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Numerical computation of an optimal control problem with homogenization in one-dimensional case

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Numerical computation of an optimal control problem with homogenization in one-dimensional case

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DEDICATION

I would like to dedicate this thesis to my wife Yanfei without whose support I would not have been able to complete this work. I would also like to my friends and family for their loving guidance and financial assistance during the writing of this work.

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ABSTRACT

We consider an optimal control problem in which the state equation has rapidly oscillating coefficients(characterized by matrix A_ϵ, where ϵ is a small parameter). Based on some important results from the paper by S. Kesavan and J. Saint Jean Paulin (1997), we convert this optimal control problem to a partial differential equation problem. Therefore, solving optimal control problem is equivalent to solving this partial differential equation problem. By several numerical examples in one dimensional case, we also show that the limit satisfies a problem of the same type but with matrix A_0(the H-limit of A_ϵ).

CHAPTER 1. Overview

This is the opening paragraph to my thesis which introduce the optimal control problem and the connection between this problem and partial differential equations. This thesis is mainly based on the results from the paper by S. Kesavan and J. Saint Jean Paulin (1997).

1.1 Introduction

We will discuss the homogenization of an optimal control problem in which the state equation (given by a second-order elliptic boundary value problem) has rapidly oscillating coefficients. We just consider the one-dimensional case in this thesis. Let $f \in L^2(\Omega)$, A and B are matrices whose entries are functions on bounded domain Ω with smooth boundary. B is also symmetric and nonnegative. $N > 0$ is a given constant. Let $\theta(x)$ be a control variable and the optimal control problem which can be found in the paper by S. Kesavan and J. Saint Jean Paulin (1997) is defined as follows,

$$\begin{cases} -\mathrm{div}(A\nabla u) = f(x) + \theta(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and the state $u = u(\theta)$ is thus defined as the weak solution in $H_0^1(\Omega)$ of above problem. Then the cost function is given by

$$J(\theta) = \frac{1}{2}\int_\Omega (B\nabla u, \nabla u)\, dx + \frac{N}{2}\int_\Omega \theta^2(x)\, dx.$$

Minimization of the above cost function is a standard minimization problem, a discussion of which can be found in the book by J. L. Lions (1968) and we obtain a reduced form by introducing a new adjoint state p,

$$\begin{cases} -\mathrm{div}(A\nabla u) = f(x) + \theta(x) & \text{in } \Omega \\ \mathrm{div}(A^t\nabla p - B\nabla u) = 0 & \text{in } \Omega, \end{cases}$$

where $u, p \in H_0^1(\Omega)$, and the optimal control θ^* can be characterized by such inequality

$$\int_\Omega (p + N\theta^*)(\theta - \theta^*)\, dx \geq 0 \quad \forall \theta \in S,$$

where S is a subset of $L^2(\Omega)$.

What we are interested in is that given a parameter $\epsilon > 0$ which tends to zero, the matrices A and B above depend on ϵ. And we also have the same assumptions on A_ϵ and B_ϵ. In Kesavan's paper, there are also following conclusions. Suppose A_ϵ is matrix depending on ϵ, then θ_ϵ^* exists and is bounded in $L^2(\Omega)$. Thus, we have

$$\theta_\epsilon^* \rightharpoonup \theta^* \qquad \text{weakly in } L^2(\Omega),$$

where θ^* is also an optimal control defined by a problem of the same type with matrices A^* and B^*. That paper also gives the following theorem. The solution (u_ϵ, p_ϵ) of system

$$\begin{cases} -\mathrm{div}(A_\epsilon\nabla u_\epsilon) = f(x) + \theta(x) & \text{in } \Omega \\ \mathrm{div}(A_\epsilon^t\nabla p_\epsilon - B\nabla u_\epsilon) = 0 & \text{in } \Omega, \\ u_\epsilon = p_\epsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

is bounded and also have the following weak convergence result in $(H_0^1(\Omega))^2$,

$$u_\epsilon \rightharpoonup u_0, \text{ as } \epsilon \to 0$$

$$p_\epsilon \rightharpoonup p_0, \text{ as } \epsilon \to 0$$

where u_0, p_0 satisfy the following system of equations,

$$\begin{cases} -\mathrm{div}(A_0\nabla u_0) = f(x) + \theta(x) & \text{in } \Omega \\ \mathrm{div}(A_0^t\nabla p_0 - B\nabla u_0) = 0 & \text{in } \Omega. \end{cases}$$

