternative form for the \(r^{th}\) equation of motion in the modal coordinates

\[
\ddot{y}_r + 2\xi_r \omega_r \dot{y}_r + \omega_r^2 y_r = \frac{1}{m_r} \phi_r^T \{f_0\} e^{j\omega t} \tag{2.28}
\]

where \(\xi_r\) is the modal damping ratio of the \(r^{th}\) mode shape and is defined in terms of the corresponding \(r^{th}\) modal mass, stiffness, and damping coefficients

\[
\xi_r = \frac{c_r}{2\sqrt{k_r m_r}} \tag{2.29}
\]

If the right hand side of Equation 2.28 is nonzero, the time domain solution of Equation 2.28 is given as

\[
y_r = Y_r e^{j\omega t} \tag{2.30}
\]

where \(Y_r\) is the unknown \(r^{th}\) modal amplitude.

\[
Y_r = \frac{1}{m_r} \frac{\phi_r^T \{f_0\}}{\omega_r^2 - \omega^2 + j2\xi_r \omega \omega} \tag{2.31}
\]
Once all \( y_r, r = 1 \ldots N \) are determined, the solution for the structure’s displacement in the physical domain \( \{x\} \) can be obtained from Equation 2.21 that can be conveniently rewritten as

\[
\{x\} = \sum_{r=1}^{N} \{\phi\}_r y_r \quad (2.32)
\]

Equation 2.32 is frequently referred to as modal superposition since it indicates that the final solution \( \{x\} \) is obtained by superimposing the contribution of each mode shape individually to the total displacement vector. Substitution of Equation 2.31 into Equation 2.32 leads to

\[
\{x\} = \sum_{r=1}^{N} \frac{\{\phi\}_r \{\phi\}^T \{f_o\}}{m_r(\omega^2 - \omega^2 + j2\xi\omega, \omega)} e^{j\omega t} \quad (2.33)
\]

From the time domain solution in the physical domain Equation 2.33 the structure’s FRF model can be obtained in terms of the Receptance FRF matrix as

\[
[R(\omega)] = \sum_{r=1}^{N} \frac{\{\phi\}_r \{\phi\}^T}{m_r(\omega^2 - \omega^2 + j2\xi\omega, \omega)} \quad (2.34)
\]

One element of \( [R] \), \( R_{pq}(\omega) \) is defined as the ratio of the Fourier transform of the structure’s displacement \( X_p(\omega) \) and input force \( F_q(\omega) \) at the \( p^{th} \) and \( q^{th} \) coordinates, respectively. From the definition of \( R_{pq}(\omega) \) and Equation 2.34, \( R_{pq}(\omega) \) is given as

\[
R_{pq} = \frac{X_p(\omega)}{F_q(\omega)} = \sum_{r=1}^{N} \frac{\phi_m \phi_r}{m_r(\omega^2 - \omega^2 + j2\xi\omega, \omega)} \quad (2.35)
\]

where \( \phi_m \) and \( \phi_p \) are the \( p^{th} \) and \( q^{th} \) elements of the \( r^{th} \) model shape vector, respectively.

When the excitation and response coordinate points are coincident, \( p = q \), \( R_{pp} \) is called Driving Point Receptance FRF. In this case, Equation 2.39 can be rewritten as follows
\[ R_{pq} = \sum_{n=1}^{N} \frac{\phi_{np}^2}{m_n (\omega_n^2 - \omega^2 + j2\xi_n \omega, \omega)} \] (2.36)

And when \( p \neq q \), \( R_{pq} \) is called *Transfer Point Receptance FRF*. In a special case where \( p \) and \( q \) refer to different directions of the same point, \( R_{pq} \) is called *Cross Receptance FRF*.

Two additional FRF relationships that are commonly employed in vibration testing are *Mobility* and *Accelerance (Inertance)*. Mobility \( M_{pq} \) is defined as the ratio between the velocity of the structure at coordinates \( p \) and the unit force applied at coordinate point \( q \). Accelerance \( A_{pq} \) is defined as the ratio between the acceleration at coordinate \( p \) to the unit force applied at coordinate \( q \). The relationships between the mobility FRF and the accelerance FRF to the receptance FRF are given by

\[ M_{pq} = j\omega R_{pq} \] (2.37)

\[ A_{pq} = -\omega^2 R_{pq} = j\omega M_{pq} \] (2.38)

**Incomplete Model, an Experimental Concern**

All the preceding theory has been concerned with complete models, that is, the analysis has been presented for an \( N \) degree of freedom system with the implicit assumption that all the mass, stiffness and damping properties are known and the mass, stiffness, and damping matrices are complete. The eigenvalues and eigenvectors cover all the modes and all the degrees of freedom. The frequency response functions are all available given any DOF. While this is a valid approach for a theoretical investigation, experimentally it is seldom the truth [6][20].
The full accelerance and dynamic stiffness matrices are inverses of each other, making the two descriptions equivalent in the sense that knowledge of a system's behavior under applied forces (accelerance) determines its behavior under imposed accelerations (dynamic stiffness) as well. This is only true, however, if the knowledge of an $N$-DOF system is complete in the sense that one has determined the acceleration of each DOF under $N$ linearly independent sets of forces, so that one knows what each of those accelerations will be when a unit force is applied to any one DOF and no force to any of the others. In practice, however, one often deals with a partial impedance matrix of dimension lower than $N$, for instance, $m$. Such a matrix gives the acceleration (force) in each of the $m$ DOF when a unit force (acceleration) is applied to one of them and the force (acceleration) is zero for each of the remaining $N-m$ DOF. Since the conditions at the $N-m$ "excluded" DOF are different for the partial accelerance matrix and the partial dynamic stiffness matrix, it is no longer possible to obtain one from the other. Mathematically, the inverse of a sub-matrix is not a sub-matrix of the inverse of the matrix.

Experimentally we deal with the real structure. Therefore all the degrees of freedom are there and the only problem is whether we can measure all of them to construct a complete model. There are two ways in which a model can be incomplete – by the omission of some modes, or by the omission of some coordinates, or both. Consider first the complete FRF matrix, which is $N \times N$:

$$[Y(\omega)]_{N \times N}$$

and then suppose that we can only describe the system with certain coordinates only and thus we have to ignore what happens at the others. This is not to say that the others do not exist. The reduced response model is of order $n \times n$, and is written as:
Now it is clear that as we have not altered the basic system, and it still has the same number of degrees of freedom even though we are not able to describe all of them. The elements that remain in the reduced mobility matrix are identical to the corresponding elements as in the full $N \times N$ matrix. In other words, the reduced matrix is a subset of the original matrix.

Problem occurs when we try to find the inverse of the FRF. Commonly it is the impedance type of data. The impedance matrix $[Z]$ that corresponds to the whole FRF matrix is defined as

$$[Z(\omega)] = [Y(\omega)]^{-1}$$

(2.39)

and the impedance matrix $[Z^R]$ that corresponds to the reduced FRF matrix will be denoted as

$$[Z^R(\omega)] = [Y^R(\omega)]^{-1}$$

(2.40)

It is clear the elements in the reduced impedance matrix are not the same quantities as the corresponding elements in the full impedance matrix and, indeed, a completely different impedance matrix applies to each specific reduction. Liu and Ewins studies this problem in more detail and tries to establish the relationship between $[Z]$ and $[Z^R]$ [Liu et al, 1998]. Chapter 4 applies this relationship to the motion transmissibility formula to study the composition of the DOF deficiency error.
Definitions Used in Substructure Testing

Three structures are involved in the process of substructure testing: the Test Item, the Vehicle and the Combined Structure [19][42][43][53]. Each structure is defined by its input-output relationship as follows:

*Test Item* is the structure under investigation that presents an input – output relationship that is given by

\[ \{X\} = [T]\{F\} \]  \hspace{1cm} (2.41)

where \( \{X\} \) denotes the test item output motion vector, either displacement, velocity, or acceleration when it is standing alone, \([T]\) is the test item's FRF response model (receptance, mobility, or accelerance) without any attachments on the test item, and \( \{F\} \) is the test item's input vector. For the test item, all the connections to the vehicle are disconnected and thus the interface constraints are exposed as interface forces and moments. Therefore \( \{F\} \) consists of the internal, interface, and external excitations. Usually the test item serves certain primary functions or it is the item to be studied. The chances to perform a design change to facilitate its testing are quite restricted. Our goal is to find \([T]\) without any contamination from the vehicle.

*Vehicle* is the structure that the test item is attached to while in service or the test fixture that is attached to the test item in order to apply the excitation to it. The vehicle’s input-output relationship is given as

\[ \{Y\} = [V]\{P\} \]  \hspace{1cm} (2.42)

where \( \{Y\} \) denotes the vehicle output motion vector when the vehicle stands alone, \([V]\) is the vehicle's FRF response model and it reflects the dynamics of the bare vehicle behavior, and
\{P\} is the vehicle's input vector. Usually the vehicle serves certain secondary functions or it is designed in order to ease the study of the test item. Thus it is more flexible to change its design.

**Combined Structure** is the structure that incorporates the test item and the bare vehicle. When in real working conditions, the test item and the vehicle cannot be separated, or in certain test setup, the vehicle has to be attached to the test item to transmit excitation, the response of the combined structure, instead that of the individual structure, is measured. The FRF obtained in the combined structure is not the FRF of the pure test item. This problem is especially important for the lightweight test item mounted on heavier vehicles.

Since field and laboratory dynamic environments deal with combined structures, i.e., structures that result from the combination of two or more substructures, frequency domain substructure concepts are employed to study the structural interactions that occur when the test item is connected to the vehicle in the field or to some test attachments in the laboratory.

In Equation 2.41, the interface DOF are differentiated from external DOF by using the following equation:

\[
\begin{bmatrix}
\{X_i\} \\
\{X_e\}
\end{bmatrix} =
\begin{bmatrix}
[H_a] & [H_e] \\
[H_a] & [H_e]
\end{bmatrix}
\begin{bmatrix}
\{F_i\} \\
\{F_e\}
\end{bmatrix} 
\tag{2.43}
\]

Equation 2.43 can be used for the single test structure, the vehicle or any number of independent structures that are coupled at a finite number of locations. Subscripts \(i\) and \(e\) in Equation 2.43 refer to degrees of freedom of interface points and external points, respectively. Interface points are points on the structure that are directly connected to another structure. Since the interface points have multiple degrees of freedom and there can be multiple interface connectors in many applications, it is important to treat \(i\) not as a single
value, but as a set of values. External points are points on the structure that are not directly involved in the coupling process. Interface linear and/or angular motions \( \{X_i\} \) occur at connecting points while external motions \( \{X_e\} \) occur at the remaining points on the structure. Interface forces and/or moments \( \{F_i\} \) occur at interface points and are due to coupling effects only. The external forces vector \( \{F_e\} \) contains all remaining forces applied to the structure.

It is important to distinguish between motions caused by interface forces from those caused by external forces. So the FRF matrix \([H]\) is partitioned into four sub-matrices, as seen from Equation 2.43. In this case, \([H_{ii}]\) defines the input-output FRF for interface points. \([H_{ee}]\) defines input-output FRF for external points, and \([H_{ie}] = [H_{ei}]^T\) defines the FRF between interface and external points, respectively.

In reality, since the interface is not a point but a small surface, there are generally two ways to model the interface. The first one is to use multiple points and for each point, only the linear degrees of freedom for interface forces and interface linear motions are concerned. These forces and linear motions are used as nodal values in finite element method to interpolate the values between them. Since the nodal values are different, the interpolation gives a distribution of forces and motions on the interface. The resultant actions have both forces and moments and the resultant motions have both linear and angular motions. The FRF matrix only has linear degrees of freedom. At a glance it seems easier since we can find the elements of the FRF matrix by measuring forces and linear motions directly. The advantage of this method is that it can use the current instrument such as force transducer and linear accelerometers, or their combination, the impedance head to obtain the measurements with good quality. Because three directional forces and three directional motions need to be measured at a single point and at least two points are needed to find the sum and difference,
two units of three axial accelerometers need to be placed on the interface. The limited space of the interface is always a problem for installing more instruments. Besides since the interface must be modeled with multiple points in the finite element model, the modeling process is obviously more complicated.

The second method is to use a single point to represent the small connecting surface and introduce the rotational degrees of freedom (RDOF) to handle moments and angular acceleration into the model. The error introduced in this simplification process is limited only to the vicinity of the connection area. The advantage is the simplicity of the model but the difficulty lies on the moment and angular acceleration measurement if we have to obtain the elements related to RDOF in the FRF matrix through direct measurement of moment and angular acceleration. This work provides a new method to solve this issue without directly measuring moment and angular acceleration.

One concern in the determination of the interface forces \( \{F_i\} \) and motions \( \{X_i\} \) requires definition of appropriate boundary conditions for the coupling points between test item and vehicle when these structures are connected in the field. Two approaches can be used to define boundary conditions at interface points. In the first approach connectors are independent coupling structures. In the second approach, a simpler interface boundary is used, where the connectors are assumed to be part of either one structure or the other. The last approach is used in this work to define boundary conditions at the interface points between the test item and the vehicle. Connectors are considered part of the test item.

Since test item and vehicle are connected through a finite number of discrete points \( N_h \), compatibility of motions at interface connecting points require

\[
\{X_i\} - \{Y_i\} = \{0\}
\]  \hspace{1cm} (2.44)
where the $N_i \times 1$ vector $\{X_i\}$ and $\{Y_i\}$ define test item and vehicle motions at the interface points, respectively. Both $\{X_i\}$ and $\{Y_i\}$ are assumed to be positive in the same direction.

The interface forces and moments must satisfy the balance condition.

$$\{F_i\} + \{P_i\} = \{0\}$$  \hspace{1cm} (2.45)

where $\{F_i\}$ and $\{P_i\}$ are $N_i \times 1$ vectors that represent the test item and vehicle interface forces, respectively. All matching forces are positive in the same direction.
CHAPTER 3 VARIOUS ERRORS IN SUBSTRUCTURE TESTING

This chapter uses the frequency domain substructure method to derive the equations for motion transmissibility of a coupled structure based on the interface frequency response functions (FRF) of the test item and vehicle that form the coupled structure. The interface conditions are developed with/without the degrees of freedom that corresponds to the shear force and moment. Numerical investigation shows that only under the condition that both the DOF of the shear force and the moment are included in the interface can the above substructure method find the same motion transmissibility result as that obtained from direct modeling of the complete structure. The restriction on moment and shear force information presents serious challenges to experimentally apply the substructure method to unknown structures since there are no known and/or effective ways to find moment FRF from measurement. The bias and noise errors in implementing the substructure testing method are also investigated. Errors due to the rocking motion from the exciter and due to the difference between the measurement point and the excitation point in the interface driving point accelerance test are also discussed. Different characteristics of the five different kinds of errors are illustrated. The results show that the DOF-deficiency error causes the most catastrophic errors that cannot be simply corrected. On the contrary, the other four errors have a less severe consequence [19].
Theoretical Development for Motion Transmissibility

Partition of Governing Equations

Without loss of generality, assume the structure is made up of two parts. One is the test item and the other is the vehicle (See Figure 3.1). The test item and the vehicle are connected at the interface points only. The governing equation for this linear elastic structure under dynamic load is

$$[M][\dot{x}]+[C][\dot{x}]+[K][x]=\{f\}$$

(3.1)

where $[M]$, $[C]$, and $[K]$ stand for the mass, damping and stiffness matrices and $\{f\}$ the external forces and moments. All of the structure’s points are classified into one of three categories, i.e., points of interest (called $i$), grounded points (called $g$), and the other points (called $o$). The $M$, $C$ and $K$ matrices can be partitioned accordingly so that Equation 3.1 becomes

$$
\begin{bmatrix}
M_{ee} & M_{eo} & M_{eg} \\
M_{oe} & M_{oo} & M_{og} \\
M_{ge} & M_{go} & M_{gg}
\end{bmatrix}
\begin{bmatrix}
\dot{x}_e \\
\dot{x}_o \\
\dot{x}_g
\end{bmatrix}
+
\begin{bmatrix}
C_{ee} & C_{eo} & C_{eg} \\
C_{oe} & C_{oo} & C_{og} \\
C_{ge} & C_{go} & C_{gg}
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_e \\
\ddot{x}_o \\
\ddot{x}_g
\end{bmatrix}
+
\begin{bmatrix}
K_{ee} & K_{eo} & K_{eg} \\
K_{oe} & K_{oo} & K_{og} \\
K_{ge} & K_{go} & K_{gg}
\end{bmatrix}
\begin{bmatrix}
x_e \\
x_o \\
x_g
\end{bmatrix}
=
\begin{bmatrix}
f_e \\
f_o \\
f_g
\end{bmatrix}
$$

(3.2)

Equation 3.2 can be written in the frequency domain as

$$
\begin{bmatrix}
M_{ee} & M_{eo} & M_{eg} \\
M_{oe} & M_{oo} & M_{og} \\
M_{ge} & M_{go} & M_{gg}
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_e \\
\ddot{x}_o \\
\ddot{x}_g
\end{bmatrix}
+
\begin{bmatrix}
C_{ee} & C_{eo} & C_{eg} \\
C_{oe} & C_{oo} & C_{og} \\
C_{ge} & C_{go} & C_{gg}
\end{bmatrix}
\begin{bmatrix}
\dot{x}_e \\
\dot{x}_o \\
\dot{x}_g
\end{bmatrix}
+
\begin{bmatrix}
K_{ee} & K_{eo} & K_{eg} \\
K_{oe} & K_{oo} & K_{og} \\
K_{ge} & K_{go} & K_{gg}
\end{bmatrix}
\begin{bmatrix}
x_e \\
x_o \\
x_g
\end{bmatrix}
=
\begin{bmatrix}
f_e \\
f_o \\
f_g
\end{bmatrix}
$$

(3.3)

where $\{X\}$ and $\{F\}$ are the complex frequency domain equivalence that correspond to $\{x\}$ and $\{f\}$. 
Ground Excitation on the Entire Structure

Now suppose the structure is excited only by known ground motion \( \{X_g\} \) so that both \( \{F_e\} \) and \( \{F_o\} \) are zero while \( \{F^*\} \) is unknown. Then, Equation 3.3 reduces to

\[
\begin{align*}
\begin{bmatrix} X_e \\ X_o \end{bmatrix} &= \begin{bmatrix} -\omega^2 M_{ee} & M_{eo} \\ M_{oe} & M_{oo} \end{bmatrix} + j\omega \begin{bmatrix} C_{ee} & C_{eo} \\ C_{oe} & C_{oo} \end{bmatrix} + \begin{bmatrix} K_{ee} & K_{eo} \\ K_{oe} & K_{oo} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -\omega^2 M_{ee} \\ -\omega^2 M_{oe} \end{bmatrix} + j\omega \begin{bmatrix} C_{ee} \\ C_{oe} \end{bmatrix} + \begin{bmatrix} K_{ee} \\ K_{oe} \end{bmatrix}^{\text{Trans}} \{X_g\} 
\end{align*}
\]

(3.4)

from which the motion transmissibility matrix \([\text{Trans}]\) is given by

\[
[\text{Trans}] = \begin{bmatrix} -\omega^2 M_{ee} & M_{eo} \\ M_{oe} & M_{oo} \end{bmatrix} + j\omega \begin{bmatrix} C_{ee} & C_{eo} \\ C_{oe} & C_{oo} \end{bmatrix} + \begin{bmatrix} K_{ee} & K_{eo} \\ K_{oe} & K_{oo} \end{bmatrix}^{-1} \\
= \begin{bmatrix} -\omega^2 M_{ee} \\ -\omega^2 M_{oe} \end{bmatrix} + j\omega \begin{bmatrix} C_{ee} \\ C_{oe} \end{bmatrix} + \begin{bmatrix} K_{ee} \\ K_{oe} \end{bmatrix}^{\text{Trans}}
\]

(3.5)

Multiplying both sides of Equation 3.4 by \(-\omega^2\) shows that the acceleration transmissibility matrix is the same as the displacement transmissibility matrix. Hence, Equation 3.4 can be partitioned as

\[
\begin{bmatrix} -\omega^2 X_e \\ -\omega^2 X_o \end{bmatrix} = [\text{Trans}_{eg}] \begin{bmatrix} -\omega^2 X_g \end{bmatrix} 
\]

(3.6)

Ignoring the \(-\omega^2 X_o\) term and its transmissibility, we have the required relationship of

\[
\begin{bmatrix} -\omega^2 X_e \end{bmatrix} = [\text{Trans}_{eg}] \begin{bmatrix} -\omega^2 X_g \end{bmatrix}
\]

(3.7)

between the input ground acceleration and the output acceleration at the points of interest. The element in the \(i^{th}\) row and the \(m^{th}\) column of \([\text{Trans}_{eg}]\) represents the \(i^{th}\) directional acceleration at the measurement point due to the \(m^{th}\) grounding DOF.
Modeling the Test Item under Interfacial Loading

Equation 3.3 can be used to describe the test item when the matrices of \([M], [C], [K], \{x\}, \{f\}, \text{ and } \{F\}\) for the entire structure are renamed \([Mt], [Ct], [Kt], \{xt\}, \{ft\}, \text{ and } \{Fr\}\) for the test item, respectively. Thus, Equation 3.3 becomes

\[
\begin{bmatrix}
-Mt_{ee} & Mt_{eo} & Mt_{ei} \\
Mt_{oe} & Mt_{oo} & Mt_{oi} \\
Mt_{ie} & Mt_{io} & Mt_{ii}
\end{bmatrix}
\begin{bmatrix}
0 \\
jw \\
0
\end{bmatrix}
\begin{bmatrix}
Ct_{ee} & Ct_{eo} & Ct_{ei} \\
Ct_{oe} & Ct_{oo} & Ct_{oi} \\
Ct_{ie} & Ct_{io} & Ct_{ii}
\end{bmatrix}
\begin{bmatrix}
Kt_{ee} & Kt_{eo} & Kt_{ei} \\
Kt_{oe} & Kt_{oo} & Kt_{oi} \\
Kt_{ie} & Kt_{io} & Kt_{ii}
\end{bmatrix}
\begin{bmatrix}
Xt_e \\
Xt_o \\
Xt_i
\end{bmatrix}
=
\begin{bmatrix}
Ft_e \\
Ft_o \\
Ft_i
\end{bmatrix}
\tag{3.8}
\]

where \(e\) denotes for the point where acceleration measurements are made, \(i\) the interface points and \(o\) the other points. Then, defining the test item's accelerance matrix \([T]\) as

\[
[T] = -\omega^2
\begin{bmatrix}
Mt_{ee} & Mt_{eo} & Mt_{ei} \\
Mt_{oe} & Mt_{oo} & Mt_{oi} \\
Mt_{ie} & Mt_{io} & Mt_{ii}
\end{bmatrix}
+ jw
\begin{bmatrix}
Ct_{ee} & Ct_{eo} & Ct_{ei} \\
Ct_{oe} & Ct_{oo} & Ct_{oi} \\
Ct_{ie} & Ct_{io} & Ct_{ii}
\end{bmatrix}
+ \begin{bmatrix}
Kt_{ee} & Kt_{eo} & Kt_{ei} \\
Kt_{oe} & Kt_{oo} & Kt_{oi} \\
Kt_{ie} & Kt_{io} & Kt_{ii}
\end{bmatrix}
\tag{3.9}
\]

allows us to write Equation 3.8 as

\[
\begin{bmatrix}
-Mt_{ee}Xt_e \\
-Mt_{oe}Xt_o \\
-Mt_{ie}Xt_i
\end{bmatrix}
= \begin{bmatrix}
T_{ee} & T_{eo} & T_{ei} \\
T_{oe} & T_{oo} & T_{oi} \\
T_{ie} & T_{io} & T_{ii}
\end{bmatrix}
\begin{bmatrix}
Ft_e \\
Ft_o \\
Ft_i
\end{bmatrix}
\tag{3.10}
\]

Now, assume that only interface forces \(\{Ft_i\}\) are nonzero so that \(\{Ft_e\} = \{Ft_o\} = 0\). Also, assume that we are interested only in the motion of Points \(e\) and \(i\). Then, Equation 3.10 reduces to

\[
\begin{bmatrix}
-Mt_{ei}Xt_e \\
-Mt_{ii}Xt_i
\end{bmatrix}
= \begin{bmatrix}
T_{ei} \\
T_{ii}
\end{bmatrix}
\begin{bmatrix}
Ft_e \\
Ft_i
\end{bmatrix}
\tag{3.11}
\]
where $[T_{ii}]$ stands for the interfacial driving point accelerance matrix and $[T_{ei}]$ stands for the transfer accelerance matrix between Points $e$ and $i$.

**Modeling the Vehicle under Interfacial Loading and Ground Excitation**

The governing equation of the vehicle under interfacial load and ground motion is obtained from Equation 3.3 with the mass, damping and stiffness matrices being that of the vehicle; $[Mv]$, $[Cv]$, and $[Kv]$ and $\{Xv\}$ being the vehicle’s displacement. Thus, Equation 3.3 becomes

$$
\begin{pmatrix}
-Mv_{ii} & Mv_{io} & Mv_{ig} \\
Mv_{oi} & Mv_{oo} & Mv_{og}
\end{pmatrix}
\begin{pmatrix}
Cv_{ii} & Cv_{io} & Cv_{ig} \\
Cv_{oi} & Cv_{oo} & Cv_{og}
\end{pmatrix}
\begin{pmatrix}
Kv_{ii} & Kv_{io} & Kv_{ig} \\
Kv_{oi} & Kv_{oo} & Kv_{og}
\end{pmatrix}
\begin{pmatrix}
Xv_i \\
Xv_o
\end{pmatrix}
= \begin{pmatrix}
Fv_i \\
Fv_o
\end{pmatrix}
$$

(3.12)

Assume the vehicle is excited by the interfacial load $\{Fv_i\}$ and the ground motion $\{Xv_g\}$, and there are no external forces on Point $o$, then Equation 3.12 reduces to

$$
\begin{pmatrix}
Xv_i \\
Xv_o
\end{pmatrix}
= \left(-\omega^2 \begin{pmatrix} Mv_{ii} & Mv_{io} \\ Mv_{oi} & Mv_{oo} \end{pmatrix}
+ j\omega \begin{pmatrix} Cv_{ii} & Cv_{io} \\ Cv_{oi} & Cv_{oo} \end{pmatrix}
+ \begin{pmatrix} Kv_{ii} & Kv_{io} \\ Kv_{oi} & Kv_{oo} \end{pmatrix}\right)^{-1}
\begin{pmatrix}
Fv_i \\
0
\end{pmatrix}
$$

(3.13)

Now, we define the accelerance matrix of the grounded vehicle to be

$$
\begin{pmatrix}
Mv_{ii} & Mv_{io} \\
Mv_{oi} & Mv_{oo}
\end{pmatrix}
\begin{pmatrix}
Cv_{ii} & Cv_{io} \\
Cv_{oi} & Cv_{oo}
\end{pmatrix}
\begin{pmatrix}
Kv_{ii} & Kv_{io} \\
Kv_{oi} & Kv_{oo}
\end{pmatrix}
\begin{pmatrix}
Xv_i \\
Xv_o
\end{pmatrix}
= \begin{pmatrix}
Fv_i \\
Fv_o
\end{pmatrix}
$$

(3.14)

and using a procedure similar to Equation 3.5, we define the motion transmissibility matrix of the grounded bare vehicle to be
Then, Equation 3.13 reduces to

\[
\begin{align*}
\{X_v, X_v\} &= \left[\begin{array}{cc} V_{ii} & V_{io} \\
V_{oi} & V_{oo}
\end{array}\right]\{F_v\} + \left[\begin{array}{c} \text{Trans}^*_{ig} \\
\text{Trans}^*_{oe}
\end{array}\right]\left(-\omega^2 X_v\right)
\end{align*}
\]

(3.16)

Since we are only interested in points \(i\), Equation 3.16 can be further reduced to

\[
\{X_v\} = \left[\begin{array}{c} V_{ii} \\
V_{oi}
\end{array}\right]\{F_v\} + \left[\begin{array}{c} \text{Trans}^*_{ig} \\
\text{Trans}^*_{oe}
\end{array}\right]\left(-\omega^2 X_v\right)
\]

(3.17)

where \([V_{ii}]\) stands for the interface driving point accelerance of the vehicle. In fact, Equation 3.17 shows the interface motion of the vehicle alone is the superposition of two motions, one caused by the ground motion \(\{X_v\}\) and the other by the interface load \(\{F_v\}\).

**Complete Coupling of the Test Item and the Vehicle**

Equations 3.10 and 3.17 govern the behavior of the test item under the interface forces and moments, and the behavior of the vehicle under the ground motion and interface loading. Now, we need to combine these two substructure equations for the case when the two structures are connected together at the interface. In this section, we shall build models for complete coupling of all forces, moments, and motions at the interface. Next section an incomplete coupling where one or more of the interface forces or moments are ignored is discussed. This is called a DOF deficiency error.
Let us take all the interface degrees of freedom into account. The interface conditions include continuity of all displacements (or accelerations) and reciprocity of force and moments so that

\[ \{X_t\} = \{X_v\} = \{X_i\}, \quad \{F_t\} = \{-F_v\} = \{F_i\} \quad (3.18) \]

Putting the interface conditions Equation 3.18 into Equation 3.11 and Equation 3.17, gives

\[ \{-\omega^2 X_t\} = [T_u] [F_i], \quad \{-\omega^2 X_v\} = [T_u] [-F_i] + [\text{Trans}_{eg}] \{-\omega^2 X_g\} \quad (3.19) \]

If \{X_t\} is replaced with \{X_e\}, and \{X_v\} is replaced with \{X_g\}, we obtain

\[ \{-\omega^2 X_e\} = [T_{ei}] [T_u + V_u]^{-1} [\text{Trans}_{eg}] \{-\omega^2 X_g\} = [\text{Trans}_{eg}] \{-\omega^2 X_g\} \quad (3.20) \]

Equation 3.20 is the equation that relates the motion at the measurement point on the test item to the vehicle's input ground motion. Recall the definition of \([\text{Trans}_{eg}]\) in Equation 3.7 where we find that

\[ [\text{Trans}_{eg}] = [T_{ei}] [T_u + V_u]^{-1} [\text{Trans}']_{eg} \quad (3.21) \]

where it is obvious that the two transmissibilities are the same. It is clear that the motion transmissibility matrix of the combined system \([\text{Trans}_{eg}]\) is related to the bare vehicle motion transmissibility matrix \([\text{Trans}']_{eg}\), the interface driving point accelerance matrices of the bare vehicle \([V_u]\) and the test item \([T_{ei}]\), and the transfer accelerance matrix of the test item \([T_{ei}]\). If we can measure the right hand side elements in Equation 3.21, we can obtain its left hand side through computation.
Error Analysis

To experimentally implement Equation 3.21, we need to measure three accelerance matrices \([T_{el}], [T_{in}], [V_{il}]\) and one motion transmissibility matrix \([\text{Trans}_{pi}^*]\). However, errors cannot be completely avoided in experiments. Five kinds of error are examined here, namely, DOF deficiency errors at the interface, bias errors, noise errors, rocking motion errors, and different excitation and measurement points errors.

