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Stresses developed by impact in a beam

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STRESSES DEVELOPED BY IMPACT IN A BEAM

by

Richard T. Othmer

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Theoretical and Applied Mechanics

Approved:

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1954
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I. INTRODUCTION

The determination of the stresses, strains, and accompanying phenomena produced in a structural member by the sudden application of a force, displacement, or moment has been the goal of many theoretical and experimental investigations of the past century and a half. Theoretically it should be possible to set up the equations of motion for the medium involved and, with the aid of the stress-strain and compatibility relationships, solve the equations contingent upon the prescribed boundary conditions. Solutions have been obtained for specific cases in which certain assumptions were made restricting the material and the type and application of load. The problem of the point disturbance in a homogeneous, isotropic medium and those of longitudinal and transverse impact on slender rods have been the most often investigated and for them several limited solutions have been obtained.

The assumptions that have been imposed generally have fallen into one or more of three categories: those concerned with the structure of the material itself, those concerned with the stress-strain relationships of the material, and those concerned with the boundaries of the material and the method of disturbing those boundaries.
A solution free of all restricting assumptions would be too general and immediately upon its application to a given problem restrictions would arise when an attempt was made to describe the material and boundary conditions. In this dissertation a homogeneous, isotropic material is considered; the stress-strain relationships are assumed to be the same as for the static plane stress situation satisfying Hooke's law; the material is in the form of a rectangular beam and the disturbances are suddenly applied. The approach to the solution is through the equations of motion expressed in terms of displacement functions.

Much of the work that has been done concerning impact on beams has dealt with the assumption that the elementary theory of bending of beams (sometimes modified with corrections for rotatory inertia and vertical shear) applied. A more direct approach is the determination of a solution to the differential equations of motion of an incremental element of the material without the assumption of beam action. The boundary conditions, however, provide the means for the evaluation of the action in the member produced by the disturbance. This latter procedure is followed herein and some of the problems encountered are discussed.
II. REVIEW OF THE LITERATURE

Early in the development of the theory of vibrations the phenomenon of impact was being discussed. Inasmuch as the application of a disturbance that varies rapidly with time will set up vibratory motions within the member on which the disturbance acts, the close association of vibrations and impact would naturally be expected. Directly related to impact, then, is the propagation of the vibratory waves in the material involved. Navier, Poisson, and Cauchy were among the early contributors toward a theoretical understanding of the theory of vibrations. Navier (29) was the first to set up the equations of motion for the vibration of elastic bodies in terms of displacements. His equations were based on a homogeneous, isotropic material and contained a constant of the same nature as Young's modulus. Poisson (32) extended the work of Navier by a more comprehensive discussion. He considered the propagation of waves originating from a point disturbance through an isotropic elastic medium. He recognized that at some distance from the disturbance two waves existed: a longitudinal wave and a transverse one. These are now generally called waves of dilatation and distortion respectively. Cauchy (5) also obtained the equations of motion in terms of displacements and like Poisson applied them to numerous
problems in vibrations but it was some time before adequate experiments were performed to test the theory.

Hodgkinson (19) was interested in the action of cast iron beams under impact loads. He performed impact tests on many beams by allowing a cast iron ball supported by a long cable to swing through an arc and strike the beam at midspan or quarter-span. He obtained maximum deflections by measuring the depth which a long peg, initially touching the beam, was driven into a mass of clay.

With the exception of a few investigators like Hodgkinson most of the work that was done at that time was theoretical and the relationship between the differential equations of motion of particles and those of light waves were noted by many among whom was Green (17). It was past the middle of the nineteenth century before Cox (9) performed his tests on elastic beams. His results indicated that, for slender beams and an assumption that the elastic curve is the same in the dynamic case as in the static, slightly less than one-half of the mass of the beam acts initially to resist the impact. He concluded, as did Hodgkinson, that the deflection was directly proportional to the velocity of the striking mass at impact.

Pochhammer (31) and later Chree (6) set up the differential equations of motion in cylindrical coordinates for a slender circular cylinder with its surfaces free of forces.
Solutions were obtained for longitudinal and torsional vibrations and were discussed for flexural vibrations but the resulting equations are complicated and a numerical solution was not given.

Lord Rayleigh's "Theory of Sound" (34) was first published about the time of Pochhammer and Chree and contained a summary of most of the work on vibrations then in existence. He (v 1,p 258) had the differential equations of motion for flexural vibrations of a long, slender beam and included the effect of "rotatory inertia." He (33) also discussed Hertz's theory of impact which relates the force developed at the surface of contact between two bodies and the relative approach of the bodies. The relative approach of the two bodies is the difference between the deflections of the bodies under the action of the force.

Another important text, Love's "Mathematical Theory of Elasticity," was first published in 1877 and included, more completely in later editions, discussions on the theory of vibrations of elastic bodies. The historical introduction to the fourth edition contains the development of the theory of vibrations up to the early part of the twentieth century interwoven with the theory of elasticity.

By 1921 Timoshenko (38) had shown theoretically that the impact of a mass on a beam of appreciable mass was composed of many sub-impacts and (37) had included additional
terms in the flexural wave equation to provide for the effect of cross-shear. In 1936 Mason (26) verified the existence of the sub-impacts by use of electrical contacts and a recording oscillograph. The development of electronic instruments capable of recording the effects of disturbances over a few microseconds has been one of the factors producing an increase in the investigations of impact and vibration.

Since 1940 the theoretical and experimental analyses insofar as impact and vibrations of beams are concerned have dealt with the flexural wave equation with and without cross-shear and rotatory inertia; the equations of Pochhammer and Chree; Hertz's law of impact; and photoelastic, electronic, and photographic equipment.

The general flexural wave equation mentioned above was derived by Timoshenko (p 337) (38). In the derivation he assumed a homogeneous isotropic elastic material, a uniform cross-section, small deflections, a plane section remains plane during bending, and a linear stress-strain relationship. The general equation is

\[
EI \frac{\partial^4 y}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} = \left[ I \rho + \frac{E l}{k} \frac{\partial^4 y}{\partial x^2 \partial t^2} \right] + \frac{\partial^2 I}{\partial x^2} \frac{\partial^4 y}{\partial t^4} = 0
\]

in which

- \(x\) - longitudinal coordinate
- \(y\) - transverse coordinate in direction of displacement
- \(t\) - time
- \(E\) - modulus of elasticity of material in beam
- \(I\) - moment of inertia of cross-section with respect to the neutral axis
If the first two terms only are considered the elementary beam vibration theory is obtained. Dohrenwend, Drucker, and Moore (12) made use of the elementary beam theory in their work on transients. Primarily they were interested in the use of high speed photography and the strain-resistance gage in conjunction with a cathode ray oscilloscope. The beams investigated were slender (15 ft by 1/4 in. deep by 1/2 in. wide) and the results obtained checked the stated theoretical strain values.

Duwez, Clark, and Bohnenblust (14) used the elementary theory and were interested in the elastic curve of a long beam under impact conditions. They obtained a theoretical deflection curve at the end of impact based on an approximate bending moment vs curvature curve which compared with their experimental deflection curve. The curve was obtained by means of a spark photograph of a white line on the side of the black beam.

Eringen (15) also used the elementary beam theory and evaluated the flexural stresses as the static modulus of elasticity times the unit strain. The unit strain was assumed proportional to its distance from the neutral surface. The bulk of Eringen's work was concerned with the determination
of the contact force which he proceeded to obtain through
a solution to the Hertz impact problem.

Jones (21) used the elementary beam theory in conjunc-
tion with a series of image loads on an infinite beam to
solve the problem of a concentrated load suddenly applied
to a simply supported beam. The conditions of his solution
required a slender beam.

Others (1, 18, 28, 40) have dealt with the elementary
form of the flexural wave equation to determine or discuss
frequencies of vibrations, the modulus of elasticity,
deflections, and stresses for various beam conditions but
have all required slender beams.

If only the first three terms (i.e. those terms inde­
dependent of $k'$) of the general wave equation are considered,
rotatory inertia is included in the production of displace­
ments. Rayleigh (v 1, p 293) (33) was about the only one
to discuss a correction of the elementary flexural equation
to provide for rotatory inertia alone. His correction
was based on an addition to the kinetic energy of the bar
and he showed that for a slender bar it could be neglected.

In the development of the general flexural vibration
equation for a beam Timoshenko made use of the elementary
static theory of bending to obtain relations for the bend­
ing moment and shearing force. He wrote the differential
equations of motion for rotation and translation of an
element of the beam. He compared the effect on the frequencies of vibration of a simple beam of the rotatory inertia and shear and found the effect due to shear to be about four times that due to rotatory inertia.

A few investigators (10, 11, 30) considered the general wave equation. Prescott (30) set up the general wave equation for flexural vibrations of a beam from energy considerations and proceeded to discuss the velocities of waves propagated in a slender beam. Davies (10) compared the velocities of wave propagation in a beam as obtained from the elementary theory, the inclusion of rotatory inertia, the general wave equation, and an "exact" theory based on the equations of Pochhammer and Chree. His results showed that for wave lengths that are large in comparison with the bar radius there was good agreement between the four, but for wave lengths small in comparison with the bar radius the Timoshenko and the "exact" theory agree very well with each other and differ appreciably from the other two.

Dengler and Goland (11) made use of the general wave equation to solve for the bending moment in an infinite beam when subjected to an impulsive impact at midspan. They performed a Fourier transformation and a Laplace transformation with respect to the length and time coordinates respectively and showed the moment disturbance dispersed
as it progressed along the beam. However, the results
were for an admittedly unrealistic selection of a particular
term in the equation which they called "effective shear
modulus." They did consider the case of a general value
for the "effective shear modulus" but left the moment
expressed in terms of improper integrals.

Leonard and Budiansky (24) presented solutions by
various procedures to several specific problems of transient
loading of uniform beams of finite length based on the
application of the laws of motion to an element of the
beam and elementary beam relationships. They considered
only cases in which the propagation velocities of the
bending and shear disturbances were equal.

Some investigators (2,14,20,25) made use of the equations
of Pochhammer and Chree to obtain solutions for, generally,
the propagation of longitudinal and flexural disturbances
in a given member. Bancroft (2) was concerned primarily
with longitudinal waves and set up a numerical solution for
the equations. He indicated a form for the solution in the
case of flexural waves but also indicated that difficulty
existed in completing the solution. Hudson (20) showed that
the Pochhammer and Chree equations yielded dispersion of
the disturbance as it progressed from the point of origin.

The determination of the effects produced by an impact
load on a beam has been attempted by many investigators and
insofar as a beam has been concerned the investigations have invariably assumed the beam to be long and slender and to respond immediately to the impact as a beam. There must be some small increment of time between the application of the impact and the transmission of the disturbance at least to the opposite boundary if not further (i.e. reflected from the boundary) in which beam action will not take place. On this basis the correct approach should be one of solution of the equations of motion for the appropriate boundary conditions. Prescott (30) and Cooper (8) proceeded in this manner but their assumptions and the selection of their boundary conditions led them to a slender beam and the propagation of flexural waves along the beam.