1.2 One-dimensional case

For the one-dimensional case,

$$\begin{cases} -\frac{d}{dx}\left(a_\epsilon \frac{du_\epsilon}{dx}\right) = f(x) + \theta(x) & \text{in } (0,1), \\ \frac{d}{dx}\left(a_\epsilon \frac{dp_\epsilon}{dx} - b_\epsilon \frac{du_\epsilon}{dx}\right) = 0 & \text{in } (0,1), \end{cases}$$

we also have the similar results. Suppose $(0,1) \subset \mathcal{R}$, if $\frac{1}{a_\epsilon} \rightharpoonup \frac{1}{a_0}$ weakly in $L^\infty(0,1)$ and $b^* = \frac{a_0^2}{g_0}$ where

$$\frac{1}{g_0} = \frac{b_\epsilon}{a_\epsilon^2} \rightharpoonup \frac{1}{g_0} \text{ weakly in } L^\infty(0,1).$$

Then we have the following weak convergence in $H_0^1(0,1)$,

$$u_\epsilon \rightharpoonup u_0, \text{ as } \epsilon \to 0$$

$$p_\epsilon \rightharpoonup p_0, \text{ as } \epsilon \to 0$$

where u_0, p_0 satisfy the equations

$$\begin{cases} -\frac{d}{dx}\left(a_0 \frac{du_0}{dx}\right) = f(x) + \theta(x) & x \in (0,1), \\ \frac{d}{dx}\left(a_0 \frac{dp_0}{dx} - b^* \frac{du_0}{dx}\right) = 0 & x \in (0,1), \\ u_0 = p_0 = 0 & x = 0, 1. \end{cases}$$

In this thesis, we rewrite this optimal control problem and consider the following forms,

$$\begin{cases} -\frac{d}{dx}(a_\epsilon(x) \frac{du_\epsilon}{dx}) = f(x) + c(x) & x \in (0,1), \\ u_\epsilon = 0 & x = 0, 1. \end{cases}$$

And the cost function is

$$J(c) = \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 \, dx + \frac{1}{2} \int_0^1 c^2(x) \, dx,$$

where $\beta \gg 1$ and $U(x)$ is a given function. $a_\epsilon(x)$ is a function defined on $[0,1]$. Thus, the optimal control c^* is the function in $[0,1]$ which minimizes $J(c)$ for $c(x) \in L^2(0,1)$.

CHAPTER 2. Optimal Control Problem and Partial Differential Equation

It is difficult to solve this optimal problem directly. From the numerical analysis viewpoint, it is advantageous to convert this problem to an equivalent PDE problem. Then we are able to analyze it by finite element or finite difference methods on numerical analysis. Let's consider the following optimal control problem

$$\begin{cases} -\frac{d}{dx}(a_\epsilon(x)\frac{du_\epsilon}{dx}) = f(x) + c(x) & x \in (0,1), \\ u_\epsilon = 0 & x = 0, 1, \\ \min \frac{\beta}{2}\int_0^1 |u_\epsilon - U|^2\,dx + \frac{1}{2}\int_0^1 c^2(x)\,dx. \end{cases}$$

Let $v_\epsilon(x) \in L^2(0,1)$ and $v_\epsilon = 0,$ if $x = 0, 1,$ then

$$\begin{aligned} L(u_\epsilon, c) &= \frac{\beta}{2}\int_0^1 |u_\epsilon - U|^2\,dx + \frac{1}{2}\int_0^1 c^2(x)\,dx \\ &= \frac{\beta}{2}\int_0^1 |u_\epsilon - U|^2\,dx + \frac{1}{2}\int_0^1 c^2(x)\,dx - \int_0^1 v_\epsilon(-\frac{d}{dx}(a_\epsilon(x)\frac{du_\epsilon}{dx}) - f(x) - c(x))dx. \end{aligned}$$

By the integration by parts and $v_\epsilon = 0, u_\epsilon = 0$ if $x = 0, 1$, we find that

$$\begin{aligned} \int_0^1 v_\epsilon \frac{d}{dx}(a_\epsilon(x)\frac{du_\epsilon}{dx})dx &= v_\epsilon a_\epsilon(x)\frac{du_\epsilon}{dx}|_0^1 - \int_0^1 a_\epsilon(x)\frac{du_\epsilon}{dx}\frac{dv_\epsilon}{dx}\,dx \\ &= -\int_0^1 a_\epsilon(x)\frac{du_\epsilon}{dx}\frac{dv_\epsilon}{dx}\,dx \\ &= -u_\epsilon a_\epsilon(x)\frac{dv_\epsilon}{dx}|_0^1 + \int_0^1 u_\epsilon \frac{d}{dx}(a_\epsilon(x)\frac{dv_\epsilon}{dx})dx \\ &= \int_0^1 u_\epsilon \frac{d}{dx}(a_\epsilon(x)\frac{dv_\epsilon}{dx})dx. \end{aligned}$$