**DOF Deficiency Error**

DOF deficiency error occurs when some DOF’s are ignored. Suppose in Equation 3.21 that the interface DOF’s are three local coordinates set on the interface called, \(x', y'\) and \(\theta'\). The corresponding general forces are shear force along \(x'\) axis, normal force along \(y'\) axis and moment about \(\theta\) axis. Without loss of generality, assume \(x', y'\) and \(\theta\) correspond to the global coordinates, \(x, y\) and \(\theta\), otherwise a coordinate transform matrix is needed which makes the problem more complicated. If the measured accelerations are along \(x, y\) and \(\theta\), and ground motion contains only \(y\) direction component, Equation 3.21 becomes

\[
\begin{bmatrix}
-\omega^2 X_1^e \\
-\omega^2 X_2^e \\
-\omega^2 X_3^e
\end{bmatrix} = 
\begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33}
\end{bmatrix} 
\begin{bmatrix}
T_{11}^* + V_{11}^* \\
T_{21}^* + V_{21}^* \\
T_{31}^* + V_{31}^*
\end{bmatrix} 
\begin{bmatrix}
T_{12}^* + V_{12}^* \\
T_{22}^* + V_{22}^* \\
T_{32}^* + V_{32}^*
\end{bmatrix} 
\begin{bmatrix}
T_{13}^* + V_{13}^* \\
T_{23}^* + V_{23}^* \\
T_{33}^* + V_{33}^*
\end{bmatrix}^{-1} \begin{bmatrix}
\text{Trans}_{12}^* \\
\text{Trans}_{22}^* \\
\text{Trans}_{32}^*
\end{bmatrix} 
\begin{bmatrix}
-\omega^2 X_1^e \\
-\omega^2 X_2^e \\
-\omega^2 X_3^e
\end{bmatrix}
\]

(3.22)

where \(X_1^e\) corresponds to the \(x\) motion, \(X_2^e\) corresponds to the \(y\) motion, and \(X_3^e\) corresponds to the \(\theta\) motion at each point \(e\), respectively. The \([\text{Trans}_{pi}^*]\) term represents
the vehicle transmissibility between points \( i \) and \( g \) for the output direction \( p \) and input direction \( q \). The \( \left[ Trans_{pi}^{pg} \right] \) term represents the overall transmissibility between the output point \( e \) and ground input point \( g \).

If we ignore the interface rotational degree of freedom (the \( \theta \) DOF), then Equation 3.22 becomes

\[
\begin{align*}
- \omega^2 X_1^e &= \left[ T_{11} \quad T_{12} \right] \begin{bmatrix} T_{11}^{ii} + V_{11}^{ii} & T_{12}^{ii} + V_{12}^{ii} \end{bmatrix}^{-1} \left[ Trans_{12}^{pg} \right] \begin{bmatrix} - \omega^2 X_2^f \end{bmatrix} \\
- \omega^2 X_2^e &= T_{21} \begin{bmatrix} T_{22}^{ii} + V_{22}^{ii} \end{bmatrix}^{-1} \left[ Trans_{22}^{pg} \right] \begin{bmatrix} - \omega^2 X_2^f \end{bmatrix} \\
- \omega^2 X_3^e &= \left[ T_{31} \quad T_{32} \right] \begin{bmatrix} T_{31}^{ii} + V_{31}^{ii} \end{bmatrix}^{-1} \left[ Trans_{32}^{pg} \right] \begin{bmatrix} - \omega^2 X_2^f \end{bmatrix}
\end{align*}
\]

Equation 3.23

If we ignore the interface horizontal translation degree of freedom (the \( x \) DOF), then Equation 3.22 becomes

\[
\begin{align*}
- \omega^2 X_1^e &= \left[ T_{12} \quad T_{13} \right] \begin{bmatrix} T_{22}^{ii} + V_{22}^{ii} & T_{23}^{ii} + V_{23}^{ii} \end{bmatrix}^{-1} \left[ Trans_{22}^{pg} \right] \begin{bmatrix} - \omega^2 X_2^f \end{bmatrix} \\
- \omega^2 X_2^e &= T_{22} \begin{bmatrix} T_{22}^{ii} + V_{22}^{ii} \end{bmatrix}^{-1} \left[ Trans_{22}^{pg} \right] \begin{bmatrix} - \omega^2 X_2^f \end{bmatrix} \\
- \omega^2 X_3^e &= \left[ T_{32} \quad T_{33} \right] \begin{bmatrix} T_{32}^{ii} + V_{32}^{ii} \end{bmatrix}^{-1} \left[ Trans_{32}^{pg} \right] \begin{bmatrix} - \omega^2 X_2^f \end{bmatrix}
\end{align*}
\]

Equation 3.24

If only the \( y \) DOF is considered at the interface, then Equation 3.23 and Equation 3.24 can be further reduced to

\[
\begin{align*}
- \omega^2 X_1^e &= \left[ T_{12} \right] \begin{bmatrix} T_{22}^{ii} + V_{22}^{ii} \end{bmatrix}^{-1} \left[ Trans_{22}^{pg} \right] \begin{bmatrix} - \omega^2 X_2^f \end{bmatrix} \\
- \omega^2 X_2^e &= \left[ T_{22} \right] \begin{bmatrix} T_{22}^{ii} + V_{22}^{ii} \end{bmatrix}^{-1} \left[ Trans_{22}^{pg} \right] \begin{bmatrix} - \omega^2 X_2^f \end{bmatrix} \\
- \omega^2 X_3^e &= \left[ T_{32} \right]
\end{align*}
\]

Equation 3.25

A comparison of Equations 3.23, 3.24, and 3.25 with Equation 3.22 clearly shows the magnitude of the problem of using a small subset of the entire interface dynamic matrices. It is no small wonder that experimental modal analysis and substructure have produced poor results when attempting to predict the behavior of two structures when they are combined from an inadequate set of measured data.
**Bias Error**

The bias error is most likely caused by poor force transducer calibration so it affects all measured data. It is characterized by a uniform shift in the FRF data. By multiplying each element of measured vehicle and test item FRF matrix with the same constant, we try to simulate the effect of a bias error on the overall transmissibility.

**Noise Error**

If the measured FRF matrices \([T_{ii}], [V_{ii}], [T_{ei}]\) and the bare vehicle transmissibility matrix \([\text{Trans}_{ig}]\) are noisy, the computed global transmissibility matrix \([\text{Trans}_{eg}]\) loses its accuracy. To simulate the noise effect on one of the above matrices, the RMS value of each FRF in the matrix is computed using Parsaval's formula first. Then random noise is generated and added to that FRF frequency by frequency. The magnitude and the phase of the noise are all random. Its RMS value is a fraction of the RMS value of the original FRF. The same process is repeated until every FRF of that matrix is contaminated by noise.

Let the noises with respect to \([T_{ei}], [T_{ii}], [V_{ii}], \) and \([\text{Trans}_{ig}]\) be written as \([NT_{ei}], \)

\([NT_{ii}], [NV_{ii}], \text{ and } [N\text{Trans}_{ig}]\). Then, Equation 3.21 becomes

\[
\begin{align*}
\{-\omega^2 X_s\} &= \left[T_{ei} + N_{ei}\right]^{-1}[T_{ii} + NT_{ii} + V_{ii} + NV_{ii}]^{-1}[\text{Trans}_{ig} + N\text{Trans}_{ig}]\{-\omega^2 X_s\} \\
&= \left[\text{Trans}_{eg}'\right]^{-1}[\text{Trans}_{eg}]\{-\omega^2 X_s\}
\end{align*}
\]

(3.26)

\([\text{Trans}_{eg}']\) is compared with \([\text{Trans}_{eg}]\) to see the noise effects.
Rocking Motion Error

In transmissibility test the theoretical excitation should be in the vertical direction only. However, in reality, very few exciters do not have rocking motion at certain frequencies. If the vehicle is excited at a single point, we might use a stinger to decrease the moment transmitted to the test setup in order to reduce the rocking motion. However, if we have to excite the structure at more than two points with a single exciter, rocking motion is unavoidable. We can model the rocking motion either by introducing a rotational DOF at the ground or by using two points of excitation \( g_1 \) and \( g_2 \) with different magnitudes of \( y \) motion.

Since in Equation 3.22, \( g \) stands for the grounding points, \( g \) can be greater that unity. Let \( g \) be \( g_1 \) and \( g_2 \). With the rocking motion included, Equation 3.22 is changed to

\[
\begin{align*}
\begin{bmatrix}
-\omega^2 X_1' \\
-\omega^2 X_2' \\
-\omega^2 X_3'
\end{bmatrix} &=
\begin{bmatrix}
T'_{11} & T'_{12} & T'_{13} \\
T'_{21} & T'_{22} & T'_{23} \\
T'_{31} & T'_{32} & T'_{33}
\end{bmatrix}
\begin{bmatrix}
TV'_{11} & TV'_{12} & TV'_{13} \\
TV'_{21} & TV'_{22} & TV'_{23} \\
TV'_{31} & TV'_{32} & TV'_{33}
\end{bmatrix}^{-1}
\begin{bmatrix}
\text{Trans}^{g_1}_{12} & \text{Trans}^{g_1}_{13} \\
\text{Trans}^{g_1}_{22} & \text{Trans}^{g_1}_{23} \\
\text{Trans}^{g_1}_{32} & \text{Trans}^{g_1}_{33}
\end{bmatrix}
\begin{bmatrix}
-\omega^2 X_1'^1 \\
-\omega^2 X_2'^1 \\
-\omega^2 X_3'^1
\end{bmatrix} \\
= &
\begin{bmatrix}
\text{Trans}^{g_1}_{12} & \text{Trans}^{g_1}_{13} \\
\text{Trans}^{g_1}_{22} & \text{Trans}^{g_1}_{23} \\
\text{Trans}^{g_1}_{32} & \text{Trans}^{g_1}_{33}
\end{bmatrix}
\begin{bmatrix}
-\omega^2 X_1'^1 \\
-\omega^2 X_2'^1 \\
-\omega^2 X_3'^1
\end{bmatrix}
\end{align*}
\]  

(3.27)

Suppose the rocking motion is within the x-y plane, that is, it can be described by \( \theta \), we have

\[
X_2'^1 = X_2'^c + b_{c \rightarrow g_1} \theta, \quad X_2'^2 = X_2'^c + b_{c \rightarrow g_2} \theta
\]

(3.28)

where \( X_2'^c, X_2'^1, \) and \( X_2'^2 \) are the y directional motions at the center \( c \) of the excitation table, point \( g_1 \) and point \( g_2 \), respectively, \( b_{c \rightarrow g_1} \) and \( b_{c \rightarrow g_2} \) are the distances from the center \( c \) to \( g_1 \) and \( g_2 \), and \( \theta \) is the rotating angle. Substitution of Equation 3.28 into Equation 3.27 gives
\begin{align*}
\begin{pmatrix}
-\omega^2 X_1^x \\
-\omega^2 X_2^x \\
-\omega^2 X_3^x 
\end{pmatrix} &= \begin{bmatrix} T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} \begin{pmatrix} TV_{11} & TV_{12} & TV_{13} \end{pmatrix} \\
\begin{pmatrix} TV_{21} & TV_{22} & TV_{23} \end{pmatrix} \\
\begin{pmatrix} TV_{31} & TV_{32} & TV_{33} \end{pmatrix} \end{pmatrix}^{-1} \begin{bmatrix}
\text{Trans}^{\text{eq1}}_{12} + \text{Trans}^{\text{eq1}}_{13} \\
\text{Trans}^{\text{eq1}}_{22} + \text{Trans}^{\text{eq1}}_{23} \\
\text{Trans}^{\text{eq1}}_{32} + \text{Trans}^{\text{eq1}}_{33} \end{bmatrix} \\
&\begin{bmatrix}
\text{Trans}^{\text{eq1}}_{12} b_{c_{x1}} + \text{Trans}^{\text{eq1}}_{13} b_{c_{x1}} \\
\text{Trans}^{\text{eq1}}_{22} b_{c_{x1}} + \text{Trans}^{\text{eq1}}_{23} b_{c_{x1}} \\
\text{Trans}^{\text{eq1}}_{32} b_{c_{x1}} + \text{Trans}^{\text{eq1}}_{33} b_{c_{x1}} \end{bmatrix} \\
&\begin{bmatrix}
-\omega^2 X_1^x \\
-\omega^2 X_2^x \\
-\omega^2 X_3^x \end{bmatrix}
\end{align*}

or

\begin{align*}
\begin{pmatrix}
-\omega^2 X_1^x \\
-\omega^2 X_2^x \\
-\omega^2 X_3^x 
\end{pmatrix} &= \begin{bmatrix} \text{Trans}^{\text{eq21}}_{12} + \text{Trans}^{\text{eq22}}_{12} \\
\text{Trans}^{\text{eq21}}_{22} + \text{Trans}^{\text{eq22}}_{22} \\
\text{Trans}^{\text{eq21}}_{32} + \text{Trans}^{\text{eq22}}_{32} \end{bmatrix} \begin{bmatrix}
\text{Trans}^{\text{eq21}}_{12} b_{c_{x1}} + \text{Trans}^{\text{eq22}}_{12} b_{c_{x1}} \\
\text{Trans}^{\text{eq21}}_{22} b_{c_{x1}} + \text{Trans}^{\text{eq22}}_{22} b_{c_{x1}} \\
\text{Trans}^{\text{eq21}}_{32} b_{c_{x1}} + \text{Trans}^{\text{eq22}}_{32} b_{c_{x1}} \end{bmatrix} \\
&\begin{bmatrix}
-\omega^2 X_1^x \\
-\omega^2 X_2^x \\
-\omega^2 X_3^x \end{bmatrix}
\end{align*}

In Equation 3.29 and Equation 3.30, the terms related to \( \theta \) are due to the base rocking motion. When rocking motion exists, we simply cannot measure Trans\(_{12}^{\text{eq1}}\) etc. in Equation 3.29 because the input motion is not a pure \( y \) directional ground motion.

In a special case, assume \( \theta \) is proportional to \( X_2^x \), i.e.,

\[ \theta = kX_2^x \tag{3.31} \]

where \( k \) is a complex constant, then

\begin{align*}
\begin{pmatrix}
-\omega^2 X_1^x \\
-\omega^2 X_2^x \\
-\omega^2 X_3^x 
\end{pmatrix} &= \begin{bmatrix} T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} \begin{pmatrix} TV_{11} & TV_{12} & TV_{13} \end{pmatrix} \\
\begin{pmatrix} TV_{21} & TV_{22} & TV_{23} \end{pmatrix} \\
\begin{pmatrix} TV_{31} & TV_{32} & TV_{33} \end{pmatrix} \end{pmatrix}^{-1} \begin{bmatrix}
k_1\text{Trans}^{\text{eq1}}_{12} + k_2\text{Trans}^{\text{eq1}}_{13} \\
\text{Trans}^{\text{eq21}}_{22} + \text{Trans}^{\text{eq22}}_{23} \\
\text{Trans}^{\text{eq21}}_{32} + \text{Trans}^{\text{eq22}}_{33} \end{bmatrix} \\
&\begin{bmatrix}
k_1\text{Trans}^{\text{eq1}}_{12} b_{c_{x1}} + k_2\text{Trans}^{\text{eq1}}_{13} b_{c_{x1}} \\
\text{Trans}^{\text{eq21}}_{22} b_{c_{x1}} + \text{Trans}^{\text{eq22}}_{23} b_{c_{x1}} \\
\text{Trans}^{\text{eq21}}_{32} b_{c_{x1}} + \text{Trans}^{\text{eq22}}_{33} b_{c_{x1}} \end{bmatrix} \\
&\begin{bmatrix}
-\omega^2 X_1^x \\
-\omega^2 X_2^x \\
-\omega^2 X_3^x \end{bmatrix}
\end{align*}

\[ \begin{align*}
\begin{pmatrix}
\text{Trans}^{\text{eq1}}_{12} + \text{Trans}^{\text{eq1}}_{13} \\
\text{Trans}^{\text{eq1}}_{22} + \text{Trans}^{\text{eq1}}_{23} \\
\text{Trans}^{\text{eq1}}_{32} + \text{Trans}^{\text{eq1}}_{33} \end{pmatrix} &= \begin{bmatrix} k_1\text{Trans}^{\text{eq21}}_{12} + k_2\text{Trans}^{\text{eq22}}_{12} \\
\text{Trans}^{\text{eq21}}_{22} + \text{Trans}^{\text{eq22}}_{22} \\
\text{Trans}^{\text{eq21}}_{32} + \text{Trans}^{\text{eq22}}_{32} \end{bmatrix} \\
&\begin{bmatrix}
-\omega^2 X_1^x \\
-\omega^2 X_2^x \\
-\omega^2 X_3^x \end{bmatrix}
\end{align*} \tag{3.32} \]
\[ k_1 = 1 + k_b_{c-x}, \quad \text{and} \quad k_2 = 1 + k_b_{c-y}, \] (3.33)

and

\[
\begin{bmatrix}
  k_1 \text{Trans}_{10}^{\text{eq}} + k_2 \text{Trans}_{12}^{\text{eq}} \\
  k_1 \text{Trans}_{22}^{\text{eq}} + k_2 \text{Trans}_{22}^{\text{eq}} \\
  k_1 \text{Trans}_{32}^{\text{eq}} + k_2 \text{Trans}_{32}^{\text{eq}} 
\end{bmatrix} =
\begin{bmatrix}
  T_{11} & T_{12} & T_{13} \\
  T_{21} & T_{22} & T_{23} \\
  T_{31} & T_{32} & T_{33}
\end{bmatrix}
\begin{bmatrix}
  T_{11} & T_{12} & T_{13} \\
  T_{21} & T_{22} & T_{23} \\
  T_{31} & T_{32} & T_{33}
\end{bmatrix}^{-1}
\begin{bmatrix}
  k_1 \text{Trans}_{10}^{\text{eq}} + k_2 \text{Trans}_{12}^{\text{eq}} \\
  k_1 \text{Trans}_{22}^{\text{eq}} + k_2 \text{Trans}_{22}^{\text{eq}} \\
  k_1 \text{Trans}_{32}^{\text{eq}} + k_2 \text{Trans}_{32}^{\text{eq}}
\end{bmatrix}
\] (3.34)

The assumption here is that the rocking motion remains the same for both the complete structure and the bare vehicle cases. However, this new transmissibility matrix is different from the original transmissibility matrix defined on pure \( y \) directional motion.

\[
\begin{bmatrix}
  \text{Trans}_{12}^{\text{eq}} \\
  \text{Trans}_{22}^{\text{eq}} \\
  \text{Trans}_{32}^{\text{eq}}
\end{bmatrix} =
\begin{bmatrix}
  \text{Trans}_{12}^{\text{eq}} + \text{Trans}_{12}^{\text{eq}} \\
  \text{Trans}_{22}^{\text{eq}} + \text{Trans}_{22}^{\text{eq}} \\
  \text{Trans}_{32}^{\text{eq}} + \text{Trans}_{32}^{\text{eq}}
\end{bmatrix} =
\begin{bmatrix}
  k_1 \text{Trans}_{10}^{\text{eq}} + k_2 \text{Trans}_{12}^{\text{eq}} \\
  k_1 \text{Trans}_{22}^{\text{eq}} + k_2 \text{Trans}_{22}^{\text{eq}} \\
  k_1 \text{Trans}_{32}^{\text{eq}} + k_2 \text{Trans}_{32}^{\text{eq}}
\end{bmatrix}
\] (3.35)

**Driving Point Difference Error**

In order to conduct driving point tests for \([T_{ii}]\) and \([V_{ii}]\), theoretically we need to apply a force and/or a moment, and measure the linear and/or angular accelerations at the same point. However, experimentally it is very difficult to put the accelerometers at a point and hit at the same point. The common practice is to hit at a nearby point that is very close to the accelerometer and treat the two points as the same point. It is interesting to see how much error in \([T_{ii}]\) and \([V_{ii}]\) is caused by a driving point difference. We first find the global transmissibility matrix \([\text{Trans}_{eq}]\) using the complete structure test. This result does not have driving point difference problem. Then the substructure method is used to compute \([\text{Trans}_{eq}]\) with the experimental \([T_{ii}]\) and \([V_{ii}]\) matrices. Here \([T_{ii}]\) and \([V_{ii}]\) have driving point
difference problem, that is, the measurement point is not the same point as the impact point. We numerically investigate the distorting effect on [Transeq] based on the distance of the hit point and the measurement point.

Suppose we put a transducer on the vehicle, and we hit in the vicinity of the transducer, about half an inch away, which is reasonable for testing. \([V_i]\) changes to \([V_{ii'}]\), where \(i\) is the measurement point and \(i'\) is the excitation point. Expanding Equation 3.14 by differentiating point \(i'\) from the rest of points \(o\), we find

\[
[V] = -\omega^2 \left[ \begin{bmatrix} M_{ii} & M_{ii'} & M_{io} \\ M_{i'i} & M_{i'i'} & M_{i'o} \\ M_{io} & M_{i'o} & M_{oo} \end{bmatrix} \right]^{-1} \begin{bmatrix} C_{ii} & C_{ii'} & C_{io} \\ C_{i'i} & C_{i'i'} & C_{i'o} \\ C_{io} & C_{i'o} & C_{oo} \end{bmatrix} \begin{bmatrix} K_{ii} & K_{ii'} & K_{io} \\ K_{i'i} & K_{i'i'} & K_{i'o} \\ K_{io} & K_{i'o} & K_{oo} \end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix} V_{ii} & V_{ii'} & V_{io} \\ V_{i'i} & V_{i'i'} & V_{i'o} \\ V_{io} & V_{i'o} & V_{oo} \end{bmatrix}
\]

(3.36)

Structurally it is easy to give explicit expressions to \(K_{ii'}\), as it has only several contributors, i.e., connecting elements between \(i\) and \(i'\). However, it is difficult to find explicit expressions for \(V_{ii'}\) due to the inverse involved. As a result, \(V_{ii'}\) involves the contribution from the whole structure. Numerical simulation can be used to evaluate the significance of this problem.

**Numerical Investigation**

A portal frame with an L shaped bracket attached at its mid-point interface as shown in Figure 3.1 is used to demonstrate the effects of ignoring interface forces and moments, measurement bias error, and measurement noise. Figure 3.2 shows the substructures isolated
for measurement of $[V_u]$, $[T_u]$, and $[T_{ui}]$. Figure 3.3 shows the test arrangement for measuring the vehicle’s interface-ground motion transmissibility $[Trans_{ig}^*]$. The overall physical dimensions are shown in Figure 3.4. The material is assumed to be steel with a nominal cross-section of 6.35 mm thick x 25.4 mm deep normal to the page (1/4 x 1.0 inches).

The finite element model of the structure uses 2-D frame element whose stiffness matrix and mass matrix are superimposed by that of a 2-D beam element and a 2-D truss element. Each element has two nodes with three degrees of freedom, i.e., axial, transverse and angular at each node. The damping is assumed to be proportional to the mass and stiffness distribution so there is no modal coupling through damping. The damping ratios for the first two elastic modes are assumed to be 0.02 and the proportional damping coefficients are determined by a method described by [4].

![Diagram of test arrangement and connection](image)

Figure 3.1. Overall transmissibility of the combined structure from a direct test
Figure 3.2 Driving point accelerance and transfer point accelerance measurement

Figure 3.3 Transmissibility test of the bare vehicle

Figure 3.4 Physical dimensions of structures analyzed
Results of Simulations

Complete Model Using Direct Analysis and Substructure Analysis

Figure 3.5 shows the transmissibility of output motions \(x, y, \theta\) at point \(e\) due to input ground motion \(y\), when computed by the direct analysis with Equation 3.9 and the complete substructure coupling analysis of Equation 3.20. Obviously when both shear force and moment DOF are considered, the transmissibility calculated from the substructure method is exactly the same as that acquired from the direct analysis method.

DOF Deficiency Error

Figure 3.6 shows a comparison of transmissibility of the \(x, y, \theta\) motions at point \(e\) due to input ground motions \(y\) when the \(x\) and/or \(\theta\) DOF are absent. The curves obtained from substructure analysis deviates greatly from that obtained from the complete structure analysis without neglecting any interfacial DOF. For instance, in the first row where the \(x\) DOF is ignored, we found that in the first plot for the \(x\) motion, the peaks are shifted 5 - 20 Hz to the right, and the shifting is not uniform. In the second plot for the \(y\) motion, two peaks and a valley become one peak and the other peaks and valleys are also shifted to the right. In the

![Figure 3.5. Comparison of transmissibilities for \(x, y, \theta\) motions at point \(e\) due to input ground motion at point \(f\) when using the direct and complete substructure models.](image-url)
third plot for the $\theta$ direction, two peaks are higher, two peaks are lower, and all the peaks and valleys are shifted right. It seems the transmissibility curves that are obtained from incomplete coupling are, at best, casually related to those for complete coupling. In the second row where the $\theta$ DOF is ignored, and in the third row where both $x$ and $\theta$ DOF are absent, we obtained similar trends. Obviously the DOF deficiency problem is such a serious problem that without including all interfacial DOF the results are meaningless.

Figure 3.6. Comparison of transmissibilities for $x$, $y$, $\theta$ motions at point $e$ due to $y$ input ground motion at point $g$ when ignoring the $x$ DOF, $\theta$ DOF, $x$ and $\theta$ DOF.
Bias Error

Figure 3.7 shows the transmissibility of the $x, y, \theta$ motions at point $e$ due to the input ground motion $y$ at point $f$ with each accelerance element in $[T_{il}], [V_{il}], [T_{el}]$ multiplied by 1.2. In the first row, we can see that the peaks and valleys are shifted a little RIGHT and are a little lower, except in the third plot, one peak is a little higher. In the second row, the peaks

![Graphs showing transmissibilities for x, y, \theta motions at point e due to y directional ground motion at point g with each element in [T_{il}], [V_{il}], and [T_{el}] multiplied by 1.2]
and valleys are shifted a little LEFT and are a little lower, except in the third plot, one peak is a little higher. In the third row, all the peaks and valleys are a little higher. It simply lifts the peak and valleys without shifting the peak and valley frequencies. The effect of 1.2 times $[T_{el}]$ is the same as 1.2 times $[Trans'_{el}]$. Generally, the bias error is not as severe as the DOF deficiency error. In addition, a 20% bias error is considered quite large in reality.

**Noise Error**

Figure 3.8 shows the comparison of transmissibility for $x$, $y$, and $\theta$ motions due to $y$ input ground motion with each element in $[T_{u}]$, $[V_{u}]$, $[T_{el}]$, $[Trans'_{el}]$ having 5% noise added. 100 averages are applied to remove the noise effect. In the first row, the noise contaminated the 0 to 50 Hz range by smearing the first peak and the first valley in each case. Several new peaks and valleys are created with lower noisy peaks and higher noisy valleys. The high frequency range appears to show little effect from the noise. The second row observes a similar trend. It seems the effect of noise on $[T_{u}]$, and $[V_{u}]$ is limited to the low frequencies as well.

The third row shows something interesting for the noise contamination is essentially nonexistent over the entire range for the first two plots. However, in the third plot, the noise contamination is severe below 50 Hz while it is almost nonexistent at frequencies above than 50 Hz. There is no significant noise contamination in the fourth row. Hence, it appears that $[T_{el}]$ and $[Trans'_{el}]$ are relatively insensitive to noise contamination when compared to $[T_{u}]$ and $[V_{u}]$. The reason that $[T_{u}]$ and $[V_{u}]$ are more noise sensitive is that the summation of these two terms need to go through the inverse process, a process that is noise sensitive.
Figure 3.8. Comparison of transmissibilities for $x$, $y$, and $\theta$ motions at point $e$ due to $y$-directional ground motion, each element in $[T_{ul}]$, $[V_{ul}]$, $[T_{ul}]$, $[\text{Trans}_{eg}]$ is added 5% noise.
**Rocking Motion Error**

Numerically we assume the rocking motion is in proportional to the linear $y$ motion.

In the first case, $k_1$ and $k_2$ in Equation 3.35 are set to 0.9 and 1.1 respectively. $g_1$ and $g_2$ are at the opposite sides of the center and their distance to the center are the same so the rocking motion part of $g_1$ and $g_2$ are anti-symmetric with respect to the center. In the second case, $k_1$ and $k_2$ are set to 0.9 and 1.15 respectively. The rocking motions of $g_1$ and $g_2$ are no longer anti-symmetric with respect to the center of the exciter. Figure 3.9 compares the transmissibility for $x, y, \theta$ motions at point $e$ due to pure $y$ input ground motion, and $y$ and $\theta$ motion. From the plots, it is clear the peaks have little inferences from the rocking motions and the valleys suffer more from it. Generally the rocking motion is not a significant error to the final results.

![Graphs](image)

*Figure 3.9 Comparison of transmissibilities for $x, y, \theta$ motions at point $e$ due to $y$-directional ground motion and $y-\theta$ rocking motions*
Driving Point Difference Error

Four cases of the driving point difference error are studied. Figure 3.10 compares the $x$, $y$, and $\theta$ motion transmissibility vs. $y$ ground motion obtained from two methods. The first method is the complete structure analysis, which is immune to the driving point difference error. The second method is the substructure analysis where the driving point accelerance matrix $[V_{tr}]$ is used to replace $[V_{tr}]$. The acceleration measurement point is $i$ and the impact point is $i'$. $i'$ is placed 0.5", 1" to the left of $i$, and 0.5", 1" to the right of $i$ respectively.

The transmissibility results show that for the general trend of all the plots are the same, and all the peaks computed from two methods are close. The difference of peaks happens mainly at the $\theta$ motion transmissibility plots in each row. The second and fourth peaks of these plots are changed up to 5 dB. Also affected is the fourth peak of the $y$ motion transmissibility plots in each row. The fourth peak is changed up to 5 dB too. The frequency shift of all the peaks is barely noticeable except for the fourth peak in the last two plots of the fourth row.

When the distance between $i'$ and $i$ increases, the notch differences grows larger as well as some of the peak differences. The notch discrepancy is even larger than the peak discrepancy. For the $x$ motion transmissibility, the last notch is shifted as much as 20 Hz in each direction from the original 118 Hz due to the driving point difference problem. For the $y$ motion transmissibility, the magnitude of the third notch is changed less than 5 dB. The $\theta$ motion transmissibility plot in the fourth row shows the shapes of the first notch and the second notch exchange. Since 1" of driving point difference is realistic in a 24" test structure, the driving point difference is an error source in this test. However it is still rather
Figure 3.10 Comparison of $x$, $y$, and $\theta$ motion transmissibilities vs. $y$ ground motion from a complete structure analysis and a substructure analysis where $[V_u']$ is in place of $[V_u]$
“mild” compared with DOF deficiency errors. For general structure, the impact of driving point difference error on the test results needs further research. It is anticipated that the larger the structure, the larger the distance limit is.

Chapter Summary

In a finite element analysis, with all the material properties known and proportional damping ratio assumption made, it is an easy "forward" problem to derive the FRF and the transmissibility of all pieces of substructures and couple them with interfacial displacement compatibility and force balance conditions. In reality, experiments are performed to determine the FRF and transmissibility of these pieces of substructures without sufficient prior knowledge on the material property and the characteristics of damping. Because the FRF and transmissibility data might be contaminated because of DOF deficiency, measurement system bias, random noise, exciter rocking motion, and driving point difference errors, their precision is often questionable. Consequently, the credibility of the global transmissibility matrix that is computed using these measured FRF and the bare vehicle transmissibility is at stake. From the above study, we conclude

1. Moments and shear forces are crucial components in the interface actions. Ignoring the DOF corresponding to moments and shear forces at the substructure interface will invalidate the substructure procedure for predicting coupled structure’s behavior. In reality, ALL INTERFACE DOF and their corresponding interface forces (and moments) and motions (linear and angular) must be considered unless there is a special case that can be proven otherwise. Ignoring the effects of shear forces and/or moments at the
interface connection points will lead to incorrect results, which are impossible to correct due to the lack of fundamental information.

2. Measurement bias error in the driving point accelerance matrix and transfer accelerance matrix of the vehicle and the test item causes error in the substructure analysis. However, this error appears to be far less than the DOF deficiency error and can be corrected once the source is recognized and evaluated.