Sneddon (36) set up the partial differential equations of motion in terms of displacement for the case of plane strain. He used two-dimensional Fourier transforms to obtain a transformed solution for the case of a variable pressure applied to the boundary of a semi-infinite elastic medium.
III. THEORETICAL INVESTIGATION

A. List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A, B, C, k, N</td>
<td>constants, with and without subscripts</td>
</tr>
<tr>
<td>b</td>
<td>width of beam</td>
</tr>
<tr>
<td>$c_1^2$</td>
<td>$E/2\ell(1+\mu)$</td>
</tr>
<tr>
<td>$c_2^2$</td>
<td>$E/\ell(1-\mu^2)$</td>
</tr>
<tr>
<td>$D_n(t)$, $D_n(x,t)$</td>
<td>functions of $t$ and of $x$ and $t$ used for convenience when integration with respect to $y$ is to be performed</td>
</tr>
<tr>
<td>E</td>
<td>modulus of elasticity</td>
</tr>
<tr>
<td>e</td>
<td>$2.718\ldots$</td>
</tr>
<tr>
<td>F</td>
<td>force, subscript indicates direction in which it acts</td>
</tr>
<tr>
<td>h</td>
<td>one-half depth of beam</td>
</tr>
<tr>
<td>i</td>
<td>$\sqrt{-1}$</td>
</tr>
<tr>
<td>L</td>
<td>one-half length of beam</td>
</tr>
<tr>
<td>$\mathcal{L}^{-1}{}$</td>
<td>indicates inverse Laplace transformation to be performed on function within brackets</td>
</tr>
<tr>
<td>m, n, p, and q</td>
<td>designations of numbers in various series</td>
</tr>
<tr>
<td>M</td>
<td>moment</td>
</tr>
<tr>
<td>s</td>
<td>parameter used in Laplace transform of $t$</td>
</tr>
<tr>
<td>$s_1$ and $s_2$</td>
<td>$s/c_1$ and $s/c_2$ respectively</td>
</tr>
<tr>
<td>$S_x$, $S_y$, and $S_{xy}$</td>
<td>unit stresses; single subscript denotes normal stress and indicates direction of stress, double subscript denotes shear-stress—first subscript denotes direction, second denotes axis perpendicular to plane in which stress acts</td>
</tr>
</tbody>
</table>
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t time variable

$T(t)$, $X(x)$, and $Y(y)$ functions of $t$, $x$, and $y$ only

$u$ and $v$ displacements in the $x$- and $y$-direction respectively

$w$ parameter used in Fourier transform of $x$

$x$ and $y$ rectangular coordinates

$V$ vertical shear

$\Delta$ small increment of distance

$\delta$ small increment of time

$\gamma$ unit shearing strain

$\epsilon$ unit normal strain, subscript denotes direction of strain

$\phi(x,y,t)$ and $\psi(x,y,t)$ displacement functions

$\rho$ density

$\omega$ arbitrary circular frequency

B. Equations of Motion for Plane Stress

If a condition of plane stress exists within a body subjected to loads that vary with time the equations of motion can be used to obtain stress-inertia equations. A free-body-diagram of an incremental section of the body is presented in Fig. 1.

The equations of motion applied to the force system give

$$-S_x \Delta y \Delta z - S_{xy} \Delta x \Delta z + (S_{xy} + \frac{\partial S_{xy}}{\partial y} \Delta x \Delta z) + (S_x + \frac{\partial S_x}{\partial x} \Delta y \Delta z) = \frac{\Delta x \Delta y \Delta z}{t^2} \frac{\partial^2 \psi}{\partial t^2}$$
Fig. 1 Free-Body-Diagram of Incremental Section of Body

and

\[-S_y \Delta x \Delta z - S_{xy} \Delta y \Delta z + (S_{xy} \frac{\partial S_{xy}}{\partial x} \Delta x) (\Delta y \Delta z) + (S_y + \frac{\partial S_y}{\partial y} \Delta y) (\Delta x \Delta z) = \varepsilon_{\Delta x \Delta y \Delta z} \frac{\partial^2 u}{\partial t^2}\]

which reduce to

\[\frac{\partial S_{xy}}{\partial y} + \frac{\partial S_x}{\partial x} = \frac{\partial^2 v}{\partial t^2}\]

and

\[\frac{\partial S_{xy}}{\partial x} + \frac{\partial S_y}{\partial y} = \frac{\partial^2 u}{\partial t^2}\]  \hspace{1cm} (1)

Since too many unknowns exist in the above equations for a solution to be obtained, the relationships between the strains and the displacements at the point are needed. These relationships are defined as

\[\varepsilon_x = \frac{\partial u}{\partial x}; \quad \varepsilon_y = \frac{\partial v}{\partial y}; \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\]  \hspace{1cm} (2)
The combination of these relations with those obtained from the equations of motion requires a mathematical expression involving the properties of the material. If the material is homogeneous and isotropic, and if the proportional limit is not exceeded the following equations apply for the plane stress case

\[ S_x = \frac{E}{1-\mu^2} (\epsilon_x + \mu \epsilon_y) \]

\[ S_y = \frac{E}{1-\mu^2} (\epsilon_y + \mu \epsilon_x) \]  \hspace{1cm} (3)

\[ S_{xy} = \frac{E}{2(1+\mu)} \]

The substitution of equations 2 into equations 3 give

\[ S_x = \frac{E}{1-\mu^2} \left[ \frac{\partial u}{\partial x} + \mu \frac{\partial v}{\partial y} \right] \]

\[ S_y = \frac{E}{1-\mu^2} \left[ \frac{\partial v}{\partial y} + \mu \frac{\partial u}{\partial x} \right] \]  \hspace{1cm} (4)

\[ S_{xy} = \frac{E}{2(1+\mu)} \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] \]

and the substitution of equations 4 into equations 1 give

\[ \frac{1-\mu}{2} \frac{\partial}{\partial y} \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] + \frac{\partial}{\partial x} \left[ \frac{\partial u}{\partial x} + \mu \frac{\partial v}{\partial y} \right] = \frac{\epsilon (1-\mu^2)}{2E} \frac{\partial^2 u}{\partial t^2} \]  \hspace{1cm} (5)

\[ \frac{1-\mu}{2} \frac{\partial}{\partial x} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] + \frac{\partial}{\partial y} \left[ \frac{\partial v}{\partial y} + \mu \frac{\partial u}{\partial x} \right] = \frac{\epsilon (1-\mu^2)}{2E} \frac{\partial^2 v}{\partial t^2} \]

With the addition and subtraction of \( \frac{\partial}{\partial x} \left[ \frac{\partial u}{\partial y} \right] \) in the first of equations 5 and the addition and subtraction of \( \frac{\partial}{\partial y} \left[ \frac{\partial u}{\partial x} \right] \) in the second of equations 5 they can be written
The equations 6 are the differential equations of motion expressed in terms of the displacements \( u \) and \( v \). The particular solution to these equations that satisfies the boundary conditions of the problem is the desired solution.

C. Displacement Functions

It is possible to define the displacements in terms of two functions \( \phi(x,y,t) \) and \( \psi(x,y,t) \). The use of these functions gives two new differential equations but each new equation contains only one unknown function instead of two. Let

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \\
\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y} &= \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2}
\end{align*}
\]

Equations 6 will be satisfied if

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\phi(1-\mu^2)}{E} \frac{\partial^2 \phi}{\partial t^2} = \frac{1}{c_1^2} \frac{\partial^2 \phi}{\partial t^2}
\]
Equations 9 and 10 are similar to those of Prescott (30), Cooper (8), Kolsky (22), and Sneddon (36). The coefficients \( c_1 \) and \( c_2 \) vary with the assumptions involved in each particular case. These equations are generally called the wave equations in which \( c_1 \) and \( c_2 \) are the velocities of propagation of the waves. It can be shown that \( \phi \) is associated with the dilatation produced in the medium and \( \psi \) with the rotation.

Since the stresses are expressed in terms of the displacements \( u \) and \( v \), they can also be expressed in terms of the functions \( \phi \) and \( \psi \).

\[
\begin{align*}
S_x(x,y,t) &= \frac{E}{1-\mu^2} \left[ \frac{\partial^2 \phi}{\partial x^2} + \mu \frac{\partial^2 \phi}{\partial y^2} + (1-\mu) \frac{\partial^2 \psi}{\partial x \partial y} \right] \\
S_y(x,y,t) &= \frac{E}{1-\mu^2} \left[ \frac{\partial^2 \phi}{\partial y^2} + \mu \frac{\partial^2 \phi}{\partial x^2} - (1-\mu) \frac{\partial^2 \psi}{\partial x \partial y} \right] \\
S_{xy}(x,y,t) &= \frac{E}{2(1+\mu)} \left[ \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} + 2 \frac{\partial^2 \phi}{\partial x \partial y} \right]
\end{align*}
\]

Solutions to equations 9 and 10 that satisfy the boundary conditions imposed on the stresses, displacements, moments, and loads should produce the solutions to equations 6.
D. Problems Considered

In this section several different sets of boundary conditions are specified for a given member. Particular solutions to equations 9 and 10 are investigated to determine if the boundary conditions are satisfied. With one exception the solutions are obtained by assuming the variables to be separable. The last situation considered involves the use of the Fourier and Laplace transformations.

1. **Transient increasing moment and shear on ends of beam**

The investigation of the effects of a rapidly applied load or displacement on the stresses, strains, and displacements developed with a beam is first made on the beam of Fig. 2. It is subjected to end moments and shears which vary with time and the boundary conditions are expressed as

\[ S_y(x, \pm h, t) = 0 \]
\[ S_{xy}(x, \pm h, t) = 0 \]
\[ u(0, y, t) = 0 \]

![Fig. 2 Rectangular Beam with Applied Moments and Shears](image-url)
\[
\int_{-h}^{+h} S_x(x_0, y, t) \, dy = 0
\]
\[
\int_{-h}^{+h} S_{xy}(x_0, y, t) \, dy = V_x \pm \dot{t}_L
\]
\[
\int_{-h}^{+h} y S_x(x_0, y, t) \, dy = M_x \pm \dot{t}_L
\]

Particular solutions to equations 9 and 10 can be obtained by assuming the variables separable which, for equation 9, produces

\[\phi(x, y, t) = X(x) \cdot Y(y) \cdot T(t)\]

and on substitution into 9 gives

\[\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} = \frac{1}{c_1^2} \frac{T}{T}\]

in which \(\ddot{X}, \ddot{Y}, \ddot{T}\) indicate the second derivative with respect to \(x, y,\) and \(t\) respectively. The three terms must be constant so

\[\frac{\ddot{X}}{X} = k_2^2 ; \frac{\ddot{Y}}{Y} = k_3^2\]

from which

\[\frac{\ddot{T}}{T} = c_1^2(k_2^2 + k_3^2)\]

and possible solutions are

\[X = A_1 \cos k_2x; \quad Y = A_2 \sin k_3y; \quad T = A_3 \sin c_1\sqrt{k_2^2 + k_3^2} \quad t\]

hence
\[ \phi(x,y,t) = A_0 \cos k_2x \sin k_3y \sin \left( \frac{k_2^2 + k_3^2}{c_1^2} \right) t \quad (14) \]

and similarly
\[ \phi'(x,y,t) = B_0 \sin k_4x \cos k_5y \sin \left( \frac{k_4^2 + k_5^2}{c_2^2} \right) t \quad (15) \]

in which the A's, B's, and k's are constants to be determined. The substitution of equations 14 and 15 into the boundary condition on \( u \) gives
\[ u(0,y,t) = A_0 k_2 \sin k_2(0) \sin k_3y \sin \left( \frac{k_2^2 + k_3^2}{c_1^2} \right) t 
+ B_0 k_5 \sin k_4(0) \sin k_5y \sin \left( \frac{k_4^2 + k_5^2}{c_2^2} \right) t = 0 \]

The displacement \( u \), then, is an antisymmetric function with respect to \( x \). Obviously the selection of \( \phi \) containing a \( \sin kx \) or \( \phi' \) containing a \( \cos kx \) would not have made \( u(0,y,t) = 0 \). The substitution into the boundary condition on \( S_y \) gives
\[ S_y(x,\hat{x},t) = (-\mu k_2/k_3) A_0 \cos k_2x \sin(\hat{x}k_3) \sin \left( \frac{k_2^2 + k_3^2}{c_1^2} \right) t 
+ (1-\mu) k_4 k_5 B_0 \cos k_4x \sin(\hat{x}k_5) \sin \left( \frac{k_4^2 + k_5^2}{c_2^2} \right) t=0 \]

To satisfy this condition \( A_0 \) and \( B_0 \), or \( k_4 \), or \( k_5 \), or \( (1-\mu) \) could equal zero but such values would produce trivial solutions. Consequently
\[ \sin (\hat{x}k_2) = 0 \quad \text{and} \quad \sin (\hat{x}k_3) = 0 \]
\[ \text{or} \quad (k_2^2 + \mu k_3^2) = 0 \]