Therefore,

$$\begin{aligned} L(u_\epsilon, c) &= \frac{\beta}{2}\int_0^1 |u_\epsilon - U|^2\,dx + \frac{1}{2}\int_0^1 c^2(x)\,dx \\ &+ \int_0^1 v_\epsilon \frac{d}{dx}(a_\epsilon(x)\frac{du_\epsilon}{dx})\,dx + \int_0^1 v_\epsilon f(x)\,dx + \int_0^1 v_\epsilon c(x)\,dx \\ &= \frac{\beta}{2}\int_0^1 |u_\epsilon - U|^2\,dx + \frac{1}{2}\int_0^1 c^2(x)\,dx \\ &+ \int_0^1 u_\epsilon \frac{d}{dx}(a_\epsilon(x)\frac{dv_\epsilon}{dx})\,dx + \int_0^1 v_\epsilon f(x)\,dx + \int_0^1 v_\epsilon c(x)\,dx. \end{aligned}$$

Then for any $t(x), w(x) \in L^2(0,1)$, we should have

$$\begin{aligned} \langle \frac{\partial L}{\partial u_\epsilon}, w\rangle &= \beta\int_0^1 (u_\epsilon - U)w\,dx + \int_0^1 w\frac{d}{dx}(a_\epsilon(x)\frac{dv}{dx})\,dx \\ &= \int_0^1 \left(\beta(u_\epsilon - U) + \frac{d}{dx}\left(a_\epsilon(x)\frac{dv_\epsilon}{dx}\right)\right) w\,dx = 0, \\ \langle \frac{\partial L}{\partial c}, t\rangle &= \int_0^1 c(x)t(x)\,dx + \int_0^1 v_\epsilon(x)t(x)\,dx \\ &= \int_0^1 (c(x) + v_\epsilon(x))\,t(x)\,dx = 0, \quad . \end{aligned}$$

Therefore,

$$\begin{aligned} \beta(u_\epsilon - U) + \frac{d}{dx}\left(a_\epsilon(x)\frac{dv_\epsilon}{dx}\right) &= 0, \\ c(x) + v_\epsilon(x) &= 0. \end{aligned}$$

i.e.

$$\begin{aligned} -\frac{d}{dx}\left(a_\epsilon(x)\frac{dv_\epsilon}{dx}\right) - \beta u_\epsilon &= -\beta U(x), \\ v_\epsilon(x) &= -c(x). \end{aligned}$$

Hence, $v(x) = -c(x)$ and the optimal problem is equivalent to the following partial differential equation problem,

$$\begin{cases} -\frac{d}{dx}(a_\epsilon(x)\frac{du_\epsilon}{dx}) + v_\epsilon(x) = f(x) & x \in (0,1), \\ -\frac{d}{dx}\left(a_\epsilon(x)\frac{dv_\epsilon}{dx}\right) - \beta u_\epsilon(x) = -\beta U(x) & x \in (0,1) \\ u_\epsilon = 0 & x = 0, 1, \\ v_\epsilon = 0 & x = 0, 1. \end{cases}$$

Now let's look several numerical examples. We solve this partial differential equation with finite difference. The finite difference for this problem is as follows,

$$-\frac{a_{\epsilon,i+\frac{1}{2}}u_{i+1}-(a_{\epsilon,i+\frac{1}{2}}+a_{\epsilon,i-\frac{1}{2}})u_i+a_{\epsilon,i-\frac{1}{2}}u_{i-1}}{h^2}+v_i=f_i, i=1,2,\cdots n-1$$
$$-\frac{a_{\epsilon,i+\frac{1}{2}}v_{i+1}-(a_{\epsilon,i+\frac{1}{2}}+a_{\epsilon,i-\frac{1}{2}})v_i+a_{\epsilon,i-\frac{1}{2}}v_{i-1}}{h^2}-\beta u_i=-\beta U_i, i=1,2,\cdots n-1$$
$$u_0=u_n=0$$
$$v_0=v_n=0,$$

where for any function $g(x)$, $g_i=g(x_i)$ and $0=x_0<x_1<\cdots<x_n=1$ is a uniform grid, with grid spacing $\Delta x=h=1/n$. We will choose a_ϵ from paper by Greéoire Allaire and Robert Brizzi (2004). Given $a_\epsilon=\dfrac{1}{2+1.8\sin(\frac{2\pi x}{\epsilon})}, \beta=100,000$ and $U(x)=\sin(2\pi x), f(x)=x^2$, we will look at several examples with different values of ϵ.

