Pulse Propagation in Cylindrically Wound Thick Composite Specimens

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Introduction

A mathematical model for the propagation of elastic disturbances through a cylindrically wound thick composite specimen is presented. The primary purpose of the modelling effort is to provide a means by which the elastic constant tensor corresponding to the specimen can be determined by in-situ measurements, in which case the effect of the curved fibre geometry must be addressed in the theoretical model. Hence the material is modelled as a cylindrically anisotropic medium, which is of orthotropic symmetry. Ray theory techniques give rise to the identification of curved ray paths along which a disturbance propagates in any one of three normal modes with a constant velocity of propagation. Thus the determination of the elastic constants from in-situ time-of-flight measurements has been reduced to the level of simplicity involved in ascertaining elastic constants from measurements made on a media possessing the usual cartesian symmetry. Finally, it will be shown that the eikonal equations for the cylindrically orthotropic media give rise to the following simple description of the ray geometry: The rays propagate so as to maintain a constant angle of attack with respect to the surfaces which describe the symmetry of the medium.

Ray theory is an asymptotic (high frequency) theory which has provided an invaluable tool for the study of wave motion in an elastic media in which the wavelength is small compared to the geometrical length scales inherent in the problem. The work presented in this article represents an extension to curvilinear anisotropy of some of the concepts that are currently well-understood for isotropic media and anisotropic media of cartesian symmetry, as representative examples we cite [1], [2], [3], and [4].

This manuscript is structured as follows. In the first section the derivation of the field equations for a cylindrically orthotropic media is sketched. In the following section a ray theory ansatz is made, and the solution to the resulting eikonal equation is presented and discussed. Then a section is devoted to a geometrical descriptions of the ray paths. Finally, we end with a short concluding section.

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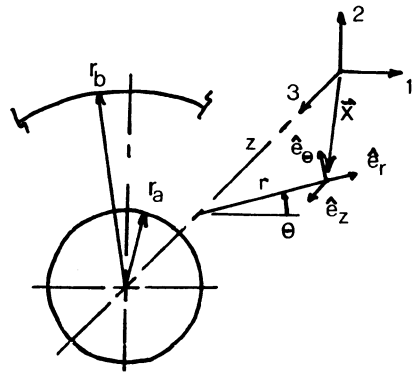

FIG. 1. SPECIMEN GEOMETRY.

FIELD EQUATIONS

In this section we outline the derivation of the displacement field equations for a cylindrically anisotropic media with orthotropic symmetry. The starting point is with the usual stress- displacement form of the field equations in polar $((r, \theta, z))$ coordinates, see for example [5],

$$\partial_r \sigma_{rr} + r^{-1} \partial_\theta \sigma_{r\theta} + \partial_z \sigma_{rz} + r^{-1}(\sigma_{rr} - \sigma_{\theta\theta}) - \rho\omega^2 u_r = 0 \tag{1a}$$

$$\partial_r \sigma_{r\theta} + r^{-1} \partial_\theta \sigma_{\theta\theta} + \partial_z \sigma_{\theta z} + r^{-1} 2\sigma_{r\theta} - \rho\omega^2 u_\theta = 0 \tag{1b}$$

$$\partial_r \sigma_{rz} + r^{-1} \partial_\theta \sigma_{\theta z} + \partial_z \sigma_{zz} + r^{-1} \sigma_{rz} - \rho\omega^2 u_z = 0 \ , \tag{1c}$$

where we have assumed time-harmonic $(e^{-i\omega t})$ wave motion, and ρ is the mass density per unit volume of the medium. The field equations above involve the (polar) components of stress, $\sigma_{\alpha\beta}$, and displacement, u_γ; in this article greek indices take on any one of the three polar indices $\alpha = r, \theta, z$. The summation convention will *not* be implied over repeated greek indices. Finally, in Eqs. (1a-c) by ∂_α we mean the operator $\partial_\alpha \equiv \partial/\partial\alpha$; also $\partial_\alpha^2 \equiv \partial^2/\partial\alpha^2$ and $\partial_\alpha^0 \equiv 1$. We note that in these equations each field quantity is taken to be a function of the field point (r, θ, z) which will be denoted by the vector $\vec{x} = r\hat{e}_r + z\hat{e}_z$, here $(\hat{e}_r, \hat{e}_\theta, \hat{e}_z)$ is the usual polar orthonormal basis, see Fig. 1. In this analysis it is assumed that either the medium is infinite in extent or all wave motion takes place sufficiently far from the medium boundaries, in which we case the former assumption serves as a useful approximation. If the medium does possess boundaries another useful restriction is to require $\lambda << r_a$, λ is the wavelength and r_a is the inner radius of a cylindrical specimen, this ensures not only that ray theory will be valid throughout the domain, but also that the transmission problem into and out of the specimen from a water medium can be approximated by treating the solid medium as a planar halfspace.