The substitution of \( \phi \) and \( \phi' \) into the boundary condition on \( S_{xy} \) produces additional limitations.
\[ S_{xy}(x, t) = (-k_3^2 + k_4^2) B_0 \sin k_4 x \cos(\pm k_5 h) \sin c_2 \sqrt{k_4^2 + k_2^2} t - 2k_2 k_3 A_0 \sin k_2 x \cos(\pm k_3 h) \sin c_1 \sqrt{k_2^2 + k_3^2} t = 0 \]

Here again trivial solutions would result if \( B_0 \) and \( A_0 \), or \( k_2 \) or \( k_3 \) were made equal to zero. Consequently

\[ \cos(\pm k_3 h) = \cos k_3 h = 0 \quad \text{and} \quad (-k_2^2 + k_4^2) = 0 \]

or \( \cos(\pm k_3 h) = \cos k_3 h = 0 \)

In both of these situations there are two conditions that must be satisfied

\[ \cos k_3 h = 0 \quad \text{and} \quad \sin(\pm k_5 h) = 0 \]

hence

\[ k_3 h = \frac{2n + 1}{2} \pi \quad \text{and} \quad \pm k_5 h = \pm m \pi \quad n = 0, 1, 2, \ldots \]

\[ m = 1, 2, 3, \ldots \]

so

\[ k_3 = \frac{2n + 1}{2h} \pi \quad \text{and} \quad k_5 = \frac{m \pi}{h} \]

Therefore

\[ \sin(\pm k_3 h) = \sin(\pm \frac{2n + 1}{2h} \pi h) = 0 \quad \text{and} \quad \cos k_5 h = \cos \frac{m \pi h}{h} = 0 \]

and as a result

\[ -k_3^2 + \mu k_2^2 = 0 \quad \text{and} \quad -k_2^2 + k_4^2 = 0 \]

Hence

\[ k_2 = \pm \sqrt{\frac{k_3}{\mu}, \quad k_3 = \pm \frac{2n + 1}{2h} \pi, \quad \text{and} \quad k_4 = \pm \frac{m \pi}{h} \}

These values substituted into equations 14 and 15 give
\[ \phi_0(x,y,t) = \sum_{n=0}^{\infty} A_n \cosh \frac{2n + 1}{2} \frac{c_1}{h} \frac{1}{\sqrt{1+k^2}} t \sinh c_1 \frac{2n + 1}{2h} \frac{1-k}{\sqrt{1-k^2}} \]  

(14a)

and

\[ \psi_0(x,y,t) = \sum_{m=1}^{\infty} B_{m0} \sin \frac{m \pi x}{h} \cos \frac{m \pi y}{h} \sin \sqrt{2} \frac{m \pi c_2}{h} t \]  

(15a)

Other possible solutions to the wave equations are

\[ \phi_1(x,y,t) = A_1 \cos k_6 x \cos k_7 y \sin c_1 \sqrt{k_6^2 + k_7^2} t \]  

(16)

and

\[ \psi_1(x,y,t) = B_1 \sin k_8 x \sin k_9 y \sin c_2 \sqrt{k_8^2 + k_9^2} t \]  

(17)

When these functions are substituted into the boundary conditions that have been used before there results

\[ k_6 = \pm \frac{1}{\sqrt{1+k^2}} \; ; \; k_8 = k_9 \; ; \; k_7 = \frac{m \pi}{h} \; ; \; k_9 = \frac{2n + 1}{2h} \pi \]

and

\[ \phi_1(x,y,t) = \sum_{m=1}^{\infty} A_{m1} \cosh \frac{m \pi x}{h} \cos \frac{m \pi y}{h} \sinh c_1 \frac{m \pi}{h} \frac{1-k}{\sqrt{1-k^2}} t \]  

(16a)

\[ \psi_1(x,y,t) = \sum_{n=0}^{\infty} B_{n1} \sin \frac{2n + 1}{2h} \frac{c_1}{\sqrt{1+k^2}} t \sin \frac{2n + 1}{2h} \frac{1-k}{\sqrt{1-k^2}} \]  

(17a)

The addition of equations 14a and 16a, and 15a and 17a produces functions which satisfy the boundary conditions considered. These functions satisfy the wave equations in \( \phi \) and \( \psi \); they produce zero displacement in the x direction along the y-axis; and they give zero shearing stress along
and zero normal stress perpendicular to the top and bottom boundaries of the beam. The use of hyperbolic functions of $y$ in the expressions for $\phi$ and $\psi$ would be undesirable since both sinh ky and cosh ky would have values for $y = \pm h$.

The existence of a resultant normal force due to $S_x$ at the left and right end boundaries of the beam can be checked by an integration:

$$
\int_{-h}^{h} S_x(t, y, t) \, dy = \int_{-h}^{h} \frac{E}{E_0} \left[ \frac{\partial^2 \phi}{\partial x^2} + \mu \frac{\partial^2 \phi}{\partial y^2} + (1-\mu) \frac{\partial^2 \psi}{\partial x \partial y} \right] \, dy
$$

Since the integration is with respect to $y$, it is necessary to consider only those terms involving $y$ so that there results, with the remaining terms carried in a general function $D(t)$,

$$
\int_{-h}^{h} S_x(t, y, t) \, dy = D_1(t) \int_{-h}^{h} \sin \frac{2n+1}{2h} y \, dy + D_2(t) \int_{-h}^{h} \cos \frac{m\pi}{h} y \, dy
$$

$$
+ D_3(t) \int_{-h}^{h} \sin \frac{m\pi}{h} y \, dy + D_4(t) \int_{-h}^{h} \cos \frac{2n+1}{2h} y \, dy
$$

which on integration becomes

$$
D_1(t) \left[ \frac{-2h}{(2n+1)\pi} (\cos \frac{2n+1}{2} - \cos \frac{2n+1}{2}) \right] + D_2(t) \left[ \frac{h}{m\pi} (\sin \frac{m\pi}{h} + \sin \frac{m\pi}{h}) \right]
$$

$$
+ D_3(t) \left[ \frac{-h}{m\pi} (\cos \frac{m\pi}{h} - \cos \frac{m\pi}{h}) \right] + D_4(t) \left[ \frac{2h}{(2n+1)\pi} (\sin \frac{2n+1}{2} - \sin \frac{2n+1}{2}) \right] + \sin \frac{2n+1}{2} \right]
$$

and reduces to

$$
D_4(t) \frac{4h}{(2n+1)\pi} (-1)^n
$$
The function that produces a value for the normal force at the ends of the beam is \( \psi_1 \). The elimination of this contributory factor can be effected by letting \( B_{hl} = 0 \), or \( k_9 = 0 \), or \( k_8 = 0 \). None of these substitutions would change \( \phi \) but \( \psi \) would reduce to equation 15a.

The evaluation of the coefficients is made by the substitution of \( \phi \) and \( \psi \) into the prescribed moment and shear conditions on the ends of the beam. The total vertical shears on the ends are obtained by integration:

\[
V_{x=L} = \int_{-h}^{h} S_{xy}(L, y, t) \, dy = b \int_{-h}^{h} \frac{E}{2(1+\mu)} \left[ \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} + 2 \frac{\partial^2 \phi}{\partial x \partial y} \right] \, dy
\]

Here again integration is with respect to \( y \) so that remaining terms are carried in a general function \( D(t) \)

\[
V_{x=L} = D_5(t) \int_{-h}^{h} \cos \frac{2n+1}{2h} y \, dy + D_6(t) \int_{-h}^{h} \sin \frac{m}{h} y \, dy
\]

which on integration becomes

\[
V_{x=L} = D_5(t) \left[ \frac{2h}{(2n+1)\pi} \right] \left[ \sin \frac{2n+1}{2} \right] + D_6(t) \frac{h}{m} \left[ \cos \frac{m}{h} - \cos \frac{m}{h} \right]
\]

The substitution for \( D_5(t) \) and \( D_6(t) \) produces

\[
V_{x=L} = \sum_{n=0}^{\infty} \frac{(-1)^n E b A o (2n+1) \pi}{(1+\mu) \sqrt{\nu} h} \sinh \frac{2n+1}{2} L \sinh \frac{c_1}{2h} \frac{1-\mu}{\mu} t
\]

(20)

\[
V_{x=-L} = \sum_{n=0}^{\infty} \frac{(-1)^n E b A o (2n+1) \pi}{(1+\mu) \sqrt{\nu} h} \sinh \frac{2n+1}{2} L \sinh \frac{c_1}{2h} \frac{1-\mu}{\mu} t
\]

(20a)
The opposite signs on the two shears on the ends of the beam indicate that they have the same sense along their line of action. The total vertical shear at any cross-section throughout the beam is dependent upon the function \( \phi \) only. Further the values of the shear as indicated by equations 20 and 20a increases with an increase in time and is zero at \( t = 0 \). Hence the applied vertical shear must be increasing with time or of such a value as to permit an approximation over a finite period of time. It is also noted that a vertical shear is developed at every cross-section simultaneously with the introduction of a vertical shear at \( x = \pm L \).

The moments at the ends are also obtained by integration

\[
M_{x = \pm L} = b \int_{-h}^{h} S_x(\pm L, y, t) \, dy = \frac{Eb}{1 - \mu^2} \int_{-h}^{h} \left[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{(1 - \mu)}{2} \frac{\partial^2 \psi}{\partial x \partial y} \right] \, dy
\]

All terms not containing \( y \) are again lumped into a convenient function \( D(t) \) and

\[
M_{x = \pm L} = D_7(t) \int_{-h}^{h} y \sin \frac{2n + 1}{2h} y \, dy + D_8(t) \int_{-h}^{h} y \cos \frac{m \pi}{n} y \, dy + D_9(t) \int_{-h}^{h} y \sin \frac{m \pi}{n} y \, dy
\]

which on integration becomes
The consolidation of the above expressions and the substitution for $D_7(t)$ and $D_9(t)$ produce

$$M_{x=\pm L} = D_7(t) \left[ \frac{(2h)^2}{(2n+1)2\eta^2} \left( 2\sin^2 \frac{n+1}{2} - \frac{2n+1}{2} \cos \frac{2n+1}{2} \cos \frac{n+1}{2} \right) \right]$$

$$+ D_8(t) \left[ \frac{h^2}{(m\eta)^2} \left( \cos m\eta + m\eta \sin m\eta - \cos m\eta - m\eta \sin m\eta \right) \right]$$

$$+ D_9(t) \left[ \frac{h^2}{(m\eta)^2} \left( \sin m\eta - m\eta \cos m\eta + \sin m\eta - m\eta \cos m\eta \right) \right]$$

The like signs on the moments indicate opposite senses of rotation on either end of the beam. The moment is dependent upon both functions $\phi$ and $\psi$. It is not limited to be only increasing with an increase in time but conceivably could oscillate between positive and negative values due to the existence of the trigonometric sine in the second term of the expression for the moment. The moment, like the shear, is developed at every cross-section simultaneously with the introduction of a moment at $x = \pm L$.

The second term of $\phi$, i.e. $\phi_1$, does not enter the expressions on any cross-section for the resultant moment, the vertical shear, the resultant longitudinal normal
force, or the normal stress in the y-direction, but it does affect the shearing stress, normal stress in the x-direction, x and y displacements throughout the depth of the beam. The use of the term requires the end boundaries to be each subjected to equal and opposite moments and shears: one moment and shear acting on the upper half of the beam, the other acting on the lower half of the beam. Consequently \( \phi \) should be reduced to only the \( \phi \) term to satisfy the given boundary conditions.

