Numerical computation of an optimal control problem with homogenization in one-dimensional case

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Numerical computation of an optimal control problem with homogenization in one-dimensional case

by

Zhen Li

A thesis submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

Major: Applied Mathematics

Program of Study Committee:
L. Steven Hou, Major Professor
Jue Yan
Ananda Weerasinghe

Iowa State University
Ames, Iowa
2008

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DEDICATION

I would like to dedicate this thesis to my wife Yanfei without whose support I would not have been able to complete this work. I would also like to my friends and family for their loving guidance and financial assistance during the writing of this work.
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I would like to take this opportunity to express my thanks to those who helped me with various aspects of conducting research and the writing of this thesis. First and foremost, Dr. Steven Hou for his guidance, patience and support throughout this research and the writing of this thesis. His insights and words of encouragement have often inspired me and renewed my hopes for completing my graduate education. I would also like to thank my committee members for their efforts and contributions to this work: Dr. Jue Yan and Dr. Ananda Weerasinghe.
ABSTRACT

We consider an optimal control problem in which the state equation has rapidly oscillating coefficients (characterized by matrix $A_\epsilon$, where $\epsilon$ is a small parameter). Based on some important results from the paper by S. Kesavan and J. Saint Jean Paulin (1997), we convert this optimal control problem to a partial differential equation problem. Therefore, solving optimal control problem is equivalent to solving this partial differential equation problem. By several numerical examples in one dimensional case, we also show that the limit satisfies a problem of the same type but with matrix $A_0$ (the H-limit of $A_\epsilon$).
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CHAPTER 1. Overview

This is the opening paragraph to my thesis which introduce the optimal control problem and the connection between this problem and partial differential equations. This thesis is mainly based on the results from the paper by S. Kesavan and J. Saint Jean Paulin (1997).

1.1 Introduction

We will discuss the homogenization of an optimal control problem in which the state equation (given by a second-order elliptic boundary value problem) has rapidly oscillating coefficients. We just consider the one-dimensional case in this thesis. Let $f \in L^2(\Omega)$, $A$ and $B$ are matrices whose entries are functions on bounded domain $\Omega$ with smooth boundary. $B$ is also symmetric and nonnegative. $N > 0$ is a given constant. Let $\theta(x)$ be a control variable and the optimal control problem which can be found in the paper by S. Kesavan and J. Saint Jean Paulin (1997) is defined as follows,

\[
\begin{cases}
-\text{div}(A \nabla u) = f(x) + \theta(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

and the state $u = u(\theta)$ is thus defined as the weak solution in $H_0^1(\Omega)$ of above problem. Then the cost function is given by

\[
J(\theta) = \frac{1}{2} \int_{\Omega} (B \nabla u, \nabla u) \, dx + \frac{N}{2} \int_{\Omega} \theta^2(x) \, dx.
\]

Minimization of the above cost function is a standard minimization problem, a discussion of which can be found in the book by J. L. Lions (1968) and we obtain a reduced form by
introducing a new adjoint state $p$,\[
\begin{aligned}
-\text{div}(A\nabla u) &= f(x) + \theta(x) \quad \text{in } \Omega \\
\text{div}(A^t\nabla p - B\nabla u) &= 0 \quad \text{in } \Omega,
\end{aligned}
\]
where $u, p \in H^1_0(\Omega)$, and the optimal control $\theta^*$ can be characterized by such inequality\[
\int_\Omega (p + N\theta^*)(\theta - \theta^*) \, dx \geq 0 \quad \forall \theta \in S,
\]
where $S$ is a subset of $L^2(\Omega)$.

