CYLINDRICALLY GUIDED WAVES IN A TRANSVERSELY ISOTROPIC SHAFT

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INTRODUCTION

The propagation of a longitudinal wave in an isotropic cylinder has been used in the inspections of pumps and shafts of a nuclear power plant [1]. An ultrasonic technique called the "cylindrically guided wave technique" (CGWT) has been developed that can detect simulated circumferential defects through long metal paths in metallic materials being used for bolts and studs [1]. In the recent development of the sizing process for the cylindrically guided wave technique, the wave scattering at the circumferential crack can be formulated in terms of the guided cylindrical waves which mathematically result from the cylinder frequency equation. Therefore, a detailed investigation of the cylinder frequency equation is in order in the sizing process for bolt and pump-shaft inspections.

Many metallic substances and fiber-reinforced composites are described as being transversely isotropic and have five elastic constants [2,3]. A metallic shaft tends to produce material anisotropy after being placed in service in a radioactive environment. The vibrational modes of crystalline rods under torsion have been studied for a transversely isotropic material [4]. In the present work, the effects that material anisotropy have on the propagation of a longitudinal wave in a transversely isotropic rod are investigated. Solutions are obtained in exact expressions and reveal different dispersive natures of the longitudinal waves for different sample materials.

GUIDED WAVE IN A TRANSVERSELY ISOTROPIC CYLINDER

A harmonic wave is guided to propagate in the longitudinal direction of a transversely isotropic cylinder. The stress-strain relation in cylindrical coordinate (r,θ,z) for a transversely isotropic medium can be written in the following form [2,3]:

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$$\sigma_{rr} = c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz}$$
$$\sigma_{\theta\theta} = c_{12}e_{rr} + c_{11}e_{\theta\theta} + c_{13}e_{zz}$$
$$\sigma_{zz} = c_{13}e_{rr} + c_{13}e_{\theta\theta} + c_{33}e_{zz} \qquad (1)$$
$$\sigma_{rz} = c_{44}e_{rz}, \quad \sigma_{\theta z} = c_{44}e_{\theta z}$$
$$\sigma_{r\theta} = \frac{1}{2}\left(c_{11} - c_{12}\right)e_{r\theta}$$

The harmonic wave is assumed to be propagating in z-direction, which is along the axis of symmetry of the material. The problem is axisymmetrical and the displacement field can be described as $(U_r, 0, U_z)$. Using the strain-displacement relations, the equations of motion can be written in terms of the displacement components as follows:

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rU_r\right)\right] + \frac{c_{13}+c_{44}}{c_{11}}\frac{\partial^2 U_z}{\partial r\,\partial z} + \frac{c_{44}}{c_{11}}\frac{\partial^2 U_r}{\partial z^2} = \frac{\rho}{c_{11}}\frac{\partial^2 U_r}{\partial t^2} \qquad (2)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_z}{\partial r}\right) + \frac{c_{13}+c_{44}}{c_{44}}\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rU_r\right)\right] + \frac{c_{33}}{c_{44}}\frac{\partial^2 U_z}{\partial z^2} = \frac{\rho}{c_{44}}\frac{\partial^2 U_z}{\partial t^2} \qquad (3)$$

where ρ is the density of the medium and t is the time variable. The cylinder considered has radius a and is free from stresses in its surface. Therefore, the boundary conditions can be written as:

$$\sigma_{rr} = 0 \text{ and } \sigma_{rz} = 0 \text{ at } r = a. \qquad (4)$$

The equations of motion and the boundary conditions can be satisfied if the displacement components have the following forms:

$$U_z = \sum_{j=1}^{2} B_j J_0(\gamma q_i r)\exp[i(\gamma z+\omega t] \qquad (5)$$

$$U_r = \sum_{j=1}^{2} A_j J_1(\gamma q_j r)\exp[i(\gamma z+\omega t)] \qquad (6)$$