The boundary conditions specified are fulfilled with the use of the series for \( \phi \) and \( \psi \). These expressions

\[
\phi(x,y,t) = \sum_{n=0}^{\infty} A_n \cosh \frac{2n+1}{2} \frac{x \sin \frac{2n+1}{2h} y \sinh c_1 \frac{2n+1}{2h} \sqrt{1-x^2}}{h^3} \ t
\]

\[
\psi(x,y,t) = \sum_{m=1}^{\infty} B_m \sin \frac{m \pi x}{h} \cos \frac{m \pi y}{h} \sin c_2 \sqrt{2} \frac{m \pi}{h} \ t
\]

for \( \phi \) and \( \psi \) require that the moment, shear, all stresses and all displacements be zero when \( t = 0 \) and simultaneously acquire magnitudes for \( t > 0 \). It is evident too that the stresses, strains, and displacements within the beam can be predicted only over the time period for which it is possible to match the applied shear and moment to the equations 20, 20a, and 21. The various stresses and displacements can be expressed in terms of the expressions for \( \phi \) and \( \psi \).
\[ S_x(x,y,t) = \sum_{n=0}^{\infty} \frac{E(2n+1)^2}{4n^2h^2} A_n \cosh \frac{n+1}{2\sqrt{\mu}} x \sin \frac{2n+1}{2h} y \]
\[ \sinh c_1 \frac{2n+1}{2h} \sqrt{1-\frac{\mu}{\lambda}} t - \sum_{m=1}^{\infty} \frac{E_m^2}{(1+\mu)h^2} \cos \frac{m\mu}{h} \sin c_2 \sqrt{\frac{m\mu}{h}} t \] (24)

\[ S_y(x,y,t) = \sum_{m=1}^{\infty} \frac{E_m^2}{(1+\mu)h^2} \cos \frac{m\mu}{h} \sin c_2 \sqrt{\frac{m\mu}{h}} \sin c_2 \sqrt{\frac{m\mu}{h}} t \] (25)

\[ S_{xy}(x,y,t) = \sum_{n=0}^{\infty} \frac{E(2n+1)^2}{4n^2h^2} A_n \sinh \frac{2n+1}{2\sqrt{\mu}} x \cos \frac{2n+1}{2h} y \]
\[ \sinh c_1 \frac{2n+1}{2h} \sqrt{1-\frac{\mu}{\lambda}} t \] (26)

\[ v(x,y,t) = \sum_{n=0}^{\infty} \frac{2n+1}{2h} A_n \cosh \frac{2n+1}{2\sqrt{\mu}} x \cos \frac{2n+1}{2h} y \sinh c_1 \frac{2n+1}{2h} \sqrt{1-\frac{\mu}{\lambda}} t \]
\[ + \sum_{m=1}^{\infty} \frac{m\mu}{h} B_m \cos \frac{m\mu}{h} \sin \sqrt{\frac{m\mu}{h}} \sin c_2 \sqrt{\frac{m\mu}{h}} t \] (27)

\[ u(x,y,t) = \sum_{n=0}^{\infty} \frac{(2n+1)^2}{2n^2h^2} A_n \sinh \frac{2n+1}{2\sqrt{\mu}} x \sin \frac{2n+1}{2h} y \sinh c_1 \frac{2n+1}{2h} \sqrt{1-\frac{\mu}{\lambda}} t \]
\[ - \sum_{m=1}^{\infty} \frac{m\mu}{h} B_m \sin \frac{m\mu}{h} \cos \frac{m\mu}{h} \sin c_2 \sqrt{\frac{m\mu}{h}} t \] (28)

Although the function \( \phi \) is associated with waves of dilatation the shearing stress is seen to be entirely dependent upon \( \phi \) whereas the normal stress in the \( y \)-direction is dependent only upon \( \psi \) which in turn is associated with the rotation. The normal stress in the \( y \)-direction can have
zero values with respect to all variables. It is zero for $y = \pm h$ and $y = 0$. If the boundary conditions are such that only one term of the series, say $m = 1$, is required then $S_y$ is also zero when $x = \frac{h}{2}, \frac{3h}{2}, \frac{5h}{2}, \ldots$ and when $t = \frac{h}{c_2^2 \tau}$.

$$\frac{2h}{c_2^2 \tau} \frac{3h}{c_2^2 \tau} \ldots$$

The normal stress in the $x$-direction at a given cross-section has its maximum value at the top and bottom of the beam and a minimum value (zero) at the middle surface of the beam. At a given elevation in the beam the normal stress in the $x$-direction is very much dependent upon the magnitudes and signs of the constants $A_n$ and $B_m$ and the elapsed time. For a case in which the boundary conditions could be met with $n = 0$, $m = 1$ and $B_1$, say, $10^5$ times $A_0$, then at a given elevation and very small values of time the second term of $S_x$ could be large enough to produce a change in the sense of $S_x$ at a given elevation in the beam.

The shearing stress $S_{xy}$ is, of course, zero on the top and bottom of the beam and reaches its maximum value for any given time and cross-section when $y = 0$.

The displacement $u$ in the $x$-direction is zero for $x = 0$ but in general differs from zero at most other points in the beam. For $y = 0$ it is a function of the expression involving $\psi$ only and if only one term again is required then $u$ is also zero for the same values of $x$ and $t$ that made
30

\[ S_y = 0. \] The expression for \( u \) is such that the displacement is not the same in the upper portion as the lower portion of the beam.

The displacement \( v \) in the \( y \) direction is the same at the top and bottom of the beam and is a function there of the expression for \( \psi \) only. If the coefficients \( A_n \) and \( B_m \) have the same sign then the maximum displacement \( v \) at a given cross-section occurs midway between the upper and lower surface of the beam at \( y = 0 \).

2. Transient decreasing moment and shear on ends of beam

Some of the possible combinations of sine, cosine, hyperbolic sine, and hyperbolic cosine of \( x \) and \( y \) have been considered in order to obtain the values of \( \phi \) and \( \psi \) in the previous section. The assumption was made that no disturbance exists at \( t = 0 \). It is a possibility that the shear and moment exist at \( t = 0 \) and decrease with increasing time. A solution of the wave equation that involves \( e \) raised to a negative power of \( t \) can take care of this contingency insofar as \( \phi \) is concerned; for the function \( \psi \) a shift from the \( \sin t \) to \( \cos t \) provides for the existence of a disturbance at \( t = 0 \). Hence take

\[
\phi_3 = \sum_{n=0}^{\infty} A_n \cosh\left(\frac{2n+1}{\mu \sqrt{\frac{\pi}{2}}} x\right) \left(\sin\frac{2n+1}{2n} y\right) e^{-\sigma_1 \frac{2n+1}{2n} \sqrt{\frac{1-\mu}{\nu}} t} \tag{29}
\]
These values of $\psi_3$ and $\psi_3$ again provide for zero shearing stress along and zero normal stress perpendicular to the top and bottom of the beam, zero displacement in the x-direction for $x = 0$, and zero resultant force perpendicular to the two ends. Since the expressions for $\phi_3$ and $\psi_3$ are the same as $\phi$ and $\psi$ of the previous section except for the change produced by replacing $\sinh \frac{2n+1}{2n} t$ by $e^{-\frac{2n+1}{2n} \sqrt{1-\mu^2} t}$ and $\sin \frac{2m}{n} t$ by $\cos \frac{2m}{n} t$, the expressions for the moment, vertical shear, stresses, strains, and displacements will be the same as before with the indicated replacement on $t$. The remarks made at the end of section 1 involving the stresses and displacements are the same insofar as $x$ and $y$ are concerned. However, $t$ must be equal to $\frac{h}{2c_2}$, $\frac{3h}{2c_2}$, $\frac{5h}{2c_2}$, etc. if zero values are to be obtained for the normal stress in the y-direction and the displacement $u$ in the x-direction under the condition that only one term of the series be required to match the applied moment.

The influence of the sinh $t$ in the first section can soon become a predominant factor in the expression for the applied moment or shear and can limit the prediction of the
stresses to an extremely narrow time period. However, the situation that develops might well include the combination of the results of section 1 and section 2. If need be, the results of the first could satisfy the conditions while the moment is building up to its maximum value and the results of the second could be used while the disturbance decays to zero.

3. Transient force applied at midspan of simple beam

a. Residual end moments. Since the development of the expressions for stress and displacement in terms of \( \phi \) and \( \psi \) is independent of the boundary conditions it would seem possible to obtain a solution to a simply supported beam subjected to an impact load or displacement at midspan. The beam of Fig. 3 is considered.

\[
\begin{align*}
F(x,t) & \\
V(t)_{x=-L} & \\
L & \\
v(t)_{x=L} & \\
\end{align*}
\]

**Fig. 3** Rectangular Beam with Impact Force at Midspan

The boundary conditions for the beam are
\[
\begin{align*}
u(0,y,t) &= 0 \\
v(\pm L,h,t) &= 0
\end{align*}
\]
\[ S_{xy}(x, \pm h, t) \]  
\[ S_x(\pm L, y, t) = 0 \]  
\[ S_y(x, \pm h, t) = 0 \]  
\[ S_y(x, -h, t) = 0 \quad x \leq -\Delta; \quad x \geq \Delta; \quad t > \delta \]  
\[ S_y(x, -h, t) = \frac{F(x, t)}{2\Delta b} \quad -\Delta \leq x \leq \Delta; \quad 0 < t < \delta \]  
\[ V_{x=\pm L} = \int_{-h}^{h} S_{xy}(\pm L, y, t) \, dy \]

\[ \phi \]  and  \[ \psi \]  are selected as before

\[ \phi(x, y, t) = A \cos k_2x \sin k_3y \sin c_1\sqrt{k_2^2 + k_3^2} t \]

\[ \psi(x, y, t) = B \sin k_4x \cos k_5y \sin c_2\sqrt{k_4^2 + k_5^2} t \]

Here again \( \phi \) is a function of \( \cos k_2x \) and \( \psi \) of \( \sin k_3x \) so that \( u(0, y, t) = 0 \). The substitution of \( \phi \) and \( \psi \) into the boundary condition on \( S_{xy} \) gives

\[ S_{xy}(x, \pm h, t) = (-k_2^2 + k_4^2) B \sin k_4x \cos k_5h \sin c_2\sqrt{k_4^2 + k_5^2} t 
-2 k_2 k_3 A \sin k_2x \cos k_3h \sin c_1\sqrt{k_2^2 + k_3^2} t = 0 \]

To satisfy this condition \( B \) and \( A \), or \( k_2 \) or \( k_3 \) could equal zero but would produce a trivial solution. Consequently

\[ (-k_2^2 + k_4^2) = 0 \quad \text{and} \quad \cos k_3h = 0 \]

or \( \cos k_5h = 0 \) for which \( k_3 = \frac{2n+1}{2h}; \ n = 0, 1, 2... \)

The substitution of \( \phi \) and \( \psi \) into \( S_x(\pm L, y, t) \) provide additional limitations.
Here it appears that
\[ k_2^2 + \mu k_3^2 = 0 \]
and \( \cos k_4 L = 0 \)
or \( \cos k_2 L = 0 \)
for which \( k_4 = \frac{2m+1}{2L} \), \( m = 0, 1, 2 \ldots \)
satisfies the boundary condition but does not produce a
definite value for \( k_5 \) or \( k_2 \). If the values are substituted
into \( S_y(x, h, t) = 0 \) there results
\begin{align*}
S_y(x, h, t) &= (-k_3^2 - \mu k_2^2) A \cos k_2 x \sin k_3 h \sin c_1 \sqrt{k_2^2 + k_3^2} t \\
&\quad - (1 - \mu) k_4 k_3 B \cos k_4 x \sin k_5 h \sin c_2 \sqrt{k_4^2 + k_5^2} t = 0
\end{align*}
Since
\[ k_3 = \frac{2m+1}{2h}, \quad \sin k_3 h \neq 0 \]
Therefore
\[ k_3^2 + \mu k_2^2 = 0 \]
and as a result \( k_2 = \frac{1}{\sqrt{\mu}} k_3 \)
In the second portion of the equation \( \sin k_5 h = 0 \) so that
\[ k_5 = \frac{p \pi}{h}, \quad (p = 1, 2, 3, \ldots), \]
but \( k_4 = \frac{2m+1}{2L} \), hence \( k_5 \nless k_4 \)
which, to satisfy \( S_{xy}(x, t, h, t) = 0 \), requires \( \cos k_5 h = 0 \)
but if \( k_5 = \frac{p \pi}{h} \) this condition cannot be met. The chosen
functions of \( \phi \) and \( \psi \) are insufficient to satisfy the
boundary conditions specified.
In order to meet the condition on $u(0,y,t)$ the adjustments of $\phi$ and $\psi$ are made on the $y$ variable.