3. Noise error effects appear to be most severe in the low frequency range. The vehicle and test item driving point accelerance matrices appear to be more sensitive to the noise contamination than the transmissibility matrices since these driving point matrices must pass through the noise sensitive matrix inversion process.

4. Rocking motion of the exciter leads to incorrect bare vehicle transmissibility matrix and global transmissibility matrix. The substructure method still works in the sense that the measured global transmissibility matrix is the same as the one calculated from the substructure method, provided that the rocking motions remain unchanged when the whole structure and the bare vehicle are tested respectively. If the rocking motion is proportional to the vertical motion, generally rocking motion error is not a severe error compared to the DOF deficiency error.

5. Differences between the impact point and the measurement point in the interface driving point accelerance test deteriorate the quality of the global transmissibility matrix that is computed using the substructure method when the distance between the two points grows a bit larger. This presents the closeness requirement of locating instruments at the interface in driving point accelerance tests. For the simulated structure, the largest dimension is 24" and 1" driving point difference already has apparent impact on the test
results. For more general structures, further research is needed to find the distance limit requirement. Generally the larger the structure, the larger the distance limit is.

6. The measurement of all required DOF in the interface driving point accelerance matrices represents a significant challenge for the modal testing community.
CHAPTER 4 DOF DEFICIENCY ERROR

The long time negligence of moment and angular acceleration components in the interface created an illusion that the only important interface force is the normal force. In fact, the neglected interface loading components are not only the moment, but also the shear force tangent to the surface in the 2D case. In the 3D case, the neglected terms also include out-of-plane forces and moments. However, compared with other errors, such as bias error and noise error, the DOF deficiency error is the most catastrophic. This chapter studies in detail the DOF deficiency error. Based on the motion transmissibility equation for a coupled structure under ground motion derived in Chapter 3, two error scenarios are studied, namely, single point or multiple point contact in a 2D model, and single point or multiple point contact in a 3D model. Explicit error expressions are derived and sources of errors are identified. It shows that in certain special cases some of the interfacial DOF can be ignored without altering the original transmissibility functions. To serve as an intermediate stage of the test apparatus development, a specific kind of structure consists of all 3D beam elements are proposed, because its in-plane motion and out-of-plane motion are de-coupled so that only 9 FRF's (6 of them are independent) are to be determined. However, it seems that in reality, the 36 FRF's (21 of them are independent) in the interfacial driving point accelerance and the transfer point accelerance matrices must be considered without neglecting a single FRF. This strict requirement presents a serious challenge for the modal testing community when they apply the substructure method in model updating and assembly analysis with experimental data.
General Ideas

Suppose the structure consists of two parts, one is the test item, and the other is the vehicle. The test item and the vehicle are connected at the interface points only. We want to study the ground motion transmissibility to a point of interest located on the test item. A direct formulation of the complete structure including the test item and the vehicle is possible. However, for practical purpose, it is easier to apply substructure method to formulate the test item and the vehicle separately, and then use the interface conditions to couple the two sets of equations to solve for the global transmissibility. The motion transmissibility equation is as follows:

\[
\{-\omega^2 X_e\} = \left[T_{el} \mathbf{I} + V_{ii} \right]^{-1} \left[\text{Trans}_{ii}^* \right] \{-\omega^2 X_g\}
\]

\[
= \left[T_{el} \mathbf{I} + TV_{ii} \right]^{-1} \left[\text{Trans}_{ii}^* \right] \{-\omega^2 X_g\}
\]

\[
= \left[\text{Trans}_{eg} \right] \{-\omega^2 X_g\}
\]

(4.1)

where \{-\omega^2 X_e\} is the linear and angular acceleration vector for a point of interest on the test item, \{-\omega^2 X_g\} is the linear and angular acceleration vector of the ground, \([T_{el}]\) is the transfer point accelerance matrix of the test item from the interface to the point of interest, \([T_{ii}]\) is the driving point accelerance matrix of the test item at the interface, \([V_{ii}]\) is the driving point accelerance matrix of the vehicle at the interface, and \([\text{Trans}_{ii}^* ]\) is the motion transmissibility matrix of the vehicle from the ground to the test item-vehicle interface. \([\text{Trans}_{eg}]\) is the motion transmissibility matrix of the combined structure that includes the test item and the vehicle. It is defined from the ground to the point of interest on the test item.

The fact that experimentally, \([T_{el}], [T_{ii}], [V_{ii}]\) and \([\text{Trans}_{eg}]\) can be measured ultimately gives us two unique advantages. One is to understand the behavior of each
substructure without going through a time consuming process of model each with finite element. The other is to use a simple formula to obtain the in-situ behavior of the test item without actually assembling the test item with the vehicle. Therefore, the proposed method is partly experimental, and partly analytical in nature and utilizes the benefit of both methods. However the drawback of the substructure method is now we have to deal with the interface FRF of both test item and vehicle. For a complete structure analysis or testing, the interface needs not to be exposed.

If we only consider part of the interface DOF of a given problem, DOF deficiency errors occur. Chapter 3 shows that other errors, such as bias error and noise error, are also possible, however, the most catastrophic error is DOF deficiency error and its effects can not be easily purged.

Two scenarios of the DOF deficiency error are studied in this chapter as follows:
1. Neglect of rotational DOF or tangential DOF in a 2D problem with single point or multiple point contact.
2. Neglect out-of-plane DOF in a 3D problem with single point or multiple point contact.

**Scenario 1: Single Point or Multiple Point Contact in 2D**

In a single contact point case, the subscript \( i \) represents one point, and thus for each DOF, there is only one force or acceleration value. In multiple contact point case, however, the subscript \( i \) no longer represents a single point, and thus for each DOF there are several forces or acceleration values. The ways to obtain the explicit expression of the inverse of the \([T_n]+[V_n]\) matrix are different for the two cases. For the single point case where each element of a DOF represents a value, the normal method using the determinant is adequate to
find an explicit expression of the inverse. For the multiple point case, where each element in
the matrix represents a small block itself, the determinant method is no longer feasible. Liu
and Ewins [Liu et al, 1998] propose a formula to solve the inverse problem. The current
work extends their method to study the DOF deficiency errors in motion transmissibility
problems.

Suppose the inverse of a matrix is defined as follows:

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}^{-1} = \begin{bmatrix}
A' & B' \\
C' & D'
\end{bmatrix}
\]

(4.2)

where the superscript \(i\) designates for the element in the final inverse matrix. Then we find

\[
A' = A^{-1} + A^{-1}BD'CA^{-1}, \quad B' = -A^{-1}BD', \quad C' = B'^T, \quad D' = (D - CA^{-1}B)^{-1}
\]

(4.3)

Similarly we have

\[
A' = (A - BD^{-1}C)^{-1}, \quad B' = C'^T, \quad C' = -D^{-1}CA', \quad D' = D^{-1} + D^{-1}CA'BD^{-1}
\]

(4.4)

Let

\[
\begin{bmatrix}
TV_{11} & TV_{12} & TV_{13} \\
TV_{21} & TV_{22} & TV_{23} \\
TV_{31} & TV_{32} & TV_{33}
\end{bmatrix}^{-1} = \begin{bmatrix}
TV'_{11} & TV'_{12} & TV'_{13} \\
TV'_{21} & TV'_{22} & TV'_{23} \\
TV'_{31} & TV'_{32} & TV'_{33}
\end{bmatrix}
\]

(4.5)

Since \(TV_{ki}\) is a term in the “combined driving point accelerance matrix”, it represents
the acceleration on the \(k^{th}\) DOF when there is a unit force/moment at the \(i^{th}\) DOF. \(TV'_{ki}\) is a
term in the “apparent mass” matrix which stands for the required force/moment on the \(k^{th}\)
DOF when there is a unit linear/rotational acceleration at the \(i^{th}\) DOF and no accelerations at
the other DOF’s. We try to find the errors associated with incomplete coupling by studying
three cases.
Case 1 Neglecting Tangential DOF

If we delete the shear force and its related tangential DOF, the \( [TV] \) can be written as

\[
[TV] = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} [TV_{11}] & [TV_{12} & TV_{13}] \\ [TV_{21}] & [TV_{22} & TV_{23}] \\ [TV_{31}] & [TV_{32} & TV_{33}] \end{bmatrix}
\]

(4.6)

With Equation 4.4 and Equation 4.5, we find

\[
[TV]^{-1} = \begin{bmatrix} [TV_{11}'] \\ [TV_{21}'] \\ [TV_{31}'] \end{bmatrix} = \begin{bmatrix} [TV_{12} & TV_{13}'] \\ [TV_{22} & TV_{23}'] \\ [TV_{32} & TV_{33}'] \end{bmatrix},
\]

(4.7)

where

\[
[TV_{11}'] = \left( [TV_{11}] - [TV_{12} & TV_{13}] [TV_{22} & TV_{23}]^{-1} [TV_{21}] \right)^{-1},
\]

\[
[TV_{12}' & TV_{13}'] = [TV_{21}]^{-1},
\]

\[
[TV_{21}' & TV_{31}'] = -[TV_{22} & TV_{23}]^{-1} [TV_{21}] [TV_{11}],
\]

\[
[TV_{22}' & TV_{32}'] = [TV_{22} & TV_{23}]^{-1} + [TV_{22} & TV_{23}]^{-1} [TV_{21}] [TV_{12} & TV_{13}] [TV_{32} & TV_{33}]^{-1}
\]

(4.8)

The global transmissibility can be written into block form as:

\[
\begin{bmatrix} \text{Trans}^x_{1y} \\ \text{Trans}^x_{2y} \\ \text{Trans}^x_{3y} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} [TV_{11}] & [TV_{12} & TV_{13}] \\ [TV_{21}] & [TV_{22} & TV_{23}] \\ [TV_{31}] & [TV_{32} & TV_{33}] \end{bmatrix} \begin{bmatrix} \text{Trans}^x_{1y} \\ \text{Trans}^x_{2y} \\ \text{Trans}^x_{3y} \end{bmatrix}
\]

(4.9)

Expanding the above equation, we find
The incomplete modeling equation is as follows:

\[
\begin{bmatrix}
T_{12}^{\text{eq}} & T_{22}^{\text{eq}} & T_{32}^{\text{eq}} \\
T_{12}^{\text{eq}} & T_{22}^{\text{eq}} & T_{32}^{\text{eq}} \\
T_{12}^{\text{eq}} & T_{22}^{\text{eq}} & T_{32}^{\text{eq}}
\end{bmatrix}
\begin{bmatrix}
TV_{22} & TV_{23} \\
TV_{22} & TV_{23} \\
TV_{32} & TV_{33}
\end{bmatrix}
\begin{bmatrix}
\text{Trans}_{12}^{\text{eq}} \\
\text{Trans}_{22}^{\text{eq}} \\
\text{Trans}_{32}^{\text{eq}}
\end{bmatrix}
\]

\[
\begin{bmatrix}
T_{11}^{\text{eq}} \\
T_{21}^{\text{eq}} \\
T_{31}^{\text{eq}}
\end{bmatrix}
\begin{bmatrix}
TV_{12} & TV_{13} \\
TV_{12} & TV_{13} \\
TV_{12} & TV_{13}
\end{bmatrix}
\begin{bmatrix}
\text{Trans}_{12}^{\text{eq}} \\
\text{Trans}_{22}^{\text{eq}} \\
\text{Trans}_{32}^{\text{eq}}
\end{bmatrix}
\]

\[
(4.10)
\]

Comparing Equation 4.10 and Equation 4.11, four errors are identified. The first error originates from the difference between the elements in the reduced apparent mass matrix as illustrated in Equation 4.8. It is clear that the elements in the reduced apparent mass matrix

\[
\begin{bmatrix}
TV_{22} & TV_{23} \\
TV_{32} & TV_{33}
\end{bmatrix}^{-1}
\]

are not the same quantities as the corresponding elements

\[
\begin{bmatrix}
TV_{22} & TV_{23} \\
TV_{32} & TV_{33}
\end{bmatrix}
\]

extracted from the full apparent mass matrix

\[
\begin{bmatrix}
TV_{11} & TV_{12} & TV_{13} \\
TV_{21} & TV_{22} & TV_{23} \\
TV_{31} & TV_{32} & TV_{33}
\end{bmatrix}
\]. With Equation 4.8 we find the explicit expression of the first error,
\[
[\text{Error}_1] = \begin{bmatrix}
T_{12}^e & T_{13}^e \\
T_{22}^e & T_{23}^e \\
T_{32}^e & T_{33}^e \\
\end{bmatrix}
\left(\begin{bmatrix}
TV_{22} & TV_{23} \\
TV_{32} & TV_{33} \\
\end{bmatrix} - \begin{bmatrix}
TV_{22} & TV_{23} \\
TV_{32} & TV_{33} \\
\end{bmatrix}\right)^{-1} \cdot \begin{bmatrix}
\text{Trans}_{2y}^n \\
\text{Trans}_{3y}^n \\
\end{bmatrix}
\]

(4.12)

The second error arises from the deletion of the coupling term \(\begin{bmatrix} TV_{12} \end{bmatrix}^i \) between shear force and normal motions, shear force and rotation at the interface.

\[
[\text{Error}_2] = \begin{bmatrix}
T_{11}^e \\
T_{21}^e \\
T_{31}^e \\
\end{bmatrix}
\begin{bmatrix}
TV_{12} & TV_{13} \\
\end{bmatrix} \cdot \begin{bmatrix}
\text{Trans}_{2y}^n \\
\text{Trans}_{3y}^n \\
\end{bmatrix}
\]

(4.13)

The third error comes from the deletion of the coupling term \(\begin{bmatrix} TV_{21} \end{bmatrix}^i \) between normal force and tangential motion, moment and tangential motion at the interface.

\[
[\text{Error}_3] = \begin{bmatrix}
T_{12}^e & T_{13}^e \\
T_{22}^e & T_{23}^e \\
T_{32}^e & T_{33}^e \\
\end{bmatrix}
\begin{bmatrix}
TV_{21} \end{bmatrix}^i \cdot \begin{bmatrix}
\text{Trans}_{1y}^n \\
\end{bmatrix}
\]

(4.14)

The fourth error is introduced by the deletion of the coupling term \(\begin{bmatrix} TV_{11} \end{bmatrix}^i \) between shear force and tangential motion at the interface.

\[
[\text{Error}_4] = \begin{bmatrix}
T_{11}^e \\
T_{21}^e \\
T_{31}^e \\
\end{bmatrix}
\begin{bmatrix}
TV_{11} \end{bmatrix}^i \cdot \begin{bmatrix}
\text{Trans}_{1y}^n \\
\end{bmatrix}
\]

(4.15)

It is evident that the DOF deficiency problem is by no means a simple one that can be compensated by easy solution.
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**Case 2 Neglecting Rotational DOF**

Similarly, to study the error when the rotational DOF is ignored, the \([TV]\) matrix can be written as

\[
[TV] = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
TV_{11} & TV_{12} \\
TV_{21} & TV_{22} \\
TV_{31} & TV_{32} \\
TV_{33} & \end{bmatrix}
\]

(4.16)

Using Equation 4.3 and Equation 4.5, we find

\[
[TV]^{-1} = \begin{bmatrix}
TV_{11}^{-1} & TV_{12}^{-1} \\
TV_{21}^{-1} & TV_{22}^{-1} \\
TV_{31}^{-1} & TV_{32}^{-1} \\
TV_{33}^{-1} & \end{bmatrix}
\]

(4.17)

where

\[
\begin{bmatrix}
TV_{11}^{-1} & TV_{12}^{-1} \\
TV_{21}^{-1} & TV_{22}^{-1}
\end{bmatrix} = \begin{bmatrix}
TV_{11}^{-1} & TV_{12}^{-1} \\
TV_{21}^{-1} & TV_{22}^{-1}
\end{bmatrix}^{-1} + \begin{bmatrix}
TV_{11}^{-1} & TV_{12}^{-1} \\
TV_{21}^{-1} & TV_{22}^{-1}
\end{bmatrix}^{-1} \begin{bmatrix}
TV_{13}^{-1} \quad TV_{33}^{-1}
\end{bmatrix} \begin{bmatrix}
TV_{31}^{-1} \\
TV_{32}^{-1}
\end{bmatrix} \begin{bmatrix}
TV_{11}^{-1} & TV_{12}^{-1} \\
TV_{21}^{-1} & TV_{22}^{-1}
\end{bmatrix}^{-1}
\]

\[
\begin{bmatrix}
TV_{13}^{-1} \\
TV_{23}^{-1}
\end{bmatrix} = -\begin{bmatrix}
TV_{11}^{-1} & TV_{12}^{-1} \end{bmatrix}^{-1} \begin{bmatrix}
TV_{13}^{-1} \\
TV_{23}^{-1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
TV_{31}^{-1} \\
TV_{32}^{-1}
\end{bmatrix} = \begin{bmatrix}
TV_{13}^{-1}
\end{bmatrix}^{-T}
\]

\[
[TV]_{33}^{-1} = \left( [TV]_{33} - [TV]_{31} \quad TV_{32} \begin{bmatrix}
TV_{11}^{-1} & TV_{12}^{-1} \end{bmatrix}^{-1} [TV]_{13}^{-1} \right)^{-1}
\]

(4.18)

The global transmissibility equation can be partitioned as:

\[
\begin{bmatrix}
\text{Trans}_{xy}^{1x} \\
\text{Trans}_{xy}^{2x} \\
\text{Trans}_{xy}^{3x}
\end{bmatrix} = \begin{bmatrix}
T_{11}^a & T_{12}^a \\
T_{21}^a & T_{22}^a \\
T_{31}^a & T_{32}^a
\end{bmatrix} \begin{bmatrix}
TV_{11}^{-1} & TV_{12}^{-1} \\
TV_{21}^{-1} & TV_{22}^{-1} \\
TV_{31}^{-1} & TV_{32}^{-1} \\
TV_{33}^{-1}
\end{bmatrix} \begin{bmatrix}
TV_{13}^{-1} \\
TV_{23}^{-1} \\
TV_{33}^{-1}
\end{bmatrix} \begin{bmatrix}
\text{Trans}_{xy}^{1y} \\
\text{Trans}_{xy}^{2y} \\
\text{Trans}_{xy}^{3y}
\end{bmatrix}
\]

(4.19)

Expanding Equation 4.19, we find
The incomplete modeling equation is as follows:

\[
\begin{pmatrix}
\text{Trans}_{xy}^1 \\
\text{Trans}_{xy}^2 \\
\text{Trans}_{xy}^3
\end{pmatrix} = 
\begin{pmatrix}
T_{11}^{ei} & T_{12}^{ei} & TV_{11}^i & TV_{12}^i & Trans_{xy}^1 \\
T_{21}^{ei} & T_{22}^{ei} & TV_{21}^i & TV_{22}^i & Trans_{xy}^2 \\
T_{31}^{ei} & T_{32}^{ei} & TV_{31}^i & TV_{32}^i & Trans_{xy}^3
\end{pmatrix}
\]

\]

(4.20)

Compare Equation 4.20 and Equation 4.21, four errors can be identified. The first error originates from the difference between \(TV_{11}^i\) and \(TV_{11}^o\), illustrated in Equation 4.18. It is clear that the elements in the reduced apparent mass matrix

\[
\begin{bmatrix}
TV_{11}^i & TV_{12}^i \\
TV_{21}^i & TV_{22}^i
\end{bmatrix}
\]

are not the same quantities as the corresponding elements

\[
\begin{bmatrix}
TV_{11}^o & TV_{12}^o \\
TV_{21}^o & TV_{22}^o
\end{bmatrix}
\]

extracted from the full apparent mass matrix

\[
\begin{bmatrix}
TV_{11}^i & TV_{12}^i & TV_{13}^i \\
TV_{21}^i & TV_{22}^i & TV_{23}^i \\
TV_{31}^i & TV_{32}^i & TV_{33}^i
\end{bmatrix}
\]

Using Equation 4.18, we find,
The second error comes from the deletion of the coupling term \( [TV_{31}, TV_{32}] \) between moment and tangential motion, moment and normal motion at the interface.

\[
[Error_2] = \begin{bmatrix} T_{11}^u & T_{12}^u \\ T_{21}^u & T_{22}^u \\ T_{31}^u & T_{32}^u \end{bmatrix} \left( \begin{bmatrix} TV_{11}^i & TV_{12}^i \\ TV_{21}^i & TV_{22}^i \end{bmatrix} - \begin{bmatrix} TV_{11} & TV_{12} \\ TV_{21} & TV_{22} \end{bmatrix}^{-1} \right) \begin{bmatrix} \text{Trans}_{1y}^u \\ \text{Trans}_{2y}^u \end{bmatrix}
\]

\[(4.22)\]

The third error arises from the deletion of the coupling term \( [TV_{13}, TV_{23}] \) between tangential force and rotation, normal force and rotation at the interface.

\[
[Error_3] = \begin{bmatrix} T_{11}^u & T_{12}^u \\ T_{21}^u & T_{22}^u \\ T_{31}^u & T_{32}^u \end{bmatrix} \begin{bmatrix} TV_{31}^i & TV_{32}^i \end{bmatrix} \begin{bmatrix} \text{Trans}_{1y}^u \\ \text{Trans}_{2y}^u \end{bmatrix}
\]

\[(4.23)\]

The fourth error comes from the deletion of the coupling term \( [TV_{33}] \) between moment and rotation at the interface.

\[
[Error_4] = \begin{bmatrix} T_{13}^u \\ T_{23}^u \\ T_{33}^u \end{bmatrix} \begin{bmatrix} TV_{31}^i & TV_{32}^i \end{bmatrix} \begin{bmatrix} \text{Trans}_{1y}^u \\ \text{Trans}_{2y}^u \end{bmatrix}
\]

\[(4.24)\]
Case 3 Neglecting Tangential and Rotational DOF's

In order to apply Equation 4.3 or Equation 4.4, we need to rearrange the order of the DOF from \{x, y, \theta_1\} to \{x, \theta_2, y\} in the transmissibility equation. The equation of global motion transmissibility becomes:

\[
\begin{bmatrix}
\text{Trans}_{y}^x \\
\text{Trans}_{y}^z \\
\text{Trans}_{y}^y
\end{bmatrix} = 
\begin{bmatrix}
T_{11} & T_{13} & T_{12} \\
T_{31} & T_{33} & T_{32} \\
T_{21} & T_{23} & T_{22}
\end{bmatrix}
\begin{bmatrix}
TV_{11} & TV_{13} & TV_{12} \\
TV_{31} & TV_{33} & TV_{32} \\
TV_{21} & TV_{23} & TV_{22}
\end{bmatrix}
\begin{bmatrix}
\text{Trans}_{y}^x \\
\text{Trans}_{y}^z \\
\text{Trans}_{y}^y
\end{bmatrix}
\]

Equation 4.26 is partitioned to leave out \[TV_{22}\]:

\[
\begin{bmatrix}
\text{Trans}_{y}^x \\
\text{Trans}_{y}^z \\
\text{Trans}_{y}^y
\end{bmatrix} = 
\begin{bmatrix}
T_{11} & T_{13} \\
T_{31} & T_{33} \\
T_{21} & T_{23}
\end{bmatrix}
\begin{bmatrix}
TV_{11} & TV_{13} \\
TV_{31} & TV_{33} \\
TV_{21} & TV_{23}
\end{bmatrix}
\begin{bmatrix}
\text{Trans}_{y}^x \\
\text{Trans}_{y}^z \\
\text{Trans}_{y}^y
\end{bmatrix}
\]

(4.27)

Since the order of inverse and reordering of rows and columns can be changed in \[TV\], (See Appendix for details),

\[
\begin{bmatrix}
TV'_{11} & TV'_{13} & TV'_{12} \\
TV'_{31} & TV'_{33} & TV'_{32} \\
TV'_{21} & TV'_{23} & TV'_{22}
\end{bmatrix} = \begin{bmatrix}
TV_{11} & TV_{13} & TV_{12} \\
TV_{31} & TV_{33} & TV_{32} \\
TV_{21} & TV_{23} & TV_{22}
\end{bmatrix}^{-1}
\]

(4.28)

we have

\[
\begin{bmatrix}
TV'_{11} \\
TV'_{31}
\end{bmatrix} = \left( \begin{bmatrix}
TV_{11} & TV_{13} \\
TV_{31} & TV_{33}
\end{bmatrix} - \begin{bmatrix}
TV_{12} \\
TV_{32}
\end{bmatrix} \begin{bmatrix}
TV_{21} & TV_{23}
\end{bmatrix} \right)^{-1}
\]

\[
\begin{bmatrix}
TV'_{12} \\
TV'_{32}
\end{bmatrix} = \begin{bmatrix}
TV'_{21} \\
TV'_{23}
\end{bmatrix}
\]

\[
\begin{bmatrix}
TV'_{21} & TV'_{23}
\end{bmatrix} = -\begin{bmatrix}
TV_{21}
\end{bmatrix} \begin{bmatrix}
TV_{21} & TV_{23}
\end{bmatrix} \begin{bmatrix}
TV'_{11} & TV'_{13}
\end{bmatrix}
\]

(4.29)
Expanding Equation 4.27, we find

$$[TV_{22}] = [TV_{22}]^{-1} + [TV_{22}]^{-1}[TV_{21} TV_{23} \begin{bmatrix} TV_{11}^i & TV_{12}^i \\ TV_{31}^i & TV_{32}^i \end{bmatrix} [TV_{22}]^{-1}$$  

(4.29)

The transmissibility equation from the incomplete model ignoring both shear force and moment is as follows.

$$\begin{align*}
\begin{bmatrix} Trans_{xy}^e \\ Trans_{xy}^v \\ Trans_{xy}^m \end{bmatrix} &= \begin{bmatrix} T_{12}^e \\ T_{32}^e \\ T_{22}^e \end{bmatrix} + \begin{bmatrix} T_{11}^e & T_{13}^e \\ T_{21}^e & T_{23}^e \\ T_{31}^e & T_{33}^e \end{bmatrix} \begin{bmatrix} TV_{11}^i & TV_{12}^i \\ TV_{31}^i & TV_{32}^i \end{bmatrix} \begin{bmatrix} Trans_{2y}^e \\ Trans_{2y}^v \end{bmatrix} \\
&+ \begin{bmatrix} T_{11}^e & T_{13}^e \\ T_{21}^e & T_{23}^e \\ T_{31}^e & T_{33}^e \end{bmatrix} \begin{bmatrix} TV_{11}^i & TV_{13}^i \\ TV_{31}^i & TV_{33}^i \end{bmatrix} \begin{bmatrix} Trans_{2y}^v \\ Trans_{2y}^m \end{bmatrix} \\
&+ \begin{bmatrix} T_{11}^e & T_{13}^e \\ T_{21}^e & T_{23}^e \\ T_{31}^e & T_{33}^e \end{bmatrix} \begin{bmatrix} TV_{11}^i & TV_{12}^i \\ TV_{31}^i & TV_{32}^i \end{bmatrix} \begin{bmatrix} Trans_{2y}^e \\ Trans_{2y}^v \end{bmatrix}
\end{align*}$$

(4.30)

The error analysis is almost the same as the previous two cases. Compare Equation 4.30 and Equation 4.31, four errors can be identified. The first error originates from the difference between $[TV_{22}^i]$ and $[TV_{22}]^{-1}$, as illustrated in Equation 4.29. It is clear that the elements in the reduced apparent mass matrix $[TV_{22}]^{-1}$ are not the same quantities as the
corresponding elements \( [TV'_{22}] \) extracted from the full apparent mass matrix

\[
\begin{bmatrix}
TV'_{11} & TV'_{12} & TV'_{13} \\
TV'_{21} & TV'_{22} & TV'_{23} \\
TV'_{31} & TV'_{32} & TV'_{33}
\end{bmatrix}
\]

Using Equation 4.29, we find,

\[
[Error_1] = \begin{bmatrix}
T'_{12} \\
T'_{32} \\
T'_{22}
\end{bmatrix}
\left( [TV'_{22}] - [TV'_{22}]^{-1} \right) \{Trans_{2y}^{xy} \}
\]

\[
= \begin{bmatrix}
T'_{12} \\
T'_{32} \\
T'_{22}
\end{bmatrix}
\left( [TV'_{22}] - [TV'_{22}]^{-1} \right) \{Trans_{2y}^{xy} \}
\]  

The second error comes from the deletion of the coupling term \( [TV'_{12}] \) between shear force and normal motion, moment and normal motion, at the interface.

\[
[Error_2] = \begin{bmatrix}
T'_{11} & T'_{13} \\
T'_{31} & T'_{33} \\
T'_{21} & T'_{23}
\end{bmatrix}
\{Trans_{2y}^{xy} \}
\]  

The third error arises from the deletion of the coupling term \( [TV'_{21}, TV'_{23}] \) between normal force and tangential motion, normal force and moment at the interface.

\[
[Error_3] = \begin{bmatrix}
T'_{12} \\
T'_{32} \\
T'_{22}
\end{bmatrix}
\{Trans_{2y}^{xy} \}
\]  

The fourth error comes form the deletion of the coupling term \( [TV'_{11}, TV'_{13}] \) between shear force and tangential motions, shear force and rotation, moment and tangential motion, moment and rotation at the interface.
Scenario 2 Neglect Out-of-Plane DOF's in 3D

2D models are simplified models that restrict the motion of the object in a confined space. Though it has theoretical merit, its practical use is very limited. One problem is that the out-of-plane general forces may interfere with the in-plane motions so that a 2D model suffers from the DOF deficiency problem. When unknown input forces exist, it is difficult to make correct measurement of the FRF.