The wave number γ is equal to $\omega/v=2\pi/\Lambda$ where v is the wave speed and Λ is the wave length. If Eqs. (5) and (6) are substituted into Eq. (2), the parameters q_j must satisfy the following characteristic equation:

$$\beta q_j^4 - bq_j^2 - (\alpha-e^2)(e^2-1) = 0, \quad b = (e^2-\alpha)\beta+(e^2-1)+(1+\delta)^2 \qquad (7)$$

$$c_2^2 = c_{44}/\rho, \quad e^2 = v^2/c_2^2, \quad \alpha = c_{33}/c_{44}, \quad \beta = c_{11}/c_{44}, \quad \delta = c_{13}/c_{44} \qquad (8)$$

The roots of Eq. (7) are the parameters q_1 and q_2 which can be written as

$$q_1^2 = (b-\psi)/2\beta, \quad q_2^2 = (b+\psi)/2\beta \qquad (9)$$

$$\psi^2 = [(1+\beta)e^2-\bar{\gamma}]^2+4\beta(\alpha-e^2)(e^2-1) \qquad (10)$$

$$\bar{\gamma} = 1+\alpha\beta-(1+\delta)^2 \qquad (11)$$

The quantities A_j and B_j in Eqs. (5) and (6) are related in the following form:

$$(\beta q_j^2+1-e^2)A_j+i\gamma(1+\delta)q_jB_j = 0 \qquad (12)$$

The equation holds for both j=1 and 2.

The stress components in Eq. (1) can be calculated in terms of the displacement components in Eqs. (5) and (6) if the strain-displacement relations are used. The shear stress σ_{rz} is first calculated and then made equal to zero at r=a in order to satisfy one of the stress boundary conditions in Eq. (4). The calculations lead to the following relation:

$$[\beta q_1^2-\delta-e^2]J_1(kq_1)A_1+[\beta q_2^2-\delta-e^2]J_1(kq_2)A_2 = 0 \qquad (13)$$

where the new quantity is $k=2\pi a/\Lambda$. The radial stress σ_{rr} is also calculated. The stress-free boundary conditions at r=a in Eq. (4) are now completely satisfied. The results yield in terms of Eq. (13) the following exact frequency equation.

$$G_{11}G_{22}kq_1J_1(kq_1)J_0(kq_2) - G_{12}G_{21}kq_2J_1(kq_2)J_0(kq_1) + (\beta-\varepsilon)\psi(1+\delta)q_1q_2J_1(kq_1)J_1(kq_2) = 0 \qquad (14)$$

$$G_{11} = \beta q_1^2-\delta-e^2,\ G_{12} = \beta q_2^2-\delta-e^2,\ \varepsilon = c_{12}/c_{44} \qquad (15)$$

$$G_{21} = \beta q_1^2-\delta(1-e^2),\ G_{22} = \beta q_2^2-\delta(1-e^2) \qquad (16)$$

The frequency equation determines the wave speed as a function of the cylinder radius, the wave length and the material constants. When the cylinder radius is small compared to the wave length, i.e., $a/\Lambda \to 0$, the frequency equation reduces in the limit to the following wave-speed equation:

$$v_0^2/c_2^2 = \alpha-2\delta^2/(\beta+\varepsilon) \qquad (17)$$

For an isotropic material, the material constants become $c_{11}=c_{33}=\lambda+\mu$, $c_{12}=c_{13}=\lambda$ and $c_{44}=\mu$. The quantities λ and μ are the Lame constants. The wave speed in Eq. (17) becomes the isotropic rod wave speed $v_0=(E/\rho)^{1/2}$ [5].

If the wave length is much smaller than the cylinder radius, the frequency equation in Eq. (14) reduces in the limit for $a/\Lambda\to\infty$ to the following form:

$$\delta(1-e^2)(b-\delta-e^2)-\beta^2q_1^2q_2^2-\beta e^2(1+\delta)q_1q_2 = 0 \qquad (18)$$

This equation predicts the surface wave speed for both isotropic and transversely isotropic materials [5,6].