$$\phi(x,y,t) = A \cos k_2 x \sin k_3 (y-y_1) \sin c_1 \sqrt{k_2^2 + k_3^2} t \quad (33)$$

$$\psi(x,y,t) = B \sin k_4 x \cos k_5 (y-y_2) \sin c_2 \sqrt{k_4^2 + k_5^2} t \quad (34)$$

Substitution of $\phi$ and $\psi$ into the boundary conditions produces for $S_{xy}$

$$S_{xy}(x,x_0,t) = (-k_2^2 + k_4^2) B \sin k_4 x \cos k_5 (x-h-y_2)$$

$$\sin c_2 \sqrt{k_4^2 + k_5^2} t - 2 k_2 k_3 A \sin k_2 x \cos k_3 (x-h-y_1)$$

$$\sin c_1 \sqrt{k_2^2 + k_3^2} t = 0$$

which requires

$$-k_2^2 + k_4^2 = 0$$

or

$$\cos k_5 (x-h-y_2) = 0$$

for which

$$k_3 (x-h-y_1) = \frac{2n+1}{2} \pi; \quad n = 0,1,2,3,...$$

For the normal stress perpendicular to the lower surface of the beam

$$S_y(x,x_0,t) = (-k_2^2 - k_3^2) A \cos k_2 x \sin k_3 (h-y_1) \sin c_1 \sqrt{k_2^2 + k_3^2} t$$

$$+ (1-\mu) k_4 k_5 B \cos k_4 x \sin k_5 (h-y_2) \sin c_2 \sqrt{k_4^2 + k_5^2} t = 0$$

from which
\[
(k_3^2 + \mu k_2^2) = 0
\]

or

\[\sin k_3(h-y_1) = 0\]

and \(\sin k_5(h-y_2) = 0\)

hence

\[k_5(h-y_2) = p\pi \quad p = 1, 2, 3, \ldots\]

or

\[y_2 = h \text{ which satisfies } \sin k_5(h-y_2) = 0 \text{ for any } k_5\]

For the displacements of the supports (or the ends of the beam)

\[v(tL, h, t) = k_3A \cos k_2L \cos k_3(h-y_1) \sin c_1\sqrt{k_2^2 + k_3^2} \quad t\]

\[+ k_4B \cos k_4L \cos k_5(h-y_2) \sin c_2\sqrt{k_4^2 + k_5^2} \quad t = 0\]

which requires

\[\cos k_2L = 0\]

or

\[\cos k_3(h-y_1) = 0\]

and \(\cos k_4L = 0\)

or

\[\cos k_5(h-y_2) = 0\]

For the normal stress perpendicular to the ends of the beam

\[S_x(tL, y, t) = (-k_2^2 - \mu k_3^2)A \cos k_2L \sin k_3(y-y_1) \sin c_1\sqrt{k_2^2 + k_3^2} \quad t\]

\[-(1-\mu)k_4k_5B \cos k_4L \sin k_5(y-y_2) \sin c_2\sqrt{k_4^2 + k_5^2} \quad t = 0\]

so that
\[ k_2^2 + \mu k_3^2 = 0 \]

or

\[ \cos k_2L = 0 \]

and

\[ \cos k_4L = 0 \]

Since \( \cos k_3(h-y_1) = 0 \), \( k_3 = \frac{2n+1}{2h} \pi \) and \( y_1 = 0 \),

\( n = 0, 1, 2, 3, \ldots \), hence \( \sin k_3(h-y_1) = 0 \) and therefore

\( k_3^2 + \mu k_2^2 = 0 \). With \( \sin k_5(h-y_2) = 0 \) and \( \cos k_5(h-y_2) = 0 \)

then \( -k_2^2 + k_4^2 = 0 \). Hence, to get \( v(\pm L, h, t) = 0 \), \( \cos k_4L = 0 \)

so that \( k_4 = \frac{2m+1}{2L} \pi \), \( m = 0, 1, 2, 3, \ldots \).

With these combinations the \( k \)'s and \( y_1 \) and \( y_2 \) are determined.

\[ k_3 = \frac{2n+1}{2h} \pi; \quad k_2 = \frac{1}{\sqrt{\mu}}; \quad k_3 = \frac{2n+1}{2\sqrt{\mu h}} \pi \]

\[ k_4 = k_5 = \frac{2m+1}{2L} \pi; \quad y_2 = h \text{ or } h - \frac{2Lp}{2m+1} \]

There is no advantage in using the more complicated value of \( y_2 \), hence \( y_2 = h \). With the above relationships it is not possible to have either \( \cos k_2L = 0 \) or \( k_2^2 + \mu k_3^2 = 0 \)

and consequently \( S_x(\pm L, y, t) = 0 \). Therefore the expressions for \( \phi \) and \( \psi \) do not satisfy exactly the boundary conditions prescribed.

\[ \phi(x, y, t) = \sum_{n=0}^{\infty} A_n \cosh\frac{2n+1}{2h}x \sin\frac{2n+1}{2h}y \sinh c_1 \frac{2n+1}{2h} \sqrt{1-\mu} t \]

\[ \psi(x, y, t) = \sum_{m=0}^{\infty} B_m \sin\frac{2m+1}{2L}x \cos\frac{2m+1}{2L}y \sin c_2 \sqrt{2m+1} \frac{2m+1}{2L} t \]
The existence of a resultant force in the x-direction is investigated through the following integration.

\[ F_x(x,t) = \int_{-h}^{h} bS_x(x,y,t) \, dy \]

\[ = D_1(x,t) \int_{-h}^{h} \sin \frac{2n+1}{2h} \pi y \, dy \]

\[ + D_2(x,t) \int_{-h}^{h} \sin \frac{2m+1}{2L} \pi (y-h) \, dy \]

in which \( D_1(x,t) \) and \( D_2(x,t) \) are the functions of \( x \) and \( t \) contained in \( S_x(x,y,t) \) and are not pertinent to the integration. The integration produces

\[ F_x(x,t) = 0 + D_2(x,t) \left[ \frac{2L}{(2m+1)^2} \left( \frac{2m+1}{L} \pi h \right) \right] \]

\[ = \frac{B_n bE(2m+1)}{(1+\nu)(2L)} (\cos \frac{2m+1}{2L} \pi x) (\sin c_2 \sqrt{\frac{2m+1}{2L} \pi t}) (1- \cos \frac{2m+1}{L} \pi h) \]

which at \( x = \pm L \) is zero. Consequently there is no resultant force in the x-direction existing on the ends of the beam.

The resultant moment on any cross-section is also obtained by integration

\[ M(x,t) = \int_{-h}^{h} bS_x(x,y,t) \, dy = D_1(x,t) \int_{-h}^{h} y \sin \frac{2n+1}{2h} \pi y \, dy \]

\[ -D_2(x,t) \int_{-h}^{h} y \sin \frac{2m+1}{2L} \pi (y-h) \, dy \]

which on integration produces
\[
M(x,t) = \sum_{n=0}^{\infty} (-1)^n \frac{2A_n b E}{\mu} \cosh \frac{2n+1}{2\sqrt{\mu} h} x \sinh \frac{c_1}{2h} \sqrt{\frac{1-\mu}{\mu}} t \\
+ \sum_{m=0}^{\infty} \left( \frac{2m+1}{2L} \sinh \frac{2m+1}{2L} h - \frac{2m+1}{2L} \sinh \frac{2m+1}{2L} h \right)
\]

(35)

At \( x = \pm L \) the above equation reduces to

\[
M(\pm L, t) = \sum_{n=0}^{\infty} (-1)^n \frac{2A_n b E}{\mu} \cosh \frac{2n+1}{2\sqrt{\mu} h} \pm L \sinh \frac{c_1}{2h} \sqrt{\frac{1-\mu}{\mu}} t
\]

The use of the selected expressions for \( \phi \) and \( \psi \) produces a resultant moment on each of the ends of the beam which is not provided for in the boundary conditions.

The resultant vertical shear on any cross-section is

\[
V(x,t) = \int_{-h}^{h} b S_{xy}(x,y,t) \, dy = D_3(x,t) \int_{-h}^{h} \cos \frac{2n+1}{2h} y \, dy
\]

in which \( D_3(x,t) \) is the function of \( x \) and \( t \) contained in \( S_{xy}(x,y,t) \) and is not pertinent to the integration. The integration produces

\[
V(x,t) = \sum_{n=0}^{\infty} (-1)^n b E A_n (2n+1) \sinh \frac{2n+1}{2\sqrt{\mu} h} x \sinh \frac{c_1}{2h} \sqrt{\frac{1-\mu}{\mu}} t
\]

(36)

The vertical shear at \( x = \pm L \) can be used to represent the reactions at the ends of the beam. The expressions for the stresses and displacements for any point throughout the beam are given in the following equations.
\[
S_{xy}(x,y,t) = \sum_{n=0}^{\infty} \frac{E A_n (2n+1)^2 \sinh \frac{2n+1}{2h} x \cos \frac{2n+1}{2h} y}{(1+\mu)(2h)^2} \sinh \frac{2n+1}{2h} t
\]

\[
S_{y}(x,y,t) = \sum_{m=0}^{\infty} \frac{E B_m (2m+1)^2 \sinh \frac{2m+1}{2L} x \sin \frac{2m+1}{2L} y}{\mu (2h)^2} \sin \frac{2m+1}{2L} t
\]

\[
S_{x}(x,y,t) = \sum_{n=0}^{\infty} \frac{E A_n (2n+1)^2 \cosh \frac{2n+1}{2h} x \sin \frac{2n+1}{2h} y}{\mu (2h)^2} \sinh \frac{2n+1}{2h} t
\]

\[
u(x,y,t) = \sum_{n=0}^{\infty} \frac{A_n (2n+1)^2 \sinh \frac{2n+1}{2h} x \sin \frac{2n+1}{2h} y}{\mu (2h)^2} \sinh \frac{2n+1}{2h} t
\]

\[
v(x,y,t) = \sum_{n=0}^{\infty} \frac{A_n (2n+1)^2 \cosh \frac{2n+1}{2h} x \cos \frac{2n+1}{2h} y}{\mu (2h)^2} \sinh \frac{2n+1}{2h} t
\]

The boundary condition concerning the distribution of the normal stress over the upper surface of the beam due to the application of the load has not been used. If it is assumed that the distribution of the vertical normal stress is represented by Fig. 4, where 4a indicates the
variation in the x-direction, and 4b indicates the time of application, the stress $S_y$ can be represented in the form

$$\frac{F}{2Ab} = S$$

**Fig. 4a** Assumed Stress Distribution Along Top of Beam

**Fig. 4b** Assumed Stress Duration with Respect to Time

of double Fourier series in $x$ and $t$.

$$S_y(x,-h,t) = \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} C_{pq} \cos \frac{qnx}{L} \sin p\omega t$$

$C_{pq}$ can be evaluated in terms of the assumed distributions by the usual procedure for the determination of the general coefficient in Fourier series.

$$C_{pq} = (1-\cos p\omega \delta) \frac{2S}{n^2 q p} \sin \frac{q^{n\lambda}}{L}$$

and
Hence for a given value of the applied force the distribution of normal stress along the upper surface of the beam is given. It was indicated previously that for the chosen expressions for \( \phi \) and \( \psi \), moments exist on the ends of the beam, consequently the boundary conditions are not satisfied exactly.

The shearing stress \( S_{xy} \) is again a function of the expression for \( \phi \) only. For a condition in which only the first term of the series was required its maximum value occurs when \( y = 0 \) and its minimum (zero) occurs at the upper and lower surface of the beam for any term of the series.

The maximum value of the normal stress \( S_y \) on a given cross-section depends on the relation between \( h \) and \( L \) but becomes zero at the bottom surface.

The normal stress \( S_x \) on a given cross-section is also dependent on the ratio of \( h \) to \( L \) and is not necessarily zero at the middle surface of the beam.

The displacement \( u \) is zero for \( x = 0 \) but is also dependent on the ratio of \( h \) to \( L \) throughout the rest of the beam.