(i) $\epsilon=0.01, \Delta x=\frac{1}{2000}$, the graphs of u and v are as follows,

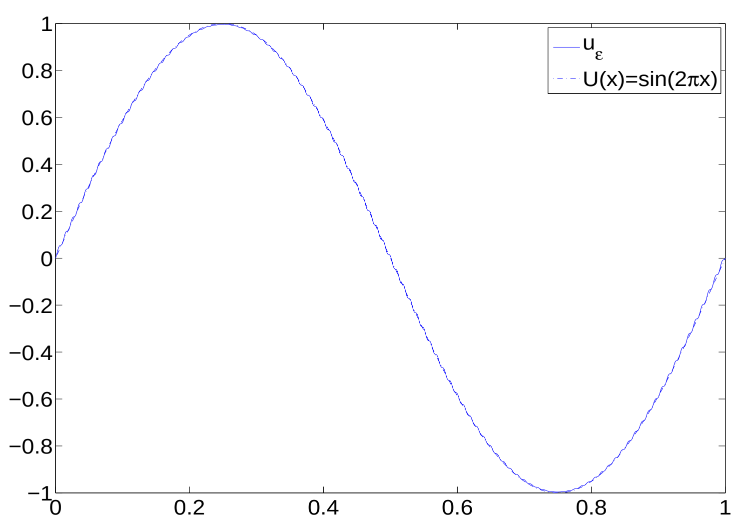

Figure 2.1 The graph of u_ϵ and $U(x)=\sin(2\pi x), 0\le x\le 1, \epsilon=0.01$

(ii) $\epsilon=0.001, \Delta x=\frac{1}{2000}$, the graphs of u and v are as follows,

From graph 2.1 and 2.3, we could find that, the shapes of function u_ϵ and $U(x)=\sin(2\pi x)$ are almost the same, when ϵ is small enough. This special case was studied by Kesavan and

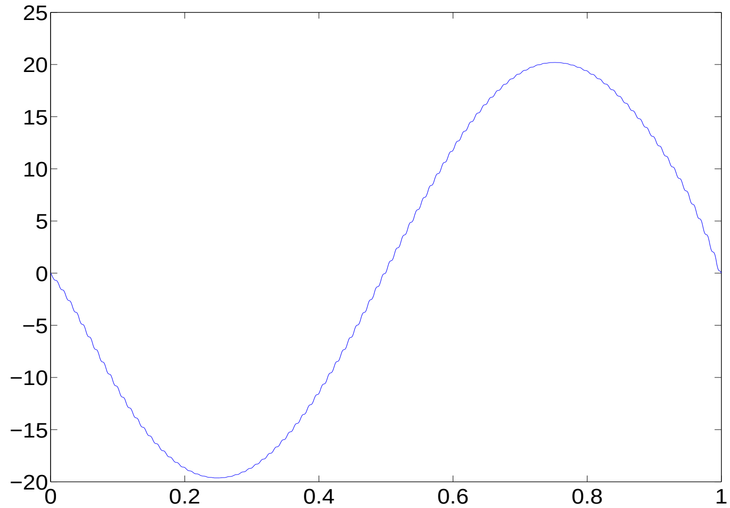

Figure 2.2 The graph of $v_\epsilon, 0 \leq x \leq 1, \epsilon = 0.01$

Vanninathan. They assume that a_ϵ is periodic. For the following problem,

$$\begin{cases} -\frac{d}{dx}(a_0\frac{du_0}{dx}) + v_0(x) = f(x) & x \in (0,1), \\ -\frac{d}{dx}\left(a_0\frac{dv_0}{dx}\right) - \beta u_0(x) = -\beta U(x) & x \in (0,1) \\ u_0 = v_0 = 0 & x = 0, 1, \end{cases}$$

where a_0 is a constant and they proved that a_0 was indeed the limit of a_ϵ in the topology of H-convergence. Also for the periodic a_ϵ of the one-dimensional case, they also gave its limit of H-convergence, which is

$$a_0 = \left[m\left(\frac{1}{a}\right)\right]^{-1},$$

where $m(h) = \int_0^1 h(y)\,dy$ for a periodic function h on [0,1].