Case 1 DOF Deficiency Error in General 3D Formulations

The motion transmissibility equation for a 3D model, Equation 4.36, is similar to its counterpart in 2D space. The only difference is that the subscripts $i$ and $g$ now stands for $x$, $y$, $z$, $\theta_x$, $\theta_y$, and $\theta_z$. 

\[
\text{Error}_i = \begin{bmatrix} T_{11}^u & T_{13}^u \\ T_{31}^u & T_{33}^u \\ T_{21}^u & T_{23}^u \end{bmatrix} \begin{bmatrix} TV_{11}^{i} & TV_{13}^{i} \\ TV_{31}^{i} & TV_{33}^{i} \end{bmatrix} \begin{bmatrix} \text{Trans}_{xy}^{\text{in}} \\ \text{Trans}_{xy}^{\text{out}} \end{bmatrix} \tag{4.35}
\]
Suppose the orders of DOF are exchanged to \( \{x, y, \theta_1, \theta_2, \theta_3, z\} \), so that in-plane terms \( \{x, y, \theta_1\} \), and out-of-plane terms \( \{\theta_2, \theta_3, z\} \), are partitioned, we have

\[
\begin{bmatrix}
\text{Trans}^a_{\text{in-in}} & \text{Trans}^a_{\text{in-out}} & \text{Trans}^a_{\text{out-in}} & \text{Trans}^a_{\text{out-out}}
\end{bmatrix}^{\text{T}} \begin{bmatrix}
\gamma^\text{iv}_{\text{in-in}} & \gamma^\text{iv}_{\text{in-out}} & \gamma^\text{iv}_{\text{out-in}} & \gamma^\text{iv}_{\text{out-out}}
\end{bmatrix} \begin{bmatrix}
\text{Trans}^a_{\text{in-in}} & \text{Trans}^a_{\text{in-out}} & \text{Trans}^a_{\text{out-in}} & \text{Trans}^a_{\text{out-out}}
\end{bmatrix}^{-1}
\]

(4.37)

Using a similar procedure in 2D case, we find

\[
[\text{Trans}^a_{\text{in-in}}] = [T^\text{iv}_{\text{in-in}}] [TV^\text{iv}_{\text{in-in}}] [\text{Trans}^a_{\text{in-in}}] + [T^\text{iv}_{\text{in-out}}] [TV^\text{iv}_{\text{in-out}}] [\text{Trans}^a_{\text{in-out}}] + [T^\text{iv}_{\text{out-in}}] [TV^\text{iv}_{\text{out-in}}] [\text{Trans}^a_{\text{out-in}}] + [T^\text{iv}_{\text{out-out}}] [TV^\text{iv}_{\text{out-out}}] [\text{Trans}^a_{\text{out-out}}]
\]

(4.38)

where

\[
[TV^\text{iv}_{\text{in-in}}] = [TV^\text{iv}_{\text{in-in}}]^{-1} + [TV^\text{iv}_{\text{in-out}}]^{-1} [TV^\text{iv}_{\text{out-in}}] [TV^\text{iv}_{\text{out-out}}] [TV^\text{iv}_{\text{in-in}}]^{-1}
\]

\[
[TV^\text{iv}_{\text{in-out}}] = [-[TV^\text{iv}_{\text{in-in}}]^{-1} [TV^\text{iv}_{\text{in-out}}] [TV^\text{iv}_{\text{out-out}}]
\]
The transmissibility equation from the 2D model is

\[
\begin{align*}
[T_{V_{out-in}}] &= [TV_{in-out}]^T, \\
[TV_{out-out}] &= ([TV_{out-out}] - [TV_{out-in}][TV_{in-in}]^T[TV_{in-out}])^T
\end{align*}
\]  

(4.39)

The first error originates from the difference between the former and the latter, where the former is a sub-matrix extracted from the inverse of the \([TV]\) matrix. The latter is the inverse of a reduced \([TV]\) matrix. The second error comes from the cross coupling between out-of-plane forces and in-plane motions in the inverse of the \([TV]\) matrix, and the third error comes from the cross coupling between in-plane forces and out-of-plane motions in the inverse of the \([TV]\) matrix. The last error arises from coupling of the out-of-plane forces and out-of-plane motions in the inverse of the \([TV]\) matrix, and it is brought into the in-plane motion transmissibility \([Trans_{in-in}^{eg}]\) by the cross coupling terms in transfer accelerance \([T_{in-out}^e]\) and the cross coupling terms in vehicle motion transmissibility \([Trans_{out-in}^{eg}]\).
The implication of Equation 4.38 is that in a general 3D structure, a 2D model can not yield correct transmissibility results because it neglects out-of-plane DOF. However, out-of-plane ground motion always exists and cross coupling terms such as \([TV_{in-out}]\) and \([TV_{out-in}]\) will bring the influence of out-of-plane motions into the in-plane motion. Therefore for industry use, a 3D model is needed and the \([TV]\) matrix is 6 by 6. For the 36 interfacial acceleration, 21 out of them are unique.

Equation 4.38 reveals that this 3D coupling problem is so complex that all the six interface DOF must be measured to obtain the correct result. None of them can be neglected. This requirement presents a strenuous task for designing testers.

**Case 2 Special 3D Structures with De-coupled DOF's**

As a first trial, it might be useful to deal with something less difficult than a 3D model. Can we find a specific kind of structure that a 2D model suffices? If that is feasible, the \([TV]\) matrix is 3 by 3. Six FRF's are unique. Then we must find a structure that all the four errors shown in Equation 4.41 vanish. In other words, the out-of-plane motion and the in-plane motion are de-coupled.

It is clear that for the first three errors to vanish, there should be no cross coupling between the in-plane and out-of-plane DOF in the inverse of the \([TV]\) matrix. To diminish the last error, either the cross coupling terms in transfer acceleration \([T'_{in-out}]\) and the cross coupling terms in vehicle motion transmissibility \([Trans'_{out-in}]\) need to be zero.

A typical case is a frame made of 3D beam elements. For all the elements, their longitudinal symmetric planes, which are orthogonal to the neural planes, are required to be
within the same plane. The interface points and the interested points are within this plane. Since the symmetrical plane of the structure passes all principal inertia axes of the beams, the out-of-plane forces can only induce out-of-plane motion of the beam and have no effect on in-plane motion. Thus the in-plane motion and out-of-plane motion are de-coupled. If we put transducers on the ground and on the interface to measure the in-plane motion, the interference of out-of-plane ground motion can be avoided.

**Numerical Examples**

Figure 4.1 shows a 2D test setup where the global transmissibility is obtained through a complete structure test. It is the same as Figure 3.1. The test item is an L-shaped frame and the vehicle is a portal frame. The ground motion and the motion at the tip of the L-shaped frame are measured.

Figure 4.2 shows the test item and the vehicle under impact tests, respectively. It is the same as Figure 3.2. \([T_{ii}], [V_{ii}],\) and \([T_{ei}]\) are measured.

Figure 4.3 shows the grounded bare vehicle under vertical floor motion. It is the same as Figure 3.3.

Figure 4.4 shows a similar test setup, but the L-shaped frame is twisted so its horizontal part is 45 degrees out of the x-y plane and its vertical part remains unchanged. This test setup is 3D in nature.

Several finite element analyses on the two test setups are performed. The first FEA is on the first test setup. It models the structure with 2D frame elements. The mass and stiffness matrices of a 2D frame element are superimposed by a 2D beam element with a 2D
Figure 4.1 Transmissibility from a direct test

Figure 4.2 Driving point acceleration and transfer point acceleration measurement

Figure 4.3 Transmissibility test of the bare vehicle
truss element. Each frame element has two nodes. Each node has three DOF, i.e., axial (x), transverse (y), and angular (θ). Proportional damping is assumed and the damping ratios of the first two elastic modes are assumed to be 0.02. The ground motion is in y direction and thus within the 2D symmetrical plane.

We computed the global motion transmissibility from ground to the point of interest e on the test item using a complete structure model. By doing this the complete interface DOF are assumed automatically. Then we use the incomplete interface DOF, to compute the global transmissibility matrices again with the substructure method. By comparing the two sets of results, the DOF deficiency errors can be identified. The different terms of the errors are plotted to show their relative importance with respect to the correct curves.

The second FEA is also on the first test setup. It models the structure with 3D frame elements. The mass and stiffness matrices of a 3D frame element are superimposed by that of two 2D beam elements, i.e. a 1D truss element, and a 1D shaft element. Each element has two nodes, and each has six DOF, i.e., axial DOF (x), two transverse DOF (y, z) and three
rotational DOF \((\theta_x, \theta_y, \theta_z)\). Lumped mass matrix is adopted and the same Raleigh proportional damping coefficients as the first FEA are used. The natural frequencies of the 2D and 3D model of the same structure are compared to show the difference due to modeling. The ground motion is at \(y\) and \(z\) directions with same magnitude. So in-plane motion and out-of-plane motion exist simultaneously.

We try to verify that the out-of-plane motion has no effect on the in-plane motion for this special test setup. The substructure method is employed. The first computation takes into account all six interfacial DOF. The second computation deliberately neglects the out-of-plane interface DOF, i.e., \(z\), \(\theta_z\) and \(\theta_y\). If both calculations yield correct answers for the in-plane motion transmissibility, we should have

\[
[\text{Trans}^{\theta_x}_{\theta_y}] = [\text{Trans}^{\theta_x}_{\theta_y}] + [\text{Trans}^{\theta_x}_{\theta_y}]
\]

\[
[\text{Trans}^{\theta_y}_{\theta_y}] = [\text{Trans}^{\theta_y}_{\theta_y}] + [\text{Trans}^{\theta_y}_{\theta_y}]
\]

\[
[\text{Trans}^{\theta_z}_{\theta_y}] = [\text{Trans}^{\theta_z}_{\theta_y}] + [\text{Trans}^{\theta_z}_{\theta_y}]
\]

(4.42)

The left-hand side terms of Equation 4.42 are in-plane motion transmissibility computed with reduced interface DOF \((x, y, \theta_x)\) by the substructure method. The right-hand side terms of Equation 4.42 are in-plane transmissibility computed with complete interface DOF by the substructure method. If Equation 4.42 is correct, the coupling between \(x\), \(y\), \(\theta_x\) and \(z\) has to vanish, and thus the out-of-plane motion has no effect on the in-plane motion for this special test setup.

The third FEA is performed on the second test setup with a rotated L-shaped frame. 3D frame elements and lumped mass assumption are used. The Raleigh coefficients are set to be the same as the 3D model for the first setup. The ground motion is in \(y\) direction. We
use substructure method to discover the global transmissibility. In the first case, the interfacial DOF are complete, all $x, y, z, \theta_x, \theta_y, \theta_z$ are included. In the second case, reduced interfacial DOF are used, i.e., $x, y,$ and $\theta_z$. We try to show that in general 3D case, all six DOF in the interface have to be considered simultaneously, or otherwise the analysis would lead to incorrect results.

Results of Simulations

DOF Deficiency Error in a 2D Model

Figure 4.5 compares the $x, y, \theta_z$ motion transmissibility vs. $y$ ground motion of the first setup in both complete interface DOF and incomplete interface DOF cases. Errors are plotted as well. The calculation is done using the 2D model. Since in Equations 4.14, 4.15, 4.24, 4.25, 4.34, 4.35, the terms $[\text{Trans}_{xy}^y], [\text{Trans}_{yz}^y]$ are essentially zero due to the symmetry of the portal frame, Error 3 and Error 4 for all case are several hundred dB down than Error 1 and Error 2.

In the $x$ DOF deficiency case, Error 1 is generally larger than Error 2. In the $\theta$ DOF deficiency case, Error 2 is generally larger than Error 1. In the $x$ and $\theta$ DOF deficiency case, Error 2 is generally larger than Error 1. Error 2 represents the coupling terms between the missing DOF and remaining DOF in the inverse of $[TV]$, while Error 1 is caused by the difference between the elements extracted from the full apparent mass matrix and the elements in the reduced apparent mass matrix. It seems the coupling of DOF in the interface is the key player that causes the DOF deficiency error.
Figure 4.5 Transmissibility and error of the first setup with complete interface coupling, and incomplete interface coupling.
Generally the magnitude of the larger one of Error 1 and Error 2 is in the same order of the magnitude of the transmissibility itself. Error 3 and Error 4 are significantly smaller, and thus can be neglected.

**Influence of Out-of-Plane DOF's**

Table 4.1 shows the natural frequencies of the first setup from a 2D model and a 3D model. Due to the symmetric nature of the setup, the 2D model captures the in-plane modes only, and the 3D model captures both the in-plane and out-of-plane modes. Figure 4.6 shows the $x$, $y$, $z$, $\theta_x$, $\theta_y$, and $\theta_z$ vs. $y$ and $z$ ground motion transmissibility. The motion transmissibility is calculated in two ways. First the substructure method uses all interfacial DOF. Second, reduced interfacial DOF, i.e., only $x$, $y$, and $\theta_z$ are used, the same as 2D case.
Table 4.1. Natural Frequencies of the test setup using 2D and 3D model

<table>
<thead>
<tr>
<th>Natural Frequency (Hz)</th>
<th>2D model</th>
<th>3D model</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st mode</td>
<td>17.3</td>
<td>17.3</td>
</tr>
<tr>
<td>2nd mode</td>
<td>32.0</td>
<td>29.7</td>
</tr>
<tr>
<td>3rd mode</td>
<td>58.1</td>
<td>32.0</td>
</tr>
<tr>
<td>4th mode</td>
<td>81.1</td>
<td>58.1</td>
</tr>
<tr>
<td>5th mode</td>
<td>185.2</td>
<td>81.1</td>
</tr>
<tr>
<td>6th mode</td>
<td>299.2</td>
<td>113.9</td>
</tr>
<tr>
<td>7th mode</td>
<td>324.8</td>
<td>185.2</td>
</tr>
<tr>
<td>8th mode</td>
<td>573.2</td>
<td>189.1</td>
</tr>
<tr>
<td>9th mode</td>
<td>710.2</td>
<td>299.2</td>
</tr>
<tr>
<td>10th mode</td>
<td>788.9</td>
<td>322.5</td>
</tr>
</tbody>
</table>

From Figure 4.6, it is clear that both methods yield identical results for the in-plane motion transmissibility. Hence it proves that due to the symmetry of the structure, the out-of-plane and in-plane forces and motions are de-coupled. The in-plane transmissibility is determined by the in-plane forces and moments only and it is the same as in 2D case. The substructure formulation cannot predict the $z$, $\theta_z$, and $\theta_y$ transmissibility. This interesting result verifies the possibility to use the setup as a first experimental attempt to find $[T_{ill}]$ and $[V_u]$. There are only three general accelerations ($a_x$, $a_y$, $a_z$) and three general forces ($F_x$, $F_y$, $M_z$), thus the matrices are 3 by 3 instead of 6 by 6. The result greatly simplifies the instrumentation requirement.

Figure 4.7 shows the $x$, $y$, $z$, $\theta_z$, $\theta_y$, and $\theta_z$ vs. y ground motion transmissibility for the second test setup. The motion transmissibility is calculated in two ways. First the substructure method uses the complete interfacial DOF, i.e., $x$, $y$, $z$, $\theta_z$, $\theta_y$, and $\theta_z$. Second, the reduced interfacial DOF, i.e., only $x$, $y$, and $\theta_z$ are used, the same as 2D case. As the second test setup does not have a symmetrical plane, it is truly 3D. The cross coupling terms
Figure 4.6 $x, y, z, \theta_x, \theta_y,$ and $\theta_z$ vs. $y$ and $z$ ground transmissibility from the 3D model of the first setup using complete structure formulation and substructure formulation with reduced interfacial DOF's.

Figure 4.7 $x, y, z, \theta_x, \theta_y,$ and $\theta_z$ vs. $y$ ground motion transmissibility for the second test setup using complete structure formulation and substructure formulation with reduced interfacial DOF.
in $[TV]$ cannot be neglected. Therefore it is predicted that the two sets of results have significant difference. It shows in general case, 3D model has to be used and testing method that can measure the $6 \times 6$ $[TV]$ has to be developed. To deal the 36 elements in the $[TV]$ matrix puts challenges on both the test method and the test equipment.

**Chapter Summary**

By looking into the details of the various dimensional coupling, it is evident that DOF deficiency problem is a notoriously difficult one to handle and is not simple to correct by conventional means. It appears to be an All or None problem.

1. Moments, shear forces and their respective DOF are crucial components in the interface in 2D structure models. The lack of considering these DOF will invalidate the substructure procedure to predict the behavior of coupled structure by obtaining totally different frequency response functions of the coupled structure and thus making the extraction of natural frequencies and modes information impossible.

2. The DOF deficiency error in a 2D model is caused by several sources. The first error comes from the fact that the elements in the reduced apparent mass matrix are not the same quantities as the corresponding elements extracted from the full apparent mass matrix. The second error comes from the deletion of the cross coupling term between the force in the neglected DOF and the motions in the DOF’s that remain. The third error comes from the deletion of the cross coupling term between the motion in the neglected DOF and the forces in the DOF’s that remain. The fourth error comes from the deletion of the cross coupling term between the motion and force in the neglected DOF.
3. The DOF deficiency error in a 3D model is brought about by several sources. The first error comes from the fact that the extracted elements from the inverse of a 3D interface model and the elements of the inverse of a 2D interface model are different. The former is a sub-matrix extracted from the inverse of the 3D $[TV]$ matrix. The latter is the inverse of the 2D $[TV]$ matrix. The second error comes from the deletion of the cross coupling terms between out-of-plane forces and in-plane motions. The third error comes from the deletion of the cross coupling terms between in-plane forces and out-of-plane motions. The last error arises from the deletion of the cross coupling terms between the out-of-plane forces and out-of-plane motion.

4. In a general structure, a 2D model does not yield correct transmissibility results because it neglects out-of-plane DOF’s. Therefore for industrial application, a 3D model and 36 interface FRF’s are needed. None of them can be neglected.

5. Currently, moment and angular acceleration measurements are inadequate for determining the required angular FRF’s for doing substructure modeling. Numerically, we showed a special test setup where the $[TV]$ matrix can be reduced to a 3 by 3 size so that only 9 FRF’s (6 FRF’s are unique, theoretically) need to be measured. This special case can serve as the first step for developing the technology that might be applicable in the more general cases.
CHAPTER 5 METHODS FOR EXTRACTING MULTIDIRECTIONAL FRF MATRICES – 2D SCENARIOS

Overview

As illustrated in Chapter 4, experimentally \([T_d], [T_u], [V_u] \) and \([Trans^*_a]\) can be measured. Among them \([T_u]\) and \([V_u]\) are most important because they are the keys to most substructure coupling problems. Since in reality it is very difficult to apply and measure moments directly to obtain multidirectional driving point and transfer point accelerance matrices, an indirect way has to be used.

Consider application of substructure concepts to the test item where an “Instrument Cluster” has been attached at point “T” as shown in Figure 5.0.1\(^1\). Assume that the “Instrument Cluster” can be adequately modeled using finite element method and also assume that one or more linear or angular accelerometers can be attached to or built into the “Instrument Cluster”. Now we shall explore how multiple measurements can be made so that the test item’s driving point and transfer point accelerances can be extracted from the combined structure.

The accelerometer installed on the “Instrument Cluster” is TAP. TAP is a linear and angular accelerometer manufactured by Kistler Instrument, Inc. It measures the normal and rotational accelerations with respect to its base. In 2D case, another accelerometer is used to measure the tangential motion at the same place so that a TAP-accelerometer combination is

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\(^1\) The numbering of figures, tables, and equations in Chapter 5 will correspond to the scenario numbering and the figures, tables and equations in Overview will be numbered as 5.0.x.
formed. In 3D cases, if three TAP units are put in orthogonal positions, we can find three linear accelerations as well as three angular accelerations.

Chapters 5, 6, and 7 are the essence of this work. In Chapter 5, a special implementation of the conceptual “Instrument Cluster” in 2D case, a T-bar, is proposed. In Chapter 7, a special implementation of the conceptual “Instrument Cluster” in 3D case, a C-bar, is proposed. Ten original methods to extract the multidirectional driving point and transfer point accelerance are developed in Chapter 5 and Chapter 7. All these methods work well with noise-free data in FE simulations. Their robustness are studied with noise polluted data to simulate the real test environment. If a method is found noise sensitive, it will not have good performance in real tests since there are more errors in reality than just the noise problem. Chapter 6 shows an experiment that implements one of the extraction methods in Chapter 5. It reveals the feasibility of the test method and also illustrated the difficulties in the test due to the cross axis sensitivity problem of the TAP.
Figure 5.0.2 shows the dimensions and the FE model of the test item. The test item is a cold rolled steel beam that is 24" long, 1" wide and 0.25" thick. Its Young's Modules is 2.11x 10^{11} Pa and its density is 7850 kg/m^3. Its FE model consists of 48 3D spatial beam elements, and three truss elements are used to represent the rubber band suspension. Lumped mass assumption is employed in these elements. The rigid body modes are below 1 Hz.

Table 5.0.1 shows the natural frequency of the test item above the rigid body modes. For all the simulation work in this chapter we study up to 200 Hz and include one natural frequency above the rigid body modes.

Table 5.0.1 Natural frequency of the test item

<table>
<thead>
<tr>
<th>Order</th>
<th>Natural Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>91</td>
</tr>
<tr>
<td>2</td>
<td>251</td>
</tr>
<tr>
<td>3</td>
<td>492</td>
</tr>
<tr>
<td>4</td>
<td>814</td>
</tr>
<tr>
<td>5</td>
<td>1215</td>
</tr>
</tbody>
</table>
Figure S.0.3 shows the dimensions and the FE model of the T-bar. Its Young's Modules is $2.11 \times 10^{11}$ Pa and its density is 7850 kg/m$^3$. Its cross section is 0.5" x 0.3125" for both the horizontal part and the vertical part. Its FE model consists of 4 nodes and 3 elements. The connecting point between the test item and the T-bar is Point 1. The driving point accelerance $[V_i]$ is obtained at $x_1$, $y_1$ and $\theta_1$. The proportional damping coefficients $\alpha = 3.175$ and $\beta = 2.888 \times 10^{-5}$. $\alpha$ and $\beta$ are chosen to be the same for both the test item and the T-bar so that it is easier to model the combined structure.

Table 5.0.2 shows the natural frequency of the T-bar. Obviously the lowest natural frequency of the T-bar is much higher than the range of investigation so that we can treat it as a rigid bar. However the nice property of the method proposed here is that since we have the finite element model of the T-bar, we have its frequency response functions at hand. No matter if it is rigid or flexible, the methods proposed in Chapter 5 and Chapter 7 always work.

In Scenarios 2, 3, and 4 of Chapter 5, we model the TAP-accelerometer combination. In Scenario 4, if two identical TAP-accelerometer combinations are used, there is numerical difficulty in the computations. For simplification we assume the dimension of the first TAP-accelerometer combination is 0.5" x 0.25" x 0.25", the second TAP-accelerometer combination is 2" x 0.5" x 0.5". Both of them are made of steel.

Figure 5.0.4 shows the driving point accelerance of the test item up to 200 Hz. For each FRF, the left graph is the magnitude in dB (Ref: 1). The right graph is the phase in degrees. The total number of nodes in the FE model of the beam is 48. The driving point and the response point are the same, Point 2, the second node to the left end. The DOF of $x_2$, $y_2$, and $\theta_2$ are considered as the DOF belonging to $i$. Among various graphs, what is very
Figure 5.0.3 Dimension and finite element model of T-bar

Table 5.0.2 Natural frequency of the T-bar

<table>
<thead>
<tr>
<th>Order</th>
<th>Natural Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5128</td>
</tr>
<tr>
<td>2</td>
<td>8047</td>
</tr>
<tr>
<td>3</td>
<td>23092</td>
</tr>
<tr>
<td>4</td>
<td>33947</td>
</tr>
<tr>
<td>5</td>
<td>40224</td>
</tr>
</tbody>
</table>

familiar is $T_{22}$, $y$ directional acceleration over $y$ directional excitation, which is classical in textbooks. $T_{11}$ is also more predictable since the axial resonance is much higher than the lateral resonance for a beam. Therefore the $T_{11}$ curve essentially represents the rigid body type of behavior of the beam in this direction. What cannot be seen in various vibration textbooks are $T_{23}$ (and its equivalent $T_{32}$) and $T_{33}$. $T_{23}$ represents the $y$ directional acceleration
under a unit moment on the $\theta$ direction. $T_{33}$ represents the $\theta$ directional acceleration under the unit excitation on the $\theta$ direction, a unit moment.

In 3D beam elements, the axial stiffness is not coupled with the lateral and rotational motions and vice versa, therefore $T_{12}(\omega) = T_{13}(\omega) = T_{21}(\omega) = T_{23}(\omega) = 0$. The Maxwell reciprocity is reflected at $T_{23}(\omega) = T_{32}(\omega)$.

Figure 5.0.4 Driving point accelerance [$T_{ii}$] of Test Item without T-bar attached
Figure 5.0.4 (continued)
Figure 5.0.5 shows the multidirectional transfer point accelerance of the test item. For each FRF, the left graph is the magnitude in dB (ref: 1). The right graph is the phase in degrees. The response point in the transfer point accelerance is Point 48, the second node to the right end. $x_{48}$, $y_{48}$, and $\theta_{48}$ are considered as the DOF's that belongs to $e$. Since for 3D beam element, its axial force is not coupled with the lateral and rotational motions and vice versa, therefore $T_{e_{112}}(\omega) = T_{e_{113}}(\omega) = T_{e_{212}}(\omega) = T_{e_{311}}(\omega) = 0$. 

Figure 5.0.5 Transfer point accelerance of Test Item $[T_{ei}]$ without T-bar attached
Figure 5.0.5 (continued)
Noise is always a problem in real test. A feasible method must be robust with noise contamination. The noise generation mechanism is as follows:

Since the measured acceleration, for instance, $X_i$, is a function of frequency, we can compute the root-mean-square (RMS) value of the signal of in $X_i$ using Parseval's Formula [43]. The noise generated by a random noise generator and is thus scaled so that its RMS value is 5% of the RMS value of the signal. 30 averages are used to improve the data quality.

Scenario 1: Driving Point Accelerance with Test Item and T-bar

Theoretical development

For 2D, there are three unknown interface forces, i.e., shear force, normal force, and moment. Assume that the T-bar has a built-in TAP-accelerometer combination that is part of the T-bar's dynamic characteristics and that the TAP-accelerometer combination can measure tangential, normal, and angular acceleration at the interface. For this scenario, we have three interface accelerations and three impact forces measured on the "Instrument Cluster" and the impact force directions are known. Since the T-bar has known values of input force as well as linear and angular accelerations, it is possible to solve for the interface motions and the interface driving point acceleration $[T_{il}]$ if the T-bar dynamic characteristics are known.

Figure 5.1.1 shows the combined structure and separate items of the test setup. For convenience the test item is depicted with a line.
Figure 5.1.1. Combined Structure and Separated Components

The nomenclature used in this scenario is as follows. \( \{X\} \) is the frequency domain description of the acceleration vector of the test item. Similarly \( \{Y\} \) is the acceleration vector of the T-bar. \( \{F\} \) is the force and moment vector acting on the test item. \( \{P\} \) is the force and moment vector acting on the T-bar. \( [T] \) is the driving point acceleration matrix of the test item. \( [V] \) is the driving point acceleration matrix of the T-bar.

We need to work out the relationships to solve for \([T_{il}]\) in terms of the measured quantities and the T-bar characteristics. The frequency domain motions and forces can be written as

\[
\{X\} = \begin{bmatrix} X_i \\ X_e \end{bmatrix}, \quad \{Y\} = \begin{bmatrix} Y_i \\ Y_e \end{bmatrix}, \quad \{P\} = \begin{bmatrix} P_i \\ P_e \end{bmatrix}, \quad \{F\} = \begin{bmatrix} F_i \\ F_e \end{bmatrix}
\]

(5.1.1)

where \( i \) is the DOF's related to driving points and \( e \) is the DOF related to points other than \( i \).
The interface conditions are motion continuity while the force and moment must satisfy Newton's Third Law so that

\[ \{X_t\} = \{Y_t\}, \{F_t\} + \{P_t\} = \{0\} \] (5.1.2)

The governing equations for the test item become

\[ \begin{bmatrix} X_t \\ X_e \end{bmatrix} = \begin{bmatrix} T_{u} & T_{s} \\ T_{s} & T_{ee} \end{bmatrix} \begin{bmatrix} F_t \\ F_e \end{bmatrix} \] (5.1.3)

The governing equations for the T-bar become

\[ \begin{bmatrix} Y_t \\ Y_e \end{bmatrix} = \begin{bmatrix} V_{u} & V_{u} \\ V_{s} & V_{ee} \end{bmatrix} \begin{bmatrix} P_t \\ P_e \end{bmatrix} \] (5.1.4)

A test condition is that no external forces are applied to the test item so that

\[ \{F_e\} = \{0\} \] (5.1.5)

Now if we ignore \{Y_e\} and use Equation 5.1.5, the governing equations become:

\[ \{X_t\} = [T_{u}][F_t] \] (5.1.6)

\[ \{X_e\} = [T_{s}][F_t] \] (5.1.7)

\[ \{Y_t\} = [V_{u}][P_t] + [V_{s}][P_e] \] (5.1.8)

Then we apply interface condition into Equations 5.1.6, 5.1.7, and 5.1.8 to obtain

\[ \{X_t\} = [T_{u}][-P_t] \] (5.1.9)

\[ \{Y_t\} = [V_{u}][P_t] + [V_{s}][P_e] \] (5.1.10)

\[ \{X_e\} = [T_{s}][-P_t] \] (5.1.11)

For each impact, \{X_e\} has only one non-zero element at the excitation DOF \( q \) and \( P_q \) is frequency dependent. The elements in other DOF's are zero.
If we divide \( \{ X_i \}, \{ X_e \}, \{ P_i \}, \) and \( \{ P_e \} \) by \( P_q(\omega) \) at each frequency \( \omega \), \( \{ X_i \} \) and \( \{ X_e \} \) become FRF's, \( \{ \bar{X}_i \} \) and \( \{ \bar{X}_e \} \), \( \{ P_i \} \) becomes force ratio function \( \{ \bar{F} \} \) and \( \{ P_e \} \) becomes a constant vector \( \{ \bar{F} \} \) where \( P_q(\omega) \) is replaced by 1.

Equations 5.1.9, 5.1.10, and 5.1.11 become

\[
\{ \bar{X}_i \} = [T_{ui}][-\bar{F}_i], \tag{5.1.13}
\]

\[
\{ \bar{V}_i \} = [V_{ui}][[\bar{F}]] + [V_{ue}][\bar{F}_e] \tag{5.1.14}
\]

\[
\{ \bar{X}_e \} = [T_{ue}][-\bar{F}] \tag{5.1.15}
\]

The essential effect of \( [V_{ue}][\bar{F}_e] \) is to extract elements in \( [V_{ue}] \) to form a vector \( \{ V_{ue} \} \) where \( q \) is the excitation DOF. \( \{ V_{ue} \} \) is computed directly and it has nothing to do with measurement.

\[
[V_{ue}][\bar{F}_e] = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \end{bmatrix} = \{ V_{ue} \} \tag{5.1.16}
\]

From Equation 5.1.14 we solve for \( \{ \bar{F} \} \)

\[
\{ \bar{F}_i \} = [V_{ui}]^{-1}(\{ \bar{X}_i \} - [V_{ue}][\bar{F}_e]) \tag{5.1.17}
\]

Putting Equation 5.1.17 into Equation 5.1.13 yields
\{\bar{X}_i\} = [T_u] V_u \{P_i\} - \{\bar{X}_i\}

(5.1.18)

Expand the excitation and response vectors \{\bar{X}_i\} and \{P_i\} to the square matrix by including several test cases. These test cases must be independent and collectively they should excite motions of all the DOF, i.e., the tangential, normal and angular motions.

Finally we have the driving point FRF matrix

\[ [T_u] = [\bar{X}_i][V_u] [P_i] - [\bar{X}_i]^{-1} [V_u] \]

(5.1.19)

Now we try to solve for \[T_u\], the transfer accelerance. If another TAP-accelerometer combination with negligible mass is installed on any of the \(e\) point, both the accelerations and the impact forces can be measured. Then \{\bar{X}_e\} is known. Equation 5.1.15 can be used to solve for \[T_u\].

Putting Equation 5.1.17 into Equation 5.1.15 gives

\[ \{\bar{X}_e\} = [T_u] V_u \{P_e\} - \{\bar{X}_i\} \]

(5.1.20)

Expand to square matrices, \[T_u\] can be solved.

\[ [T_u] = [\bar{X}_e][V_u] [P_e] - [\bar{X}_i]^{-1} [V_u] \]

(5.1.21)

The solution of \[T_u\] is to move the T-bar to the point that needs to be studied and treat the new point as a new driving point.

**Numerical Simulation**

Figure 5.1.2 shows the excitation and measurement points on the combined structure. The T-bar is connected to the beam at Point 2. When testing the combined
system, three impacts are used at \(x_{50}, y_{50}\) and \(y_{51}\) in order to obtain the square FRF matrix \([\mathbf{X}_i]\). \([V_\alpha]\) for the T-bar alone is computed accordingly.

The impact at \(x_{50}\) introduces a moment and a horizontal force with respect to the interface. The impact at \(y_{50}\) brings a moment and a vertical force. The impact at \(y_{51}\) introduces a pure vertical force. The method proposed also applies to the elastic bodies. What is important is that we must excite the response in each DOF. For instance, if we impact \(y_{50}, y_{51}\), and \(y_{52}\), though vertical forces and moment are introduced, the horizontal response is unavailable. The first row of \([\mathbf{X}_i]\) is zero and no useful results can be obtained.