The displacement \( v \), too, is dependent on the \( h \) to \( L \) ratio but at \( x = \frac{1}{2}L \), and \( y = \pm h \) it is zero. The displacement

\[
S_y(x,-h,t) = \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} \frac{2S}{(1 - \cos p \omega)} \sin \frac{q\pi \Delta}{L} \cos \frac{\pi x}{L} \sin p \omega t
\]
of the upper and lower surfaces in the y-direction is not necessarily the same except at the points indicated above.

b. **Zero end moments.** The moments on the ends of the beam can be reduced to zero if the boundary conditions of the previous problem are changed slightly. The normal stress \( S_x(x, y, t) \) is required to be zero and the normal stress \( S_y(x, h, t) \) is no longer required to be zero. These requirements can be met if

\[
\mu k_3^2 + k_2^2 = 0 \quad \text{but} \quad \mu k_2^2 + k_3^2 \neq 0
\]

The use of the above change in the boundary conditions produces a noticeable effect on the expressions for \( \phi \) and \( \psi \).

\[
\phi(x, y, t) = \sum_{n=0}^{\infty} A_n \cosh\left[\frac{\mu a}{2h} x \right] \sin \left[ \frac{2n+1}{2h} y \right] \sin \left[ \frac{c_1}{2h} n \sqrt{1 - \mu^2} t \right]
\]

\[
(x, y, t) = \sum_{m=0}^{\infty} B_m \sin \left[ \frac{2m+1}{2L} x \right] \sin \left[ \frac{2m+1}{2L} c_2 \sqrt{\mu^2 - 1} (y-h) \right] \sin \left[ \frac{c_2}{2L n} \sqrt{\mu^2 - 1} t \right]
\]

The substitution of these values for \( \phi \) and \( \psi \) into the various expressions for the stresses and displacements gives
\[
S_y(x,y,t) = \sum_{n=0}^{\infty} \frac{EB_m \left[ \frac{2n+1}{2L} \right]^2}{1+\mu} \frac{\sinh \sqrt{\mu}}{2h} \left( \frac{2n+1}{2h} \right) \sin c_1 \frac{2n+1}{2h} \sqrt{1-\mu^2} t \\
+ \sum_{m=0}^{\infty} \frac{EB_m \left[ \frac{2m+1}{2L} \right]^2}{1+\mu} \cos \frac{2m+1}{2L} x \cos \frac{2m+1}{2L} (y-h) \sin c_2 \frac{2m+1}{2L} \sqrt{2} t
\]

\[
S_x(x,y,t) = \sum_{n=0}^{\infty} \frac{EB_m \left[ \frac{2m+1}{2L} \right]^2}{1+\mu} \cos \frac{2m+1}{2L} x \cos \frac{2m+1}{2L} (y-h) \sin c_2 \frac{2m+1}{2L} \sqrt{2} t
\]

\[
S_{xy}(x,y,t) = \sum_{n=0}^{\infty} \frac{EB_m \left[ \frac{2n+1}{2L} \right]^2}{1+\mu} \sinh \sqrt{\mu} \left( \frac{2n+1}{2h} \right) \cos \frac{2n+1}{2h} y \sin c_1 \frac{2n+1}{2h} \sqrt{1-\mu^2} t
\]

\[
u(x,y,t) = \sum_{n=0}^{\infty} \frac{EB_m \left[ \frac{2m+1}{2L} \right]^2}{1+\mu} \sin \frac{2m+1}{2L} x \cos \frac{2m+1}{2L} (y-h) \sin c_2 \frac{2m+1}{2L} \sqrt{2} t
\]

\[
v(x,y,t) = \sum_{n=0}^{\infty} \frac{EB_m \left[ \frac{2n+1}{2L} \right]^2}{1+\mu} \cos \frac{2n+1}{2h} x \cos \frac{2n+1}{2h} y \sin c_1 \frac{2n+1}{2h} \sqrt{1-\mu^2} t
\]

The normal stress \( S_y \) now depends on both the expression for \( \phi \) and for \( \psi \). Neither the upper nor the lower surface is free from stress and the applied disturbances must of necessity be of a form that can at least be approximated by these series.
The normal stress \( S_x \) depends now only on the expression for \( \psi \). Its maximum value on a given cross-section occurs at \( y = h \). It is interesting to note that if the cos
\[
\frac{2m+1}{2L} \cos(y-h)
\]
were used in \( \psi \) instead of the sin term the normal stress \( S_x \) at \( y = h \) would be equal to zero regardless of the value of \( x \) or \( t \). This situation would not be expected to exist since both the displacements in the \( x \) and \( y \)-directions occur even if the cos term were used.

The displacement \( u \) is zero for \( x = 0 \) as prescribed but does not necessarily equal zero at the middle surface of the beam.

The displacement \( v \) is zero for \( x = \frac{1}{2}L \) and \( y = \frac{1}{2}h \). It has different values at corresponding points along the upper and lower surface of the beam.

Since the normal stress \( S_x \) is zero at both ends of the beam, there is neither a resultant moment nor a resultant normal force acting on the ends.

c. Use of Fourier and Laplace transformations. In the previous situations the solutions indicate an existence of stress and displacement throughout the beam simultaneously with the application of the disturbance. The actual condition must be one of transmittal of the effect of the initial shock from the point or surface of occurrence through the material of the beam to the other boundaries. It is not until after the effect of the initial shock has been
registered throughout the beam that the beam can respond as a unit. In this section then the use of the Fourier and Laplace transforms to obtain a solution to the stress distribution during this transient period is discussed.

The wave equations, 9 and 10, are transformed on the variable t through the Laplace transformation as indicated on page 6 of Churchill (7) and on x through the Fourier transformation as indicated on page 19 of Sneddon (36). If w is the Fourier operator then the transform of \( \phi(x,y,t) \) is

\[
\bar{\phi}(w,y,t) = \int_{-\infty}^{\infty} \phi(x,y,t) e^{iwx} \, dx
\]

and if s is the Laplace operator then the transform of \( \bar{\phi}(w,y,t) \) is

\[
\bar{\phi}(w,y,s) = \int_{0}^{\infty} \bar{\phi}(w,y,t) e^{-st} \, dt
\]

and similarly

\[
\bar{\psi}(w,y,s) = \int_{0}^{\infty} \int_{-\infty}^{\infty} \psi(x,y,t) e^{iwx} e^{-st} \, dx \, dt
\]

The transformations of the wave equations produce

\[
\frac{d^2 \bar{\phi}(w,y,s)}{dy^2} - (w^2 + s_1^2) \bar{\phi}(w,y,s) = 0 \quad (50)
\]

\[
\frac{d^2 \bar{\psi}(w,y,s)}{dy^2} - (w^2 + s_2^2) \bar{\psi}(w,y,s) = 0 \quad (51)
\]

The necessary assumptions involved in the above equations are that \( \phi(x,y,t), \frac{\partial \phi(x,y,t)}{\partial x}, \psi(x,y,t) \) and \( \frac{\partial \psi(x,y,t)}{\partial x} \)
are zero when \( x \) approaches \( t = \infty \) and that the member is quiescent up to \( t = 0 \).

A beam with an impact force applied at midspan is again considered. In this case, however, the \( x \)-axis is contained in the upper surface of the member and the origin exists directly under the load. The boundary conditions of immediate concern are those existing at the top of the beam where it is assumed that the impact is applied in such a manner that no shearing stress is developed, hence

\[
S_{xy}(x,0,t) = 0
\]

The impact is represented in a manner indicated by Dengler and Goland (11) in which two Dirac functions are used. The stress on the upper surface is given by

\[
S_y(x,0,t) = \begin{cases} 
0; & t < 0, \ t > \delta, \ |x| > \Delta \\
\frac{F}{6\Delta^2}; & 0 < t < \delta, \ -\Delta < x < \Delta
\end{cases}
\]

so that the impulse of the applied load is

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_y(x,0,t) \, dx \, dt = \frac{F}{b}
\]

and the Laplace and Fourier transformations can be found as indicated on page 35 of Sneddon (36). The effect of the boundary conditions at the bottom of the beam are not felt until the disturbance has had time to reach the bottom of the beam so that initially the solutions to equations
50 and 51 are taken as

\[ \varphi(w, y, s) = A \exp(-\sqrt{w^2 + s_1^2} y) \]  
\[ \psi(w, y, s) = B \exp(-\sqrt{w^2 + s_2^2} y) \]  

in which \( \exp(a) = e^a \).

The solution for the stresses from these equations produces values for the stresses along the boundary \( y = 2h \). Some of these stresses should be zero, consequently equal and opposite stresses to those indicated should be assumed to be applied and their effect superimposed on the initial effect in time and position.

To evaluate \( A \) and \( B \) the boundary conditions are transformed. The transformation of \( S_{xy}(x, 0, t) \) produces

\[ 0 = \frac{d^2 \psi(w, 0, s)}{dy^2} + w^2 \varphi(w, 0, s) - 2iw \frac{d\varphi(w, 0, s)}{dy} \]

which with the substitution of equations 52 and 53 becomes

\[ 0 = (2w^2 + s_2^2) B + 2iw\sqrt{w^2 + s_1^2} A \]  

The transformation of \( S_y(x, 0, t) \) produces

\[ \left(1 - \mu^2\right) \frac{F}{E_b} = \frac{d^2 \varphi(w, 0, s)}{dy^2} - \mu w^2 \varphi(w, 0, s) + (1 - \mu) \frac{w d\varphi(w, 0, s)}{dy} \]

which become when equations 52 and 53 are substituted

\[ \left(1 - \mu^2\right) \frac{F}{E_b} = (w^2 - \mu w^2 + s_1^2) A - i(1 - \mu) w\sqrt{w^2 + s_2^2} B \]
Equations 54 and 55 are solved for $A$ and $B$.

\[
A = \frac{(1-\mu^2) F (2w^2 + s_2^2)}{E_b \left[ (2w^2 + s_2^2)(w^2 - \mu w^2 + s_1^2) - 2(1-\mu^2)\frac{w^2}{w^2 + s_2^2} \right] - \frac{s_1}{w^2 + s_1^2}}
\]

and

\[
B = \frac{-2(1-\mu^2) F \frac{w}{w^2 + s_1^2}}{E_b \left[ (2w^2 + s_2^2)(w^2 - \mu w^2 + s_1^2) - 2(1-\mu^2)\frac{w^2}{w^2 + s_2^2} \right] - \frac{s_1}{w^2 + s_1^2}}
\]

Expressions for $\phi(x,y,t)$ and $\psi(x,y,t)$ are obtained by performing the inverse transformations of equations 52 and 53. The various stresses and displacements can be expressed in terms of the transformed functions, for example,

\[
S_y(x,y,t) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{0}^{1} A(w^2 - \mu w^2 + s_1^2) \exp(-y\frac{w^2 + s_1^2}{w^2 + s_1^2}) \exp(-iB(1-\mu)\frac{w}{w^2 + s_2^2} \exp(-y\frac{w^2 + s_2^2}{w^2 + s_2^2}) e^{-iwx} dw
\]

The expressions to be transformed are not available in any of the common tables of Laplace or Fourier transforms. It is not the intention here to perform the complete inverse transformations but rather to discuss the form of the functions that will be produced. $\phi(x,y,t)$ is obtained by performing the inverse transformations and the inverse Laplace transform is considered first:
\[
\int_{s_1}^{s_2} \phi(w, y, s) \, ds = \frac{(1-\mu^2)}{E \beta}
\]

\[
\int_{s_1}^{s_2} \left\{ \left( \frac{(w^2 - \mu w^2 + s_1^2) \exp \left( -\sqrt{w^2 + s_1^2} \right)}{(w^2 - \mu w^2 + s_1^2)^2 - \sqrt{(1-\mu)^2 w^2}} \right) \right\} \, ds
\]

in which the substitution \( s_2^2 = 2 \frac{s_1^2}{1-\mu} \) has been made.

The inverse transform is not readily obtainable, however, it would be expected to exist since \( s \) is raised to a higher power in the denominator than it is in the numerator. To obtain an indication of the function formed an approximate solution is examined: first, an approximation is written for the radical in which \( s_1 \) appears in the denominator; second, the numerator and denominator are divided by \((w^2 - \mu w^2 + s_1^2)\); and third, an inverse transformation is performed on the approximation obtained in the second step.