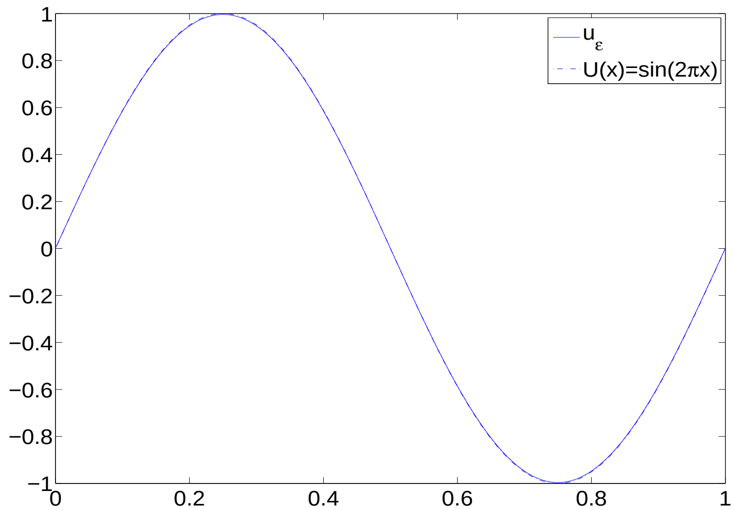

Figure 2.3 The graph of u_ϵ and $U(x) = \sin(2\pi x), 0 \le x \le 1, \epsilon = 0.001$

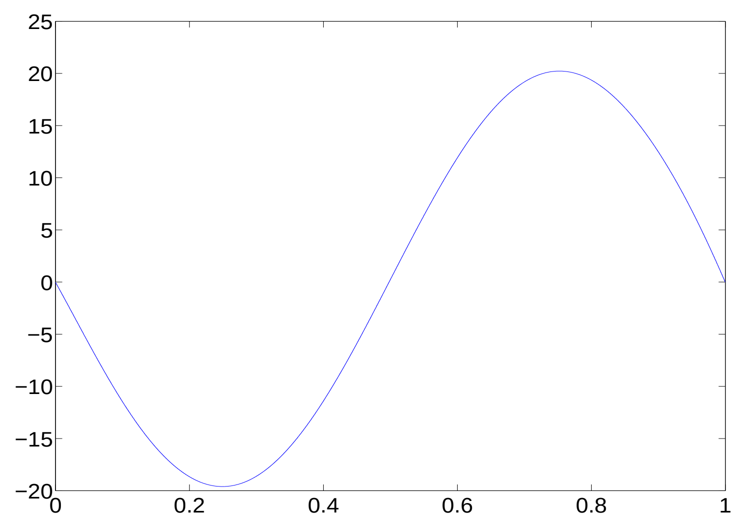

Figure 2.4 The graph of $v_\epsilon, 0 \le x \le 1, \epsilon = 0.001$

CHAPTER 3. Compare the Results for a_ϵ and a_0

We will compare the relationship between

$$\begin{cases} -\frac{d}{dx}(a_\epsilon(x)\frac{du_\epsilon}{dx}) + v_\epsilon(x) = f(x) & x \in (0,1), \\ -\frac{d}{dx}\left(a_\epsilon(x)\frac{dv_\epsilon}{dx}\right) - \beta u_\epsilon(x) = -\beta U(x) & x \in (0,1) \\ u_\epsilon = v_\epsilon = 0 & x = 0, 1, \end{cases}$$

and

$$\begin{cases} -\frac{d}{dx}(a_0\frac{du_0}{dx}) + v_0(x) = f(x) & x \in (0,1), \\ -\frac{d}{dx}\left(a_0\frac{dv_0}{dx}\right) - \beta u_0(x) = -\beta U(x) & x \in (0,1) \\ u_0 = v_0 = 0 & x = 0, 1, \end{cases}$$

with two numerical examples. Like the prior example, let's suppose $a_\epsilon = \dfrac{1}{2 + 1.8\sin(\frac{2\pi x}{\epsilon})}, \beta = 100,000$ and $U(x) = \sin(2\pi x), f(x) = x^2$. Let u_ϵ, v_ϵ denote the numerical solutions of partial differential equations with a_ϵ and u_0, v_0 denote the numerical solutions of partial differential equations with a_0. Hence,

$$a_0 = \left[m\left(\frac{1}{a}\right)\right]^{-1} = \left[\int_0^1 (2 + 1.8\sin(2\pi y))\,dy\right]^{-1} = \frac{1}{2}.$$

We will give several graphs to illustrate the errors between u_ϵ and u_0, v_ϵ and v_0 for different values of ϵ.