Figure 5.1.3 shows the extracted \([\mathbf{T}_{ij}]\) with 5% RMS noise and 30 averages. For each FRF, the left graph is the magnitude in dB (ref: 1) and the right graph is the corresponding phase in degrees. \(T_{11}\) represents \(x\) directional acceleration vs. \(x\) directional force. \(T_{22}\) represents \(y\) directional acceleration vs. \(y\) directional force. \(T_{23}\) represents \(y\) directional acceleration vs. \(\theta\) directional moment. \(T_{32}\) represents \(\theta\) directional acceleration vs. \(y\)
directional force. \( T_{33} \) represents \( \theta \) directional acceleration vs. \( \theta \) directional moment. The method is robust under noise in maintaining the major trends. \( T_{22} \) seems most robust and the problem is at the low frequency valleys where the signal-noise ratio is low. Curve fitting might eliminate the noise effect since the majority of the curve is clear. The rapid change in the phase plot is typical for experimental data where 180° is the same as -180°.

![Figure 5.1.3](image)

(a) Extracted vs. theoretical \([T_{ij}]\) in Scenario 1, with 5% noise and 30 averages

(b)
Figure 5.1.3 (continued)
Scenario 2: Test Item, T-bar and TAP-Accelerometer Combination

Theoretical Development

This problem is restricted to 2D. The difference of Scenario 2 and Scenario 1 is that the TAP-accelerometer combination is not built in the T-bar and the mass of the TAP-accelerometer combination is not negligible. The TAP-accelerometer is put at the interface and has the ability to measure tangential, normal, and angular acceleration. Figure 5.2.1 shows the combined structure and separate components of the test setup.

The nomenclature used in this scenario is as follows. \( \{Z\} \) is the frequency domain acceleration vector of the accelerometer. Similarly \( \{X\} \) is the acceleration vector of the test item. \( \{Y\} \) is the acceleration vector of the T-bar. \( \{Q\} \) is the force and moment vector acting on the TAP-accelerometer combination. \( \{F\} \) is the force and moment vector acting on the test item. \( \{P\} \) is the force and moment vector acting on the T-bar. \( \{7\} \) is the FRF matrix of the test item. \( \{A\} \) is the FRF matrix of the TAP-accelerometer combination. \( \{V\} \) is the FRF matrix of the T-bar.

Figure 5.2.1 Combined structure and separate components
We need to work out the relationships to solve for $[T_{ii}]$ in terms of measured quantities, the T-bar characteristics and the TAP-accelerometer characteristics. The frequency domain motions and forces can be written as

$$
\{Z\} = \{Z_i\}, \{X\} = \{X_i\}, \{Y\} = \{Y_i\}, \{Q\} = \{Q_i\}, \{P\} = \{P_i\}, \{F\} = \begin{bmatrix} F_{in} \\ F_{i2} \\ F_e \end{bmatrix} \tag{5.2.1}
$$

where $i$ is the driving point and $e$ represents points other than $i$ for the test item, T-bar and TAP-accelerometer combination. $F_{ii}$ and $F_{i2}$ represent different forces acting on the interface from the TAP-accelerometer combination and the T-bar respectively.

The interface conditions include the motion continuity and general force balance that are expressed as

$$
\{X_i\} = \{Y_i\} = \{Z_i\}, \{F_{in}\} + \{Q_i\} = \{0\}, \{F_{i2}\} + \{P_i\} = \{0\} \tag{5.2.2}
$$

No external forces are applied to both the test item and the TAP-accelerometer combination so that

$$
\{F_e\} = \{0\}, \{Q_e\} = \{0\} \tag{5.2.3}
$$

The governing equation for the test item is

$$
\begin{bmatrix} X_i \\ X_e \end{bmatrix} = \begin{bmatrix} T_{ii} & T_{ie} \\ T_{ei} & T_{ee} \end{bmatrix} \begin{bmatrix} F_{in} + F_{i2} \\ F_e \end{bmatrix} \tag{5.2.4}
$$

The governing equation for the T-bar is

$$
\begin{bmatrix} Y_i \\ Y_e \end{bmatrix} = \begin{bmatrix} V_{ii} & V_{ie} \\ V_{ei} & V_{ee} \end{bmatrix} \begin{bmatrix} P_i \\ P_e \end{bmatrix} \tag{5.2.5}
$$

The governing equation for the TAP-accelerometer combination is
where \( \{Z_i\}, \{P_i\}, \{V_u\}, \{V_w\}, \{A_u\}, \{A_w\}, \) and \( \{A_{ew}\} \) are known.

To simplify the above equations, we apply Equation 5.2.3 and ignore the external motions \( \{Y_i\} \), and \( \{Z_e\} \). Then the governing equations become

\[
\begin{align*}
\{X_i\} &= \left[ T_u \right] [F_n + F_{f2}] \\
\{Y_i\} &= \left[ V_u \right] [P_i] + \left[ V_w \right] [P_e] \\
\{Z_i\} &= \left[ A_u \right] [Q_i] \\
\{X_e\} &= \left[ T_u \right] [F_n + F_{f2}]
\end{align*}
\]

(5.2.7) \hspace{1cm} (5.2.8) \hspace{1cm} (5.2.9) \hspace{1cm} (5.2.10)

where \( \{Z_i\}, \{P_i\}, \{V_u\}, \{V_w\}, \) and \( \{A_u\} \) are known. Then we apply the interface condition to these governing equations to obtain

\[
\begin{align*}
\{X_i\} &= \left[ T_u \right] [-Q_i - P_i] \\
\{X_i\} &= \left[ V_u \right] [P_i] + \left[ V_w \right] [P_e] \\
\{X_i\} &= \left[ A_u \right] [Q_i] \\
\{X_e\} &= \left[ T_u \right] [-Q_i - P_i]
\end{align*}
\]

(5.2.11) \hspace{1cm} (5.2.12) \hspace{1cm} (5.2.13) \hspace{1cm} (5.2.14)

Using a similar division procedure as described in detail in Scenario 1, we can change Equation 5.2.11 through Equation 5.2.14 to the forms of measured FRF and force ratio.
functions. \{X_i\} and \{X_s\} become FRF's \{\overline{X_i}\} and \{\overline{X_s}\}. \{P_i\} and \{Q_i\} become force ratio functions \{\overline{P_i}\} and \{\overline{Q_i}\}. \{P_s\} becomes a constant vector \{\overline{P_s}\} where \(P_s(\omega)\) is replaced by 1.

\[
\{\overline{X_i}\} = [T_i]([-\overline{Q_i} - \overline{F_i}]) \quad (5.2.15)
\]

\[
\{\overline{X_s}\} = [V_s][\overline{P_s}] + [V_s][\overline{P_s}] \quad (5.2.16)
\]

\[
\{\overline{X_i}\} = [A_i][\overline{Q_i}] \quad (5.2.17)
\]

\[
\{\overline{X_s}\} = [T_s]([-\overline{Q_i} - \overline{F_i}]) \quad (5.2.18)
\]

From Equation 5.2.17 we have

\[
\{\overline{Q_i}\} = [A_i]^{-1}\{\overline{X_i}\} \quad (5.2.19)
\]

From Equation 5.2.16 we have

\[
\{\overline{P_i}\} = [V_s]^{-1}((\{\overline{X_s}\} - [V_s][\overline{P_s}]) \quad (5.2.20)
\]

From Equation 5.2.15 we have

\[
\{\overline{X_i}\} = [T_i] - [A_i]^{-1}\{\overline{X_i}\} - [V_s]^{-1}((\{\overline{X_i}\} - [V_s][\overline{P_s}])) \quad (5.2.21)
\]

Rewrite Equation 5.2.21, we have

\[
\{\overline{X_i}\} = [T_i][V_s]^{-1}[V_s][\overline{P_s}] - ([A_i]^{-1} + [V_s]^{-1})[\overline{X_i}] \quad (5.2.22)
\]

Expand the impact and response vectors \{\overline{X_i}\} and \{\overline{P_s}\} to be square matrices by including several test cases. These test cases must be independent and collectively they should excite all motions of the DOF's, i.e., the tangential, normal and angular motions.

\[
[\overline{X_i}] = [T_i][V_s]^{-1}[V_s][\overline{P_s}] - ([A_i]^{-1} + [V_s]^{-1})[\overline{X_i}] \quad (5.2.23)
\]

Finally the driving point FRF matrix is

\[
[T_i] = [\overline{X_i}][V_s]^{-1}[V_s][\overline{P_s}] - ([A_i]^{-1} + [V_s]^{-1})[\overline{X_i}]^{-1} \quad (5.2.24)
\]
**Numerical Simulations**

Figure 5.2.2 shows the excitation and measurement points on the combined structure. The T-bar is first connected to the beam at Point of the beam and also of the combined system. The accelerometer is connected at Point 2 too. Point 1 of the T-bar coincides with Point 2 of the combined system. Points 2, 3, and 4 of the T-bar become Points 50, 51, and 52 for the combined system. The TAP-accelerometer is modeled by a beam element whose nodes are 2 and 53. For the combined system, three impacts are given at $x_{50}$, $y_{50}$ and $y_{51}$. Acceleration measurements are obtained at $x_2$, $y_2$ and $\theta_2$ to find $[X_t]$ for the combined structure.

Figure 5.2.3 shows the extracted $[T_h]$ using noisy data. The left plot of each sub-graph is the magnitude and the right plot is the phase. Obviously under the noise, the method is robust in maintaining the major trends. Again the low frequency valleys have some problem due to the poor signal noise ratio.

Figure 5.2.2 Impact and measurement points
Figure 5.2.3 Extracted vs. theoretical $[T_n]$ in Scenario 2, with 5% RMS noise and 30 averages
Scenario 3 Transfer Point Accelerance Measurement

Theoretical Development

This problem is restricted to 2D space. The difference between Scenario 3 and Scenario 2 is that the accelerometer is moved from the original interface point $i$ where the T-bar and the test item meet to another point $q$. The TAP-accelerometer combination has the
ability to measure tangential, normal and angular acceleration. Therefore the interface
conditions are changed since we measure the accelerations at point \( q \), not point \( i \).

The nomenclature used in this scenario is as follows. \( \{ Z \} \) is the frequency domain
acceleration of the TAP-accelerometer combination. Similarly \( \{ X \} \) is the acceleration vector
of the test item. \( \{ Y \} \) is acceleration vector of the T-bar. \( \{ Q \} \) is the force vector acting on the
TAP-accelerometer combination. \( \{ F \} \) is the force vector acting on the test item. \( \{ P \} \) is the
force vector acting on the T-bar. \( [T] \) is the FRF matrix of the test item. \( [A] \) is the FRF matrix
of the TAP-accelerometer combination. \( [V] \) is the FRF matrix of the T-bar. We tried to use
the measured quantities and the dynamics of the T-bar and the TAP-accelerometer
combination to solve for \([T_{ii}]\). The frequency domain motions and forces can be written as

\[
\{ Z \} = \begin{bmatrix} Z_i \\ Z_e \end{bmatrix}, \quad \{ X \} = \begin{bmatrix} X_i \\ X_e \end{bmatrix}, \quad \{ Y \} = \begin{bmatrix} Y_i \\ Y_e \end{bmatrix}, \quad \{ Q \} = \begin{bmatrix} Q_i \\ Q_e \end{bmatrix}, \quad \{ P \} = \begin{bmatrix} P_i \\ P_e \end{bmatrix}, \quad \{ F \} = \begin{bmatrix} F_i \\ F_e \end{bmatrix}
\]  

(5.3.1)

where \( i \) is the driving point, \( q \) is the transfer point and \( e \) is the points other than \( i \) and \( q \).

Figure 5.3.1 Combined Structure and Separate Components
The interface conditions are continuity of the accelerations and balance of the forces.

\[ \{X_i\} = \{Y_i\}, \quad \{X_q\} = \{Z_q\}, \quad \{F_i\} + \{P_i\} = \{0\}, \quad \{F_q\} + \{Q_q\} = \{0\} \quad (5.3.2) \]

The test constraints are that there are no external forces acting on either the test item or the TAP-accelerometer combination.

\[ \{F_e\} = \{0\}, \quad \{Q_e\} = \{0\} \quad (5.3.3) \]

The governing equation for the test item is

\[ \begin{bmatrix} X_i \\ X_q \\ X_e \end{bmatrix} = \begin{bmatrix} T_{ii} & T_{iq} & T_{ie} \\ T_{qi} & T_{qq} & T_{qe} \\ T_{ei} & T_{eq} & T_{ee} \end{bmatrix} \begin{bmatrix} F_i \\ F_q \\ F_e \end{bmatrix} \quad (5.3.4) \]

The governing equation for the T-bar is

\[ \begin{bmatrix} Y_i \\ Y_e \end{bmatrix} = \begin{bmatrix} V_{ii} & V_{ie} \\ V_{ei} & V_{ee} \end{bmatrix} \begin{bmatrix} P_i \\ P_e \end{bmatrix} \quad (5.3.5) \]

The governing equation for the TAP-accelerometer is

\[ \begin{bmatrix} Z_q \\ Z_e \end{bmatrix} = \begin{bmatrix} A_{qq} & A_{qe} \\ A_{eq} & A_{ee} \end{bmatrix} \begin{bmatrix} Q_q \\ Q_e \end{bmatrix} \quad (5.3.6) \]

What are known are \( \{Z_q\}, \quad \{P_e\}, \quad \{V_{ii}\}, \quad \{V_{ie}\}, \quad \{V_{ei}\}, \quad \{V_{ee}\}, \quad \{A_{ii}\}, \quad \{A_{ie}\}, \quad \{A_{ei}\}, \quad \{A_{ee}\} \).

Now apply \( \{F_e\} = \{0\} \) and \( \{Q_e\} = \{0\} \) to the forces and ignore the motions \( \{X_e\}, \quad \{Y_e\}, \)

and \( \{Z_e\} \). Then the governing equations become

\[ \{X_i\} = [T_{ii}][F_i] + [T_{iq}][F_q] \quad (5.3.7) \]

\[ \{X_q\} = [T_{qi}][F_i] + [T_{qq}][F_q] \quad (5.3.8) \]

\[ \{Y_i\} = [V_{ii}][P_i] + [V_{ie}][P_e] \quad (5.3.9) \]

\[ \{Z_q\} = [A_{qq}][Q_q] \quad (5.3.10) \]
Next, apply interface condition into the governing equations to obtain

\[ \{Y_i\} = [T_u](-P_i) + [T_{eq}](-Q_{eq}) \]  \hspace{1cm} (5.3.11)

\[ \{Z_q\} = [T_{eq}](-P_i) + [T_{eq}](-Q_{eq}) \]  \hspace{1cm} (5.3.12)

\[ \{Y_i\} = [V_i](P_i) + [V_{eq}](P_e) \]  \hspace{1cm} (5.3.13)

\[ \{Z_q\} = [A_{eq}][Q_q] \]  \hspace{1cm} (5.3.14)

In order to solve for \( [T_{eq}] \), we must assume that \( [T_u] \) and \( [T_{eq}] \) are already known based on the method in Scenario 2. Using a similar division procedure as described in detail in Scenario 1, we can change Equation 5.3.11 through Equation 5.3.14 to the forms of measured FRF and force ratio functions. \( \{Y_i\} \) and \( \{Z_q\} \) become FRF's \( \{\overline{Y}_i\} \) and \( \{\overline{Z}_q\} \). \( \{P_i\} \) and \( \{Q_q\} \) become force ratio functions \( \{\overline{P}_i\} \) and \( \{\overline{Q}_q\} \). \( \{P_e\} \) becomes a constant vector \( \{\overline{P}_e\} \) where \( P_q(\omega) \) is replaced by 1.

\[ \{\overline{Y}_i\} = [T_u](-\overline{P}_i) + [T_{eq}](-\overline{Q}_q) \]  \hspace{1cm} (5.3.15)

\[ \{\overline{Z}_q\} = [T_{eq}](-\overline{P}_i) + [T_{eq}](-\overline{Q}_q) \]  \hspace{1cm} (5.3.16)

\[ \{\overline{Y}_i\} = [V_i](\overline{P}_i) + [V_{eq}](\overline{P}_e) \]  \hspace{1cm} (5.3.17)

\[ \{\overline{Z}_q\} = [A_{eq}][\overline{Q}_q] \]  \hspace{1cm} (5.3.18)

From Equation 5.3.18 we have

\[ \{\overline{Q}_q\} = [A_{eq}]^{-1}\{\overline{Z}_q\} \]  \hspace{1cm} (5.3.19)

Eliminate \( \{\overline{Y}_i\} \) from Equation 5.3.15 and Equation 5.3.17, we have

\[ [V_i](\overline{P}) + [V_{eq}](\overline{P}_e) = [T_u][\overline{P}_i] + [T_{eq}](-[A_{eq}]^{-1}\{\overline{Z}_q\}) \]  \hspace{1cm} (5.3.20)

Putting Equation 5.3.19 into Equation 5.3.16 yields
(5.3.21)

\[ \{\mathbf{Z}_q\} + [\mathbf{T}_{qv} \mathbf{I}_{qv}]^{-1} \{\mathbf{Z}_q\} = [\mathbf{T}_{qv} \mathbf{P}_{qv}] \{\mathbf{P}_r\} \]

Expose \( \{-\mathbf{P}_r\} \) from Equation 5.3.20, we have

\[ \{\mathbf{P}_r\} = ([\mathbf{V}_u] + [\mathbf{T}_{ru}]^{-1} ([\mathbf{T}_{qu}]^\top [\mathbf{A}_{qu}]^{-1} \{\mathbf{Z}_q\} + [\mathbf{V}_u] \{\mathbf{P}_r\}) \]

Put \( \{-\mathbf{P}_r\} \) into Equation 5.3.21, we can find

\[ ([\mathbf{I}] + [\mathbf{T}_{qv} \mathbf{I}_{qv}]^{-1}) \{\mathbf{Z}_q\} = [\mathbf{T}_{qv} \mathbf{V}_u] + [\mathbf{T}_{ru}]^{-1} ([\mathbf{T}_{qu}]^\top [\mathbf{A}_{qu}]^{-1} \{\mathbf{Z}_q\} + [\mathbf{V}_u] \{\mathbf{P}_r\}) \]

where \([\mathbf{I}]\) is the identity matrix. Expand the impact and response vectors \( \{\mathbf{Z}_q\} \) and \( \{\mathbf{P}_r\} \) to the square matrix by including several test cases, Equation 5.3.23 becomes

\[ ([\mathbf{I}] + [\mathbf{T}_{qv} \mathbf{I}_{qv}]^{-1}) \{\mathbf{Z}_q\} = [\mathbf{T}_{qv} \mathbf{V}_u] + [\mathbf{T}_{ru}]^{-1} ([\mathbf{T}_{qu}]^\top [\mathbf{A}_{qu}]^{-1} \{\mathbf{Z}_q\} + [\mathbf{V}_u] \{\mathbf{P}_r\}) \]

(5.3.24)

Since in Equation 5.3.24 both \( [\mathbf{T}_{qv}] \) and \( [\mathbf{T}_{qu}]^\top \) exist, it is impossible to find an explicit expression to solve for \( [\mathbf{T}_{qv}] \). Therefore an iterative formulation is used

\[ [\mathbf{T}_{qv}] = ([\mathbf{I}] + [\mathbf{T}_{qv} \mathbf{I}_{qv}]^{-1}) \{\mathbf{Z}_q\} = [\mathbf{T}_{qv}]^\top [\mathbf{A}_{qv}]^{-1} \{\mathbf{Z}_q\} + [\mathbf{V}_u] \{\mathbf{P}_r\}]^\top ([\mathbf{V}_u] + [\mathbf{T}_{ru}]^{-1}) \]

(5.3.25)

At each frequency we put the identity matrix \([\mathbf{I}]\) as the initial trial value of \( [\mathbf{T}_{qv}] \) to the right hand side of Equation 5.3.25 to solve for \( [\mathbf{T}_{qv}] \). Then we put the solved \( [\mathbf{T}_{qv}] \) as the new trial value back to the right hand side of Equation 5.3.25 again to solve for \( [\mathbf{T}_{qv}] \). This procedure continues until the relative error of the trial and the computed \( [\mathbf{T}_{qv}] \) is very small. The numerical simulation shows that the convergence is very fast, about 3-5 iterations and the relative error of the trial and computed \( [\mathbf{T}_{qv}] \) is less than \( 10^{-6} \).
Numerical Simulations

Figure 5.3.2 shows the impact and measurement points. For numerical simulation below, we use \( e \) instead of \( q \) to denote the transfer point, Point 48. Figure 5.3.3 shows the extracted \([T_{ei}]\) using the noisy data. Similar to the \([T_{ii}]\), under the noise, the method is robust.

The methods illustrated in Scenario 2 and Scenario 3 can be used together to find the driving point and transfer point acceleration matrices. They are especially useful in the test of lightweight structures where the addition of instruments significantly changes the system dynamic characteristics. Since the extracted \([T_{ii}]\) and \([T_{ei}]\) are for the test item alone, the influence of both the T-bar and the TAP-accelerometer has been removed from the measured data.

![Figure 5.3.2 Impact and measurement points](image)
Figure 5.3.3 Extracted vs. theoretical transfer point accelerance \( T_e \) with 5% noise, 30 averages
Figure 5.3.3 (continued)

Scenario 4 Test item, T-bar and Two TAP-Accelerometer Combinations

Theoretical Development

Although using the methods in Scenario 2 and Scenario 3 we can find the driving point and transfer point accelerance, it is not very easy as we need to do three separate tests. A driving point accelerance test at the driving point is followed by a driving point accelerance test at the response point. Then a transfer point accelerance test needs to be done.
where the T-bar is put on the driving point and the accelerometer is put on the response point. What if we have two accelerometers and we put one at the driving point and the other at the transfer point? Is it possible to extract both $[T_d]$ and $[T_r]$ using this configuration? Scenario 4 answers these questions. Figure 5.4.1 shows the combined system and the separate components where interactions are exposed.

The nomenclature used in this scenario is as follows. $\{Z_i\}$ is the frequency domain acceleration vector of the TAP-accelerometer combination located at $i$. Similarly, $\{Z_q\}$ is the acceleration vector of the TAP-accelerometer combination located at $q$. $\{X\}$ is the acceleration vector of the test item. $\{Y\}$ is the acceleration vector of the T-bar. $\{Q_i\}$ is the force acting on the TAP-accelerometer combination at $i$. $\{Q_q\}$ is the force acting on the TAP-accelerometer combination at $q$. $\{F\}$ is the force acting on the test item. $\{P\}$ is the force acting on the T-bar. $[T]$ is the FRF matrix of the test item. $[A]$ is the FRF matrix of the
TAP-accelerometer combination. \( [V] \) is the FRF matrix of the T-bar. The frequency domain motions and forces can be written as

\[
\begin{align*}
\{Z_i\} &= \begin{bmatrix} Z_i \\ Z_e \end{bmatrix}, \quad \{Z_q\} = \begin{bmatrix} Z_q \\ Z_e \end{bmatrix}, \quad \{Q_i\} = \begin{bmatrix} Q_i \\ Q_e \end{bmatrix}, \quad \{Q_q\} = \begin{bmatrix} Q_q \\ Q_e \end{bmatrix} \\
\{Y_i\} &= \begin{bmatrix} Y_i \\ Y_e \end{bmatrix}, \quad \{P_i\} = \begin{bmatrix} P_i \\ P_e \end{bmatrix}
\end{align*}
\]  
(5.4.1)  

\[
\begin{align*}
\{X_i\} &= \begin{bmatrix} X_i \\ X_q \\ X_e \end{bmatrix}, \quad \{F\} = \begin{bmatrix} F_{i1} + F_{i2} \\ F_q \\ F_e \end{bmatrix}
\end{align*}
\]  
(5.4.2)  

where \( i \) is the driving point, \( q \) is the transfer point and \( e \) represents the points other than \( i \) and \( q \) on test item, T-bar and TAP-accelerometer combinations.

The interface conditions are continuity of the accelerations and the balance of forces.

\[
\begin{align*}
\{X_i\} &= \{Y_i\} = \{Z_i\}, \quad \{X_q\} = \{Z_q\} \\
\{F_{ii}\} + \{P_i\} &= \{0\}, \quad \{F_{i2}\} + \{Q_i\} = \{0\}, \quad \{F_q\} + \{Q_q\} = \{0\}
\end{align*}
\]  
(5.4.4)  

(5.4.5)

It is assumed that there are no external forces acting on the test item and on TAP-accelerometers combinations.

\[
\{F_e\} = \{0\}, \quad \{Q_e\} = \{0\}
\]  
(5.4.6)

The governing equation for the test item is

\[
\begin{align*}
\begin{bmatrix}
X_i \\
X_q \\
X_e
\end{bmatrix} &= \begin{bmatrix} T_{ii} & T_{iq} & T_{ie} \\
T_{qi} & T_{qq} & T_{qe} \\
T_{ei} & T_{eq} & T_{ee}
\end{bmatrix} \begin{bmatrix} F_{i1} + F_{i2} \\
F_q \\
F_e
\end{bmatrix}
\end{align*}
\]  
(5.4.7)

The governing equation for the T-bar is

\[
\begin{align*}
\begin{bmatrix}
Y_i \\
Y_e
\end{bmatrix} &= \begin{bmatrix} V_{ii} & V_{ie} \\
V_{ei} & V_{ee}
\end{bmatrix} \begin{bmatrix} P_i \\
P_e
\end{bmatrix}
\end{align*}
\]  
(5.4.8)
The governing equation for the TAP-accelerometer located at the driving point is

\[
\begin{bmatrix}
Z_i \\
Z_e
\end{bmatrix} =
\begin{bmatrix}
A_{ii} & A_{ie} \\
A_{ei} & A_{ee}
\end{bmatrix}
\begin{bmatrix}
Q_i \\
Q_e
\end{bmatrix}
\]

(5.4.9)

The governing equation for the TAP-accelerometer located at the driving point is

\[
\begin{bmatrix}
Z_q \\
Z_e
\end{bmatrix} =
\begin{bmatrix}
A_{aq} & A_{ae} \\
A_{aq} & A_{ee}
\end{bmatrix}
\begin{bmatrix}
Q_q \\
Q_e
\end{bmatrix}
\]

(5.4.10)

\{Z_i\}, \{Z_q\} and \{P_c\} can be measured by the TAP-accelerometer combination and the impact hammer. \{V_{ii}\}, \{V_{ie}\}, \{V_{qe}\}, \{A_{ii}\}, \{A_{ie}\}, \{A_{ee}\}, \{A_{aq}\}, \{A_{ae}\}, \{A_{eq}\}, and \{A_{ee}\} can be computed from the finite element models of the T-bar and the TAP-accelerometer combination.

Apply Equation 5.4.6 and ignore \{X_e\}, and \{Y_e\}, \{Z_e\} in the governing equations yields

\[
\begin{bmatrix}
X_i
\end{bmatrix} = \begin{bmatrix}
T_{ii} & F_{i1} + F_{i2}
\end{bmatrix} + \begin{bmatrix}
T_{iq} & F_q
\end{bmatrix}
\]

(5.4.11)

\[
\begin{bmatrix}
X_q
\end{bmatrix} = \begin{bmatrix}
T_{aq} & F_{i1} + F_{i2}
\end{bmatrix} + \begin{bmatrix}
T_{aq} & F_q
\end{bmatrix}
\]

(5.4.12)

\[
\begin{bmatrix}
Y_i
\end{bmatrix} = \begin{bmatrix}
V_{ii} & P_i
\end{bmatrix} + \begin{bmatrix}
V_{ie} & P_e
\end{bmatrix}
\]

(5.4.13)

\[
\begin{bmatrix}
Z_i
\end{bmatrix} = \begin{bmatrix}
A_{ii} & Q_i
\end{bmatrix}
\]

(5.4.14)

\[
\begin{bmatrix}
Z_q
\end{bmatrix} = \begin{bmatrix}
A_{aq} & Q_q
\end{bmatrix}
\]

(5.4.15)

Apply Equations 5.4.4, 5.4.5 to the governing equations, we have

\[
\begin{bmatrix}
Z_i
\end{bmatrix} = \begin{bmatrix}
T_{ii} & -Q_i - P_i
\end{bmatrix} + \begin{bmatrix}
T_{iq} & -Q_q
\end{bmatrix}
\]

(5.4.16)

\[
\begin{bmatrix}
Z_q
\end{bmatrix} = \begin{bmatrix}
T_{aq} & -Q_i - P_i
\end{bmatrix} + \begin{bmatrix}
T_{aq} & -Q_q
\end{bmatrix}
\]

(5.4.17)

\[
\begin{bmatrix}
Z_i
\end{bmatrix} = \begin{bmatrix}
V_{ii} & P_i
\end{bmatrix} + \begin{bmatrix}
V_{ie} & P_e
\end{bmatrix}
\]

(5.4.18)
\{Z_i\} = [A_i]\{Q_i\} \quad (5.4.19)

\{Z_q\} = [A_{qq}][Q_q] \quad (5.4.20)

Unlike Scenario 3, here we do not need to assume that \[T_{qq}\] is already known. On the contrary, we can solve for \[T_{qq}\] even without placing the T-bar at point \(q\). This is something really interesting and has promising application use.

Using a similar division procedure as described in detail in Scenario 1, we can change Equation 5.4.16 through Equation 5.4.20 to the forms of measured FRF and force ratio functions. \(Z_i\) and \(Z_q\) become FRF's \(\tilde{Z}_i\) and \(\tilde{Z}_q\). \(P_i\), \(Q_i\) and \(Q_q\) become force ratio functions \(\tilde{P}_i\), \(\tilde{Q}_i\) and \(\tilde{Q}_q\). \(P_e\) becomes a constant vector \(\tilde{P}_e\) where \(P_e(\omega)\) is replaced by 1.

\[\{\tilde{Z}_i\} = [T_i][\tilde{Q}_i - \tilde{P}_i] + [T_{iq}][\tilde{Q}_q - \tilde{P}_q]\]

(5.4.21)

\[\{\tilde{Z}_q\} = [T_q][\tilde{Q}_i - \tilde{P}_i] + [T_{qq}][\tilde{Q}_q - \tilde{P}_q]\]

(5.4.22)

\[\{\tilde{Z}_i\} = [V_i][\tilde{P}_i] + [V_{ie}][\tilde{P}_e]\]

(5.4.23)

\[\{\tilde{Z}_i\} = [A_{ii}][\tilde{Q}_i]\]

(5.4.24)

\[\{\tilde{Z}_q\} = [A_{qq}][\tilde{Q}_q]\]

(5.4.25)

From Equation 5.4.24 and Equation 5.4.25 we have

\[\{\tilde{Q}_i\} = [A_{ii}]^{-1}\{\tilde{Z}_i\}\]

(5.4.26)

\[\{\tilde{Q}_q\} = [A_{qq}]^{-1}\{\tilde{Z}_q\}\]

(5.4.27)

\(\{\tilde{P}_i\}\) is obtained from Equation 5.4.23 as

\[\{\tilde{P}_i\} = [V_i]^{-1}(\{\tilde{Z}_i\} - [V_{ie}][\tilde{P}_e])\]

(5.4.28)
Put \( \{Q_i\} \), \( \{Q_q\} \), and \( \{F_i\} \) into Equation 5.4.21 and Equation 5.4.22, and expand them to square matrices by including different sets of tests.

\[
[Z_i] = [T_{ii}]^{-1} [Z_i] - [V_{ii}]^{-1} ([Z_i] - [V_{re} \{F_i\}]) + [T_{qi}]^{T} (- [A_{qq}]^{-1} [Z_q]) \tag{5.4.29}
\]

\[
[Z_q] = [T_{qi}]^{-1} [Z_i] - [V_{qi}]^{-1} ([Z_i] - [V_{re} \{F_i\}]) + [T_{qq}]^{T} (- [A_{qq}]^{-1} [Z_q]) \tag{5.4.30}
\]

Now we solve for \( [T_{qi}]^{T} \) in Equation 5.4.29 to obtain

\[
[T_{qi}]^{T} = ([T_{ii}]^{-1} [V_{ii}] - ([F] + [T_{ii}] ([A_{ii}]^{-1} + [V_{ii}]^{-1})([Z_i] - [V_{re} \{F_i\}])) ([Z_q]^{-1} [A_{qq}]) \tag{5.4.31}
\]

Solve for \( [T_{qq}] \) using Equation 5.4.30

\[
[T_{qq}] = -([Z_q] + [T_{qi}] ([A_{ii}]^{-1} + [V_{ii}]^{-1})([Z_i] - [V_{re} \{F_i\}])) ([Z_q]^{-1} [A_{qq}]) \tag{5.4.32}
\]

The numerical simulation proved that \( [T_{qi}]^{T} \) and \( [T_{qq}] \) can be found using Equation 5.4.31 and Equation 5.4.32 using the noise-free data of \( [Z_i] \) and \( [Z_q] \).