In order to express the field equations solely in terms of the displacement components, we use the constitutive law for a cylindrically orthotropic media together with the usual strain-displacement relations to give

$$
\begin{aligned}
&\left[a_{11}\partial_r^2 + G_{r\theta}r^{-2}\partial_\theta^2 + G_{rz}\partial_z^2 + L_{rr}\right] u_r \\
&+ \left[a_{12}r^{-1}\partial_r\partial_\theta + G_{r\theta}r^{-1}\partial_\theta\partial_r + L_{r\theta}\right] u_\theta \\
&+ \left[a_{13}\partial_r\partial_z + G_{rz}\partial_z\partial_r + L_{rz}\right] u_z - \rho\omega^2 u_r = 0
\end{aligned} \tag{2a}
$$

$$
\begin{aligned}
&\left[G_{r\theta}r^{-1}\partial_r\partial_\theta + a_{21}r^{-1}\partial_\theta\partial_r + L_{\theta r}\right] u_r \\
&+ \left[G_{r\theta}\partial_r^2 + a_{22}r^{-2}\partial_\theta^2 + G_{\theta z}\partial_z^2 + L_{\theta\theta}\right] u_\theta \\
&+ \left[a_{23}r^{-1}\partial_\theta\partial_z + G_{\theta z}r^{-1}\partial_z\partial_\theta + L_{\theta z}\right] u_z - \rho\omega^2 u_\theta = 0
\end{aligned} \tag{2b}
$$

$$
\begin{aligned}
&\left[G_{rz}\partial_r\partial_z + a_{31}\partial_z\partial_r + L_{zr}\right] u_r \\
&+ \left[G_{\theta z}r^{-1}\partial_\theta\partial_z + a_{32}r^{-1}\partial_z\partial_\theta + L_{z\theta}\right] u_\theta \\
&+ \left[G_{rz}\partial_r^2 + G_{\theta z}r^{-2}\partial_\theta^2 + a_{33}\partial_z^2 + L_{zz}\right] u_z - \rho\omega^2 u_z = 0 \ .
\end{aligned} \tag{2c}
$$

The operators $L_{\alpha\beta}$ appearing in the above field equation are $L_{\alpha\beta} = L_{\alpha\beta}(\partial_\gamma, \partial_\gamma^0)$, i.e. they involve only first or zeroth order derivatives with respect to r, θ, or z. They have been separated from the second order operators, which we will later see exclusively determine the eikonal equations.

The constants a_{ij} and $G_{\alpha\beta}$ appearing in the above field equations come from the constitutive law as follows, see for example [5] or [6]. The a_{ij} give the normal polar stresses in terms of the normal polar strains, and thus are the components of a matrix whose inverse $a_{ij}^{-1} = S_{ij}$ have components $S_{11} \equiv 1/E_r$, $S_{12} \equiv -\nu_{r\theta}/E_\theta = -\nu_{\theta r}/E_r \equiv S_{21}$, etc. Here the E_α and the $\nu_{\alpha\beta}$ are the Young's modulus and Poisson's ratio for a cylindrically anisotropic media. The S_{ij}, and hence a_{ij}, is symmetric and thus contain six constants. The remaining three are the shear moduli $G_{\alpha\beta} = G_{\beta\alpha}$, $\alpha \neq \beta$.

Ray Theory

In this section we derive the eikonal equations for a cylindrically orthotropic elastic media. The solution of the eikonal equations will then be presented and discussed. This will give rise to the identification of ray paths along which a disturbance in any one of three normal modes will propagate at a constant phase velocity. The transport equations, which give the amplitude and phase information along a ray, will be neither derived nor discussed in this article. For the original purpose of extracting material constants from time of flight measurements, the eikonal equations provide exactly the information needed: (1) the ray paths, and (2) the (group) velocity along any one ray. In this section however we will speak only of phase velocities.

Unlike an anisotropic media of cartesian symmetry our media itself has a natural geometric length scale, the radius r in the (r, θ, z) description of a field point. Hence we expect the theory to perform best when applied to media whose fiber radius of curvature is large compared to a wavelength in the solid.