(i) $\epsilon = 0.01, \Delta x = \frac{1}{2000}$, the graphs of errors of u_ϵ and v_ϵ are as follows,

(ii) $\epsilon = 0.001, \Delta x = \frac{1}{2000}$, the graphs of errors of u_ϵ and v_ϵ are as follows,

From figure 3.2 and figure 3.6, we can find the oscillation of the error of u_ϵ and u_0. Therefore, we can find a test function, such that u_ϵ is weak convergent to u_0. Analogously, the error of v_ϵ and v_0 also has such oscillation, which means that v_ϵ is also weak convergent to v_0.

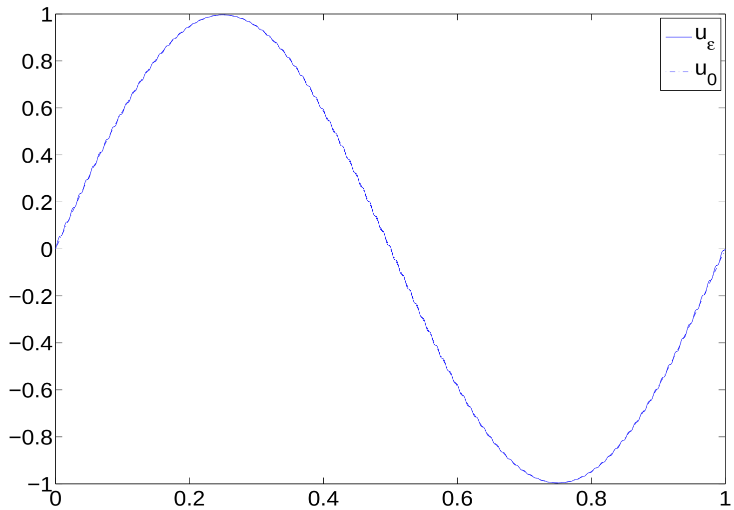

Figure 3.1 The graph of u_ϵ and $u_0, \epsilon = 0.01$

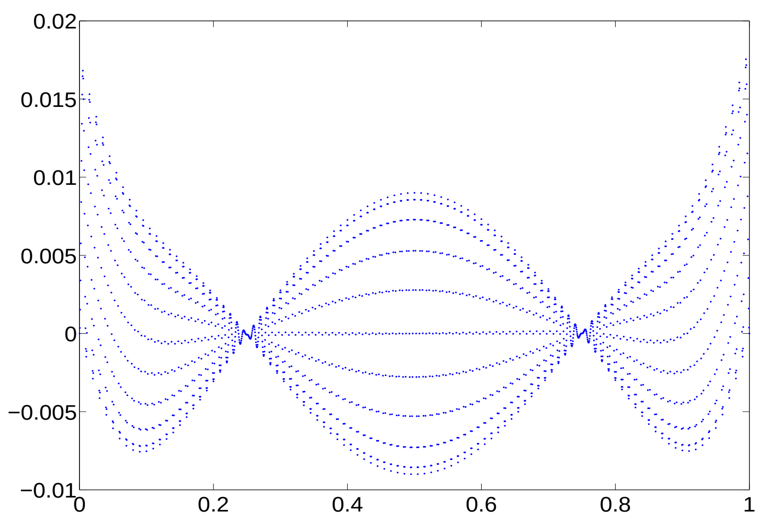

Figure 3.2 The graph of error between u_ϵ and $u_0, \epsilon = 0.01$

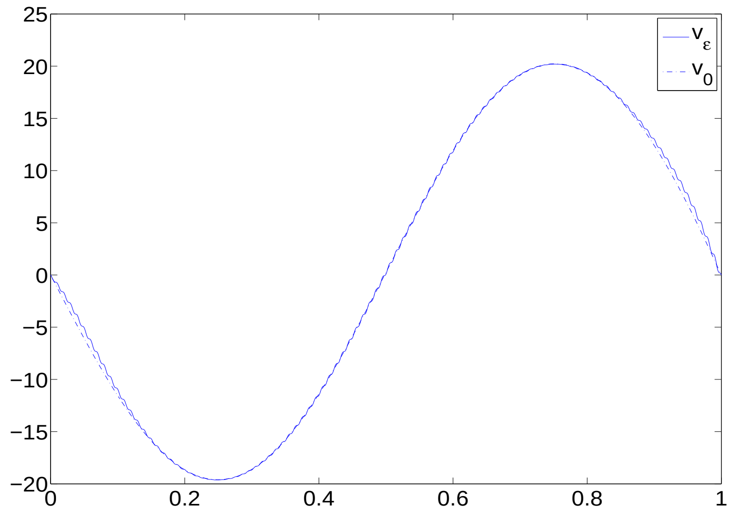

Figure 3.3 The graph of v_ϵ and $v_0, \epsilon = 0.01$

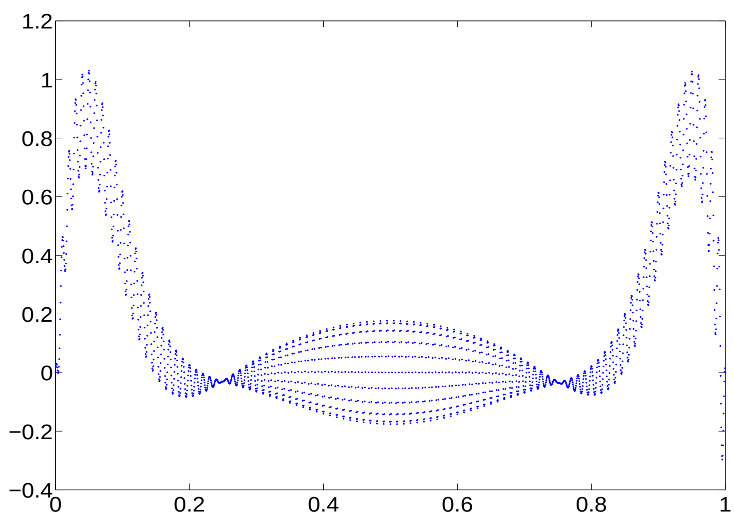