The radical written in another form is

\[
\sqrt{(s_1^2 + w^2)^2 + s_1^4 + (1-\mu) w^2 s_1^2 w^2} = (s_1^2 + w^2) + \frac{s_1^4 (1-\mu) w^2 s_1^2 - w^4}{2(s_1^2 + w^2)} - \frac{s_1^4 (1-\mu) w^2 s_1^2 - w^4}{8(s_1^2 + w^2)^3} \ldots
\]

The right hand side of the above equation is approximately equal to

\[
w^2 + s_1^2 - \mu w^2/2 + s_1^2/2 -s_1^2/8 + w^2(1-\mu)/8
\]

\[
= 11 s_1^2/8 + (9-6\mu) w^2/8
\]
When this result is divided by \((w^2 - \mu w^2 + s_1^2)\) and multiplied by \(w^2 / (1 - \lambda)^3\) there results

\[
\frac{1 - \mu}{\delta} \left[ (9 - 6\mu)w^2 + \frac{2 - 5\mu}{1 - \mu} s_1^2 - \ldots \right]
\]

in which terms containing higher powers of \(s_1\) have been dropped.

The remaining terms in the numerator and denominator of equation 58 are divided by \((w^2 - \mu w^2 + s_1^2)\) and the portion of equation 58 within the brackets becomes

\[
\frac{N_1 \exp(-N_3)s^2 + a^2}{s^2 + b^2}
\]

(59)
in which

\[
N_3 = \frac{\nu}{c_1}
\]

\[
a^2 = \frac{c_1^2 w^2}{N_2 w^2} = \frac{8 (1 - \mu)^3 - (9 - 6\mu)(1 - \lambda)}{8 \sqrt{1 - \mu} - 2 + 5\mu}
\]

\[
b^2 = N_2 w^2 = \frac{8 \sqrt{1 - \lambda} c_1^2}{8 \sqrt{1 - \mu} - 2 + 5\mu}
\]

Since \(N_1\) is not involved in the transformation it is dropped temporarily. If equation 59 has its numerator and denominator multiplied by \(\sqrt{s^2 + a^2}\) part of the term that results can be transformed using pair 89, p. 300, Churchill (7).

\[
\omega_{-1} \left\{ \frac{\exp(-N_3 s^2 + a^2)}{\sqrt{s^2 + a^2}} \right\} = \begin{cases} 0 & \text{when } 0 < t < N_3 \\ J_0 \left( a \sqrt{t^2 - N_3^2} \right) & \text{when } t > N_3 \end{cases}
\]

The remaining term, \(\sqrt{s^2 + a^2} / s^2 + b^2\), is of the form of pair 55 except \(a < b\). However, the radical can again be expanded and a
division by $s^2 + b^2$ performed. If only the first two terms from the division are considered the inverse transform is

$$\int_0^1 \left\{ \frac{1}{s} + \frac{a^2-2b^2}{2s^3} \right\} dt = 1 + \frac{(a^2-2b^2)t^2}{4}$$

where pairs 1 and 3, p. 295, Churchill (7), are used. The two inverse transforms are combined through the Faltung integral so that the resultant inverse transform is represented by the following

$$\int_0^t (1 + N_4 t^2 - 2N_4 t R + N_4 R^2) J_0(a) R^2 \frac{1}{R^2 - N^2} dR \quad (60)$$

in which $N_4 = (a^2 - 2b^2)/4$.

The integration produces a series containing increasing powers of $t$ and powers of $w$ and $y$. The inverse Fourier transformation must be performed on the result of the above integration. Only the terms containing $a^2$ and $b^2$ are involved in the inverse Fourier transformation and with the exception of the first term from the above integral only even powers of $w$ exist. The inverse Fourier transformation can be effected by use of pair 107, p. 38, Campbell and Foster (4). For example: $w^2$ transforms to $2/x^3$. The term $N_3$ contains $y$ which, on the expansion of the Bessel function, exists in even powers or not at all. Consequently the function $\phi(x,y,t)$ consists of a series of terms of the form $\frac{t^n y^m}{x^p}$, in which $n$, $m$, and $p$ are integers but not
consecutive. For convergence the function depends on the coefficients that exist in the Bessel function in equation 60 although for small values of t the function would tend to converge.

The function \( \phi(x,y,t) \) does not satisfy the wave equation exactly but, because of the limitation introduced through the inverse Laplace transform, the function does not exist for \( t < \frac{y}{c_1} \). Consequently the portions of the stresses dependent on \( \phi(x,y,t) \) do not exist until \( t > \frac{y}{c_1} \). This effect is a step in the right direction in that the previous sections have indicated a development of stress throughout the beam simultaneously with the introduction of a disturbance. The effect produced here, at least, indicates a propagation of stress in the y-direction.

The inverse transformations performed on \( \psi(w,y,s) \) produce similar results since the denominator is the same and the power of the numerator is the same. However, since \( s_2 \) exists in the exponent of e the limitation on t produced through the inverse Laplace transformation requires that the portions of the stresses dependent on \( \psi \) not exist until \( t > \frac{y}{c_2} \).
IV. EXPERIMENTAL INVESTIGATION

A. Problem Considered

The problem selected for first consideration in the experimental investigation was the case of the impact load at midspan. The sweep writing capabilities of a DuMont type 303-AH cathode-ray oscilloscope presented an excellent means for observing a transient phenomenon. Consequently, the test set up was constructed with the intention of obtaining the strains produced at various locations throughout the beam, first, by the initial strain pulse as it was propagated from the line (or area) of impact and second, by the final beam action.

B. Material, Equipment and Apparatus

The beam used in this investigation was the same beam originally used by Bowden (3) and is shown in Fig. 5 with the other equipment involved in the test. It had an overall length of 23.45 in., a width and depth of 3.94 in. and was supported as a simple beam with a span length of 20.00 in. It was made of Bethlehem structural steel containing the following: Carbon 0.18%, Manganese 0.44% Phosphorus 0.011% and Sulfur 0.025%.

Fig. 6 is a sketch of the beam set up which shows the location of gages and method of application of impact
Fig. 5 Steel Beam, Amplifier, and Oscilloscope
Fig. 6 Steel Beam and Striking Mechanism
load at midspan. The side of the beam not shown had strain
gages located directly behind those indicated in the sketch.
The strain gages used were SR-4 resistance strain gages type
C-13 with a gage factor of 3.18, a resistance of $1500 \pm 5$
ohms, and a nominal gage length of 0.25 in.

Fig. 7 is a schematic sketch of the electrical
portion of the test set up. Gages no. 1 and 2 are located
near the top of the beam immediately under the load as
indicated in Fig. 6. The Tektronix type 112 direct coupled
amplifier was connected across the two gages and the output
from the amplifier was fed to the DuMont type 303-AH
oscilloscope. Provision was made through the external
synchronization to drive the sweep on the oscilloscope
simultaneously with the contact of the hammer on the beam.

C. Preliminary Calculations and Discussion

The change in potential across the gages due to a change
in strain of the member can be calculated as indicated
by Dohrenwend (13).

$$\Delta E_g = \frac{R_g R(G.F.) I \Delta \varepsilon}{R_g + R}$$  \hspace{1cm} (61)

in which

- $\Delta E_g$ = change in voltage across gages, volts
- $R_g$ = resistance of gages, ohms
- $R$ = resistance of ballast resistor, ohms
- $(G.F.)$ = gage factor of gage
Fig. 7 Block Diagram of Circuit
\[ I = \text{current, amperes} \]
\[ \Delta \epsilon = \text{change in strain} \]

When the amplifier was connected directly to the plates of the oscilloscope and the maximum gain of the amplifier was used a calibrating signal of 5 millivolts peak to peak produced a deflection of approximately 0.6 inch on the screen of the oscilloscope. A current of 20 milliamperes was arbitrarily selected as a safe current to be used in the gages. The above equation was used to calculate the change in strain that would be needed to produce a 5 millivolt signal. When two C-13 gages were used with a ballast resistor of 23,400 ohms and a current of 20 milliamperes a change in strain of 30 microinches per inch was calculated to be needed.

In the case of an axially applied impact load the strain produced is approximately equal to the velocity of deformation divided by the velocity of propagation of the pulse of strain. The velocity of propagation was taken as \( c_1 \) which is \( 2.11 \times 10^5 \) in. per sec. when the modulus of elasticity \( E \) is \( 30 \times 10^6 \) psi, Poisson's ratio \( \mu \) is 0.30, and the density \( \rho \) is \( 7.35 \times 10^{-4} \) lb-sec\(^2\) per in.\(^4\). To obtain a strain of 30 microinches per inch the velocity of deformation (or impact) must be maintained at 6.33 in. per sec for a time period long enough to produce a strain pulse at least 1/4 in. long. Hence the
velocity would need be maintained for approximately 1 1/4 microseconds. It is expected that the velocity of deformation would need be much greater for the impact applied transversely. To even approximate the constant velocity would require a mass large in comparison with the beam. Consequently it was decided to use a small mass and provide a means of obtaining a large initial velocity. This procedure was expected to produce a strain pulse having a large value of strain at or near the leading edge of the pulse followed by a rapid decay of strain.

The spring actuated striking hammer shown in Fig. 6 was used to provide the impact. The head of the striking hammer was a 3/4-in. diameter steel rod about 4 in. long and weighed about 1/2 lb. It was supported by a 1/2-in. diameter steel rod 35 in. long and weighing about 1.9 lb. The supporting arm was pivoted at its other end so that it would be horizontal when the striking head was resting on the upper surface of the beam. This set up for striking the beam was calculated to develop a velocity of the head at contact with the beam of approximately 250 in. per sec. A velocity of deformation of this magnitude would produce a strain in the order of magnitude of 1000 microinches per inch in an axial impact situation. On the basis of this estimate it was concluded that a sufficient strain pulse would be developed to produce a measurable voltage
change across the gages. Preliminary impact blows made in
adjusting the striking mechanism seemed to justify the
conclusion because a noticeable amount of plastic deforma-
tion was produced on the upper surface of the beam.

D. Proposed Test Procedure

Each set of gages (where a set includes two gages
back to back and on opposite sides of the beam) were to
be connected in series in turn and the test performed as
indicated below.

1. With the sweep of the 303-AH oscilloscope on
"recurrent" and at the desired writing rate
several preliminary impacts to be struck to
check reproduceability of each trace.

2. Voltage and time calibration signals to be
recorded.

3. With the synchronization set on external, a
Hewlett-Packard audio-oscillator to provide the
signal to trigger the sweep, and with the sweep
on "driven" several identical impacts to be made
and recorded.

The contact of the striking hammer with the beam to be
used to provide the closed circuit to trigger the sweep
externally. The traces to be recorded by a DuMont
oscilloscope recording camera.
The pulse of strain produced by the striking hammer travels in the beam and arrives at a given set of gages. The change in strain effected at the gages produces a voltage change which causes a displacement of the trace on the oscilloscope screen. Since the theoretical investigation indicated that the stresses and, hence, the strains would be dependent upon two velocities of propagation ($c_1$ and $c_2$), it would be expected that the trace be displaced twice or have two peaks.

E. Results

The test procedure was started but did not progress beyond step 1. The impact blow produced proved to be incapable of providing a large enough strain pulse to cause a displacement on the oscilloscope screen. The sets of gages that involve gages 1 and 2, 3 and 4, and 5 and 6 were individually connected into the circuit and attempts were made to obtain displacement of the trace with the sweep at various writing rates. A discussion of the possible changes in the setup is given in the following section.
V. DISCUSSION OF RESULTS

The object of this investigation was to obtain expressions for the stresses developed in a beam due to the application of a suddenly applied disturbance. It appeared to the author that the correct approach was through the use of the equations of motion of elemental section of the member subjected to the disturbance without the specification of beam action included in the development of the equations.

The setting up of the partial differential equations of motion led to the use of displacement functions to adjust the two equations to a condition of one unknown function in each equation. The two equations produced were of the form of wave equations. Particular solutions of these wave equations were taken and constants determined so as to satisfy certain combinations of boundary conditions.