We begin by making the ansatz

$$u_\alpha = U_\alpha e^{i\omega\phi} \tag{3a}$$

$$U_\alpha = \sum_{n=0}^{\infty} (i\omega)^{-n} U_\alpha^{(n)} \ , \tag{3b}$$

where again $\alpha = r, \theta, z$, and where $U_\alpha = U_\alpha(\vec{x}), \phi = \phi(\vec{x}), \vec{x}$ denotes the field point at which the field equations (2a-c) are satisfied. Also, it is convenient to define the "gradient of phase" vector $\vec{p} \equiv \vec{\nabla}\phi$. Inserting the ansatz (3) into the field equations (2a-c) gives to leading order in ω $(O(\omega^2))$, note the $L_{\alpha\beta}$ operators do not contribute, the eikonal equations

$$\begin{aligned}
&[a_{11}p_r p_r + G_{r\theta}p_\theta p_\theta + G_{rz}p_z p_z]\,U_r^{(0)} \\
&+[a_{12}p_r p_\theta + G_{r\theta}p_r p_\theta]\,U_\theta^{(0)} \\
&+[a_{13}p_r p_z + G_{rz}p_z p_r]\,U_z^{(0)} - \rho U_r^{(0)}\omega^2 = 0 \qquad (4a)\\
&[G_{r\theta}p_r p_\theta + a_{21}p_\theta p_r]\,U_r^{(0)} \\
&+[G_{r\theta}p_r p_r + a_{22}p_\theta p_\theta + G_{\theta z}p_z p_z]\,U_\theta^{(0)} \\
&+[a_{23}p_\theta p_z + G_{\theta z}p_z p_\theta]\,U_z^{(0)} - \rho U_\theta^{(0)}\omega^2 = 0 \qquad (4b)\\
&[G_{rz}p_r p_z + a_{31}p_z p_r]\,U_r^{(0)} \\
&+[G_{\theta z}p_\theta p_z + a_{32}p_z p_\theta]\,U_\theta^{(0)} \\
&+[G_{rz}p_r p_r + G_{\theta z}p_\theta p_\theta + a_{33}p_z p_z]\,U_z^{(0)} - \rho U_z^{(0)}\omega^2 = 0\ , \qquad (4c)
\end{aligned}$$

where the p_α denote the components of the vector $\vec{\nabla}\phi \equiv \vec{p} = p_r\hat{e}_r + p_\theta\hat{e}_\theta + p_z\hat{e}_z$.

Equations (4a-c) are solved rigorously in [7]. The physics of the solution, however, can be spotted immediately by inspection. The eikonal equations (4a-c) have exactly the same form as those for an orthotropic media of cartesian symmetry, where here the role of the cartesian basis $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is played by the polar basis $(\hat{e}_r, \hat{e}_\theta, \hat{e}_z)$. This is not due to a trivial coordinate rotation; we are comparing here two different media, and, the comparison can be effected only if all the $L_{\alpha\beta}$ operators in Eqs. (2a-c) drop out, which is exactly what happens in the $w \to \infty$ limit. Thus we expect the wave motion in the cylindrically orthotropic media to behave locally like that of an orthotropic media of cartesian symmetry, in the limit that $w \to \infty$. In what follows we will see that this is exactly what we find.

The general form of ϕ for which the components of $\vec{p} \equiv \vec{\nabla}\phi$ give rise to nontrivial $\vec{u}(\vec{x})$ is, see [7],

$$\phi = \hat{p}'' \cdot \vec{x} / C_\nu(\hat{p}'')\ . \qquad (5)$$

Here the $1/C_\nu(\hat{p})$ is a slowness function analogous to that encountered in media with cartesian symmetry but with the following important difference: $C_\nu(\hat{p})$ uses as its reference the local coordinate system described by the polar $(\hat{e}_r, \hat{e}_\theta, \hat{e}_z)$ basis. Thus $C_\nu(\hat{p})$ gives the phase velocity of propagation of the ν^{th} mode ($\nu = 1-3$) in the $\hat{p}$-direction, where $\hat{p}$ is referred to the local polar basis. Thus for any given $\hat{p}$ vector, the velocity $C_\nu(\hat{p})$ will be spatially varying. The other key element of Eq. (5) involves the unit vector $\hat{p}''$, which is given by $\hat{p}'' \equiv$ ($\hat{p}$ rotated ccw by θ), $\hat{p}$ is any given vector and θ is the angular coordinate of the field point $\vec{x}$. In this way the $\vec{p}$ vector field which satisfies the eikonal equations (4a-c) are spatially varying.