What we are interested in is that given a parameter $\epsilon > 0$ which tends to zero, the matrices $A$ and $B$ above depend on $\epsilon$. And we also have the same assumptions on $A_\epsilon$ and $B_\epsilon$. In Kesavan’s paper, there are also following conclusions. Suppose $A_\epsilon$ is matrix depending on $\epsilon$, then $\theta^*_\epsilon$ exists and is bounded in $L^2(\Omega)$. Thus, we have\[
\theta^*_\epsilon \rightharpoonup \theta^* \quad \text{weakly in } L^2(\Omega),
\]
where $\theta^*$ is also an optimal control defined by a problem of the same type with matrices $A^*$ and $B^*$. That paper also gives the following theorem. The solution $(u_\epsilon, p_\epsilon)$ of system\[
\begin{aligned}
-\text{div}(A_\epsilon\nabla u_\epsilon) &= f(x) + \theta(x) \quad \text{in } \Omega \\
\text{div}(A_\epsilon^t\nabla p_\epsilon - B\nabla u_\epsilon) &= 0 \quad \text{in } \Omega,
\end{aligned}
\]
is bounded and also have the following weak convergence result in $(H^1_0(\Omega))^2$,\[
\begin{aligned}
u_\epsilon &\rightharpoonup u_0, \text{ as } \epsilon \to 0 \\
p_\epsilon &\rightharpoonup p_0, \text{ as } \epsilon \to 0
\end{aligned}
\]
where $u_0, p_0$ satisfy the following system of equations,\[
\begin{aligned}
-\text{div}(A_0\nabla u_0) &= f(x) + \theta(x) \quad \text{in } \Omega \\
\text{div}(A_0^t\nabla p_0 - B\nabla u_0) &= 0 \quad \text{in } \Omega.
\end{aligned}
\]
1.2 One-dimensional case

For the one-dimensional case,\[\begin{align*}
\frac{d}{dx} (a_\epsilon \frac{du_\epsilon}{dx}) &= f(x) + \theta(x) \quad \text{in } (0, 1), \\
\frac{d}{dx} (a_\epsilon \frac{dp_\epsilon}{dx} - b_\epsilon \frac{du_\epsilon}{dx}) &= 0 \quad \text{in } (0, 1),
\end{align*}\]
we also have the similar results. Suppose \((0, 1) \subset \mathcal{R}\), if \(\frac{1}{a_\epsilon} \rightharpoonup \frac{1}{a_0}\) weakly in \(L^\infty(0, 1)\) and \(b^* = \frac{a_0^2}{g_0}\) where
\[\frac{1}{g_0} = \frac{b_\epsilon}{a_\epsilon^2} \rightharpoonup \frac{1}{g_0} \text{ weakly in } L^\infty(0, 1).\]
Then we have the following weak convergence in \(H^1_0(0, 1),\)
\[u_\epsilon \rightharpoonup u_0, \text{ as } \epsilon \to 0\]
\[p_\epsilon \rightharpoonup p_0, \text{ as } \epsilon \to 0\]
where \(u_0, p_0\) satisfy the equations
\[\begin{align*}
- \frac{d}{dx} (a_0 \frac{du_0}{dx}) &= f(x) + \theta(x) \quad x \in (0, 1), \\
\frac{d}{dx} (a_0 \frac{dp_0}{dx} - b^* \frac{du_0}{dx}) &= 0 \quad x \in (0, 1), \\
u_0 = p_0 = 0 \quad x = 0, 1.
\end{align*}\]
In this thesis, we rewrite this optimal control problem and consider the following forms,
\[\begin{align*}
- \frac{d}{dx} (a_\epsilon(x) \frac{du_\epsilon}{dx}) &= f(x) + c(x) \quad x \in (0, 1), \\
u_\epsilon &= 0 \quad x = 0, 1.
\end{align*}\]
And the cost function is
\[J(c) = \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 \, dx + \frac{1}{2} \int_0^1 c^2(x) \, dx,\]
where \(\beta \gg 1\) and \(U(x)\) is a given function. \(a_\epsilon(x)\) is a function defined on \([0, 1]\). Thus, the optimal control \(c^*\) is the function in \([0, 1]\) which minimizes \(J(c)\) for \(c(x) \in L^2(0, 1)\).
CHAPTER 2. Optimal Control Problem and Partial Differential Equation

It is difficult to solve this optimal problem directly. From the numerical analysis viewpoint, it is advantageous to convert this problem to an equivalent PDE problem. Then we are able to analyze it by finite element or finite difference methods on numerical analysis. Let’s consider the following optimal control problem

\[
\begin{align*}
- \frac{d}{dx} (a_\epsilon(x) \frac{du_\epsilon}{dx}) &= f(x) + c(x) \quad x \in (0, 1), \\
u_\epsilon &= 0 \quad x = 0, 1, \\
\min \beta \int_0^1 |u_\epsilon - U|^2 \, dx + \frac{1}{2} \int_0^1 c^2(x) \, dx.
\end{align*}
\]