Figure 3.4 The graph of error between v_ϵ and $v_0, \epsilon = 0.01$

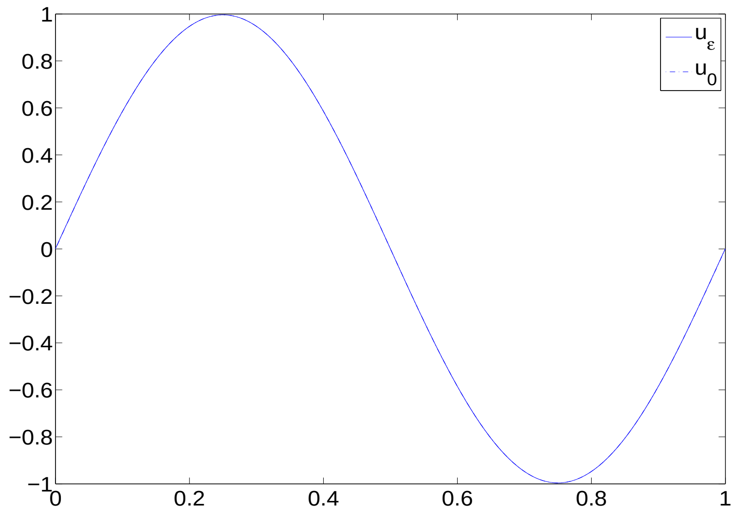

Figure 3.5 The graph of u_ϵ and $u_0, \epsilon = 0.001$

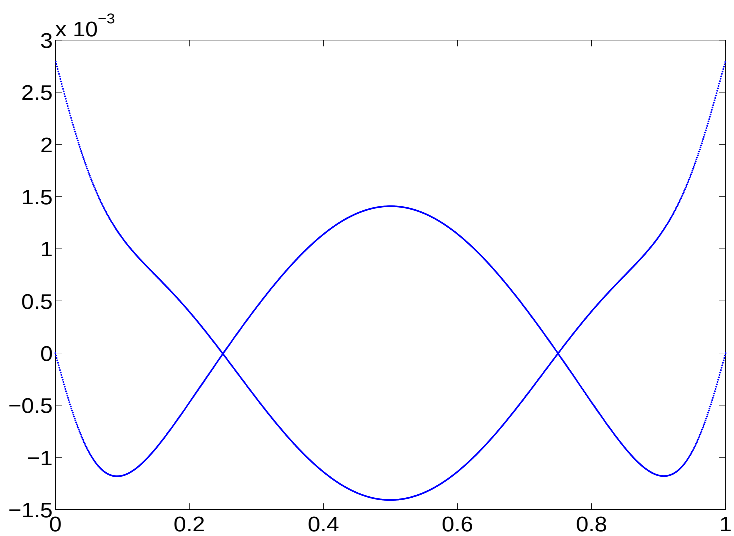

Figure 3.6 The graph of error between u_ϵ and $u_0, \epsilon = 0.001$

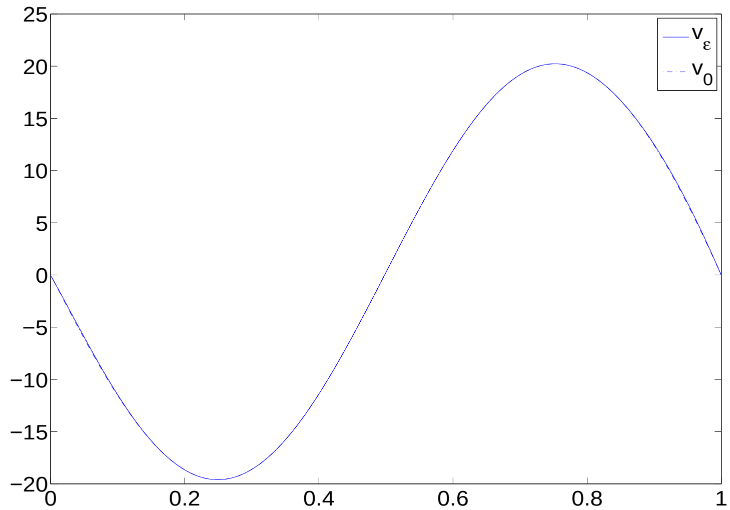

Figure 3.7 The graph of v_ϵ and $v_0, \epsilon = 0.001$

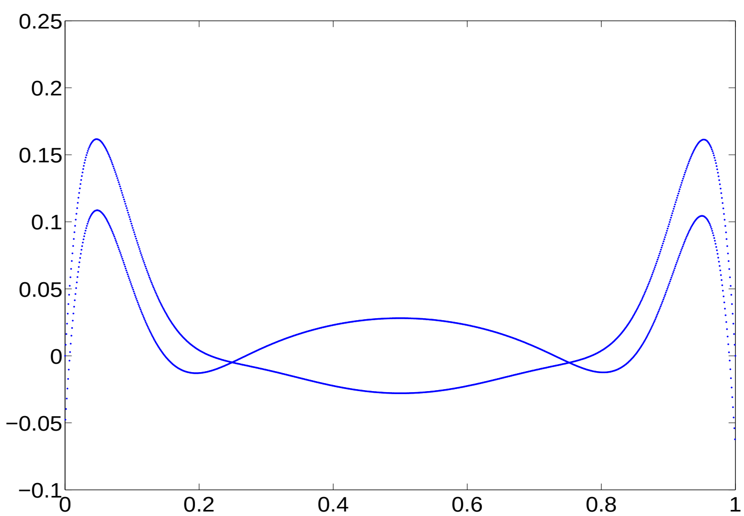

Figure 3.8 The graph of error between v_ϵ and $v_0, \epsilon = 0.001$

CHAPTER 4. Results of Optimal Control Problem

For the optimal control problem

$$-\frac{d}{dx}(a_\epsilon(x)\frac{du_\epsilon}{dx}) = f(x) + c(x), x \in (0,1)$$

for a given $c(x)$, there will be a corresponding u_ϵ. What we want to do is to find a pair of $c(x)$ and u_ϵ, such that

$$L(u,c) = \frac{\beta}{2}\int_0^1 |u_\epsilon - U|^2\,dx + \frac{1}{2}\int_0^1 c^2(x)\,dx$$

can attain its minimum. Since $c(x) = -v(x)$,

$$L(u,c) = \frac{\beta}{2}\int_0^1 |u_\epsilon - U|^2\,dx + \frac{1}{2}\int_0^1 v^2(x)\,dx.$$

For the same given $a_\epsilon = \dfrac{1}{2+1.8\sin(\frac{2\pi x}{\epsilon})}, \beta = 100,000$ and $U(x) = \sin(2\pi x), f(x) = x^2$, after solving the equivalent partial differential equations, we have the following minimum of $L(u_\epsilon, c)$.

(i) Let $\epsilon = 0.01, \Delta x = \frac{1}{2000}$. Then the minimum that $L(u_\epsilon, c)$ attains is

$$\min L(u_\epsilon, c) = \frac{\beta}{2}\int_0^1 |u_\epsilon - U|^2\,dx + \frac{1}{2}\int_0^1 v^2(x)\,dx = 93.32562206282566.$$

(i) Let $\epsilon = 0.001, \Delta x = \frac{1}{2000}$. Then the minimum that $L(u_\epsilon, c)$ attains is

$$\min L(u_\epsilon, c) = \frac{\beta}{2}\int_0^1 |u_\epsilon - U|^2\,dx + \frac{1}{2}\int_0^1 v^2(x)\,dx = 70.54072381983838.$$

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