Inserting the results (5) into the original ansatz (3) yields the asymptotic representation for the vector displacement field

$$\vec{u}(\vec{x}) \sim \vec{U}^{(0)} e^{i\omega \hat{p}'' \cdot \vec{x} / C_\nu(\hat{p}'')}, \omega \to \infty \ , \tag{6}$$

where again $\hat{p}'' = \hat{p}''(\vec{x})$ is any unit vector fixed with respect to the local $(\hat{e}_r, \hat{e}_\theta, \hat{e}_z)$ basis. Note that the phase velocity $C_\nu(\hat{p}'')$ is *not* spatially varying, although the $\hat{p}'' \cdot \vec{x}$ term *is*. Thus the ray will propagate along a curved path, but with a constant phase velocity.

RAY PATHS

In the previous section we established that the general solution to the eikonal equation for ray propagation in a cylindrically orthotropic media, Eq. (6), indicates that the ray paths are such that the direction of propagation remains fixed with respect to the local $(\hat{e}_r, \hat{e}_\theta, \hat{e}_z)$ basis. This in general will describe a family of 3-D space curves, but for the purpose of illustration we restrict ourselves here to the $(\hat{e}_r, \hat{e}_\theta)$ plane. Then it can be shown trivially that the ray paths are those functions $y = f(x)$ which satisfy the (nonlinear) ODE

$$f'(x) = \frac{Ax + f(x)}{x - Af(x)} \ , \tag{7}$$

here A is a scalar constant and the cartesian origin $(x = 0, y = 0)$ coincides with the polar origin $(r = 0, \theta)$. Thus some ray paths which lie in the (r, θ) plane are depicted in Fig. 2.

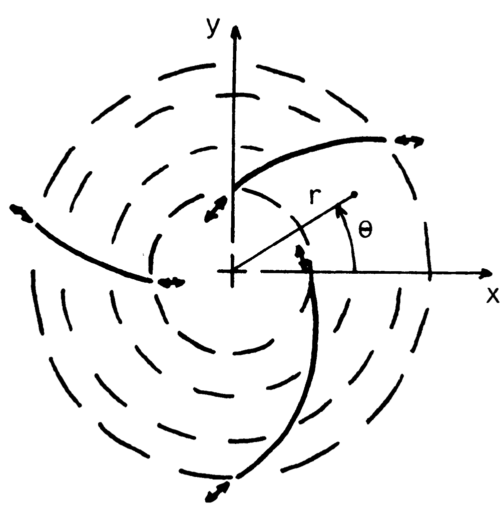

FIG. 2. RAY PATHS IN THE (r, θ) PLANE.

In an experimental situation it is the direction of energy propagation which is important. The rays depicted in Fig. 2 can be interpreted as rays along which a disturbance propagates at either the phase or the group velocity. In the former case the wavevector is at all points tangent to the ray, and in the latter case the direction of energy velocity is at all points tangent to the ray.

CONCLUSIONS

In this article the eikonal equations for ray propagation in a cylindrically orthotropic media were derived, and their solution presented and discussed. The ray paths were identified as those paths along which a disturbance propagates with a constant velocity. It was found that the ray paths are curved in such a way that the ray propagates so as to maintain a constant angle of attack with respect to the surfaces which describe the symmetry of the media.

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REFERENCES

1. F. C. Karal and J. B. Keller, "Elastic Wave Propagation in Homogeneous and Inhomogeneous Media", J. Acoust. Soc. Am., **31**(6), 1958.
2. J. D. Achenbach, A. K. Gautesen, and H. McMaken. Ray Methods for Waves in Elastic Solids, Pitman Pub., Mass., 1982.
3. A. N. Norris, "A Theory of Pulse Propagation in Anisotropic Elastic Solids", Wave Motion, 9, 1987.
4. A. N. Norris, "Elastic Gaussian Wave Packets in Isotropic Media", Acta Mechanica, 71, 1988.
5. S. A. Ambartsumyan, Theory of Anisotropic Plates, Technomic Pub., Conn., 1970.
6. S. G. Lekhnitskii, Theory of Elasticity of an Anisotropic Body, Mir Pub., Moscow, 1981.
7. D. E. Budreck, Ray Theory for Cylindrically Anisotropic Elastic Media, in preparation.