Let \( v_\epsilon(x) \in L^2(0, 1) \) and \( v_\epsilon = 0 \), if \( x = 0, 1 \), then

\[
L(u_\epsilon, c) = \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 \, dx + \frac{1}{2} \int_0^1 c^2(x) \, dx
\]

\[
= \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 \, dx + \frac{1}{2} \int_0^1 c^2(x) \, dx - \int_0^1 v_\epsilon \left( - \frac{d}{dx} (a_\epsilon(x) \frac{du_\epsilon}{dx}) - f(x) - c(x) \right) \, dx.
\]

By the integration by parts and \( v_\epsilon = 0, u_\epsilon = 0 \) if \( x = 0, 1 \), we find that

\[
\int_0^1 v_\epsilon \frac{d}{dx} (a_\epsilon(x) \frac{du_\epsilon}{dx}) \, dx = v_\epsilon a_\epsilon(x) \frac{du_\epsilon}{dx} \Big|_0^1 - \int_0^1 a_\epsilon(x) \frac{du_\epsilon}{dx} \, dv_\epsilon \, dx
\]

\[
= - \int_0^1 a_\epsilon(x) \frac{du_\epsilon}{dx} \, dv_\epsilon \, dx
\]

\[
= - u_\epsilon a_\epsilon(x) \frac{dv_\epsilon}{dx} \Big|_0^1 + \int_0^1 u_\epsilon \frac{d}{dx} (a_\epsilon(x) \frac{dv_\epsilon}{dx}) \, dx
\]

\[
= \int_0^1 u_\epsilon \frac{d}{dx} (a_\epsilon(x) \frac{dv_\epsilon}{dx}) \, dx.
\]
Therefore,

\[
L(u_\epsilon, c) = \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 \, dx + \frac{1}{2} \int_0^1 c^2(x) \, dx \\
+ \int_0^1 v_\epsilon \frac{d}{dx}(a_\epsilon(x) \frac{du_\epsilon}{dx}) \, dx + \int_0^1 v_\epsilon f(x) \, dx + \int_0^1 v_\epsilon c(x) \, dx \\
= \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 \, dx + \frac{1}{2} \int_0^1 c^2(x) \, dx \\
+ \int_0^1 u_\epsilon \frac{d}{dx}(a_\epsilon(x) \frac{dv_\epsilon}{dx}) \, dx + \int_0^1 v_\epsilon f(x) \, dx + \int_0^1 v_\epsilon c(x) \, dx.
\]

Then for any \( t(x), w(x) \in L^2(0,1) \), we should have

\[
\langle \frac{\partial L}{\partial u_\epsilon}, w \rangle = \beta \int_0^1 (u_\epsilon - U)w \, dx + \int_0^1 w \frac{d}{dx}(a_\epsilon(x) \frac{dv_\epsilon}{dx}) \, dx \\
= \int_0^1 \left( \beta(u_\epsilon - U) + \frac{d}{dx}(a_\epsilon(x) \frac{dv_\epsilon}{dx}) \right) w \, dx = 0,
\]

\[
\langle \frac{\partial L}{\partial c}, t \rangle = \int_0^1 c(x)t(x) \, dx + \int_0^1 v_\epsilon t(x) \, dx \\
= \int_0^1 (c(x) + v_\epsilon(x)) t(x) \, dx = 0.
\]

Therefore,

\[
\beta(u_\epsilon - U) + \frac{d}{dx}(a_\epsilon(x) \frac{dv_\epsilon}{dx}) = 0,
\]

\[
c(x) + v_\epsilon(x) = 0.
\]

i.e.

\[
-\frac{d}{dx}(a_\epsilon(x) \frac{dv_\epsilon}{dx}) - \beta u_\epsilon = -\beta U(x),
\]

\[
v_\epsilon(x) = -c(x).
\]

Hence, \( v(x) = -c(x) \) and the optimal problem is equivalent to the following partial differential equation problem,

\[
\begin{cases}
-\frac{d}{dx}(a_\epsilon(x) \frac{du_\epsilon}{dx}) + v_\epsilon(x) = f(x) & x \in (0,1), \\
-\frac{d}{dx}(a_\epsilon(x) \frac{dv_\epsilon}{dx}) - \beta u_\epsilon(x) = -\beta U(x) & x \in (0,1) \\
u_\epsilon = 0 & x = 0,1, \\
v_\epsilon = 0 & x = 0,1.
\end{cases}
\]
Now let’s look several numerical examples. We solve this partial differential equation with finite difference. The finite difference for this problem is as follows,