**Numerical Simulation**

Figure 5.4.2 shows the impact and measurement points. For numerical simulation below, we still use \( e \) other than \( q \) to denote the transfer point.

When noise-free data of \( [Z_i] \) and \( [Z_q] \) are used, the extracted \( [T_{ei}] \), and \( [T_{ee}] \) are the same as the theoretical values obtained from the pure beam model. At first this result is very encouraging because we have a way to use several TAP-accelerometer combinations at various response points while we put the T-bar at a single driving point. Potentially, this method gives us all the transfer accelerances related to the driving point, and all the driving point accelerances on these response points with the T-bar located at one point. This
procedure should be very useful when doing a multi-channel analysis for a larger structure.

Now polluted data is used to test the robustness of the procedure under noise. Figure 5.4.3 and Figure 5.4.4 shows the extracted $[T_{ei}]$ and $[T_{ee}]$ when using the noisy data for the case where $i$ stands for Point 2 and $e$ stands for Point 48. In $[T_{ei}]$ and $[T_{ee}]$, 1 represents $x$ direction, 2 represents the $y$ direction and 3 the $\theta$ direction.

It is found that this test method is very sensitive to noise. The pollution of noise makes the results deviate greatly from the theoretical value even when the noise level is 1% RMS instead of 5% RMS. Therefore the method is not robust. Only with $10^{-6}$ RMS noise and 100 averages will the plots look similar to those in Scenarios 2 and 3. Therefore, although this method is very attractive, it is doubtful that it can be made to work in practical cases.

The reason why this solution is sensitive to the noise may lie in the fact that two different sets of acceleration measurement at Point 2 and Point 48 are used in our
computation. Later we will find similar phenomena happen in some of the 3D scenarios where also two different sets of acceleration measurement are used. However there may be ways to overcome this noise problem when redundant data and/or pseudo-inverse techniques are used. Future research is required to see if the great noise sensitivity can be overcome.

Figure 5.4.3 Extracted vs. the theoretical $[T_d]$ with 1% noise and 30 averages
Figure 5.4.3 (continued)
Figure 5.4.4 Extracted vs. the theoretical $[T_{re}]$ with 1% noise and 30 averages
Scenario 5 Flexible Placement of TAP-Accelerometer Combination

Theoretical Development

The major difficulty in applying the extraction schemes of Scenario 1 through Scenario 4 is space limitation for transducer placement. There may be not enough space to put the TAP-accelerometer combination at the desired measurement point where both the test item and the vehicle are connected.
There are several solutions to the space limitation problem. First, we can put the TAP-accelerometer combination in the vicinity of the connecting point on the test item. We can put the TAP-accelerometer combination on top of the test item and put the T bar on the bottom of the test item as illustrated in Scenario 1 to Scenario 4. Then we can assume these two points represent the same point in the structure. However, errors do exist for this arrangement as illustrated in Chapter 3.

Second, we can put the TAP on the top of the test beam to measure the normal motion and rotation, and put another accelerometer on the T-bar to measure the tangential motion. We can assume that the tangential motion measured by the accelerometer is the same as the tangent motion at the interface point. But this arrangement still has the same errors as in the first arrangement.

Third we can put both the TAP and a second accelerometer on T-bar to build an "Instrumented T-bar". This arrangement has the advantage of putting everything on the T-bar so that the only problem is to be able to attach the T-bar to the test structure at the desired points.

Among the above three solutions, the third solution can be used as a general case and is implemented as Scenario 5. In this scenario, the impact forces are known and the interface accelerations are unknown. In this scenario the acceleration transducers are not attached directly at the interface point but are attached some where to the T-bar. Figure 5.5.1 shows the combined structure and the separate components of this scenario.

The nomenclature used in this scenario is as follows. \( \{X\} \) is the frequency domain description of the acceleration vector of the test item. Similarly \( \{Y\} \) is the acceleration vector of the T-bar. \( \{F\} \) is the force and moment vector acting on the test item. \( \{P\} \) is the
Accelerations are measured at $m_x$, $m_y$, and $m_\phi$

Figure 5.5.1 Combined Structure and Separate Components

force and moment vector acting on the T-bar. $[T]$ is the driving point accelerance matrix of the test item. $[V]$ is the driving point accelerance matrix of the T-bar.

The governing equation for the test item is

$$\{X_i\} = [T_i][F_i]$$ (5.5.1)

The governing equations for the T-bar is

$$\{Y_i\} = [V_{ii}][P_i] + [V_{ie}][P_e]$$ (5.5.2)

$$\{Y_m\} = [V_{mi}][P_i] + [V_{me}][P_e]$$ (5.5.3)

where $i$ is the DOF’s related to driving points, $e$ is the DOF’s related to points other than $i$, and $m$ is the DOF’s related to the measurement points. Note here $m \neq i$. $m$ can be a single point in T-bar where the TAP-accelerometer combination are installed. Since the number of DOF of $i$ is 3, assume that the number of DOF of $m$ is also 3. The transducers at $m$ should be able to measure tangential, normal and rotational motions. Therefore, one transducer might
be a TAP type accelerometer that measures normal and rotational motions at \( m \), and the other accelerometer is a standard linear accelerometer that measures the tangential motion at point \( m \). It is also possible that \( m \) represent two points that the normal and rotational motions are measured at one point and the tangential motion is measured at another point since \( m_x, m_y, \) and \( m_\theta \) need not to be the DOF’s of the same point \( m \).

The interface conditions are

\[
\begin{align*}
\{X_i\} &= \{Y_i\}, \{F_i\} = \{-P_i\} \\
\{Y_i\} &= [T_{ii}]\{-P_i\} \\
\{Y_i\} &= [V_{ii}]\{P_i\} + [V_{ie}]\{P_e\} \\
\{Y_m\} &= [V_{me}]\{P_i\} + [V_{me}]\{P_e\}
\end{align*}
\]

Putting the interface conditions to the above equations, we find

\[
\begin{align*}
\{Y_i\} &= [T_{ii}]\{-P_i\} \\
\{Y_i\} &= [V_{ii}]\{P_i\} + [V_{ie}]\{P_e\} \\
\{Y_m\} &= [V_{me}]\{P_i\} + [V_{me}]\{P_e\}
\end{align*}
\]

The unknown values are \( \{Y_i\}, \{T_{ii}\}, \) and \( \{P_i\} \), and the known values are \( \{V_{ii}\}, \{V_{me}\}, \{P_e\}, \{Y_m\}, \{V_{me}\}, \) and \( \{V_{me}\} \). Using a similar division procedure as described in detail in Scenario 1, we can change Equations 5.5.5 through 5.5.7 to the forms of measured FRF and force ratio functions. \( \{Y_i\} \) and \( \{Y_m\} \) become FRF’s \( \{\tilde{Y}_i\} \) and \( \{\tilde{Y}_m\} \). \( \{P_i\} \) becomes force ratio functions \( \{\tilde{F}_i\} \). \( \{P_e\} \) becomes a constant vector \( \{\tilde{F}_e\} \) where \( P_e(\omega) \) is replaced by 1.

\[
\begin{align*}
\{\tilde{Y}_i\} &= [T_{ii}]\{-\tilde{P}_i\} \\
\{\tilde{Y}_i\} &= [V_{ii}]\{\tilde{P}_i\} + [V_{ie}]\{\tilde{P}_e\} \\
\{\tilde{Y}_m\} &= [V_{me}]\{\tilde{P}_i\} + [V_{me}]\{\tilde{P}_e\}
\end{align*}
\]

Expanding to square matrices by incorporating three different sets of data gives

\[
\begin{align*}
\{\tilde{Y}_i\} = -[T_{ii}]\{\tilde{P}\}
\end{align*}
\]
\[
[Y_t] = [V_u] [P_t] + [V_{ue}] [P_e] \tag{5.5.12}
\]

\[
[Y_m] = [V_{mi}] [P_t] + [V_{me}] [P_e] \tag{5.5.13}
\]

Equation 5.5.13 gives
\[
[P_t] = [V_{mi}]^{-1} ([Y_m] - [V_{me}] [P_e]) \tag{5.5.14}
\]

which is then substituted into Equation 5.5.12 to give \([Y_t]\) as
\[
[Y_t] = [V_u] [V_{mi}]^{-1} ([Y_m] - [V_{me}] [P_e]) + [V_{ue}] [P_e] \tag{5.5.15}
\]

Substitution of Equation 5.5.15 into Equation 5.5.11 allows us to solve for \([T_u]\) as
\[
[T_u] = -[V_u] - [V_{ue}] [P_e] ([Y_m] - [V_{me}] [P_e])^{-1} [V_{mi}] \tag{5.5.16}
\]

If \(m = i\), then Equation 5.5.16 reduces to
\[
[T_u] = [Y_t] [V_{ue}] [P_e] - [Y_t]^{-1} [V_u] \tag{5.5.17}
\]

Equation 5.5.17 is the same as Equation 5.1.19, which verifies the correctness of Equation 5.5.17 in the limiting case.

**Numerical Simulations**

For Scenarios 5 and Scenario 6, a different FE model T-bar is used in order to refine the model and put more transducers on it to obtain responses on more points. The FE model of the T-bar has seven nodes and six elements. The vertical part has three nodes, 1, 2, and 5 and two elements. The horizontal part has five nodes, 3, 4, 5, 6, and 7 and four elements. The new finite element model is shown in Figure 5.5.2.
For Scenario 5, there are three impacts at location \( x_3, y_3 \) and \( y_5 \) on the T-bar to obtain
\[
[V_{ie} \parallel P_x] \quad \text{and} \quad [V_{me} \parallel P_x] \quad \text{(See Figure 5.5.2).}
\]
Figure 5.5.3 shows the position of the impact and measurement points on the combined structure. The response \([Y_m]\) is obtained at DOF \( x_{55}, y_{55} \) and \( \theta_{55} \) with a TAP-accelerometer combination. Figure 5.5.4 shows the extracted \([T_{ii}]\) at Point 2 with zero noise and a 5% RMS noise level and 30 averages. It shows this method is robust. The most advantage of this method is that the acceleration measurement can be done at any point of the T-bar. This measurement need not be done at the small interface in the vicinity of the interface since we have a FE model that relates the response at that point to the interface point.
Figure 5.5.3 Impact and measurement point at the combined structure.

Figure 5.5.4 Extracted vs. theoretical $[T_{ii}]$ with 10% RMS noise and 30 averages.
Figure 5.5.4 (continued)
Scenario 6: Two TAP-Accelerometer Combinations on a T-bar

Theoretical Development

In Scenario 5, the impact forces are obtained through the force transducer embedded in the hammer. If the impact forces are not known, is it possible to identify the forces as well as to extract the $T_{ii}$? In this scenario, the impact forces are aligned but their values are not known. We try to use the measurements from two sets of built-in TAP-accelerometer combinations to identify these forces and find the multidirectional driving point accelerance matrix $[T_{ii}]$. Figure 5.6.1 shows the combined structure and separate components of the test setup.

The nomenclature used in this scenario is very similar to that used in Scenario 5. $\{X\}$ is the frequency domain description of the acceleration vector of the test item. Similarly $\{Y\}$ is the acceleration vector of the T-bar. $\{F\}$ is the force and moment vector acting on the test item. $\{P\}$ is the force and moment vector acting on the T-bar. $[T]$ is the driving point accelerance matrix of the test item. $[V]$ is the driving point accelerance matrix of the T-bar.

The governing equation of the test item is

$$\{X_i\} = [T_{ii}]\{F_i\}$$  (5.6.1)

The governing Equations for the T-bar is

$$\{Y_i\} = [V_{ii}]\{P_i\} + [V_{ie}]\{P_e\}$$  (5.6.2)

$$\{Y_e\} = [V_{ei}]\{P_i\} + [V_{ee}]\{P_e\}$$  (5.6.3)

$$\{Y_s\} = [V_{si}]\{P_i\} + [V_{se}]\{P_e\}$$  (5.6.4)

where $i$ is the DOF’s related to driving points, $e$ is the DOF’s related to points other than $i$. 
Accelerations are measured at \( m_x, m_y, m_\theta, n_x, n_y \) and \( n_\theta \).

Figure 5.6.1 Combined Structure and Separate Components

the number of DOF of \( i \) is 3, assume that each \( m \) and \( n \) also have 3 DOF, respectively.

Besides the transducers at \( m \) and \( n \) should be able to measure tangential, normal and rotational motions. Therefore, two TAP and accelerometer combinations are used at both locations of \( m \) and \( n \) to measure each normal, angular and tangential motions.

The interface conditions are

\[
\{ X_i \} = \{ Y_i \}, \quad \{ F_i \} = \{- P_i \}
\]  

(5.6.5)

Applying the interface conditions to Equations 5.6.1, 5.6.2, 5.6.3, and 5.6.4 gives

\[
\{ Y_i \} = [T_{ii}] \{- P_i \}
\]  

(5.6.6)

\[
\{ Y_i \} = [V_{ii}] [P_i] + [V_{ai}] [P_a]
\]  

(5.6.7)

\[
\{ Y_n \} = [V_{in}] [P_i] + [V_{an}] [P_a]
\]  

(5.6.8)

\[
\{ Y_n \} = [V_{in}] [P_i] + [V_{an}] [P_a]
\]  

(5.6.9)
The unknown values are \( \{Y_i\}, \{T_{ui}\}, \{P_i\}, \) and \( \{P_e\} \). The known values are \( \{V_{ii}\}, \{V_{ne}\}, \{Y_n\}, \{V_{me}\}, \{Y_m\}, \{V_{me}\}, \{Y_m\}\), and \( \{V_{ne}\}\). Here since \( \{P_e\}\) is not known, we are not able to find the FRF's. \( \{Y_m\}\) is just measured acceleration data.

Expanding into square matrices by including different test results using multiple impacts gives

\[
\begin{align*}
[Y_i] &= -[T_{ui}][P_i] \\
[Y_i] &= [V_{ii}][P_i] + [V_{re}][P_e] \\
[Y_m] &= [V_{mi}][P_i] + [V_{me}][P_e] \\
[Y_n] &= [V_{ni}][P_i] + [V_{ne}][P_e]
\end{align*}
\]

It is noticed that Equations 5.6.12 and 5.6.13 can be used to solve for the unknown forces \( [P_i] \) and \( [P_e] \). \( [P_i] \), \( [P_e] \) and \( [Y_i] \) can be solved by combining Equations 5.6.11, 5.6.12 and 5.6.13,

\[
\begin{bmatrix}
V_{ii} & V_{ir} & -I \\
V_{mi} & V_{me} & 0 \\
V_{ni} & V_{ne} & 0
\end{bmatrix}
\begin{bmatrix}
[P_i] \\
[P_e] \\
[Y_i]
\end{bmatrix}
=
\begin{bmatrix}
0 \\
[Y_m] \\
[Y_n]
\end{bmatrix}
\]

thus

\[
\begin{bmatrix}
[P_i] \\
[P_e] \\
[Y_i]
\end{bmatrix}
=
\begin{bmatrix}
V_{ii} & V_{ir} & -I \\
V_{mi} & V_{me} & 0 \\
V_{ni} & V_{ne} & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
0 \\
[Y_m] \\
[Y_n]
\end{bmatrix}
\]

Hence the input forces \( [P_e] \) can be identified. After solving \( [P_i] \) and \( [P_e] \) and \( [Y_i] \), \( [T_{ui}] \) is found as

\[
[T_{ui}] = -[Y_i][P_i]\]

\[(5.6.16)\]
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Or we can combine Equation 5.6.12 and Equation 5.6.13 to solve \([P_i]\) and \([P_e]\) first.

\[
\begin{bmatrix}
V_{ni} & V_{ne} \\
V_{ni} & V_{ne}
\end{bmatrix}
\begin{bmatrix}
P_i \\
P_e
\end{bmatrix} = 
\begin{bmatrix}
Y_{n1} \\
Y_{n2}
\end{bmatrix}
\]  

(5.6.17)

\[
\begin{bmatrix}
P_i \\
P_e
\end{bmatrix} = 
\begin{bmatrix}
V_{ni} & V_{ne} \\
V_{ni} & V_{ne}
\end{bmatrix}^{-1}
\begin{bmatrix}
Y_{n1} \\
Y_{n2}
\end{bmatrix}
\]  

(5.6.18)

Then Equation 5.6.10 and Equation 5.6.11 are used to solve for \([T_{ii}]\).

\[
[T_{ii}] = -[V_{ii}][P_e][P_i]^{-1}
\]  

(5.6.19)

**Numerical Simulations**

First we impact the combined system with certain forces to get two sets of measurement results. For one set we mean 36 measurement data concerning all the six degrees of freedom. Then assume we do not know the forces, and try to solve for \([T_{ii}]\) and the impact forces.

Figure 5.6.2 shows the position of the impact and measurement points on the combined structure for Scenario 6. The impacts are at \(x_{51}, y_{51}\) and \(y_{53}\), the same as the impacts in Scenario 5. The first set of measurement is taken at \(x_{55}, y_{55}\), and \(\theta_{55}\), the right end of the T-bar. The second set of measurement is taken at the \(x_{50}, y_{50}\), and \(\theta_{50}\) from the middle part of the T-bar.

It is verified that for ideal data this method works. For all the frequencies, the identified forces are exactly those applied to the combined structure and the system. However, Figure 5.6.3 shows the extracted \([T_{ii}]\) at Point 2 under the 5% RMS noise and 30 averages. It shows this method is not robust.
Figure 5.6.2 Impact and measurement point at the combined structure

Efforts have also been made to input moment as well as forces input to the system and the result for moment identification is not good. Nevertheless, a smartly instrumented T-bar with multiple sensors is always very attractive. The instrumented T-bar is essentially analogous to a finger that is equipped with numerous nerve cells for feeling different forces and moments at different places.

Chapter Summary

This chapter challenges the difficult problem of determining the multiple directional driving point and transfer point accelerance matrices. The key idea of this work is to attach a properly instrumented, well-modeled vehicle “Instrument Cluster” to the test item at the interface where the driving point and transfer point accelerance matrices are sought. When the measurement on the combined system of the test item and the vehicle are obtained, and the modeling information on the vehicle is known, it is possible to indirectly derive the
Figure 5.6.3 Extracted vs. theoretical $[T_{il}]$ with 1% RMS Noise and 30 averages
interface driving point accelerance and the transfer point accelerance matrices of the original test item alone. A special implementation of the “Instrument Cluster”, called a T-bar is proposed.

This chapter studies 2D case where six scenarios are proposed and their robustness under noise are studied. First, a T-bar connected to the test item. The mass and mass moment of inertia of the TAP-accelerometer combination is ignored. Second, the T-bar is
put on one side of the interface and the modeled TAP-accelerometer combination is put in the vicinity of the interface on the test item. Third, a method of measuring the transfer point accelerance is proposed with the T-bar and the TAP-accelerometer combination. Fourth, a method to obtain the driving point and transfer point accelerance matrices using two different sets of TAP-accelerometer combination are proposed. Fifth, it is found that the TAP-accelerometer combination can be put anywhere on the T-bar without affecting the solutions. Sixth, we attempt to use redundant TAP-accelerometer combination to identify the impact forces as well as to extract the driving point accelerance matrix. The robustness study shows that Scenarios 1, 2, 3, and 5 are robust while Scenarios 4, and 6 are not.

It is noteworthy that the method proposed in Scenario 5 is most promising. It makes all the TAP and accelerometers in one piece to form an "Instrument Cluster". Within the T-bar essentially the sensors can be put anywhere. This greatly facilitates the design, manufacturing and finite element modeling of the T-bar. The effects of added stiffness and added mass of the T-bar are cancelled perfectly from the driving point accelerance measurement. This method is robust under significant noise contamination.

For the transfer accelerance measurement, it seems the method proposed in Scenario 4 works well with noise-free data. However it is also very sensitive to the noise data and its use is not recommended.
CHAPTER 6 EXPERIMENTAL INVESTIGATION ON
MULTIDIRECTIONAL FRF EXTRACTION

The implementation of any of the many acceleration extraction schemes that were
developed in Chapter 5 depends on the availability of instruments, especially the number of
rotational accelerometers such as the TAP. Since we have only one TAP, only Scenario 5 in
2D space is tested.

Experimental Arrangements

We used a cold rolled steel beam as our test item. The dimension of the beam is the
same as our numerical simulation. It is 24" x 1" x 0.25" (609.6 mm x 25.4 mm x 6.35 mm).
Its Young’s module is 2.0 x 10^{11} Pa and its density is 7520 kg/m^3.

The reason for choosing a beam as our test item is that the modeling of a beam is
simple and straightforward. We welded the T-bar to the beam to ensure good alignment
since it is very difficult to ensure that the T-bar is aligned parallel to the beam when attached
by a threaded connection.

The T-bar is the same as the T-bar used in our numerical simulations in Chapter 5. Its
FE model is the same as the second T-bar FE model that is used in Scenarios 5 and 6 in
Chapter 5. The combined structure is different from the combined structure in Chapter 5 for
the ease of experiment. The T-bar is welded at 3.5" (88.9 mm) from the right end. The
Raleigh damping coefficients were obtained from curve fitting raw data and were found to be
a = 9.9105, b = 3.5483 x 10^{-7}. 
The experimental arrangement is essentially the same as Scenario 5 in Chapter 5 and is shown in Figure 6.1. The test item is turned upside down in order to improve the quality of the impacts, especially the accuracy of the impact directions. A Kistler TAP – linear angular accelerometer Model 8696 is attached on the lower bottom left end of the horizontal bar. A PCB Model u353B16 accelerometer is mounted in the horizontal x direction on the left vertical surface. The accelerometer is automatically orthogonal to the TAP, a benefit gained from the design of the T-bar. The test beam is supported by rubber bands from the support structure. These rubber bands are attached about 1/3 of the length from each end of the beam.

For this test setup, we strike the horizontal part of the T-bar just as described in Scenario 5 of Chapter 5 with impacts called $I_1$, $I_2$ and $I_3$ as shown.

![Figure 6.1 Test setup for Scenario 5 in Chapter 5](image-url)
The data acquisition system is a Data Physics “ACE” 104 data acquisition card installed in a PC. An ACE 104 card has two output channels and two input channels. For the impact test, we use one input channel to record the force exerted by the hammer on the system while the other input channel is used to record the response signal from either the PCB accelerometer, or the TAP’s linear acceleration, or the TAP’s rotational acceleration. These input signals are processed to generate input-output FRF’s, which constitutes the 3 x 3 FRF matrix $[X_m]$ from Scenario 5. This 3 x 3 is written as

$$
[X_m] = \begin{bmatrix}
X_{m_{11}} & X_{m_{12}} & X_{m_{13}} \\
X_{m_{21}} & X_{m_{22}} & X_{m_{23}} \\
X_{m_{31}} & X_{m_{32}} & X_{m_{33}}
\end{bmatrix}
$$

where the first column corresponds to impact $I_1$, the second $I_2$, etc.

The ACE data acquisition system is set as follows. Bandwidth is 1000 Hz. The number of analysis lines is 3200. The frequency resolution is 0.3125 Hz. The acquisition time is 3.2 seconds. A rectangular window is used to reduce the noise of the input side. Since this is a lightweight structure, the response lasts longer than six seconds. Consequently, we used 3% exponential window to solve the filter leakage problem. Due to the soft impact, the auto spectrum of the force signal drops rapidly above 700 Hz. Our frequency range of interest is 0 to 600 Hz.

**Results and Discussions**

**Preliminary Results**

Figure 6.2 shows the theoretical prediction and raw measurement of $[X_m]$ for Scenario 5. Both magnitudes and phases are plotted.
Figure 6.2 Theoretical prediction and raw measurement of $[X_m]$
Figure 6.2 (continued)

(d) $X_{my1}$, y acceleration for impact at 1

(e) $X_{my2}$, y acceleration for impact at 2

(f) $X_{my3}$, y acceleration for impact at 3
(g) $X_{\theta1}$, $\theta$ acceleration for impact at 1

(h) $X_{\theta2}$, $\theta$ acceleration for impact at 2

(i) $X_{\theta3}$, $\theta$ acceleration for impact at 3

Figure 6.2 (continued)
Curve fitting the raw measurement natural frequencies are very close to their theoretical predictions. Table 6.1 shows the comparisons of measured and predicted natural frequencies.

Therefore, in general, the FE model closely represents the real structure in finding the natural frequencies. It is difficult to quantify the magnitudes of the peaks. The reason is that the peaks are highly influenced by the damping. We can extract modal damping by using the curve fitting method on the raw measured FRF's and then compute the proportional damping coefficients $\alpha$ and $\beta$ for the finite element model. The problem is the modal damping values are highly dependent on the width of the frequency range that curve fitting is applied. Different curve fitting yields different damping values and for a clearly defined mode it can be up to 20% difference, the modes that are not clearly shown on the FRF curves have much larger difference in damping predictions. Moreover, $\alpha$ and $\beta$ vary greatly when they are computed using the damping values of different modes. For instance $\alpha$ and $\beta$ can be computed using the damping of Mode 2 and Mode 3, they can also be computed using the damping of Mode 3 and Mode 4, or the damping of Mode 1 and Mode 3, etc. Experience shows that one $\beta$ can be 10 times larger than another $\beta$, while the difference of $\alpha$ is also large. After all, the idea of using $\alpha$ and $\beta$ is approximation to include the damping phenomena in the FE model and all the DOF’s have the same $\alpha$ and $\beta$. In reality, what we observed is that

<table>
<thead>
<tr>
<th>Natural Frequency</th>
<th>Measured (Hz)</th>
<th>Predicted (Hz)</th>
<th>Errors (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>86.6</td>
<td>86.2</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>237.5</td>
<td>238.0</td>
<td>0.2</td>
</tr>
<tr>
<td>3</td>
<td>450.7</td>
<td>452.1</td>
<td>0.3</td>
</tr>
<tr>
<td>4</td>
<td>744.6</td>
<td>739.8</td>
<td>0.6</td>
</tr>
</tbody>
</table>
the damping values of the rotational FRF's seem to be smaller than the damping of the linear FRF's for the same mode. Therefore to specify the difference between a peak of the measurement curve and a peak of the theoretical prediction does not have a lot of significance. Generally the peaks of all the magnitude graphs fit in the sense that most of the peaks from two curves seem to overlap in Figure 6.2.

For the Sub-graphs (a), (c), (d), (e), (f) and (g) in Figure 6.2, the corresponding notches fit well in the sense that the largest discrepancy of the notches is below 20 Hz. Sub-graph (b) shows three well-fit notches but the last notch is 40 Hz away. Sub-graphs (h) and (i) show larger valley discrepancy around 70 Hz.

Figure 6.3 shows the theoretical vs. extracted FRF \([T_{ii}]\) using the measured \([X_m]\) data shown in Figure 6.2. Due to the special beam structure, some elements of \([T_{ii}]\) are essentially zero. \(T_{xy}(\omega) = T_{yx}(\omega) = T_{xx}(\omega) = T_{\alpha\alpha}(\omega) = 0\). Hence we do not include them here. For each row in Figure 6.3, the left hand graph is the magnitude while the right graph is the corresponding phase angle.

Generally, the extractions fit their theoretical counterparts in that all the extracted \([T_{ii}]\) have the same natural frequencies as the theoretical curves. This shows the extracting method is successful in predicting natural frequencies and the peaks. It also shows a way to solve the difficult problem of extracting multidirectional driving point accelerance.

Although the magnitude of the extracted \(T_{xx}\) looks strange, its phase remains almost constant. The reason for the strange magnitude is the influence of errors from other FRF's.

The extracted \(T_{yy}\) is very similar to its theoretical counterpart. The curves are indistinguishable in many frequency ranges. The valley mismatch problem is less than 5 Hz.
The extracted $T_{xy}$ is also very similar to its theoretical counterpart. Except for the last notch, the other notches fit perfectly. The first notch from the experimental data suffers from noise problem.

The extracted $T_{xy}$ is similar to its theoretical counterpart in that the valley mismatch is less than 10 Hz.

Figure 6.3 Theoretical vs. extracted FRF $[T_{ii}]$ using measured $[X_m]$
Figure 6.3 (continued)
The valley mismatch problem is most significant in $T_{66}$. The largest discrepancy is about 60 Hz. If we use the theoretical prediction curves to replace the last three rows in the measured $[X_m]$ matrix, that is, using the theoretical $\theta$ FRF’s to replace the experimental $\theta$ FRF’s in $[X_m]$, then extracted $[T_{ii}]$ is much closer to the theoretical $[T_{ii}]$. Therefore, we tried to investigate what caused the valley mismatch problem in $[X_m]$ since the valley mismatch problem in $[X_m]$ appears to cause the valley mismatch problems in $[T_{ii}]$.

**Studies on the Valley Mismatch Problem**

We considered several possible candidates that caused the valley mismatch problem in $[X_m]$ in Figure 6.2. The first candidate is the weld that connected T-bar to the test item. The T-bar is welded to the beam as shown in Figure 6.4. The shape of the connection is complicated and difficult to model.

A refined finite element model is developed with more beam elements as shown in Figure 6.5. The beam element that is on the T-bar and also connected to the beam at the interface has a much larger area moment of inertia. The effect of increasing the area moment of inertia of that element does move the predicted valleys closer to the experimental valleys in the last three rows of Figure 6.2 while keeping the peaks and natural frequencies almost unchanged. However this move of the valleys is much smaller than required. Therefore, it appears that the weld contributes some of the valley shift, but its effect is secondary.

The second candidate is the FE modeling of the TAP and the linear transducer. The reason is that this specific T-bar is not very large compared to the TAP, and thus the TAP might alter the local structural dynamic characteristics of the T-bar significantly. So we added the mass and mass moment of inertia of the TAP to the nearby beam elements on the
T-bar, as shown in Figure 6.5. The computed results show that the influence in including the additional transducer mass only changes the natural frequency about 1-2%. The notches are almost not changed. Therefore, the FE modeling does not appear to be a major source of valley frequency error.
After the exclusion of FE modeling problems, it seems the valley mismatch problem is more related to the instrument themselves than the FE modeling of the test apparatus. There are several evidences. In the experiment, we observed that when the TAP is placed in different orientation, the FRF's change significantly. Besides the change of FRF is more significant in the $\theta$ direction than the $x$ and $y$ directions. Figure 6.5 shows a modified test setup where the TAP is put on the left end of the T-bar to measure $\theta$.

Figure 6.6 shows the measured $\theta$ FRF's vs. the theoretical $\theta$ FRF's using this modified test setup. The peaks coincide with the theoretical values. However, the notch locations are quite different than those in the last three rows in Figure 6.2. For sub-graphs (a) and (b) in Figure 6.6, there are large frequency ranges of poor matches of valleys from 0 to 350 Hz. For sub-graph (c), the frequency range of 0 – 150 Hz suffers from noise problem and there are two pseudo peaks at 60 and 120 Hz. However the second notch has a good match in that the discrepancy is only 10 Hz.