Equations 24, 25, and 26 give the normal and shearing unit stresses for the case of the transient increasing moment and shear. These equations indicate a simultaneous development of stress through the beam with the application of the disturbance. Equations 20, 20a, and 21 are the expressions for the shears and moments applied at the ends of the beam and, with the substitution of x for L,
these equations would represent the shear and moment on any cross-section. It is obvious that some amount of time, however small, must elapse before the moment (or shear) is transmitted through the beam, and that the two wave equations indicate velocities of propagation of $c_1$ and $c_2$. In view of the above statement and to provide for the propagation of the moment and shear through the beam the portion of the moment and shear dependent on $\phi$ could be specified to be zero until $t > (L \pm x)/c_1$ and then to acquire the value indicated by the equations, where $L + x$ is used for the disturbance starting at $x = -L$, and $L-x$ is used for the starting at $x = +L$. A similar limitation on time could be used for the portion of the moment and shear dependent on $\psi$; that is, no contribution until $t > (L \pm x)/c_2$. The propagation of the moment from the end of the beam toward the center, then is composed of two fronts: the leading one traveling at a velocity of $c_1$ and increasing in magnitude with time but decreasing with the distance from the end, and the second or trailing front traveling at a velocity of $c_2$ and oscillating both with time and distance from the end of the beam. If only single terms were needed to match the applied moment and shear the equations would probably be effective for several microseconds only inasmuch as $c_1 t$ becomes greater than unity rapidly and the hyperbolic sine involving $c_1 t$ increases, of course, more rapidly.
However, once sufficient time has elapsed for the wave fronts from the two ends of the beam to meet at the center of the beam it would appear that the use of the above mentioned equations would no longer be satisfactory or sufficient to describe the continued propagation (or reflection) of the disturbance front. A new set of boundary conditions which introduced the effect of the disturbance at the center of the beam would be required to provide a means of tracing the continued progress of the wave fronts.

The determination of the normal and shearing unit stresses and the displacements would also be affected by the restrictions introduced. Stresses and displacements dependent on $\phi$ would be zero until $t > (L + x)/c_1$ and those dependent on $\psi$ would be zero until $t > (L + x)/c_2$. The vertical displacement is zero at $y = \pm h$ for the particular values of $x$ that make $\cos \frac{m \pi x}{h}$ equal to zero. The locations of zero vertical displacements are unaffected by the propagation of the wave front.

For the case in which the disturbance originates at a line of width $2\Delta$ on the upper surface of the beam the wave fronts would be expected to propagate in the both $x$ and $y$ directions. To provide for the propagation in both directions the stresses and displacements could be specified to be zero until $t > \sqrt{x^2 + (h + y)^2}/c_1$ and/or $t > \sqrt{x^2 + (h + y)^2}/c_2$. 
contingent upon their dependence on $\phi$ and $\psi$. Here again, once the disturbance has arrived at a discontinuity (a free surface in this case) and a reflection would be expected, the equations would no longer indicate the continued propagation of the disturbance. A new set of boundary conditions which introduce the effect of the boundary would be required to describe the continued propagation. However, in the development of the equations for the stresses the effect of the bottom boundary of the beam was included. Hence it appears that the equations developed cover the response of the member as a unit and do not provide for the establishment of that response.

The solution for the stresses that would develop out of the use of the Fourier and Laplace transformations would indicate a propagation of the disturbance in the $y$ direction only but would indicate a simultaneous development of the stresses at a given elevation of the beam. So that, here too, the solution indicated was not one that would exactly satisfy the propagation from the line of impact.

When the applied moments, shears, and/or transverse load are specified, the various stresses can be determined from equations 25 through 26, or 37 through 39, or 45 through 47 contingent upon the series providing the means to express the applied disturbances. A method for expressing the various impact phenomena in terms of
numerical quantities is in itself a problem requiring further investigation. The expedient of expressing the impact force or moment in terms of a Fourier series or as a Dirac-delta function is satisfactory until the necessity arises for the substitution of the magnitude for the symbol F or S or M which is easily carried along in the discussions of the stresses. Timoshenko (38), Eringen (15), and others have made use of Hertz's impact equation to produce expressions for the force and time of impact for the case of a sphere striking a slender beam in which the neutral surface is located at the same position in the beam in the dynamic case as in the static case.

Inasmuch as the results of the experimental investigation are inconclusive some modification of the testing setup and procedure would be advisable. Since the equipment used required about 5 millivolts to produce an inch deflection of trace on the screen of the oscilloscope and yet no signal was received from the gages to develop even the trace of a deflection, a larger signal is required. The increased signal might be obtained by one or more of the following: 1. increase of current flowing in gage circuit; 2. increase of gage factor of gages; 3. increase of resistance of gages; 4. increase of velocity of striking hammer; 5. increase amplification of recording equipment;
6. decrease in width of beam; 7. change of beam for one of a material with a lower modulus of elasticity to density ratio.

1. The maximum current recommended to be used with the gages is about 30 ma. Consequently the current was increased to 30 ma but it was still insufficient to cause a visible signal. Conceivably the current could be increased to several times this value for a short period of time and with a cooling system to help dissipate the heat.

2. The gage factor of the gages used was in the range of the highest commercially available.

3. The type C-13 gages used have a resistance of 1500 ohms. An increase in resistance can be obtained if a type C-14 gage is used, however, this gage has a longer gage length. A "home-made" gage could be constructed which had any desired resistance and would limited only by the ability of the maker to satisfy the desired space requirements. The determination of the gage factor of the gage would also present a problem which would require an ability to construct duplicate gages.

4. The velocity of the striking hammer could be increased by the use of a greater height of fall and/or the use of stiffer springs in the starting mechanism. However, observation of surface of the beam at the line of contact indicated that plastic deformation was already
caused by the striking hammer at the velocity used. Consequently it was decided that a sufficiently large strain was developed and that refinement of method of application and pick-up was more critical.

5. The instruments used were capable of higher gain than that used. It was possible to get approximately a 1/2 in. deflection on the screen of the oscilloscope for a 0.2 millivolt peak-to-peak calibration signal but this was effected only by shorting the input of the amplifier directly to ground and even then the width of the trace was more than a tenth of an inch thick. Conceivably the noise at this high amplification could be suppressed to provide usable higher gain.

6. A decrease in the width of the beam could result in a signal. The effect would probably be manifest in better control of the application of the striking hammer, and decrease in amount of energy required to produce a given strain.

7. Probably the greatest effect on the signal would be produced by exchanging the beam for one of a different material. A material with a lower modulus of elasticity to density ratio for a given Poisson ratio would decrease the velocity of propagation of the wave through the beam and at the same time contribute a larger strain for a given velocity of impact.
An extreme example of such a material is rubber which provides the possibility of the low modulus of elasticity. A rubber beam which had been used by Moebius (27a) and was approximately of the same dimensions as the steel one discussed heretofore was available. SR-4 type A-1 strain gages with a gage factor of 2.04 and a resistance of 120 ohms were already attached to the beam in several positions. One (gage A) was in roughly the same position as gage 1, Fig. 6, six others were approximately equally spaced (4 in. between center lines) along the center of the top of the beam and the two closest to the load were located two inches from it. When the beam was turned upside down gage A was in a position roughly corresponding to gage 3, Fig. 6.

This beam was set up in the same manner as the steel beam. A current of only 6.4 ma was supplied to gage A. The striking hammer was dropped from a height of 2 1/2 in. onto a piece of steel 1/2 in. wide by 1/8 in. thick, by 3.86 in. long which was cemented to the surface of the beam. The signal received from gage A is shown in Fig. 8. The value of strain indicated by the maximum deflection of the trace in Fig. 8 from the base line is approximately 1280 microinches per inch. This deflection represented the maximum signal received from any of the gages used as a given impact. Repeated tests indicated one problem arising from the use of a line of contact rather than a point: in many cases
Horizontal Scale: 1 division = $2.0 \times 10^{-4}$ sec.
Vertical Scale: 1 division = $1.3 \times 10^{-4}$ volts
Upward displacement from initial horizontal trace indicates compressive strain.

Fig. 8 Signal from Gage A, Position 1
the hammer did not strike flush with the upper surface and when it struck first the side of the beam opposite to gage A a small tensile strain was first indicated before the compressive strain.

Fig. 9 is the signal received from gage A when the beam was turned upside down before it was struck. The damping and dispersion of the initial compressive strain wave are quite evident when a comparison of Fig. 8 and 9 is made. If it is assumed that the maximum response is produced in each case when the strain pulse has traveled the same distance along the gage, then the velocity of propagation can be estimated from the information obtainable from Fig. 8 and 9 and the distance between the gage positions. The velocity (7,200 in. per sec.) of propagation is obtained by dividing the distance between the gage positions (2.52 in.) by the time (0.00035 sec) for the peak of the wave to travel this distance. If the modulus of elasticity (6000 psi) and density (2.1 slugs per cu. ft.) are taken from Moebius (27a) and Poisson's ratio is assumed to be 0.4, then the value for $c_1$ is 8,400 in. per sec.

Fig. 10 is the signal received from one of the gages (gage B) on the upper surface of the beam 2 in. from the impact. The initial signal received indicated compression followed by a tensile indication. Apparently then a compressive pulse emanates from the line of impact followed by a
Horizontal Scale: 1 division = $2.0 \times 10^{-4}$ sec.
Vertical Scale: 1 division = $1.0 \times 10^{-4}$ volts
Upward displacement from initial horizontal trace indicates compressive strain.

Fig. 9 Signal from Gage A, Position 3
Horizontal Scale: 1 division = $2.0 \times 10^{-4}$ sec.
Vertical Scale: 1 division = $1.0 \times 10^{-4}$ sec.
Upward displacement from initial horizontal trace indicates compressive strain.

Fig. 10 Signal from Gage B
surface wave. Fig. 11 is the signal received from another gage (gage D) on the upper surface of the beam 4 in. further from the load than the gage B. Again the damping and dispersion of the wave is evident so much so that the compressive signal has been eliminated from the second gage.

The use of rubber as the material for the beam provides a successful means for obtaining recordable strains. The development of at least two different waves was indicated to exist along the upper surface, but no indication was obtained in the vertical direction directly below the load. Damping and dispersion of the disturbance were easily observed in the rubber beam but were not considered in the development of the equations for stresses. Consequently a change in the initial assumptions would be required.
Horizonal Scale: 1 division = 2.0(10)^-4 sec.
Vertical Scale: 1 division = 1.0(10)^-4 sec.
Upward displacement from initial horizontal trace indicates compressive strain.

Fig. 11 Signal from Gage D
VI. SUMMARY AND CONCLUSIONS

In this dissertation general equations for stresses and displacements developed in a beam due to suddenly applied disturbances on various boundaries of the beam are obtained. The equations are developed from the differential equations of motion for a free-body-diagram of a differential element, the strain-displacement relationships, the stress-strain relationship, and the boundary conditions. The material is assumed to be a homogeneous, isotropic one, the stress-strain relationship is assumed to be below the proportional limit of the material, and a plane stress situation is assumed to exist. The stress expressions developed are in the form of infinite series and, in themselves, do not provide for the propagation of a disturbance pulse along or through the beam but rather indicate a steady state response to the disturbance. Fourier and Laplace transformations are used to give an indication of a solution that takes into account the propagation of the disturbance in one direction through the beam and that must be modified to account for reflections of the disturbance from discontinuities.

An experimental investigation to obtain the strains developed in a simply supported steel beam was described. The difficulties encountered in attempting to produce
usable strain signals are discussed. The steel beam was replaced by a rubber beam and appreciable signals were received from the strain gages sufficient to produce a maximum deflection of more than an inch of the oscilloscope trace. The large damping capacity of the rubber and the dispersion of the wave is shown in the results obtained. The velocity of propagation of a compression was obtained and an indication of two types of waves was obtained from observations made on the upper surface of the beam.

The lack of results from the steel beam indicates the desirability of using a material with a lower modulus of elasticity and that the strain pulse is of extremely short length and/or is rapidly dispersed upon entering the beam. The damping and dispersion produced in the rubber beam indicate the advisability of the use of a material with a modulus of elasticity intermediate between steel and rubber.
VII. LITERATURE CITED


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