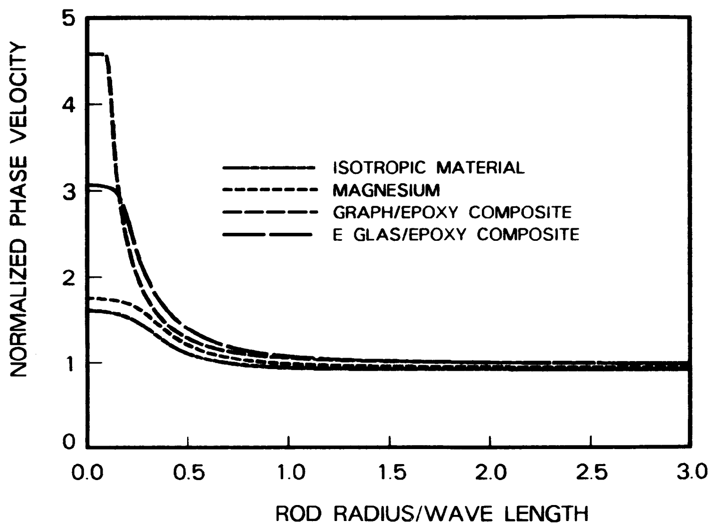

Fig. 1. Normalized phase velocity of the fundamental mode.

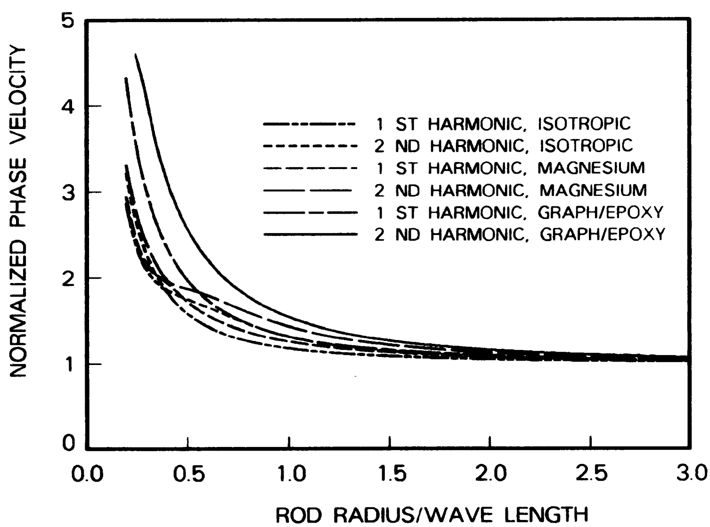

Fig. 2. Normalized phase velocity of harmonics.

WAVE DISPERSION

The value of e which is the ratio between the wave speed v and the shear wave speed c_2 has been calculated numerically from Eq. (14) as a function of the ratio a/Λ for various materials. Magnesium, graphite/epoxy composite and E glass/epoxy composite are used as the sample materials in the numerical calculations. Graphite/epoxy and E glass/epoxy composites have been described as transversely isotropic materials [7].

The material constants for graphite/epoxy composite are c_{11}=0.828, c_{33}=8.68, c_{13}=0.0285, c_{12}=0.2767, and c_{44}=0.4147; for E glass/epoxy composite they are c_{11}=1.493, c_{33}=4.727, c_{13}=0.5244, c_{12}=0.6567, and c_{44}=0.4745, all in the unit of 10^4 MN/m^2. The material constants for magnesium also in 10^4 MN/m^2 are c_{11}=5.97, c_{33}=6.17, c_{13}=2.17, c_{12}=2.62, and c_{44}=1.64 [3]. For comparison, polycrystalline magnesium with Young's modulus E=4.1x 10^6 N/cm^2 and Poisson's ratio ν=0.3 [3] is also used here as an example isotropic material.

The wave speed v normalized by the shear wave speed c_2 for the fundamental mode of wave propagation is shown in Fig. 1. The derived wave speed in Eq. (17) is seen to agree well with the numerical results in the figure. The ratios of the surface wave speed to the shear wave speed calculated from Eq. (18) are 0.988 for graphite/epoxy composite, 0.982 for E Glass/expoxy composite, 0.944 for magnesium, and 0.927 for the isotropic material. These values are also in close agreement with the wave speeds for a large value of a/Λ in Fig. 1. The spreads of the wave speeds for the sample anisotropic materials are seen to be much larger than the corresponding spread of the isotropic material. Therefore, the wave in a transversely isotropic cylinder is more dispersive than in an isotropic cylinder.

The wave speeds for some higher harmonics have also been calculated and shown in Fig. 2. It can be seen that the material anisotropy also has strong effects on the wave speeds of the harmonics.

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