\[
- \frac{a_{\epsilon,i} + \frac{1}{2}v_{i+1} - (a_{\epsilon,i} + \frac{1}{2})v_i + a_{\epsilon,i} - \frac{1}{2}v_{i-1}}{h^2} + \frac{1}{2}u_i + a_{\epsilon,i} - \frac{1}{2}u_{i-1}\]

\[
+ v_i = f_i, i = 1, 2, \cdots n - 1
\]

\[
- \frac{a_{\epsilon,i} + \frac{1}{2}v_{i+1} - (a_{\epsilon,i} + \frac{1}{2})v_i + a_{\epsilon,i} - \frac{1}{2}v_{i-1}}{h^2} - \beta u_i = -\beta U_i, i = 1, 2, \cdots n - 1
\]

\[
u_0 = u_n = 0
\]

\[
v_0 = v_n = 0,
\]

where for any function \( g(x) \), \( g_i = g(x_i) \) and \( 0 = x_0 < x_1 < \cdots < x_n = 1 \) is a uniform grid, with grid spacing \( \Delta x = h = 1/n \). We will choose \( a_{\epsilon} \) from paper by Gre’ectoire Allaire and Robert Brizzi (2004). Given \( a_{\epsilon} = \frac{1}{2 + 1.8 \sin(\frac{2\pi}{\epsilon})} \), \( \beta = 100,000 \) and \( U(x) = \sin(2\pi x) \), \( f(x) = x^2 \), we will look at several examples with different values of \( \epsilon \).

(i) \( \epsilon = 0.01, \Delta x = \frac{1}{2000} \), the graphs of \( u \) and \( v \) are as follows,

![Graph of u_\epsilon and U(x) = sin(2\pi x)](image)

Figure 2.1  The graph of \( u_\epsilon \) and \( U(x) = \sin(2\pi x) \), \( 0 \leq x \leq 1, \epsilon = 0.01 \)

(ii) \( \epsilon = 0.001, \Delta x = \frac{1}{20000} \), the graphs of \( u \) and \( v \) are as follows,

From graph 2.1 and 2.3, we could find that, the shapes of function \( u_\epsilon \) and \( U(x) = \sin(2\pi x) \) are almost the same, when \( \epsilon \) is small enough. This special case was studied by Kesavan and
Vanninathan. They assume that $a_\epsilon$ is periodic. For the following problem,

$$
\begin{align*}
\frac{d}{dx} \left( a_0 \frac{du_0}{dx} \right) + v_0(x) &= f(x) \quad x \in (0, 1), \\
\frac{d}{dx} \left( a_0 \frac{dv_0}{dx} \right) - \beta u_0(x) &= -\beta U(x) \quad x \in (0, 1) \\
u_0 = v_0 &= 0 \quad x = 0, 1,
\end{align*}
$$

where $a_0$ is a constant and they proved that $a_0$ was indeed the limit of $a_\epsilon$ in the topology of H-convergence. Also for the periodic $a_\epsilon$ of the one-dimensional case, they also gave its limit of H-convergence, which is

$$a_0 = \left[ m \left( \frac{1}{a} \right) \right]^{-1},$$

where $m(h) = \int_0^1 h(y) \, dy$ for a periodic function $h$ on $[0,1]$. 

Figure 2.2  The graph of $v_\epsilon, 0 \leq x \leq 1, \epsilon = 0.01$
Figure 2.3  The graph of $u_\epsilon$ and $U(x) = \sin(2\pi x)$, $0 \leq x \leq 1$, $\epsilon = 0.001$

Figure 2.4  The graph of $v_\epsilon$, $0 \leq x \leq 1$, $\epsilon = 0.001$
CHAPTER 3. Compare the Results for $a_\epsilon$ and $a_0$

We will compare the relationship between

$$
\begin{aligned}
- \frac{d}{dx}(a_\epsilon(x) \frac{du_\epsilon}{dx}) + v_\epsilon(x) &= f(x) \quad x \in (0, 1), \\
- \frac{d}{dx}(a_\epsilon(x) \frac{dv_\epsilon}{dx}) - \beta u_\epsilon(x) &= -\beta U(x) \quad x \in (0, 1) \\
u_\epsilon = v_\epsilon &= 0 \quad x = 0, 1,
\end{aligned}
$$