![Figure 6.5 A Modified Test Setup with TAP on the Left End](image-url)
Figure 6.6 Magnitudes of theoretical and measured $[X_m]$ related to $\theta$ with the TAP moved to the left end of the T-bar
Figure 6.7 compares the measured \( X_{m\theta/3} \) from the first test setup shown in Figure 6.2 and the \( X_{m\theta/3} \) from the second test setup shown in Figure 6.5. Their natural frequencies of the second and the third peaks are the same. However there are significant difference in the notches. The largest notch difference is about 70 - 80 Hz.

If the angular acceleration channel of the TAP just measured the \( \theta \) acceleration, the measured FRF's in Figure 6.6 and Figure 6.2 should be very close since they are all related to the rotation of the left end of the T-bar and the same set of impacts. Could there be some other reason that the angular acceleration channel reads so erroneously?

The third candidate of the valley mismatch problem is the cross axis sensitivity of the TAP between its \( x \) and \( \theta \) measurements, and the cross axis sensitivity between its \( y \) and \( \theta \) measurements. In order to investigate these cross axis sensitivity problems, the test apparatus shown in Figure 6.6 is developed using a MB Dynamics Model 50 exciter. In Figure 6.8 (a) we measure the cross axis sensitivity between \( \theta \) and \( x \), while in Figure 6.8 (b)
Figure 6.7 Comparison of the measured $X_{me/3}$ from the first the second test setups

Figure 6.8 A test setup to check the cross axis sensitivity between $x$ and $\theta$, $y$ and $\theta$
the cross axis sensitivity between $\theta$ and $y$ is measured. The motion from the exciter is vertical and we find the angular acceleration by the $\theta$ channel of the TAP. For the test in Figure 6.8 (a), the $x$ directional motion is obtained from the PCB linear accelerometer, the $\theta$ directional motion is obtained from the angular acceleration channel of the TAP. For the test in Figure 6.8 (b), the $y$ directional motion is obtained with the linear acceleration channel of the TAP and the $\theta$ directional motion is obtained with the angular acceleration channel of the TAP.

Figure 6.9 shows the magnitude and phase of the cross axis sensitivity between $x$ and $\theta$, and $y$ and $\theta$ directions. Obviously, the cross axis sensitivity between the $x$ channel and the $\theta$ channel is much larger than the cross axis sensitivity between the $y$ channel and the $\theta$ channel. The mean value of the $x$-$\theta$ cross axis sensitivity in Figure 6.9 (a) is approximately 1000 rad/s$^2$/g, while the mean value of the $y$-$\theta$ cross axis sensitivity is on the order of 50 rad/s$^2$/g. The ratio between them is around 20. It is clear from Figure 6.9(a)(b) that phase of the cross axis influence is frequency dependent. Hence, it is clear that this TAP has a significant cross axis sensitivity problem, especially between the $x$ and the $\theta$ channels. However the published cross axis sensitivity is only defined in the specification sheet to be 2% between the $y$ and $\theta$ channel.

The reason why this $x$ - $\theta$ cross axis sensitivity is larger than $y$ - $\theta$ cross axis sensitivity might come from its design. Figure 6.8 shows a schematic of the piezobeam under linear and angular motion. This figure is similar to Figure 1.3. Due to the symmetry of the piezobeam with respect to its post, a pure vertical base motion generates very little rotation at the piezobeam. However, a pure horizontal base motion brings about rotations at
Figure 6.9 Cross axis sensitivity between $x$ and $\theta$, and $y$ and $\theta$

(b) $y$ and $\theta$ cross axis sensitivity

Figure 6.10 Schematic graph of the piezobeam under vertical and horizontal motion
the end of AB, since the post is a cantilever beam sticking out from the base. To reduce the $x - \theta$ cross axis sensitivity, a possible solution is to fix the AB end so that its rotation is limited.

**Cross Axis Sensitivity Correction**

From the cross axis sensitivity point of view, since in Figure 6.1 impact I$_3$ is horizontal, and the TAP is placed to measure $y$ and $\theta$ acceleration, the cross sensitivity between $x$ and $\theta$ should significantly contaminate the $\theta$ output. When the TAP is installed on the left end of the T-bar as in Figure 6.5, the cross axis coupling is mainly between $y$ and $\theta$, which is much smaller than that between $y$ and $\theta$. Therefore we should replace the measured $Xm_{\theta3}$ with the corresponding measurement on the left end of the modified setup for further processing. The valley around 320 Hz in the measured $Xm_{\theta3}$ in Figure 6.5 ($X_{m33}$) fits well with the prediction while its counterpart in Figure 6.2 ($X_{m\theta3}$) does not.

However, moving the TAP to the left vertical surface of the T-bar also changes the test structure, and all the T-bar FRF's are computed from the first set up where the TAP and the linear accelerometer are installed. This will introduce errors. The extract $[T_{ul}]$ from using the measured $[X_m]$ with $Xm_{\theta3}$ by the new $Xm_{\theta3}$ is not very good. Figure 6.9 shows the newly extracted $[T_{ul}]$ with $Xm_{\theta3}$ replaced.

$T_{xx}$ is improved when compared to Figure 6.3 (a). The overall curve is lifted up and become closer to the theoretical curves than Figure 6.3 (a). The peaks in the extracted $T_{xx}$ are higher and look like resonance.

$T_{xy}$ is about the same up to 400 Hz, double peaks are shown in the 450 Hz range. The two curves of $T_{y\theta}$ are similar other than the double peaks around 450 Hz.
For $T_{\theta\rho}$, the notch and peak in 430 to 450 Hz range seems to change the shape of the curve a lot. Other than that, two curves are similar to those curves in Figure 6.3 (d).

$T_{\theta\rho}$ suffers from a double peak problem close to 450 Hz and also the notch before the third resonance is missing.

Figure 6.11 Theoretical vs. extracted FRF [$T_{ii}$] using raw [$X_m$] data with $X_m\theta_3$ replaced by $X_m\theta_3$ measured at the left vertical surface of the T-bar
Figure 6.11 (continued)
A cross axis sensitivity correction formula is proposed in order to reduce the all cross axis sensitivity in the measured data (with $X_{m\theta 3}$ already replaced). Assume that the relationship between the ideal $[\bar{X}_m]$ and the measured $[X_m]$ with cross axis sensitivity coupling is given by

$$
\begin{bmatrix}
X_{mz1} & X_{ma2} & X_{ma3} \\
X_{my1} & X_{my2} & X_{my3} \\
X_{m\theta 1} & X_{m\theta 2} & X_{m\theta 3}
\end{bmatrix}
= 
\begin{bmatrix}
\varepsilon_{ss} & \varepsilon_{sy} & \varepsilon_{x\theta} \\
\varepsilon_{ys} & \varepsilon_{yy} & \varepsilon_{y\theta} \\
\varepsilon_{x\theta} & \varepsilon_{y\theta} & \varepsilon_{\theta \theta}
\end{bmatrix}
\begin{bmatrix}
\bar{X}_{mz1} & \bar{X}_{ma2} & \bar{X}_{ma3} \\
\bar{X}_{my1} & \bar{X}_{my2} & \bar{X}_{my3} \\
\bar{X}_{m\theta 1} & \bar{X}_{m\theta 2} & \bar{X}_{m\theta 3}
\end{bmatrix}
$$

(6.2)

where the $[\varepsilon]$ matrix is the cross axis sensitivity matrix. However, we do not know real $[\bar{X}_m]$. So in this exercise, we use the theoretical predictions $[\bar{X}_m]$ instead. We also note that $[\varepsilon]$ is frequency dependent. However, the simplifying assumption is made that the cross axis sensitivity is a constant in our range of interest. The cross-axis sensitivities were found to vary somewhat with frequency, especially the mean value in the flat parts of the cross axis sensitivity curves is used as the nominal cross axis sensitivity.

We can use the nominal cross axis sensitivity matrix $[\bar{\varepsilon}]$ to correct the measured $[X_m]$ whose $X_{m\theta 3}$ is already replaced. Figure 6.12 shows the corrected $[X_m]$ vs. the theoretical $[\bar{X}_m]$. From Figure 6.12 we see clearly that the corrected $[X_m]$ are closer to the theoretical curves than the raw $[X_m]$ with $X_{m\theta 3}$ replaced.

Figure 6.12 (a), (b) and (c) look similar to Figure 6.3 (a) (b) and (c). Figure 6.12 (d) and (e) have major improvement over Figure 6.3 (d) and (e). The magnitude curves of the corrected measurements are a peak followed by a notch at the first resonance, the same as their theoretical counterparts in Figure 6.12 (d) and (e). In Figure 6.3 (d) and (e) the order of
peak and notch of the measurement curves at the first resonance is different from their theoretical counterparts.

Figure 6.12 (f) is better than Figure 6.3 (f) in 100 – 500 Hz range. Figure 6.12 (g) is much better than Figure 6.3 (g) since the measurement curve and the theoretical curve are almost indistinguishable in most frequencies.

Figure 6.12 (h) is better than Figure 6.3 (h) at the first notch. The first notches from both theoretical curve and the experimental curve have the same frequency while in Figure 6.3 (g) the first notches are different.

In Figure 6.12 (i), the magnitude of the first peak is 15 times higher than that of the first peak in Figure 6.6 (c). It almost reaches the theoretical value of the first peak. Therefore, even based on a simple assumption, the improvement of the measurement curves are significant.

Figure 6.13 shows the extracted $[T_{ai}]$ using the cross axis sensitivity corrected $[X_m]$. Obviously, the double peak problems are much smaller in Figure 6.13 (b), (c), (d) and (e) than in Figure 6.11 (b), (c), (d) and (e), and the third notch reappears in Figure 6.13 (e). It shows the cross axis sensitivity correction does have effects on the final results when the TAP is used as the instrument to measure angular accelerations. How to reduce the cross axis sensitivity by improving the design of the TAP and by optimizing the T-bar test setup need future research work.
Figure 6.12 Theoretical prediction and corrected raw measurement of $[X_m]$.
Figure 6.12 (continued)
Figure 6.12 (continued)
Figure 6.13 Extracted $[T_{ii}]$ with the cross axis sensitivity corrected $[X_m]$. 

(a) $T_{xx}$

(b) $T_{yy}$

(c) $T_{y\theta}$
To conclude, in this experiment, some of the extracted multidirectional driving point accelerance curves fit the theoretical predictions, and some has notch mismatch problems. We examined three sources of error, i.e., the weld stiffening effects, the modeling of the transducers as an integral part of the T-bar and the cross axis sensitivity of the TAP.

It seems the first two errors do change the notch. But their effects seem minor. Evidence shows that the cross axis sensitivity is the main cause of notch mismatch problem.
Due to the complex nature of the cross axis sensitivity, it is very difficult to remove all its effects by simple means. However a constant cross axis sensitivity assumption does bring significant improvement on both the measurement data and the extracted driving point accelerance.

It seems future research should concentrate efforts to improve the techniques of rotational acceleration measurement, i.e., the development of a better TAP with much less cross axis sensitivities between the linear and angular motion channels.
CHAPTER 7 METHODS FOR EXTRACTING MULTI-DIRECTIONAL FRF MATRICES - 3D SCENARIOS

Overview

The extraction methods developed in Chapter 5 need to be expanded in 3D applications. One possible design of the 3D conceptual “Instrument Cluster” is shown in Figure 7.0.1. It is named as a “C-bar”. Its vertical rod is used to connect it to the test item utilizing the threads at the end. The base may be two cross bars where the TAP transducers are located along with impact ball and stinger holes. When these TAP transducers are mounted in orthogonal positions, we can measure three linear and three angular accelerations at that position. In addition, six or more input locations are required to accommodate locations for either impact or stinger inputs.

Figure 7.0.2 shows the dimensions of the tilted L-bar that is similar to the one used in Chapter 4. This structure is used as the test item in the numerical example. To make the example 3D, the top horizontal bar is rotated 45° out of the paper. The material of the test item is steel. Its Young’s module is $2.11 \times 10^{11}$ Pa. Its density is 7850 kg/m$^3$.

Figure 7.0.3 shows the finite element model of the test item. It consists of 22 3D beam elements. Table 7.0.1 shows its natural frequencies when the base of the test item is free. The Raleigh damping coefficients are assumed to be the same as those that are used in Chapter 4, that is, $\alpha = 2.79$ and $\beta = 1.299 \times 10^{-4}$.

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1 The figures, tables and equations in this chapter are numbered according to scenarios. Those belong to the overview are numbered 7.0.x.
Three TAPs put into orthogonal positions to measure three linear and three angular accelerations

Figure 7.0.1. C-bar, one possible 3D implementation of the Instrument Cluster

Figure 7.0.2. Dimension of the tilted bar type of test item
Figure 7.0.3 Finite element model of the test item

<table>
<thead>
<tr>
<th>Order</th>
<th>Natural Frequency (Hz)</th>
</tr>
</thead>
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<tr>
<td>1</td>
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<tr>
<td>2</td>
<td>243.1</td>
</tr>
<tr>
<td>3</td>
<td>661.9</td>
</tr>
<tr>
<td>4</td>
<td>799.1</td>
</tr>
<tr>
<td>5</td>
<td>1124.6</td>
</tr>
</tbody>
</table>

The dimension and finite element model of the C-bar is shown in Figure 7.0.4. All of the cross sections are 0.5" x 0.5" (12.7 mm x 12.7 mm). Its Young’s Modules E is $2.11 \times 10^{11}$ Pa and its density is 7850 kg/m$^3$. In the FE model of the C-bar we use 10 3D beam elements and 11 points. Point 1 is used as the connection point. Point 2 is used as a supplementary response measurement point. Point 7 is used as the primary response measurement point. It
is equipped with three sets of TAP transducers so it can measure all 6 DOF's of accelerations when they are excited. C-bar is made of steel whose Young's Modules is $2.11 \times 10^{11}$ Pa and its density is 7850 kg/m$^3$. Table 7.0.2 shows the natural frequencies of the C-bar when it is hung free - free.

The multidirectional driving point accelerance matrix $[T_{ii}]$ is divided into four sub matrices as shown Equation 7.0.1.

![Figure 7.0.4 Dimension and finite element model of C-bar](image)

**Table 7.0.2 Natural frequency of the C-bar when it is hung free - free**

<table>
<thead>
<tr>
<th>Order</th>
<th>Natural Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3699.8</td>
</tr>
<tr>
<td>2</td>
<td>3699.8</td>
</tr>
<tr>
<td>3</td>
<td>4258.6</td>
</tr>
<tr>
<td>4</td>
<td>4258.6</td>
</tr>
<tr>
<td>5</td>
<td>5201.5</td>
</tr>
</tbody>
</table>
\[ [T_{ii}] = \begin{bmatrix} [A] & [B] \\ [C] & [D] \end{bmatrix} \]  \hspace{1cm} (7.0.1)

The four sub-matrices \([A], [B], [C]\) and \([D]\) of the theoretical \([T_{ii}]\) are shown in Figure 7.0.5. For each FRF, only magnitude in dB (ref: 1) is shown. The subscripts are defined as follows: 1 represents \(x\), 2 represents \(y\), 3 represents \(z\), 4 represents \(\theta_x\), 5 represents \(\theta_y\), and 6 represents \(\theta_z\).

Figure 7.0.5a shows \([A]\) portion of \([T_{ii}]\) matrix that consists of \(T_{11}, T_{12}, T_{13}, T_{21}, T_{22}, T_{23}, T_{31}, T_{32}, T_{33}\). Figure 7.0.5b shows the \([B]\) portion of the \([T_{ii}]\) matrix that consists of \(T_{14}, T_{15}, T_{16}, T_{24}, T_{25}, T_{26}, T_{34}, T_{35}, T_{36}\). Figure 7.0.5c shows the \([C]\) portion of the \([T_{ii}]\) matrix that consists of \(T_{41}, T_{42}, T_{43}, T_{51}, T_{52}, T_{53}, T_{61}, T_{62}, T_{63}\). Figure 7.0.5d shows the \([D]\) portion of the \([T_{ii}]\) matrix that consists of \(T_{44}, T_{45}, T_{46}, T_{54}, T_{55}, T_{56}, T_{64}, T_{65}, T_{66}\). Maxwell reciprocity is reflected by the identity for the off diagonal elements within Figure 7.0.5a and Figure 7.0.5d, and also by the identity for the elements between Figure 7.0.5b and Figure 7.0.5c.

The multidirectional transfer point accelerance matrix \([T_{ei}]\) is divided into four sub matrices as shown Equation 7.0.2.

\[ [T_{ei}] = \begin{bmatrix} [E] & [F] \\ [G] & [H] \end{bmatrix} \]  \hspace{1cm} (7.0.2)

The four sub-matrices \([E], [F], [G]\) and \([H]\) of the theoretical multidirectional transfer point matrix of the test item \([T_{ei}]\) are shown in Figure 7.0.6. In the notation used here, \(i\) stands for the driving point, i.e., Point 1 in Figure 7.0.3 while \(e\) stands for the response measurement point, i.e., Point 22 in Figure 7.0.3. For each FRF, only the magnitude is
shown in dB (ref: 1). In addition, 1 represents $x$, 2 represents $y$, 3 represents $z$, 4 represents $\theta_x$, 5 represents $\theta_y$, and 6 represents $\theta_z$.

Figure 7.0.6a shows the upper left corner of the $[T_{el}]$ matrix, i.e., $T_{e11}$, $T_{e12}$, $T_{e13}$, $T_{e21}$, $T_{e22}$, $T_{e23}$, $T_{e31}$, $T_{e32}$, and $T_{e33}$. Figure 7.0.6b shows the upper right corner of the $[T_{el}]$ matrix, i.e., $T_{e14}$, $T_{e15}$, $T_{e16}$, $T_{e24}$, $T_{e25}$, $T_{e26}$, $T_{e34}$, $T_{e35}$, and $T_{e36}$. Figure 7.0.6c shows the lower left corner of the $[T_{el}]$ matrix, i.e., $T_{e41}$, $T_{e42}$, $T_{e43}$, $T_{e51}$, $T_{e52}$, $T_{e53}$, $T_{e61}$, $T_{e62}$, and $T_{e63}$. Figure 7.0.6d shows the lower right corner of the $[T_{el}]$ matrix, i.e., $T_{e44}$, $T_{e45}$, $T_{e46}$, $T_{e54}$, $T_{e55}$, $T_{e56}$, $T_{e64}$, $T_{e65}$, and $T_{e66}$.

![Graphs showing the magnitude of driving point acceleration for different frequencies](image)

(a) [A]

Figure 7.0.5 Magnitude of Driving Point Accelerance of the Test Item
Four test scenarios are discussed. The first scenario considers three TAP's and the impact forces are already known. The second scenario considers six TAP's but the magnitude of impact forces is not known. The third scenario extends the first scenario to place the C-bar with arbitrary position and orientation while still obtaining $[T_{ii}]$ defined in global coordinates. The fourth scenario studies how to use two C-bars to find transfer acceleration $[T_{ei}]$ as well as the driving point acceleration $[T_{ee}]$ of a remote point $e$. Noise pollution is used in 3D cases to test the robustness of the methods. The noise generation mechanism is the same as that of Chapter 5.
Figure 7.0.5 (continued)
(d) $[D]$

Figure 7.0.5 (continued)
Figure 7.0.6 Magnitude of Test item Transfer Accelerance matrix \([T_{el}]\)
Figure 7.0.6 (continued)
Figure 7.0.6 (continued)
Scenario 1 Impact Force Known, Three TAP Configuration

Theoretical Development

In this scenario, three TAP’s are required to measure 3 linear accelerations and 3 rotational accelerations. It is assumed that the three TAP’s are embedded in the “Instrument Cluster”. Six different impacts are needed in order to find $[T_u]$. For simplicity, the test item is depicted as a line.
The nomenclature used in this scenario is as follows. \( \{ X \} \) is the frequency domain description of the acceleration vector of the test item. Similarly \( \{ Y \} \) is the acceleration vector of the C-bar. \( \{ F \} \) is the force and moment vector acting on the test item. \( \{ P \} \) is the force and moment vector acting on the C-bar. \( \{ T \} \) is the multidirectional driving point accelerance matrix of the test item. \( \{ V \} \) is the driving point accelerance matrix of the C-bar.

We need to work out the relationships to solve for \( \{ T_{ii} \} \) in terms of the measured quantities and the C-bar characteristics.

The governing equation of the test item is

\[
\{ X_i \} = \{ T_{ii} \} \{ F_i \}
\]

(7.1.1)

where the subscript \( i \) represents the DOF's related to driving points. The number of DOF of \( i \) is 6.
The governing equations of the "Instrument Cluster" are

\[ \{y_i\} = \left[ V_u \right] \{P\} + \left[ V_{\alpha} \right] \{P_e\} \]  \hspace{1cm} (7.1.2)

\[ \{y_m\} = \left[ V_{\alpha} \right] \{P\} + \left[ V_{me} \right] \{P_e\} \]  \hspace{1cm} (7.1.3)

\[ \{y_n\} = \left[ V_{\alpha} \right] \{P\} + \left[ V_{ne} \right] \{P_e\} \]  \hspace{1cm} (7.1.4)

\[ \{y_o\} = \left[ V_{\alpha} \right] \{P\} + \left[ V_{\alpha e} \right] \{P_e\} \]  \hspace{1cm} (7.1.5)

where \( e \) is the DOF related to points other than \( i \), and each of the subscripts \( m, n, \) and \( o \) represents a linear DOF and a rotational DOF. Physically each of \( m, n, \) and \( o \) represents a TAP's two measurements of a linear acceleration and a rotational acceleration.

The matching interface conditions between structures at point \( i \) are

\[ \{X_i\} = \{y_i\}, \quad \{F_i\} = -\{P_i\} \]  \hspace{1cm} (7.1.6)

Substituting Equation 7.1.6 into Equation 7.1.1 gives

\[ \{Y_i\} = \left[ T_{ui} \right] \{-P_i\} \]  \hspace{1cm} (7.1.7)

The known values in Equation 7.1.2 through Equation 7.1.5 and Equation 7.1.7 are \( \{P_i\}, \left[ V_u \right], \left[ V_{\alpha} \right], \{y_m\}, \left[ V_{me} \right], \{y_n\}, \left[ V_{ne} \right], \{y_o\}, \left[ V_{\alpha e} \right] \). The unknown values are \( \{y_i\}, \left[ T_{ui} \right], \) and \( \{P_i\} \).

A similar division procedure as described in Scenario 1 in Chapter 5 is used to change Equations 7.1.2, 7.1.3, 7.1.4, 7.1.5 and 7.1.7 to the forms of FRF's and force ratios.

\[ \{\bar{V}_i\} = \left[ V_u \right] \{\bar{P}\} + \left[ V_{\alpha} \right] \{\bar{P}_e\} \]  \hspace{1cm} (7.1.8)

\[ \{\bar{V}_n\} = \left[ V_{\alpha} \right] \{\bar{P}\} + \left[ V_{ne} \right] \{\bar{P}_e\} \]  \hspace{1cm} (7.1.9)

\[ \{\bar{V}_o\} = \left[ V_{\alpha} \right] \{\bar{P}\} + \left[ V_{\alpha e} \right] \{\bar{P}_e\} \]  \hspace{1cm} (7.1.10)

\[ \{\bar{V}_e\} = \left[ V_{\alpha} \right] \{\bar{P}\} + \left[ V_{\alpha e} \right] \{\bar{P}_e\} \]  \hspace{1cm} (7.1.11)
\[
\{\vec{Y}\} = [T_{ii}]^{-1}\{\vec{P}\}
\]  
(7.1.12)

In order to solve for \( [T_{ii}] \) in Equation 7.1.12, we need to solve for \( \{\vec{P}\} \) and \( \{\vec{Y}\} \) from the known information. Thus we can solve for \( \{\vec{P}\} \) by combining Equations 7.1.9, 7.1.10 and 7.1.11 to obtain

\[
\begin{bmatrix}
[V_m]_{2x6} \\
[V_n]_{2x6} \\
[V_o]_{2x6}
\end{bmatrix}
\begin{bmatrix}
\vec{P}
\end{bmatrix} = \begin{bmatrix}
\{Y_m\} - \{V_{me}\}_{2x6} \{\vec{P}\}_r \\
\{Y_n\} - \{V_{ne}\}_{2x6} \{\vec{P}\}_e \\
\{Y_o\} - \{V_{oe}\}_{2x6} \{\vec{P}\}_e
\end{bmatrix}
\]  
(7.1.13)

Now we take the inverse of Equation 7.1.13 and expand \( \{\vec{Y}\}_m, \{\vec{Y}\}_n, \{\vec{Y}\}_o \) and \( \{\vec{P}\} \) by incorporating the six different sets of test data where the input forces are different in order to square the matrices. This gives

\[
\begin{bmatrix}
[V_m]_{2x6} \\
[V_n]_{2x6} \\
[V_o]_{2x6}
\end{bmatrix} = \begin{bmatrix}
[V_m]_{2x6}^{-1} \\
[V_n]_{2x6}^{-1} \\
[V_o]_{2x6}^{-1}
\end{bmatrix}
\begin{bmatrix}
\{Y_m - V_{me} \vec{P}\}_r_{2x6} \\
\{Y_n - V_{ne} \vec{P}\}_e_{2x6} \\
\{Y_o - V_{oe} \vec{P}\}_e_{2x6}
\end{bmatrix}
\]  
(7.1.14)

Next we expand Equations 7.1.12 and 7.1.8 into square matrices by using the six different sets of data to obtain

\[
\begin{bmatrix}
\{\vec{Y}\}_i_{6x6}
\end{bmatrix} = -[T_{ii}]_{6x6} \begin{bmatrix}
\vec{P}
\end{bmatrix}_{6x6}
\]  
(7.1.15)

\[
\begin{bmatrix}
\{\vec{Y}\}_i_{6x6}
\end{bmatrix} = [V_{ii}]_{6x6} \begin{bmatrix}
\vec{P}
\end{bmatrix}_{6x6} + [V_{ie}]_{6x6} \begin{bmatrix}
\vec{P}
\end{bmatrix}_{6x6}
\]  
(7.1.16)

Finally, substitution of \( \{\vec{P}\} \) from Equation 7.1.14 into Equations 7.1.15 and 7.1.16 gives

\[
[T_{ii}]_{6x6} = -[V_{ii}]_{6x6} - [V_{ie}]_{6x6} \begin{bmatrix}
\vec{P}
\end{bmatrix}_{6x6}^{-1}
\]  
(7.1.17)

In principle, we can solve for \( [T_{ii}] \) from Equation 7.1.17.
Numerical Simulation

Figure 7.1.2 shows the impact and measurement points of Scenario 1. The L-bar is modeled by Point 1 through 22, while the C-bar is modeled by Point 1 and 23 through 28. The impact points are $x_{24}$, $y_{24}$, $z_{24}$, $y_{26}$, $y_{22}$, and $z_{29}$. The three TAP's are mounted in orthogonal directions at Point 28. The measurement directions are $x$, $y$, $z$, $\theta x$, $\theta y$, and $\theta z$ at Point 28. Intuitively, when the C-bar is a rigid body, $F_x$ at Point $x_{24}$ causes a $x$-directional force and a $z$-directional moment about Point 1. $F_y$ at Point $y_{24}$ causes a $y$-directional force and a $z$-directional moment. $F_z$ at Point $z_{24}$ causes a $z$-directional force and a $y$-directional moment. $F_y$ at Point $y_{26}$ causes a pure $y$-directional force and no moments. $F_z$ at Point $z_{29}$ causes a $z$-directional force and a $x$-directional moment. $F_y$ at Point $y_{32}$ causes a $y$-directional force and a $x$-directional moment. Therefore the responses of six DOF's are excited.

Figure 7.1.3 shows the multidirectional driving point accelerance plots of the test item at Point 1, that is, the extracted $[T_{ii}]$ with 5% RMS noise and 30 averages. The format of the plot is the same as Figure 7.0.5. Four sub-matrices $[A]$, $[B]$, $[C]$ and $[D]$ of $[T_{ii}]$ are shown. It appears that this method is robust under noise contamination.

Although ideally $[C]$ and $[B]$ are reciprocals, $[C]$ is less polluted than $[B]$. Therefore it appears that measuring $[C]$ instead of $[B]$ is a much better choice in practice. Besides $[C]$ relates angular motion and force, while $[B]$ relates linear motion and moment. Angular accelerometers are already available and thus, in principle, $[C]$ can always be measured using the input − output relationship directly. However, the device to apply a pure moment does not exist and thus, in principle, $[B]$ cannot be measured using the input − output relationship directly.
Measurements are made at Point 28

Figure 7.1.2 Impact and Measurement Points

Figure 7.1.3 Extracted vs. theoretical $[T_d]$ with 5% RMS noise and 30 averages
Figure 7.1.3 (continued)
Figure 7.1.3 (continued)
Scenario 2 Input Force Magnitudes Unknown, Six TAP Configuration

Theoretical Development

In this scenario, we assume the direction of the impact force is correct but the magnitude of the force is unknown. We try to use redundant TAP's measurements to identify these unknown force magnitudes. This is similar to Scenario 6 in Chapter 5, where we tried to find the impact force magnitudes by using two sets of measurement at different points. Thus, there are twelve unknowns, i.e., six interface general forces, plus six unknown
Magnitudes for the impact forces. Three additional TAP's are needed in order to obtain sufficient information for this test situation. As in the last test scenario, six impacts are needed in order to find $[T_u]$.

The nomenclature used in this scenario is as follows. $\{X\}$ is the frequency domain description of the acceleration vector of the test item. Similarly $\{Y\}$ is the acceleration vector of the C-bar. $\{F\}$ is the force and moment vector acting on the test item. $\{P\}$ is the force and moment vector acting on the C-bar. $[T]$ is the multidirectional driving point accelerance matrix of the test item. $[V]$ is the driving point accelerance matrix of the C-bar.

We need to work out the relationships to solve for $[T_{ui}]$ in terms of the measured quantities and the C-bar characteristics.

The s equation of the test item is
The governing equations of the "Instrument Cluster" are

\[ \{X_i\} = \{T_i\} \{F_i\} \]  (7.2.1)

\[ \{Y_i\} = [V_{ni}][P_i] + [V_{ne}][P_e] \]  (7.2.2)

\[ \{Y_n\} = [V_{mi}][P_i] + [V_{me}][P_e] \]  (7.2.3)

\[ \{Y_e\} = [V_{ne}][P_i] + [V_{ne}][P_e] \]  (7.2.4)

The interface conditions are

\[ \{X_i\} = \{Y_i\}, \{F_i\} = \{-P_i\} \]  (7.2.5)

Now combine Equations 7.2.3 through 7.2.5 to give

\[ \begin{bmatrix} \{Y_m\} \\ \{Y_n\} \end{bmatrix} = \begin{bmatrix} V_{mi} & V_{me} \\ V_{ni} & V_{ne} \end{bmatrix} \begin{bmatrix} \{P_i\} \\ \{P_e\} \end{bmatrix} \]  (7.2.6)

The inverse of Equation 7.2.6 gives

\[ \begin{bmatrix} \{P_i\} \\ \{P_e\} \end{bmatrix} = \begin{bmatrix} V_{mi} & V_{me} \\ V_{ni} & V_{ne} \end{bmatrix}^{-1} \begin{bmatrix} \{Y_m\} \\ \{Y_n\} \end{bmatrix} \]  (7.2.7)

Now we expand the vectors in Equation 7.2.7 to square matrices by incorporating six different sets of testing data. This gives

\[ \begin{bmatrix} \{P_i\} \\ \{P_e\} \end{bmatrix} = \begin{bmatrix} V_{mi} & V_{me} \\ V_{ni} & V_{ne} \end{bmatrix}^{-1} \begin{bmatrix} \{Y_m\} \\ \{Y_n\} \end{bmatrix} \]  (7.2.8)

Here \([P_i], [P_e], [V_{mi}]\) and \([V_{ni}]\) are 6 x 6. After finding \([P_i]\) and \([P_e]\), we put \([P_i]\) and \([P_e]\) into the expanded form of Equations 7.2.1 and 7.2.2, and use the interface conditions Equation 7.2.5 to obtain

\[ \{Y_i\} = -[T_i][P_i] \]  (7.2.9)

\[ \{Y_i\} = [V_{ni}][P_i] + [V_{ne}][P_e] \]  (7.2.10)
We can obtain \([T_{ii}]\) from Equations 7.2.9 and 7.2.10 as

\[
[T_{ii}] = -[V_{ii}]\left[I_P\right]^{-1}
\]  

(7.2.11)

In principle, the interface accelerance matrix \([T_{ii}]\) is determined.