and

$$
\begin{aligned}
- \frac{d}{dx}(a_0 \frac{du_0}{dx}) + v_0(x) &= f(x) \quad x \in (0, 1), \\
- \frac{d}{dx}(a_0 \frac{dv_0}{dx}) - \beta u_0(x) &= -\beta U(x) \quad x \in (0, 1) \\
u_0 = v_0 &= 0 \quad x = 0, 1,
\end{aligned}
$$

with two numerical examples. Like the prior example, let’s suppose $a_\epsilon = \frac{1}{2 + 1.8 \sin\left(\frac{2\pi x}{\epsilon}\right)}$, $\beta = 100,000$ and $U(x) = \sin(2\pi x)$, $f(x) = x^2$. Let $u_\epsilon, v_\epsilon$ denote the numerical solutions of partial differential equations with $a_\epsilon$ and $u_0, v_0$ denote the numerical solutions of partial differential equations with $a_0$. Hence,

$$a_0 = \left[ m \left( \frac{1}{a} \right) \right]^{-1} = \left[ \int_0^1 (2 + 1.8 \sin(2\pi y)) \ dy \right]^{-1} = \frac{1}{2}.$$

We will give several graphs to illustrate the errors between $u_\epsilon$ and $u_0$, $v_\epsilon$ and $v_0$ for different values of $\epsilon$.

(i) $\epsilon = 0.01, \Delta x = \frac{1}{2000}$, the graphs of errors of $u_\epsilon$ and $v_\epsilon$ are as follows,

(ii) $\epsilon = 0.001, \Delta x = \frac{1}{20000}$, the graphs of errors of $u_\epsilon$ and $v_\epsilon$ are as follows,

From figure 3.2 and figure 3.6, we can find the oscillation of the error of $u_\epsilon$ and $u_0$. Therefore, we can find a test function, such that $u_\epsilon$ is weak convergent to $u_0$. Analogously, the error of $v_\epsilon$ and $v_0$ also has such oscillation, which means that $v_\epsilon$ is also weak convergent to $v_0$. 
Figure 3.1  The graph of $u_\epsilon$ and $u_0$, $\epsilon = 0.01$

Figure 3.2  The graph of error between $u_\epsilon$ and $u_0$, $\epsilon = 0.01$
Figure 3.3  The graph of $v_\epsilon$ and $v_0$, $\epsilon = 0.01$

Figure 3.4  The graph of error between $v_\epsilon$ and $v_0$, $\epsilon = 0.01$
Figure 3.5  The graph of $u_\epsilon$ and $u_0$, $\epsilon = 0.001$

Figure 3.6  The graph of error between $u_\epsilon$ and $u_0$, $\epsilon = 0.001$
Figure 3.7  The graph of $v_\epsilon$ and $v_0$, $\epsilon = 0.001$

Figure 3.8  The graph of error between $v_\epsilon$ and $v_0$, $\epsilon = 0.001$
CHAPTER 4. Results of Optimal Control Problem

For the optimal control problem

\[-\frac{d}{dx}(a_\epsilon(x)\frac{du_\epsilon}{dx}) = f(x) + c(x), x \in (0, 1)\]

for a given \(c(x)\), there will be a corresponding \(u_\epsilon\). What we want to do is to find a pair of \(c(x)\) and \(u_\epsilon\), such that

\[L(u, c) = \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 \, dx + \frac{1}{2} \int_0^1 c^2(x) \, dx\]

can attain its minimum. Since \(c(x) = -v(x)\),

\[L(u, c) = \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 \, dx + \frac{1}{2} \int_0^1 v^2(x) \, dx.\]

For the same given \(a_\epsilon = \frac{1}{2 + 1.8 \sin(\frac{2\pi x}{\epsilon})}, \beta = 100,000\) and \(U(x) = \sin(2\pi x), f(x) = x^2\), after solving the equivalent partial differential equations, we have the following minimum of \(L(u_\epsilon, c)\).

(i) Let \(\epsilon = 0.01, \Delta x = \frac{1}{2000}\). Then the minimum that \(L(u_\epsilon, c)\) attains is

\[\min L(u_\epsilon, c) = \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 \, dx + \frac{1}{2} \int_0^1 v^2(x) \, dx = 93.32562206282566.\]

(i) Let \(\epsilon = 0.001, \Delta x = \frac{1}{2000}\). Then the minimum that \(L(u_\epsilon, c)\) attains is

\[\min L(u_\epsilon, c) = \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 \, dx + \frac{1}{2} \int_0^1 v^2(x) \, dx = 70.54072381983838.\]
BIBLIOGRAPHY