**Numerical Simulation**

Figure 7.2.2 shows the impact and measurement points for this test scenario. The impact points are the same as for the last scenario. However, this time the magnitudes of the forces are unknown. Two sets of three orthogonal TAP's are installed at Points 23 and 28. Therefore twelve signals can be obtained on one impact.

![Impact and measurement points](image_url)

*Figure 7.2.2 Impact and measurement points*
Figure 7.2.3 shows the extracted \([T_{ii}]\) at Point 1. The format of the plot is the same as Figure 7.0.5. Four sub-matrices \([A], [B], [C] \) and \([D]\) of \([T_{ii}]\) are shown. For noise-free signals, the theoretical curve and the extracted \([T_{ii}]\) curves are the same and the identified impact force is exactly the same as originally input. Unfortunately, it appears that this method is very sensitive to the noise pollution. The \([T_{ii}]\) extracted from the polluted FRF's are not similar to the theoretical curve. Therefore until further research clarifies the reason for this poor behavior, this method is not recommended for use in practical problems.

![Figure 7.2.3 Extracted vs. theoretical \([T_{ii}]\) with 1% noise and 30 averages](image-url)
Figure 7.2.3 (continued)
Figure 7.2.3 (continued)
Scenario 3 C-bar Method With Coordinate Transformation

Theoretical development

Practically, there are two concerns when using a C-bar. First, when the C-bar is connected to the interface point, we have to be able to install and remove it in a simple manner so that a threaded hole and bolt combination is a connection choice. When the C-bar is screwed into the test item, it is screwed and fastened to the end of the hole. Every time we
screw the C-bar into different holes, the orientation of the C-bar will be different relative to the global coordinate system.

Second, when the interface is not oriented in the same directions as the global coordinate system; a local coordinate system may be more convenient. Hence, the \([T_{ii}]\) is better defined in terms of the local coordinate system rather than the global coordinates in order to have a cleaner physical meaning. However, when we compare different \([T_{ii}]\) at different interfaces, or when we try to couple \([T_{ii}]\) with \([V_{ii}]\) for another system in substructure testing, a global coordinate system is required.

These two problems all require the use of a coordinate transformation. When we apply the impacts to the C-bar, we apply them in the local coordinate system defined by the C-bar instead of a global coordinate system. The measurements are obtained in the local coordinates since the TAP's are fixed in the C-bar. The FE model of the C-bar is also defined in terms of its local coordinates. These matrices have to be converted to the global coordinates before attempting the extraction and coupling process.

The derivation of this scenario is similar to Scenario 1 in this Chapter. For all the previous scenarios in 2D and 3D, all the matrices and equations are defined in the global coordinates. For convenience, the local coordinate system of the C-bar is labeled \(O'x'y'z'\) while the global coordinate system is labeled \(Oxyz\). Figure 7.3.1 shows the two coordinate systems.

The transformation matrices are defined as follows. Assume the angles between the correspondent axes of the local coordinate system and that of the global coordinate system are known, we can define a directional cosine matrix \([\lambda]\) as follows:
The transformation matrix is defined as

\[
[\lambda] = \begin{bmatrix}
\cos(x,x') & \cos(x,y') & \cos(x,z') \\
\cos(y,x') & \cos(y,y') & \cos(y,z') \\
\cos(z,x') & \cos(z,y') & \cos(z,z')
\end{bmatrix}
\]  

(7.3.1)

The directional cosine matrix between Oxyz and O'x'y'z' is \([\lambda]\), which is 3 x 3. The coordinate transformation matrix \([\beta]\) employs the use of \([\lambda]\) twice, and is 6 x 6.

The directional cosine matrix between Oxyz and O'x'y'z' is \([\lambda]\), which is 3 x 3. The coordinate transformation matrix \([\beta]\) employs the use of \([\lambda]\) twice, and is 6 x 6.

Figure 7.3.1. Local coordinate system and global coordinate system
The following matrices in Scenario 1 are to be modified. The first one is the impacting force \( \left[ P_e \right] \). Now \( \left[ P_e \right] \) is defined in the local coordinates. A strike in \( x' \) direction may have three components, one in \( x, y, \) and \( z \) directions, respectively. Thus,

\[
\left[ P_e \right] = \left[ \beta \right] \left[ P_{e'} \right]
\]

(7.3.3)

The second matrix is the measurement matrix \( \left[ X_m \right] \). The TAP's that are installed on the C-bar can only measure motions defined in terms of the C-bar's local coordinates. Thus,

\[
\left[ X_m \right] = \left[ \beta \right] \left[ X_{m'} \right]
\]

(7.3.4)

The third set of matrices are \( \left[ V_{ii} \right] \) and \( \left[ V_{mi} \right] \). Assume the impedance matrix derived from \( \left[ V_{ii} \right] \) is \( \left[ Z_{ii} \right] \), and the impedance matrix derived from \( \left[ V_{ii} \right] \) is \( \left[ Z_{ii} \right] \), i.e.,

\[
\left[ Z_{ii} \right] = \left[ V_{ii} \right]^{-1}
\]

(7.3.5)

\[
\left[ Z_{ii'} \right] = \left[ V_{ii'} \right]^{-1}
\]

(7.3.6)

Assume there are forces acting on Point \( i \), and the forces and accelerations are related by \( \left[ Z_{ii} \right] \) or its local coordinates counterpart \( \left[ Z_{ii'} \right] \).

\[
\left\{ F_i \right\} = \left[ Z_{ii} \right] \left\{ X_i \right\}
\]

(7.3.7)

\[
\left\{ F_i \right\} = \left[ Z_{ii'} \right] \left\{ X_i \right\}
\]

(7.3.8)

The invariance of virtue work requires the same work be done regardless of the coordinate system orientation used to represent the virtue motion and force. Thus, let there be two descriptions of the virtue acceleration called \( \left\{ \delta X_i \right\} \) and \( \left\{ \delta X_i \right\} \) and two descriptions of the virtual force \( \left\{ F_i \right\} \) and \( \left\{ F_i \right\} \). Then, the virtue work becomes

\[
W = \left\{ F_i \right\}^T \left\{ \delta X_i \right\} = \left\{ F_i \right\}^T \left\{ \delta X_i \right\}
\]

(7.3.9)

Now, substitution of Equations 7.3.5, 7.3.6, 7.3.7 and 7.3.8 into Equation 7.3.9 gives
The fourth set of matrices of interest is \([V_{ii}][P_e]\) and \([V_{me}][P_e]\). According to Equations 7.3.3, 7.3.10 and 7.3.11, we have

\[
[V_{ii}][P_e] = [\beta][V_{ii}][\beta]^T [\beta][P_e] = [\beta][V_{ii}][P_e]
\] (7.3.12)

\[
[V_{me}][P_e] = [\beta][V_{me}][\beta]^T [\beta][P_e] = [\beta][V_{me}][P_e]
\] (7.3.13)

Here the transformation matrix \([\beta]\) is orthogonal, i.e.,

\[
[\beta]^{-1} = [\beta]^T
\] (7.3.14)

The local coordinates have to be defined relative to the global coordinates so that the software can generate the transformation matrix before the measurement is taken. This transformation matrix is case dependent. However, the measurements can always be conducted in the local coordinate system. After the measurements \([X_m]\) are taken, they are transformed into the global coordinate system. The FE model of the C-bar is stored in the software and once the impact points and directions are given, the global description of the C-bar \([V_{ii}], [V_{me}][P_e]\), and \([V_{me}][P_e]\) can be determined immediately. Then, the software runs to find \([T_{ii}]\) defined in the global coordinate system. If necessary, the \([T_{ii}]\) defined in the local coordinate system can be computed as well.

\[
[T_{ii}] = [\beta]^T [T_{ii}] [\beta]
\] (7.3.15)

The rest of the derivation of this scenario is the same as the 3D Scenario 1 when the matrices are converted into their global coordinate forms.
Numerical Simulations

Figure 7.3.1 shows the impact and measurement points for Scenario 3. The impacts are defined in the local coordinate system \(O'x'y'z'\). For \(\varphi = 30^\circ\), the \([\lambda]\) matrix becomes

\[
[\lambda] = \begin{bmatrix}
\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\
0 & 1 & 0 \\
\frac{1}{2} & 0 & \frac{\sqrt{3}}{2}
\end{bmatrix}
\]  

Using the equations derived in this section, we can find the \([T_{il}]\) defined in the global coordinate system. This \([T_{il}]\) is the same as the theoretical curve shown in Figure 7.0.7.

Figure 7.3.2 shows the extracted \([T_{il}]\) at Point 1 of the L-bar under the coordinate transformation. The format of the plot is the same as Figure 7.0.5. Four sub-matrices \([A]\), \([B]\), \([C]\) and \([D]\) of \([T_{il}]\) are shown. For noise-free signals, the theoretical curve and the
extracted $[T_{ii}]$ curves are the same and the identified impact force is exactly the same as originally input. 5% RMS noise and 30 averages are used to pollute the measured signals. The results show that this method is robust. Similar to Scenario 1, the extracted $[C]$ sub-matrix is much cleaner than the $[B]$ matrix. This achievement is very important in practice since it solves the arbitrary orientation problem and makes the C-bar method feasible for practical use.

Figure 7.3.2 Extracted vs. the theoretical $[T_{ii}]$ with 5% noise, 30 averages and local transformation

(a) $[A]$
Figure 7.3.2 (continued)
Figure 7.3.2 (continued)
Scenario 4 Transfer Point Accelerance Measurements with Two C-bars

*Theoretical Development*

Since the output of an Instrument Cluster is a set of accelerations, an "Instrument Cluster" can be treated as a "General Accelerometer" with its own dynamic properties. If we have two "Instrument Clusters", and three TAP's are installed on "Instrument Cluster I" and "Instrument Cluster II", respectively, we are able to find the transfer accelerance and the
driving point acceleration of a remote point. This analysis is an extension of the Scenario 4 described in Chapter 5.

The following nomenclature is used. \( \{X\} \) is the acceleration vector of the test item, \( \{Y\} \) is the acceleration vector of "Instrument Cluster I", \( \{Z\} \) is the acceleration vector of "Instrument Cluster II". Then, the governing equations of the test item are

\[
\{X_i\} = \left[T_i\right]\{F_i\} + \left[T_j\right]\{F_j\} 
\]

(7.4.1)

\[
\{X_j\} = \left[T_j\right]\{F_i\} + \left[T_j\right]\{F_j\} 
\]

(7.4.2)

Now assume that the accelerations are measured by two sets of orthogonal TAP's installed at Point \( m \) and Point \( a \) in the two C-bars. The governing equations for the "Instrument Cluster I" are

\[
\{Y_i\} = \left[V_{ii}\right]\{P_i\} + \left[V_{ie}\right]\{P_e\} 
\]

(7.4.3)
\[ \{Y_m\} = [V_m]_i \{P_i\} + [V_{me}]_i \{P_e\} \]  \hfill (7.4.4)

The governing equations of the "Instrument Cluster II" are

\[ \{Z_j\} = [V_{ij}]_j \{P_j\} \]  \hfill (7.4.5)

\[ \{Z_a\} = [V_{aj}]_j \{P_j\} \]  \hfill (7.4.6)

where \([V]\) is the \([V]\) matrix for the second C-bar.

The interface conditions now involve both points \(i\) and \(j\) so that

\[ \{X_i\} = \{Y_i\}, \quad \{F_i\} = \{-P_i\}, \quad \{X_j\} = \{Z_j\}, \quad \{F_j\} = \{-P_j\}, \]  \hfill (7.4.7)

Equation 7.4.7 is substituted into Equations 7.4.1 through 7.4.6. A division similar to that in Scenario 1 in Chapter 5 is used on Equations 7.4.1 through 7.4.6 to transform the accelerations into FRF's and the forces into force ratio functions. Expanding the resulting vectors in Equations 7.4.1 through 7.4.6 into square matrices by incorporating six different test sets of data gives

\[ \{\bar{V}\} = -[T_{\alpha i}]_i \{\bar{P}_i\} - [T_{\phi j}]_j \{\bar{P}_j\} \]  \hfill (7.4.8)

\[ \{\bar{Z}\} = -[T_{\alpha j}]_j \{\bar{P}_j\} - [T_{\phi i}]_i \{\bar{P}_i\} \]  \hfill (7.4.9)

\[ \{\bar{V}\} = [V_{\alpha i}]_i \{\bar{P}_i\} + [V_{\phi j}]_j \{\bar{P}_j\} \]  \hfill (7.4.10)

\[ \{\bar{V}_m\} = [V_{m i}]_i \{\bar{P}_i\} + [V_{me}]_i \{\bar{P}_e\} \]  \hfill (7.4.11)

\[ \{\bar{Z}\} = [V_{\alpha j}]_j \{\bar{P}_j\} \]  \hfill (7.4.12)

\[ \{\bar{Z}_a\} = [V_{aj}]_j \{\bar{P}_j\} \]  \hfill (7.4.13)

The number of DOF of \(m\) and \(a\) are each 6. Since we can perform a driving point accelerance test at Point \(i\) using the methods described in Scenario 3 before the transfer
accelerance test, the known values are \([T_{ii}], [V], [V_m], [Z_a] \) and \([\bar{P}_r]\), while the unknown values are \([\bar{V}_i], [\bar{P}_f], [Z_j], [\bar{P}_j], [T_{ij}], \) and \([T_{ji}]\).

Equation 7.4.13 is used to solve for \([\bar{P}_j]\), which gives

\[
[\bar{P}_j] = [V_{aj}]^\dagger [Z_a] 
\] (7.4.14)

From Equation 7.4.12, we solve for \([Z_j]\) to obtain

\[
[Z_j] = [V_g] [\bar{P}_j] 
\] (7.4.15)

From Equation 7.4.11, we solve for \([\bar{P}_i]\) to obtain

\[
[\bar{P}_i] = [V_m]^{-1} ([V_m] - [V_{mi}] [\bar{P}_j]) 
\] (7.4.16)

Here \([\bar{P}_i], [\bar{P}_f] \) and \([V_{mi}]\) are 6 x 6. After finding \([\bar{P}_i]\) and \([\bar{P}_j]\), we put them into Equation 7.4.10 to solve for \([\bar{V}_i]\).

Suppose we have already known \([T_{ii}]\) from a previous driving point accelerance test. We can use Equation 7.4.8 to solve for \([T_{ij}]\), the test item transfer point accelerance without anything attached to it. Thus, from Equation 7.4.8 we obtain

\[
[T_{ij}] = -([\bar{V}_i] + [T_{ii}] [\bar{P}_f] [\bar{P}_j])^{-1} 
\] (7.4.17)

After the interface accelerance matrix \([T_{ij}]\) is solved, we can use Equation 7.4.9 to solve for \([T_{ji}]\), the driving point accelerance for the point that no driving point accelerance test is performed. This gives

\[
[T_{ji}] = -([Z_j] + [T_{ij}] [\bar{P}_f] [\bar{P}_j])^{-1} 
\] (7.4.18)
Note that all the measurements and the C-bar $[V]$ matrices have been transformed into global coordinates in this derivation.

**Numerical Simulation**

Figure 7.4.2 shows the impact and measurement points for the transfer point acceleration measurement. The impact points are located at $24x$, $24y$, $24z$, $26y$, $29z$ and $32y$ for the lower C-bar. The TAP measurements are made at Point 28 for the lower C-bar and Point 39 for the upper C-bar. A coordinate transformation is considered for the upper C-bar while the local coordinates of the lower C-bar coincide with the global coordinates.

Figure 7.4.3 shows the extracted multidirectional transfer point acceleration matrix $[T_{ei}]$ where $i$ represents Point 1 and $e$ represents Point 33 with 1% RMS noise and 30 averages. Without noise contamination, the extracted $[T_{ei}]$ is the same as the theoretical curves shown in Figure 7.0.6. However, similar to the Scenario 4 in Chapter 5, this method is not robust under noise contamination. Therefore it is not recommended for practical use.

![Figure 7.4.2 Impact and measurement points for L-bar with two C-bars attached](image-url)
Figure 7.4.3 Extracted vs. the theoretical \( T_{e_d} \), with 1% RMS noise and 30 averages
Figure 7.4.3 (continued)
Figure 7.4.3 (continued)
Chapter Summary

This is the application of the "Instrument Cluster" method in real 3D structure. A C-bar is proposed as special implementation of the 3D "Instrument Cluster". Four 3D test scenarios are proposed in this chapter and their robustness under noise is studied. First, the C-bar is connected to the test item where the embedded instruments are placed at some convenient places on it. Second, we attempt to use redundant TAP's to identify the magnitudes of the impact forces as well as to extract the driving point accelerance matrix.
Third, a coordinate transformation is introduced to relate the local coordinates defined by the interface C-bar orientation in order to account for the fastening and the interface orientation problem. Fourth, a method to obtain the driving point and transfer point accelerance matrices using two C-bars is proposed. The robustness study shows that Scenarios 1 and 3 are robust while Scenarios 2 and 4 are not.

Among those methods, Scenario 3 is the most promising. The rigid body restriction imposed by the previous research work has been eliminated. All of the 36 elements in the driving point accelerance matrix can be obtained in one time. The TAP accelerometer can be installed anywhere on the C-bar to form an "Instrument Cluster". The transformation matrix solves the arbitrary orientation problem so that all the measurement can be transformed into their counterparts under the global coordinate system for subsequent comparison and coupling analysis. The extracted driving point accelerance matrix can be computed in terms of either the global and/or the local coordinate system. Moreover the method is robust under noise contamination.

A possible improvement can be made by using redundant TAP's to check on how well the other TAP's are functioning and whether there is noise problem. The noise problem may occur if one of the TAP's is close to a node point at a particular range of frequencies. The redundant TAP can be used to suppress noise, and to improve results with suitable algorithms.

The practical implementation of the concepts in this chapter for industrial application requires that five tasks must be accomplished. First, the "Instrument Cluster" needs to be designed and built. Second, a finite element model of the "Instrument Cluster" needs to be formulated. Third, the finite element model needs to be carefully validated. Fourth,
appropriate software is needed to process the data, particularly calculation routines that may
be less sensitive to measurement noise. Fifth, the end user needs to be trained in the proper
use of the technique.
CHAPTER 8 CONCLUSIONS

This dissertation studies the notoriously difficult problem of experimentally determining the multidirectional driving point accelerance and transfer accelerance matrices of any structure. Each of these matrices are $6 \times 6$ for the general 3D case at each frequency. Compared to previous studies, this thesis considers an application with many more degrees of freedom, considers the case where the test attachment structure is not a rigid body, so it can be a flexible structure, and uses a specially designed attachment that measures both linear and angular accelerations. For the past 30 years, it was common practice to make many simplifications that often resulted in poor predictions for the dynamic behavior of a structure composed of two substructures when the substructure are measured independently.

A number of potential test methods for determining the multidirectional driving point and transfer accelerance matrices are proposed and analyzed from the finite element approach in order to evaluate their potential usefulness in practical 2 and 3 dimensional applications. Several potential test methods are found to be lacking the required robustness with respect to noisy data.

In the past, “rigid” attachment or “instrument cluster” structures are required when attempting to measure these accelerances. In this thesis, finite element methods are developed that allow the use of an “elastic” rather than “rigid” “instrument cluster”. Hence, a hybrid experimental and finite element method is successfully developed to achieve potentially satisfactory test results.

In the past, two accelerometers are required to measure angular acceleration. In this thesis, the TAP linear-angular accelerometer is used with promising results. The test data
indicate that accelerometer design modifications are required to reduce the present cross-axis sensitivity between \(\theta z\) and \(x\) accelerations. The biggest advantage of the TAP design is its ability to measure the angular acceleration at a point when compared to placing two accelerometers at a significant distance apart. With proper instrument development and further testing, the methodologies developed here should guide the experimental mechanics community to achieve significant test improvements.

Two studies are performed in order to compare the errors inherent in current practice. The first study looks at various errors that occur in substructure testing such as DOF deficiency, measurement system bias, random noise, exciter rocking motion, and driving point difference errors, etc. From this study, we found

- The moments and shear forces can be crucial components in the interface actions. Ignoring the DOF's of moments and shear forces at the interface can invalidate the substructure procedure so that the prediction of the coupled structure behavior is incorrect, since totally different frequency response functions of the coupled structure are obtained. Thus, the extraction of natural frequencies and modes information is impossible. In reality, ALL INTERFACIAL DOF and their corresponding interface forces (and moments) and motions (linear and angular) must be considered unless there is a special case can be proven otherwise. The DOF deficiency error is impossible to correct due to the complexity involved.

- The measurement bias error in the driving point accelerance matrix and transfer accelerance matrix of the vehicle and the test item causes errors in the substructure analysis. However, this error appears to be less important than the DOF deficiency error since it can be corrected for once the source is recognized and evaluated.
- The noise error effects appear to be most severe in the low frequency range. The vehicle and test item driving point accelerance matrices appear to be more sensitive to the noise contamination than the transmissibility matrices since these driving point matrices must pass through a noise sensitive matrix inversion process.

- The rocking motion of the exciter armature leads to both an incorrect bare vehicle transmissibility matrix and a global transmissibility matrix. The rocking motions are generally different for the bare vehicle test and the combine structure test.

- The differences between the impact point and the measurement point in the interface driving point accelerance test will deteriorate the quality of the global transmissibility matrix that is calculated with the substructure method when the distance between the two points becomes too large. The result shows that the instruments must be located close to one another during the driving point accelerance tests.

The second study looks into the details of the DOF deficiency problem and studies its composition since the DOF deficiency problem causes the largest error. This study reveals that DOF deficiency is not a simple problem that can be easily corrected for by conventional means.

- The DOF deficiency error in a 2D model is caused by several sources. First, it comes from the incompleteness of the reduced interface model so that the inverse from a complete model and the inverse from a DOF reduced model are different. Another source is the deletion of the cross coupling terms between the neglected motions and remaining forces. Similarly, another source is the deletion of the cross coupling terms between the neglected forces and the remaining motions. The last source is due to the deletion of the cross coupling terms for the neglected forces and the neglected motions.
• The DOF deficiency error in a 3D model is brought about by several sources. The first error comes from the incompleteness of the interface model; i.e., the inverse of a complete 3D interface model and the inverse from a 2D interface model are different. The former is a sub-matrix extracted from the inverse of a 3D \([TV]\) matrix. The latter is the inverse of a 2D \([TV]\) matrix. The second error is due to deletion of the cross coupling between out-of-plane forces and in-plane motions. The third error is due to deletion of the cross coupling terms between in-plane forces and out-of-plane motions. The last error is due to deletion of the coupling terms between the out-of-plane forces and out-of-plane motion.

• In a general structure, a 2D model does not yield correct transmissibility results because it neglects the out-of-plane DOF's as well as out-of-plane ground motions that always exist. Therefore for industrial use, a 3D model must be used where 36 interfacial acceleration data curves are needed. Ideally 21 of these curves are unique. It also reveals that this 3D coupling problem is so complex that all the six interfacial DOF must be measured to obtain the correct result. None of them can be neglected unless special circumstances exist.

Based on the above two studies, the notoriously difficult problem of finding the multidirectional driving point and transfer point accelerance matrices is challenged in an unconventional way. The key idea is to attach a properly instrumented and structurally well-modeled "Instrument Cluster" to the test item at the interface where the driving point and transfer point accelerance matrices are sought. When the measurements on the combined system of the test item and the "Instrument Cluster" are obtained, it is possible to obtain the interface driving point accelerance and the transfer point accelerance matrices of the original
test item alone. The experimentally verified FE model of the "Instrument Cluster" provides the FRF's of the "Instrument Cluster" that are needed for the extraction process.

Two different "Instrument Cluster" designs are considered, one for the 2D case is called the T-bar and one for the 3D case is called the C-bar. Six different test scenarios are considered for the T-bar and their applicability under noisy conditions is evaluated.

First, the T-bar is attached to the test item where the embedded instrument are not modeled. Second, the T-bar is put on one side of the interface while the modeled TAP-accelerometer combination is put in the vicinity of the interface of the test item. Third, a method of measuring the transfer point accelerance is proposed with the T-bar and a TAP-accelerometer combination. Fourth, a method to obtain the driving point and transfer point accelerance matrices using two different sets of TAP-accelerometer combinations is proposed. Fifth, it is found that a TAP and accelerometer combination can be put anywhere on the T-bar without affecting the solutions. Sixth, we attempt to use redundant TAP and accelerometer combination to identify the impact forces as well as to extract the driving point accelerance matrix. All six 2D scenarios gave perfect results from perfect noise free data. However, the robustness study shows that Scenarios 1, 2, 3, and 5 are robust while Scenarios 4, and 6 are not.

For driving point accelerance measurements, Scenario 5 is most promising. Within the T-bar essentially, the sensors can be put anywhere. This greatly facilitates the design, manufacturing and finite element modeling of the T-bar. The effects of added stiffness and added mass of the T-bar can be removed from the measurement to obtain the driving point accelerance for the pure test item. This method is also robust under signal noise contamination.
For the transfer accelerance measurement, Scenario 3 is robust under noise. The method proposed in Scenario 4 works well with noise-free data. However, Scenario 4 is also very sensitive to noisy data.

In the case of the 3D C-bar “Instrument Cluster” approach, four test scenarios are proposed and their robustness under noise is studied. First, a C-bar is connected to the test item where the embedded instruments are placed at some convenient location. Second, we attempt to use a redundant TAP combination to identify the impact forces as well as to extract the driving point accelerance matrix. Third, a coordinate transformation is introduced to relate the local coordinates defined by the interface C-bar orientation. This coordinate transformation solves the fastening problem where the C-bar’s orientation may not match some global orientation. Fourth, a method to obtain the driving point and transfer point accelerance matrices using two C-bars is proposed. The robustness study shows that Scenarios 1 and 3 are robust while Scenarios 2 and 4 are not.

Among those methods, Scenario 3 is the most promising one. The rigid body restriction imposed by the previous research work has been broken. All the 36 elements of the driving point accelerance matrix can be obtained in one test. The TAP can be installed anywhere in the C-bar to form an “Instrument Cluster”. The transformation matrix solves the screw fastening and the interface orientation problems so that all measurements can be transformed into their counterparts under the global coordinate system for subsequent comparison and coupling analysis. The extracted driving point accelerance matrix can be shown in either the global and/or the local coordinate system. Most importantly, the method is robust under noise contamination.
Experimental work was performed to implement Scenario 5 for the T-bar. A free-free beam serves as the test item. The difference between the measured natural frequencies of the test setup and its theoretical values is within 0.6%. Generally the measured FRF's fit well with the theoretically predicted FRF's. All of the peaks fit very closely while some of the notches or valleys show discrepancies; particularly those associated with the $\theta$ measurement. The reasons are that the finite element model does not model the TAP-accelerometer combination, the low frequency output of the TAP is noisy and there are significant cross axis sensitivities between the $x$ and $\theta$ channels. The noise contaminates the notches of other modes but has little effect on the peaks. The cross axis sensitivity distorts the notches in the FRF's that are related to $\theta$. The extracted driving point accelerances are close to their theoretical predictions except at some of the notches; particularly those that are controlled or heavily influenced by the rotational motion.

Future work includes several issues. First, the TAP must be improved to remove the current cross-axis sensitivity as well as reduce its low frequency noise. Second, develop methods to use a redundant TAP in order to crosscheck the angular measurements. Third, improve the design of the "Instrument Cluster" to take advantage of its elastic nature and improve its FE modeling so that all instrument effects are taken into account. Fourth, develop an instrument cluster with appropriate instruments, software, and training to make this test method feasible in an industrial setting.
STARTING FROM THE FOLLOWING EQUATION.

\[
\begin{bmatrix}
TV_{11} & TV_{12} & TV_{13} \\
TV_{21} & TV_{22} & TV_{23} \\
TV_{31} & TV_{32} & TV_{33}
\end{bmatrix}^{-1} = 
\begin{bmatrix}
TV_{11}^i & TV_{12}^i & TV_{13}^i \\
TV_{21}^i & TV_{22}^i & TV_{23}^i \\
TV_{31}^i & TV_{32}^i & TV_{33}^i
\end{bmatrix}
\]  \hspace{1cm} (a1)

Pre-multiply and post-multiply a \([Q]\) matrix defined as follows:

\[
[Q] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]  \hspace{1cm} (a2)

Pre-multiply \([Q]\) means to exchange the second and the third row. Post-multiply \([Q]\) means to exchange the second and the third column. \([Q]\) is related to its inverse as follows:

\[
[Q]^{-1} = [Q]
\]  \hspace{1cm} (a3)

Since

\[
\begin{bmatrix}
1 & 0 & 0 & TV_{11} & TV_{12} & TV_{13} \\
0 & 0 & 1 & TV_{21} & TV_{22} & TV_{23} \\
0 & 1 & 0 & TV_{31} & TV_{32} & TV_{33}
\end{bmatrix}^{-1} = 
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
= \left( \begin{bmatrix}
1 & 0 & 0 & TV_{11} & TV_{12} & TV_{13} \\
0 & 0 & 1 & TV_{21} & TV_{22} & TV_{23} \\
0 & 1 & 0 & TV_{31} & TV_{32} & TV_{33}
\end{bmatrix} \right)^{-1}
\]  \hspace{1cm} (a4)

\[
\begin{bmatrix}
TV_{11} & TV_{13} & TV_{12} \\
TV_{31} & TV_{33} & TV_{32} \\
TV_{21} & TV_{23} & TV_{22}
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
1 & 0 & 0 & TV_{11}^i & TV_{12}^i & TV_{13}^i \\
0 & 0 & 1 & TV_{21}^i & TV_{22}^i & TV_{23}^i \\
0 & 1 & 0 & TV_{31}^i & TV_{32}^i & TV_{33}^i
\end{bmatrix} = 
\begin{bmatrix}
TV_{11}^i & TV_{13}^i & TV_{12}^i \\
TV_{31}^i & TV_{33}^i & TV_{32}^i \\
TV_{21}^i & TV_{23}^i & TV_{22}^i
\end{bmatrix}
\]  \hspace{1cm} (a5)
Therefore

\[
\begin{bmatrix}
TV_{11} & TV_{13} & TV_{12} \\
TV_{31} & TV_{33} & TV_{32} \\
TV_{21} & TV_{23} & TV_{22}
\end{bmatrix}
\begin{bmatrix}
TV_{11}' & TV_{13}' & TV_{12}' \\
TV_{31}' & TV_{33}' & TV_{32}' \\
TV_{21}' & TV_{23}' & TV_{22}'
\end{bmatrix}^{-1} =
\begin{bmatrix}
TV_{11}' & TV_{13}' & TV_{12}' \\
TV_{31}' & TV_{33}' & TV_{32}' \\
TV_{21}' & TV_{23}' & TV_{22}'
\end{bmatrix}
\]
REFERENCES


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